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# Using graph concepts to assess the feasibility of a sequenced air traffic flow with low conflict rate

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**Abstract :** In this paper, we propose to study feasibility issues of a new air traffic paradigm. In this paradigm, aircraft are following immaterial moving points in such a way that no conflict (or at most few) occurs between aircraft. We provide lower and upper bounds on the maximum density of a solution. In particular, we characterize the density of the solution according to the colorability of an auxiliary graph, modelling the potential conflicts between moving points.

## 1 Introduction

The current air traffic system is based on the definition of trunk routes and predefined cruise flight levels that the aircraft have to respect, and on the use of a division of the airspace into sectors managed by the air traffic controllers. One of the controllers' main tasks is to manage the traffic and to forecast and avoid conflicts, *i.e.* situations where two or more aircraft get too close to each other. This corresponds in the en-route airspace to a distance lower than 5 Nautic Miles (NM) between two aircraft flying at the same flight level. In the current system, conflicts are mainly concentrated around the nodes of the network; the attention of the controllers is thus concentrated on these specific locations.

Research for a new air traffic paradigm is however motivated by the two following facts. First, air traffic has been facing a quasi-continuous growth over the last decades. According to the Performance Review Commission (2008), it is forecast to encounter a 4% increase each year, at least up to 2012. Due to the crisis, this growth will undoubtedly be delayed by one or two years. As a consequence, sectors are threatened with saturation which occurs when the number of aircraft intending to enter a sector during a time period exceeds its capacity. Avoiding saturation is currently achieved by splitting over-crowded sectors into smaller ones or by delaying or rerouting aircraft. Such solutions are not sustainable in the long term, notably because non saturated neighbouring sectors are planned to be rare and because small sectors are harder to monitor than larger ones. Second, the Performance Review Commission (2008) also emphasizes that using a trunk route system induces an average distance lengthening of 5.8% compared to the use of orthodromic or "great circle" routes. The resulting estimated extra distance flown in 2007 is about 450 million kilometers, corresponding to additional  $CO_2$  emissions of about 5 million tons. Such a trunk route organization is thus not compatible with the current necessity of reducing greenhouse gas emissions.

Several approaches are dedicated to the enhancement of this system. On the one hand, some work is based on the use of temporal actions to reduce the conflicts rate; on the other, research is also carried out on the design of structural networks. In this paper, we propose the study of a new air traffic paradigm that takes benefit from both

approaches. In this paradigm, aircraft are compelled to follow fictive regular moving points on routes in order to decrease fuel consumption and conflicts of the whole system. More precisely, we define and analyse a measure, called density, characterizing the traffic potentially supported by this system.

In the next section, we present more precisely the concepts of our paradigm, give a brief review of the different approaches proposed in literature to reduce the number of conflicts in air traffic systems and discuss the advantages of our paradigm. In section 3, we formalize the maximum density problem and explicit the assumptions we make. We also discuss how these assumptions can be met in practise and how we can handle operationnal constraints to apply our paradigm in a real traffic system in section 4. Section 5 introduces two graph representations of the problem: the conflicts graph and the routes graph; then it presents some definitions and properties which are useful to derive results on the maximum density problem. Section 6 is devoted to giving lower bounds on the maximum density we can expect for a graph, while in section 7 we present some upper bounds. Finally in section 8 we give a characterization of graphs admitting what we call a  $\frac{1}{2}$ -dense solution.

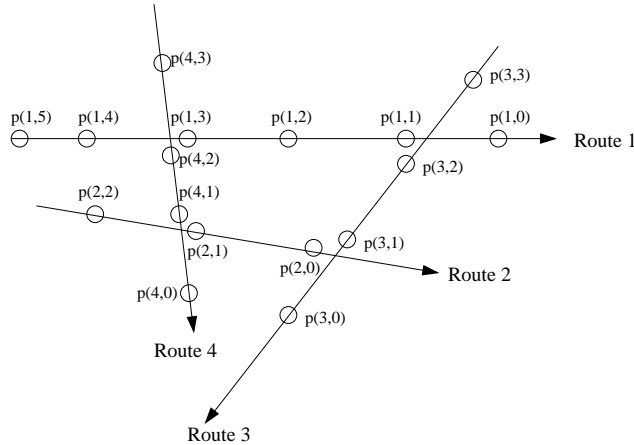
## 2 A new air traffic paradigm

### Overall organization

In this paper, we call *route* a set defined by an origin, a destination, and a 3D-profile. Our goal is to manage a given set of routes, corresponding to the air traffic of a day or a part of it, in such a way that very few conflicts appear. Typically we are interested in a set of orthodromies between airports in order to reduce the total distance and  $CO_2$  emissions. Note however that the vertical profile is not necessarily a cruise flight level, as aircraft may be allowed to fly at their optimal altitude taking advantage in their loss of weight. In the same way, the horizontal profile can be different from the great circle, as aircraft may follow the consumption-optimal route, according to winds for example. The different routes may also corresponds to a pre-specified network. Finally, given an origin and a destination, there may be different routes, mainly because aircraft of different types may not fly at the same altitude.

In this paper, for simplicity, we draw routes as straight lines. In such a system, routes intersections are far more numerous and dispatched all over space, making air traffic controllers' tasks more complex. For this reason, we propose an approach where a part of the traffic, or the totality, is completely organized and aircraft arrival times at intersection points implicitly ensure a nil or at least a very low average conflict rate. Our paradigm can be described as follows. On each route, fictive and immaterial *moving points* are generated at a regular pace. These moving points are represented by circles in figure 1. The points move regularly in the same direction so that two consecutive moving points are not in conflict and are separated by the same time period, called *moving points period*. Moving points of route  $i$  are indexed consecutively so that  $p(i, x)$  denotes the  $x^{th}$  moving point on route  $i$ . There are two types of points: some of them are forbidden, the others are allowed. Aircraft are compelled to precisely follow allowed points during their cruise, and are not authorized to stand outside these allowed moving points. From an operational point of view, a new type of automatic pilot would take care of precisely following a selected moving point. Thus the 4D profile of a moving point has to be compatible with the performances of an aircraft type.

The problem can be decomposed in many subproblems: the first one is to choose horizontal and vertical profiles for each route; the second one is to find the pace for



**Fig. 1.** Routes representation. Moving points are denoted by  $p(i, x)$  where  $i$  stands for the route  $i$  and  $x$  for the  $x^{th}$  moving point on the route.

the moving points and the phase shifts between all the routes; the last one is to assign an allowed/forbidden status to each moving point of each route in such a way that the corresponding traffic volume is in compliance with the forecast one and that the resulting conflict rate is nil (or at least as low as possible considering operational constraints). For instance, in figure 1, moving points  $p(1, 3)$  and  $p(4, 2)$  are in conflict and thus cannot both have the status “allowed”. In this paper, we only deal with the last problem, ie. we suppose that we already have a 3D-profile, a pace, and the phase shifts between the routes. We focus on assignments without conflicts between allowed moving points and we are interested in stationary solutions, since we consider assignments on an infinite time horizon.

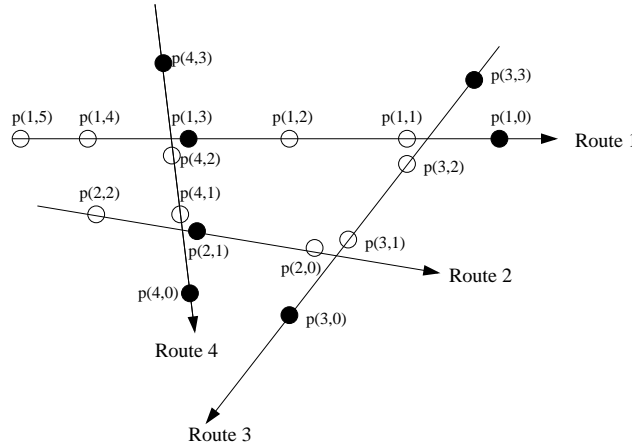
### Density of a solution

The quality of such an allowed/forbidden assignment of moving points can be evaluated in many ways. We can focus on the occupation rate, defined on a route as the number of allowed moving points over the total number of moving points on that route. For an infinite time horizon the occupation rate is defined as the lower limit of this ratio. The mean occupation rate is then the mean of all the occupation rates per route. Maximizing the mean occupation rate would then correspond to maximizing the global network capacity. However this criterion is not really appropriate as it can lead to non-realistic solutions. For instance, maximizing the mean occupation rate in figure 1 leads to an optimal solution where all the moving points of routes 1 and 2 are allowed, while all those of routes 3 and 4 are forbidden (or vice-versa). As a consequence no aircraft can fly along routes 3 and 4, which is clearly not acceptable from a practical point of view.

One may consider maximizing the lowest occupation rates over the routes. Unfortunately the same drawback occurs. Indeed an optimal solution may contain an arbitrarily long sequence of consecutive forbidden moving points, followed by an arbitrarily long sequence of consecutive allowed moving points.

To avoid these non-operational assignments, we introduce the *density* as a new criterion to evaluate a solution. More precisely, we say that a solution has a density of  $\frac{1}{k}$  (or that it is a  $\frac{1}{k}$ -dense solution) if and only if there are at most  $(k - 1)$  consecutive forbidden moving points on any route. It means that, if we consider  $k$  successive moving

points, at least one is allowed. From an operational point of view, it implies that an aircraft will not have to wait on the ground more than  $(k - 1)$  moving point periods before the next allowed moving point. We call an assignment *exactly  $\frac{1}{k}$ -dense* if on each route, each allowed moving point is followed by exactly  $(k - 1)$  forbidden moving points. For instance, the configuration in figure 1 has an exactly  $\frac{1}{3}$ -dense solution, exposed in figure 2, where allowed moving points are represented by dark-filled circles, and forbidden ones by white-filled circles. We choose the density as objective function for the moving points assignment problem in order to obtain operational solutions. The occupation rate will only be a secondary criterion. Notice however that any  $\frac{1}{k}$ -dense solution has an occupation rate of at least  $\frac{1}{k}$ .



**Fig. 2.** A  $\frac{1}{3}$ -dense solution for the routes representation depicted in figure 1.

## Related work

Some recent studies on air traffic management have proposed systems to reduce the conflicts rate. A first approach is short-term speed control. The principle is to prevent a conflict between two aircraft by slowing down or accelerating them before their intersection point. In Constans et al. (2005) the authors propose to minimize potential conflict quantity in a sliding horizon loop process by slightly modifying aircraft speeds. Villiers (2004) recommends a subliminal control approach to resolve conflicts. A possibility would be to slightly modify aircraft speeds, so that controllers do not remark the speed change and that conflicts are avoided.

Another approach consists in designing structural networks to regulate air traffic. These networks can be conflict-free by construction, and should be able to manage a high traffic volume, in the same way as motorways act for car traffic. Irvine and Hering (2007) propose a regular lattice covering the European core area. In this structure, nine consecutive flight levels (FL) are used. On each one, a couple of directions is assigned. Aircraft are compelled to follow these tracks, and thus have to change their flight level if they want to turn. In another paper, Hering (2005) creates two freeways (from Baltic Sea to Spain, and from Ireland to Turkey) with different high altitude flight levels. Aircraft have to follow specific rules to enter or leave these freeways. The purpose is to control intercontinental and a part of long haul European traffic, separately from short-distance traffic. However as aircraft do not fly directly from their origin to their destination, these approaches lead to additional travel distances.

Other approaches thus consider orthodromic routes. Some papers, see Barnett (2000), evaluate the risk of conflict in a free-flight system. In the free-flight paradigm, the role of controllers is considerably reduced in favour of a larger autonomy of pilots. However such systems can be envisaged only in low traffic areas like the Mediterranean sea, and seem unworkable over Europe. If controllers are still needed for high traffic areas, their tasks must be lightened by limiting the number of conflicts between routes. In Fondacci et al. (1998), different heuristics based on permutation graphs are proposed to assign routes to flight levels. The authors also refined the idea by allowing an aircraft to change its flight level at different points of its route to avoid conflicts. Routes of aircraft are thus cut into successive segments with the same direction but at different flight levels. The problem is then to assign segments to the set of possible flight levels so that no segments intersect, and the number of level changes for aircraft is minimized. These approaches involve well-known NP-hard problems on graphs, such as maximum clique and  $\chi$ -coloring. More specifically they can be placed in the domain of intersection graph theory that we detail next. The same approach is adopted in Barnier and Brisset (2004) where the authors propose to vertically separate intersecting aircraft flows by assigning them different flight levels. Basically if two routes intersect, they have to be assigned to different flight levels to avoid potential conflicts.

Intersection graphs have been studied now for three decades. An *intersection graph* is defined as follows: if  $M$  is a class of sets, the intersection graph of  $M$ , denoted  $IG(M)$ , is a graph with vertex set  $M$  and where an edge exists between two vertices  $M_i$  and  $M_j$  of  $IG(M)$  if and only if the intersection between  $M_i$  and  $M_j$  is not empty. According to  $M$ , intersection graphs have specific names. For example, **STRING** is the class of intersection graphs of curves in the plane, **SEG** is the class of intersection graphs of segments in the plane, **INT** is the class of intersection graphs of intervals on a line (they are also called *interval graphs*). First results on intersection graphs were developed by Ehrlich et al. (1976). They proved that **SEG** is strictly included in **STRING**, and that  $\chi$ -colorability is still NP-hard for **SEG** class. Other works on intersection graphs have been achieved by Kratochvil and Matousek (1994), who established many results on inclusion between different classes of intersection graphs. Other classes of intersection graphs have also been studied by Hlineny (1998) and de Castro et al. (2002). Many problems are still open for intersection graphs, two of them are the following: What is the complexity of CLIQUE for **SEG** graphs ? Are planar graphs in **SEG** ? Scheinerman (1984) conjectures the latter is true in his Ph.D. thesis.

By taking their projection, we can consider ODs as segments in the plane. Assigning different flight levels to ODs that intersect then corresponds to a  $\chi$ -coloring problem in **SEG** class. One drawback of this approach is that the temporality of the traffic is not taken into account: two ODs that intersect are to be assigned to different levels even if only two or three flights a day are scheduled on each route. It leads to a quite dense intersection graph to dispatch to a restricted number of flight levels, while some spatial intersections almost never generate a temporal conflict. Indeed a typical traffic day over Europe involves only about 30,000 flights on 12,000 different ODs, which gives an average of 2.5 aircraft per OD. The opposite approach would be to carefully consider the time schedule of flights on each route to prevent only existing conflicts. However, in addition to the very large size of the optimization problems it would imply, we think that such an approach would not answer to practical air traffic regulation issues. Indeed some delays quite often occur, particularly at airports, due to congestion or weather conditions, generating an important uncertainty on the effective occurrence of conflicts. Thus such a schedule should be robust enough to handle time modifications

on any flight, which seems quite unrealistic considering interactions between flights, or should be updated almost continuously to adapt to the current situation.

In the paradigm presented in this paper, we consider the temporality of the problem through the discretization of routes into moving points. We do believe that this approach is robust to handle perturbations since, in a  $\frac{1}{k}$ -dense solution, at least one allowed moving point is available every  $kT$  units of time on any route. Moreover a delay for an aircraft at take off has no impact on other flights as an allowed moving point is asserted to be conflict-free with other routes. It can only cause another delay if another flight was already planned at the next allowed moving point, which is unlikely on the same OD for quite dense solutions.

Besides we consider that only one flight level is available for each flight and hence do not address in this paper the problem of assigning flight levels to the aircraft. Compared to the flight level assignment approach developed in Fondacci et al. (1998) and Barnier and Brisset (2004), our paradigm maps several flight levels together by taking advantage of the temporality to avoid conflicts. Basically considering only two routes that intersect, the first approach would use two different flight levels, while in ours both routes can be placed at the same level, producing a  $\frac{1}{2}$ -dense solution. As the set of possible flight levels is quite small (less than 20 in practice with today's separation standards), our approach can certainly be integrated in a larger air traffic regulation system to efficiently use some flight levels.

### 3 Formalization of the maximum density problem

In this paper, we address the problem of finding a feasible assignment of maximum density for the moving points of the routes. An assignment is called feasible if no conflict occurs between any two allowed moving points. We recall that an assignment has a density of  $\frac{1}{k}$  if there are at most  $(k - 1)$  consecutive forbidden moving points on any route.

An input of the problem is given by a set of  $n$  routes, specified by the coordinates of their origin and destination and by a 3D-profile. In addition we consider in this paper that the set of moving points for each route has already been specified. We restrict our attention to periodic systems where the moving points on routes are regularly separated by a time period that may depend on the route: a moving point is present every  $T_i$  time units on route  $i$ . As a consequence the set of moving points on route  $i$  is entirely defined by the position  $p_i$  of a moving point at a given instant (say time 0), the speed profile  $v_i$  followed by point on the route, and the time period  $T_i$ . Function  $v_i$  associates the instantaneous speed  $v_i(s)$  to each curvilinear abscissa  $s$  of the trajectory. We call *routes representation* this description of an input of the maximum density problem.

#### MAXIMUM DENSITY PROBLEM

INSTANCE : A set of  $n$  routes, each route  $i$  being associated to a position  $p_i$ , a speed function  $v_i$  and a period  $T_i$ .

SOLUTION : A feasible assignment of the moving points.

MEASURE : Density of the assignement.

The complexity status of the maximum density problem is open. Notice however that we are interested in solving real instances for European air traffic, which commonly consists of more than 10,000 Origin-Destinations (ODs) couples. Hence even if determining the complexity class of the problem is of great interest, in practice we are restricted to very low time complexity algorithms, say linear, quadratic or at most cu-

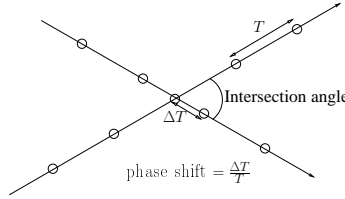
bic in the number of ODs. In addition throughout this paper we make the following assumptions:

**Assumption 1 (Periodicity)** *The moving time period is the same on all the routes. It is denoted by  $T$ .*

As a consequence, if moving points  $p(i, x)$  and  $p(j, y)$  are in conflict, then the couples  $p(i, x + z)$  and  $p(j, y + z)$  are in conflict for every  $z \in \mathbb{Z}$ . This assumption simplifies the structure of the *conflicts graph* introduced in the next section. Notice that instantaneous speeds can be different from one route to another, and even along the same route, as 3D-profile may not consist in following a simple cruise flight level.

**Assumption 2 (Unicity)** *For any two distinct routes  $i$  and  $j$ , a moving point of route  $i$  can be in conflict with at most one moving point of route  $j$ .*

To satisfy this assumption, we need to impose a sufficiently large distance between two consecutive moving points. This minimum distance depends on the angle between routes at their intersection and the phase shift between moving points of different routes, as shown in figure 3. The phase shift is expressed as a ratio of the time interval  $\Delta T$  to the moving points period  $T$ ,  $\Delta T$  being the time interval between two consecutive moving points of different routes at an intersection.



**Fig. 3.** Phase shift and intersection angle

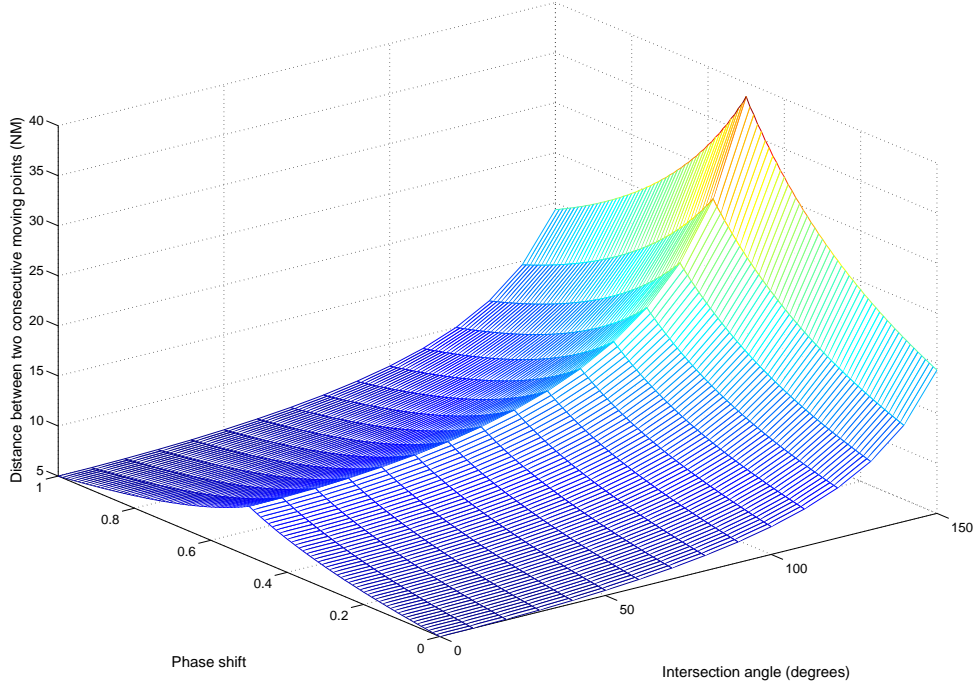
To respect the separation distance  $D_s$  of 5NM, supposing that aircraft fly at the same speed, we determine that a distance of 20NM is sufficient for angles up to  $120^\circ$ . This minimum distance increases to 30NM to deal with angles up to  $140^\circ$ . For an usual cruise speed of 500NM per hour, the corresponding time period  $T$  does not exceed 5 minutes, which seems quite reasonable in practice. Figure 4 gives the necessary distance  $d$  between two consecutive moving points in order to respect assumption 2. This distance  $d$  is a function of the intersection angle  $\theta$  and the phase shift  $\frac{\Delta T}{T}$  between the two routes, and can be expressed as follows:

$$d = \frac{D_s}{\max\left(\frac{\Delta T}{T}; 1 - \frac{\Delta T}{T}\right) \cos \frac{\theta}{2}}$$

Notice that assumptions 1 and 2 imply that if a conflict occurs at the intersection of routes  $i$  and  $j$  between moving points  $p(i, \alpha)$  and  $p(j, \beta)$ , then each moving point  $p(i, x)$  on route  $i$  is in conflict with moving point  $p(j, x + \beta - \alpha)$  on route  $j$ , and only this moving point. We denote by  $\Delta_{ij}$  the quantity  $\beta - \alpha$ , i.e. the index difference of any two moving points in conflict at the intersection of routes  $i$  and  $j$ . For instance on the routes representation of figure 1, we have  $\Delta_{1,4} = -1$  and  $\Delta_{2,3} = 1$ .

In the next section, we bring some answers to the questions relative to an operational use of our paradigm.





**Fig. 4.** Minimum distance (NM) between two consecutive moving points to respect assumption 2

## 4 Operational implementation issues

Many questions arise when we are looking at the paradigm from an operational view point. We try to answer to the most important ones in this section.

First of all, it is really important to understand that the assumptions on which our paradigm is based are realistic. First, we suppose that there is a time period  $T_i$  for each route, and the periodicity assumption implies that there is only one period  $T$  for all the routes. The fact that moving points are regularly separated on a route implies that all the moving points of a route have the same 4D-profile. This is not at all unrealistic, since given an Origin-Destination and a flight level, there are very few aircraft, and hence even less different types of aircraft; moreover, most of the time, they have quite the same performances, implying that they can follow the same routes. Finally, the routes set can be composed of several routes for a given Origin-Destination, each route using different aircraft performances. In order to satisfy the second assumption (the unicity assumption), we just need a sufficient distance  $d$  between two consecutive moving points, and this distance is computed in the previous section. The distance is related to the maximum angle of a routes intersection. Too big angles may induce the non-respect of this assumption. However, we can partition the set of routes in such a way that no angle above a threshold occurs in any set of the partition. It is obvious that two aircraft following opposite routes may be a problem. We hence put them on two different flight levels, as it is the case presently.

One may ask how this paradigm deals with uncertainty: is an aircraft able to precisely follow a moving point on which it is assigned? Many causes are sources of uncertainty. To take winds into account, one first can use the dominant winds, which

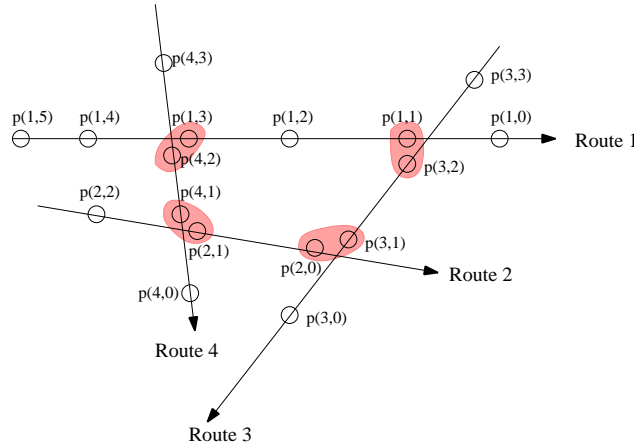
are predictable the day before, in order to adapt speeds of moving points on the routes. This does not drop all the uncertainty, but reduces it. To deal with medium amplitude uncertainty, we can define the speeds of moving points far enough from upper and lower bounds of aircraft performances, in order aircraft to accelerate or decelerate to follow moving points. For small-size uncertainty and to cope with the difficulty to follow exactly the moving points due to position incertitudes, we may define a bigger separation distance  $D_s$  between moving points. For example, if this distance is 7NM (instead of 5NM), it means that aircraft have to follow a moving point within the distance interval  $[-1NM, 1NM]$ . It is important to note that the minimum distance between two consecutive moving point is linear with the separation distance  $D_s$ . Harder to predict meteorological phenomena, like storms, are much more delicate. We may envisage that aircraft are guided by controllers through routes, or that they are forced to join a free allowed moving point on a route bypassing the storm area.

Finally, routes might be redefined between two consecutive days. We assume that air traffic controllers are in charge of transferring the traffic from one day to the next one by asking pilots to slightly modify aircraft speed and /or trajectory in order to catch (or be caught by) a free allowed moving point.

The next section presents two graph representations for the maximum density problem, namely the conflicts graph and the routes graph.

## 5 Routes graph and conflicts graph

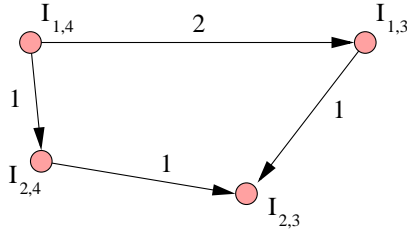
We introduce in this section two graphs associated with the maximum density problem. The *conflicts graph* represents the conflict relations between pairs of moving points, and hence is an infinite (but periodic) graph. The conflicts graph is useful to represent a solution. Opposingly the *routes graph* is a condensed representation of the problem, at most quadratic in the number of routes, which allows us to analyze the structure of the conflicts graph. In particular we exhibit a relation between the cycles in the routes graphs and the connected components in the conflicts graphs. The definitions below are illustrated by figures 6 and 7, relative to the routes representation given in figure 5.



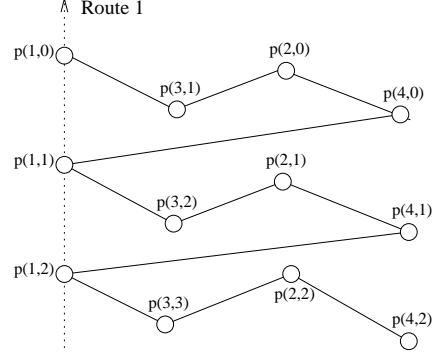
**Fig. 5.** A routes representation of a maximum density problem. The pair of reference moving points for each intersection is encircled with a dark zone.

Consider a routes representation, such as that in figure 5. For each intersection  $I_{i,j}$  where at least one conflict occurs between two moving points from routes  $i$  and  $j$ ,

we arbitrarily choose a couple of moving points in conflict at this intersection. These moving points are called *reference moving points* of route  $i$  (resp.  $j$ ) for intersection  $I_{i,j}$ . We call *routes graph* the graph  $G^A = (V^A, A^A)$  where each vertex stands for a routes intersection where a conflict between two moving points occurs. An arc  $a = (I_{i,j}, I_{i,k})$  belongs to  $A^A$  if intersections  $I_{i,j}$  and  $I_{i,k}$  are consecutive on route  $i$ . We say that arc  $a$  is associated with route  $i$ . Denoting by  $r_{i,l}$  the index of the reference moving point of route  $i$  for intersection  $I_{i,l}$ , the weight of arc  $a$  is then defined as  $w_a = r_{i,k} - r_{i,j}$ , which is the number of moving points' periods on route  $i$  from reference moving point of route  $i$  at  $I_{i,j}$  to reference moving point of route  $i$  at  $I_{i,k}$ . Figure 6 gives the routes graph of the routes representation represented in figure 5.



**Fig. 6.** Routes graph for routes representation defined in figure 5.



**Fig. 7.** Conflicts graph for routes representation defined in figure 5.

We call *conflicts graph* the graph  $G^C = (V^C, E^C)$  where  $V^C$  is the set of moving points; there is an edge between two moving points if and only if there is an instant when these two points generate a conflict. Figure 7 gives the conflicts graph of the routes representation defined in figure 5. Clearly a feasible assignment corresponds to an independent set in the conflicts graph. Notice that since we investigate in this paper solutions on an infinite time horizon, the conflicts graph associated with the routes representation is an infinite graph.

### 5.1 Cycles of the routes graph

Recall that an arc  $a = (I_{i,j}, I_{i,k})$  belongs to  $A^A$  if intersections  $I_{i,j}$  and  $I_{i,k}$  are consecutive on route  $i$ . Denoting by  $r_{i,l}$  the index of the reference moving point of route  $i$  for intersection  $I_{i,l}$ , the weight of arc  $a$  is then defined as  $w_a = r_{i,k} - r_{i,j}$ .

Given a cycle  $\mathcal{C}$  in the routes graph together with an orientation, we call *cycle weight*  $w(\mathcal{C})$  of  $\mathcal{C}$  the sum of the weights of the arcs in the cycle following the orientation, minus the sum of the weights of the arcs whose direction is opposite to the orientation. Introducing the indicator function  $\delta(a, \mathcal{C})$ , equal to 1 if arc  $a$  has the same orientation as  $\mathcal{C}$ , to  $-1$  if it has the opposite orientation, and to 0 if it does not belong to the cycle, we have:

$$w(\mathcal{C}) = \sum_{a \in A^A} \delta(a, \mathcal{C}) w_a$$

Clearly, choosing the opposite orientation for the cycle leads to the opposite weight. Thus we can define the *positive weight* of a cycle, corresponding to the orientation giving a positive weight. In figure 6, we represent the routes graph of the routes representation

of figure 5. It is an elementary cycle, whose weight in the clockwise orientation is equal to  $2 + 1 - 1 - 1 = 1$ . This notion of cycle weight has a central role for understanding the structure of the conflicts graph. Apparently the definition looks quite incorrect, as the weight of an arc in the routes graph depends on the (arbitrary) choice of the couple of reference moving points for each intersection. It appears that it is not the case:

**Lemma 1** *The cycle weight of a cycle  $\mathcal{C}$  in the routes graph is independent of the choice of the reference moving points.*

*Proof.* Let  $\mathcal{C}$  be an oriented elementary cycle of the routes graph. Let  $e_0, \dots, e_{l-1}$  be the edges of  $\mathcal{C}$  indexed consecutively following the orientation of the cycle. By definition, each edge  $e_i$  corresponds to an arc  $a_i$  of  $G^A$ . In short we denote  $\delta_i$  the value of the indicator function  $\delta(a_i, \mathcal{C})$ . With our notation we thus have  $w(\mathcal{C}) = \sum_{i=0}^{l-1} \delta_i w(a_i)$ . For the sake of simplicity we assume that each arc  $a_i$  is associated to route  $i$ . Now let  $r_{i,i+1}$  and  $r_{i+1,i}$  be the indices of the reference moving points for intersection  $I_{i,i+1}$  on routes  $i$  and  $i+1$ , respectively (indices will be written modulo  $l$  to lighten the notations). If arc  $a_i$  follows the orientation of  $\mathcal{C}$ , its weight is by definition  $r_{i,i+1} - r_{i,i-1}$ . Otherwise it is the opposite quantity. Thus we simply have  $w(a_i) = \delta_i(r_{i,i+1} - r_{i,i-1})$ . Replacing this expression in the weight of the cycle, we get:

$$w(\mathcal{C}) = \sum_{i=0}^{l-1} \delta_i^2 (r_{i,i+1} - r_{i+1,i}) = \sum_{i=0}^{l-1} (r_{i,i+1} - r_{i+1,i})$$

By definition, moving points  $p(i, r_{i,i+1})$  and  $p(i+1, r_{i+1,i})$  are in conflict. Due to hypotheses 1 and 2 of periodicity and unicity,  $p(i, r_{i,i+1})$  is in conflict on route  $i+1$  only with moving point  $p(i+1, r_{i,i+1} + \Delta_{i,i+1})$ . Thus  $w(\mathcal{C}) = \sum_{i=0}^{l-1} -\Delta_{i,i+1}$ , and hence is independent of the choice of the reference moving points.

Let us focus in particular on the smallest strictly positive cycle weight in the routes graph. We introduce the following definition:

**Definition 1.** *We define  $w(G^A)$  as the smallest strictly positive cycle weight in routes graph  $G^A$ .*

We show in the next section that  $w(G^A)$  is nothing but the number of connected components of the conflicts graph.

## 5.2 Connected components of the conflicts graph

In this section, we highlight the relationship between the routes graph  $G^A$  and the conflicts graph  $G^C$ . For the sake of simplicity we assume that the routes graph is connected. Otherwise we can restrict our discussion to a connected component of  $G^A$ . Due to assumption 1 the conflicts graph is an infinite and periodic graph. In particular we can make the two following remarks:

**Remark 1** *If there exists a path in  $G^C$  between moving points  $p(i, x)$  and  $p(j, y)$ , then there exists a path in  $G^C$  between  $p(i, x + z)$  and  $p(j, y + z)$  whatever  $z \in \mathbb{Z}$ .*

*Proof.* Consider a path  $(p(i_1, x_1), \dots, p(i_k, x_k))$  in the conflicts graph. As all the routes have the same moving point period  $T$ , if  $p(i_1, x_1)$  and  $p(i_2, x_2)$  are in conflict, then  $p(i_1, x_1 + 1)$  and  $p(i_2, x_2 + 1)$  are also in conflict. Applying that property to the whole path, it proves that there exists a path in  $G^C$  between  $p(i_1, x_1 + 1)$  and  $p(i_k, x_k + 1)$ . The result immediately follows.

**Remark 2** *There exists a path in the conflicts graph between two moving points  $p(i, x)$  and  $p(i, y)$  for any route  $i$  if and only if there exists a path between two moving points  $p(i_0, x)$ ,  $p(i_0, y)$  of a given route  $i_0$ .*

*Proof.* The necessary part is obvious, we only have to prove the sufficient one. Let  $P$  be a path between  $p(i_0, x)$  and  $p(i_0, y)$ . We know that there exists  $z$  so that there is a path  $P_1$  from  $p(i_0, x)$  to  $p(i, z)$  because the routes graph is connected. Taking the isomorphic path to  $P_1$  starting from  $p(i_0, y)$ , we find a path  $P_2$  from  $p(i_0, y)$  to  $p(i, y + z - x)$ . Combining  $P_1$ ,  $P$  and  $P_2$ , we obtain a path from  $p(i, z)$  to  $p(i, y + z - x)$ . Applying remark 1, we find a path from  $p(i, x)$  to  $p(i, y)$ .

Without loss of generality, we assume that there exists a cycle of weight  $w(G^A)$  including a moving point of route 1. We then denote by  $C_x$  the connected component of moving point  $p(1, x)$  in the conflicts graph, whatever  $x$ . Due to the periodicity of moving points, all components  $C_x$  are isomorphic copies of  $C_0$ . This is stated in the following property:

**Property 1** *All connected components  $C_x$  are isomorphic.*

*Proof.* Let  $x$  be an integer. Consider the bijection  $\varphi : p(i, y) \mapsto p(i, y - x)$ . Due to assumption 1, two moving points  $p$  and  $q$  are adjacent if and only if  $\varphi(p)$  and  $\varphi(q)$  are adjacent. To conclude that  $C_x \equiv C_0$ , we simply use remark 1 which implies that  $p(i, y) \in C_x$  if and only if  $p(i, y - x) \in C_0$ .

If all  $C_x$  are isomorphic copies of  $C_0$ , we can wonder how many connected components are present in the conflicts graph. Recall that  $G^C$  is an infinite graph, and thus answering this question is not immediate. The following property is the keystone of our analysis:

**Property 2** *There exists a path in the conflicts graph between two moving points  $p(i, x)$  and  $p(i, y)$  for any route  $i$  if and only if there exists a cycle in the routes graph of positive weight  $|y - x|$ .*

*Proof.* The proof is illustrated by figure 8. Using lemma 1, the cycle weights are independent of the choice of the reference moving points in the routes graph. First consider a path of length 2,  $(p(a, \alpha), p(b, \beta), p(c, \gamma))$ , in the conflicts graph  $G^C$ . Since  $p(a, \alpha)$  and  $p(b, \beta)$  are in conflict, there exists an intersection  $I_{ab}$  in the routes graph. For the same reason there also exists an intersection  $I_{bc}$ . Notice that these 2 intersections may not be consecutive on route  $b$ . However, by definition, an arc exists in  $G^A$  between two consecutive intermediate intersections, and hence we have a path  $\mathcal{P}$  from  $I_{ab}$  to  $I_{bc}$ . The weight of  $\mathcal{P}$ , defined as the summation of its arcs, depends only on our choice of reference moving points for intersections  $I_{ab}$  and  $I_{bc}$ , since indices of intermediate references eliminate 2 by 2 in the summation. We choose the couple  $(p(a, \alpha), p(b, \beta))$  as reference moving points for intersection  $I_{ab}$  and the couple  $(p(b, \beta), p(c, \gamma))$  for intersection  $I_{bc}$ . The weight of  $\mathcal{P}$  is then  $p(b, \beta) - p(b, \beta) = 0$ .

More generally, let us consider an elementary path in the conflicts graph between  $p(i, x)$  and  $p(i, y)$ , say  $\mathcal{Q} = (p(i, x), p(j_1, z_1), p(j_2, z_2), \dots, p(j_n, z_n), p(i, y))$ . By analogy from the case of a path of length 2, there exists a path  $\mathcal{P}$  in the routes graph between  $I_{i, j_1}$  and  $I_{j_n, i}$  whose weight is null. This corresponds for each intersection  $I_{j_k, j_{k+1}}$  to choose  $p(j_k, z_k)$  and  $p(j_{k+1}, z_{k+1})$  as reference moving points. For intersection  $I_{i, j_1}$  (resp.  $I_{j_n, i}$ ) reference moving points are  $p(i, x)$  and  $p(j_1, z_1)$  (resp.  $p(j_n, z_n)$  and  $p(i, y)$ ). Let us then form a cycle  $\mathcal{C}$  by adding to  $\mathcal{P}$  the arcs along route  $i$  from  $I_{i, j_n}$  to  $I_{i, j_1}$ . By

definition the weights of these arcs sum up to the difference of time periods  $T$  between the associated reference moving points on route  $i$ , which are  $p(i, x)$  and  $p(i, y)$ . The weight of  $\mathcal{C}$  is thus equal to  $(x - y)$ . Taking the opposite orientation leads to a cycle of weight  $(y - x)$ , which permits to conclude that there is a cycle of weight  $|y - x|$ .

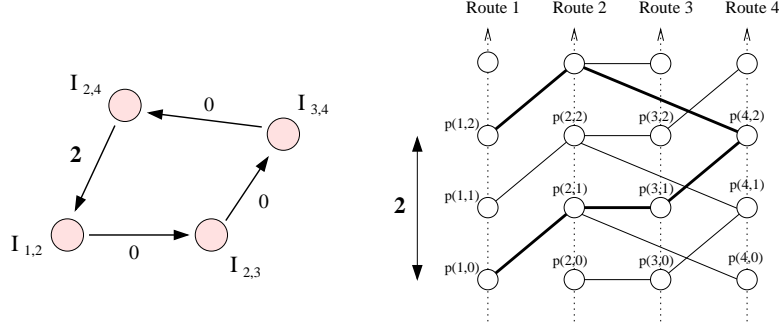


Fig. 8. Property 2

Conversely consider an elementary cycle  $\mathcal{C}$  of positive weight  $w(\mathcal{C})$  in  $G^A$  including an arc associated with a given route  $i_0$ . For the sake of simplicity we can reindex the routes in such a way that  $\mathcal{C} = (I_{i_0, i_0+1}, \dots, I_{k, i_0})$ . Consider now moving point  $q_{i_0} = p(i_0, 0)$ . Since intersection  $I_{i_0, i_0+1}$  exists in  $G^A$ , this point is in conflict with  $q_{i_0+1} = p(i_0 + 1, \Delta_{i_0, i_0+1})$ . In its turn  $q_{i_0+1}$  is in conflict with  $q_{i_0+2} = p(i_0 + 2, \Delta_{i_0, i_0+1} + \Delta_{i_0+1, i_0+2})$ . We exhibit in this way a path  $\mathcal{Q} = (q_{i_0}, \dots, q_{k+1})$  in  $G^C$  with  $q_{k+1}$  the moving point on route  $i_0$  of index  $\Delta_{i_0, i_0+1} + \dots + \Delta_{k-1, k} + \Delta_{k, i_0}$ . Proof of lemma 1 established that this is precisely  $w(\mathcal{C})$ . Thus there exists a path between  $p(i_0, 0)$  and  $p(i_0, w(\mathcal{C}))$  in the conflicts graph. Remark 1 permits us to conclude that there exists a path in  $G^C$  between  $p(i_0, x)$  and  $p(i_0, x + w(\mathcal{C}))$ . Remark 2 proves that there exists a path in  $G^C$  between  $p(i, x)$  and  $p(i, x + w(\mathcal{C}))$  for any route  $i$ .

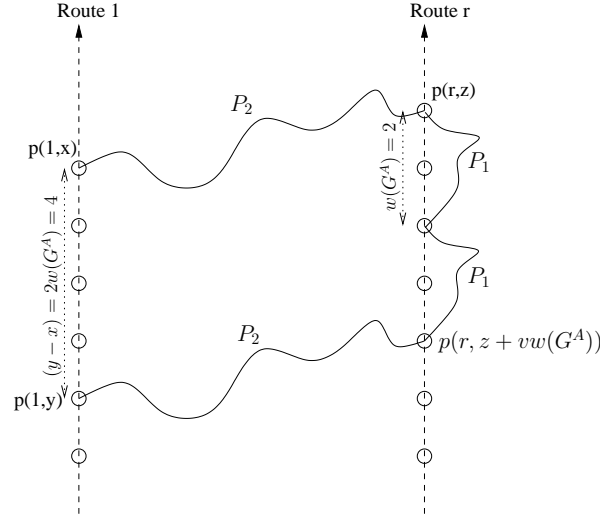
As a direct consequence the conflicts graph  $G^C$  has exactly  $w(G^A)$  different connected components including a moving point of route 1:

**Theorem 1** *For any indices  $x$  and  $y$ , connected components  $C_x = C_y$  if and only if  $x$  equals  $y$  modulo  $w(G^A)$ . Hence the conflicts graph admits exactly  $w(G^A)$  connected components.*

*Proof.* From property 2, we know that there exists a path in the conflicts graph between any moving points  $p(1, x)$  and  $p(1, x + w(G^A))$ . Thus  $C_x = C_y$  if  $x = y$  modulo  $w(G^A)$ .

Conversely if  $C_x = C_y$  for some indices  $x < y$ , by definition there exists a path  $P_0$  between  $p(1, x)$  and  $p(1, y)$ . Let route  $r$  be a route belonging to the smallest strictly positive weight cycle in routes graph  $G^A$ . For clarity of figure 9 we assume that  $r \neq 1$ . Using property 2, we can deduce that there is a path  $P_1$  in the conflicts graph between any moving points  $p(r, z)$  and  $p(r, z + w(G^A))$ , for all  $z \in \mathbb{Z}$ . Since there exists in routes graph  $G^A$  a cycle (one of weight  $w(G^A)$ ) passing by route 1 and  $r$ , there exists a path  $P_2$  in conflicts graph  $G^C$  between  $p(1, x)$  and some moving point  $p(r, z)$  of route  $r$ . Let  $d = \gcd((y - x), w(G^A))$  be the greatest common divisor of the two weights. Applying Bezout's identity, there exists 2 integers  $u$  and  $v$  such that  $d = u(y - x) + vw(G^A)$ . We now construct the following path in the conflicts graph (see figure 9):

- Starting from  $p(1, x)$ , we take the path  $P_2$  to  $p(r, z)$ .



**Fig. 9.** Case  $w(G^A) = 2$  and  $y - x = 4$  of theorem 1

- We go to  $p(r, z + vw(G^A))$  using  $v$  times path  $P_1$ .
- We go back on route 1 using the converse of path  $P_2$ , which means that we are at the moving point  $p(1, x + vw(G^A))$ .
- We go to  $p(1, x + d)$  using  $u$  times path  $P_0$ .

In this way, we obtained a path between  $p(1, x)$  and  $p(1, x + d)$  in the conflicts graph, which implies, due to property 2, the existence of a cycle of weight  $d$  in the routes graph. On the one hand, by definition of  $w(G^A)$  as the smallest positive weight, we have  $w(G^A) \leq d$ . On the other hand, as  $d$  is the greatest common divisor of  $w(G^A)$  and  $(y - x)$ , it is strictly inferior to  $w(G^A)$ , unless  $(y - x)$  is a multiple of  $w(G^A)$ . Therefore we have  $d = w(G^A)$  and thus  $(y - x)$  is a multiple of  $w(G^A)$ . That is to say,  $x$  equals  $y$  modulo  $w(G^A)$ .

We know that  $w(G^A)$  represents the number of (isomorphic) connected components in a connected conflicts graph. For this reason this quantity will also be denoted as  $w(G^C)$ . If routes representation has more than one connected component, we can work separately on each connected component and reduce to our case.

## 6 Lower bounds for the maximum density of a graph

The aim of this section is to exhibit feasible assignments on general instances of the maximum density problem, which provide lower bounds for the maximum density we can expect. A first, quite straightforward, result is given in theorem 2:

**Theorem 2** *For any routes representation with  $n$  routes, there exists a  $\frac{1}{n}$ -dense solution.*

*Proof.* The proof uses a construction. We exhibit an exact  $\frac{1}{n}$ -dense solution, that is to say a solution where, for each route, 1 moving point out of  $n$  has the allowed status, the others having the forbidden status. Such a solution is entirely determined by giving one allowed moving point per route in the index interval  $[1, n]$ . Indeed an exact  $\frac{1}{k}$ -dense solution is necessarily periodic, of period  $k$ . We just need to check that the chosen allowed moving points are conflict-free. We proceed as follows:

- We choose (arbitrarily) an exactly  $\frac{1}{n}$ -dense solution for route 1. For the sake of simplicity, let us choose moving point  $p(1, 1)$ , and subsequently all moving points  $p(1, 1 + nz)$  for  $z \in \mathbb{Z}$  as allowed moving points. Each route  $j$  intersecting route 1 has then forbidden moving points. If  $p(1, 1)$  is in conflict with  $p(j, x)$ , the set of moving points of route  $j$  in conflict with an allowed moving point of route 1 is  $\{p(j, x + nz) \mid z \in \mathbb{Z}\}$  due to assumptions 1 and 2. Thus an exactly  $\frac{1}{n}$ -dense solution for route 1 may forbid at most one index in interval  $[1, n]$  per route.
- More generally for the  $i$ th route we define a  $\frac{1}{n}$ -dense solution by choosing (arbitrarily) an available index  $x \in [1, n]$ . Notice that the  $i - 1$  previous steps may have forbidden at most  $i - 1$  moving points in index interval  $[1, n]$ , thus such a choice is always possible.

This procedure constructs an exactly  $\frac{1}{n}$ -dense solution for a routes representation with  $n$  routes.

Theorem 2 gives a lower bound on the density depending on the number of routes in the problem. We now underline the relation between the  $\frac{1}{k}$ -density and the chromatic number  $\chi$  of the conflicts graph.

**Theorem 3** *If the conflicts graph is  $\chi$ -colorable and  $w(G^A) \geq \chi$ , then there exists a  $\frac{1}{2\chi-1}$ -dense solution.*

*Proof.* Assume that there exists a proper coloring  $\mu : V^C \rightarrow [0, \chi - 1]$  for conflicts graph  $G^C$ . We use this coloring  $\mu$  to color connected component  $C_0$ . Figure 10 shows an example of coloration of  $C_0$  for a 3-colorable conflicts graph, where color 0 appears as black, color 1 as dashed gray and color 2 as light gray. Notice that, by definition of a connected component, we can choose independently a proper coloring  $\lambda_\alpha$  on each  $C_\alpha$  to obtain a valid coloring  $\lambda$  for the whole conflicts graph. Recall that all  $C_\alpha$  are isomorphic to  $C_0$ , and more precisely that  $\varphi_\alpha : p(i, x) \mapsto p(i, x - \alpha)$  is a bijection from  $C_\alpha$  to  $C_0$ . Thus for any permutation  $\pi$  of  $[0, \chi - 1]$ , the function  $\pi \circ \mu \circ \varphi_\alpha$  is a proper coloring of  $C_\alpha$ . We choose for permutation  $\pi$  the translation  $\varphi_\alpha^{-1}$  modulo  $\chi$ :

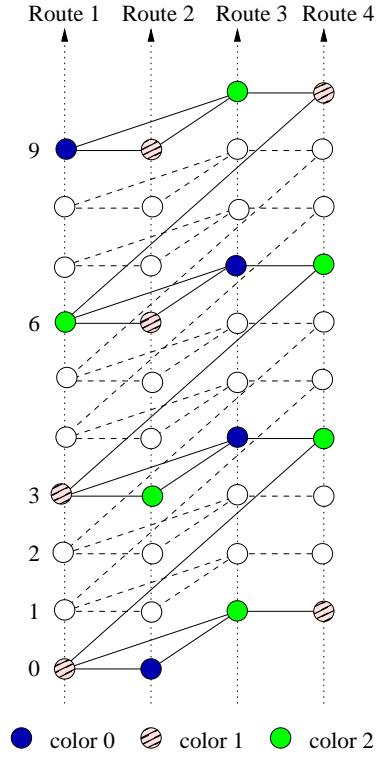
$$\text{For all } p(i, x) \in C_\alpha, \text{ we set } \lambda(p(i, x)) = [\mu(p(i, x - \alpha)) + \alpha] \bmod \chi$$

In simpler words, color  $(c + \alpha) \bmod \chi$  in component  $C_\alpha$  plays the same role as color  $c$  in component  $C_0$ . Figure 11 illustrates the resulting coloring of  $C_1$  and  $C_2$  with respect to the coloring of  $C_0$  in figure 10.

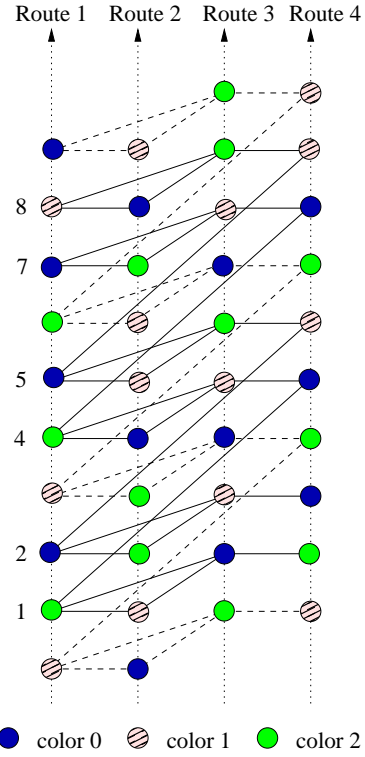
To define an assignment, we only pick one color  $c$ , i.e. we give the allowed status to all the moving points colored in  $c$ . By definition this assignment is valid since each color defines an independent set of  $G^C$ . For instance in figure 11 we can choose the moving points of color 0. The resulting solution is  $\frac{1}{3}$ -dense on routes 1 and 4, but only  $\frac{1}{4}$ -dense on routes 2 and 3. Choosing color 1 leads to a solution only  $\frac{1}{5}$ -dense, due to route 1.

We show that such an assignment is at least  $\frac{1}{2\chi-1}$ -dense whatever the color  $c$  we choose. A sufficient condition is that each color is present at least once for any sequence of  $(2\chi - 1)$  consecutive moving points. We can make the following remark: consider an index interval  $[y, y + \chi - 1]$  of length  $\chi$ . Then coloring  $\lambda$  on the sequence  $p(i, y), \dots, p(i, y + \chi - 1)$  is bijective if no moving point, except possibly  $p(i, y)$ , belongs to  $C_0$ . Indeed we have simply  $\lambda(p(i, u + 1)) = [\lambda(p(i, u)) + 1] \bmod \chi$  for all  $u > y$ . As a consequence there is one moving point of each color in the sequence.





**Fig. 10.** A 3-coloration of  $C_0$



**Fig. 11.** Coloration of the other connected components

Now let us consider a sequence  $p(i, x), \dots, p(i, x + 2\chi - 2)$  of  $(2\chi - 1)$  consecutive moving points. If no moving point belongs to  $C_0$ , the previous remark permits an immediate conclusion. Otherwise let  $y_0$  be the index of the first moving point of  $C_0$  in the sequence. In the same way if  $y_0 > x + \chi - 1$ , we can conclude that each moving point of  $p(i, x), \dots, p(i, x + \chi - 1)$  is assigned to a different color. Finally, let us consider the case where  $y_0 \leq x + \chi - 1$ . Notice that the interval  $I = [y_0, x + 2\chi - 2]$  is at least  $\chi$  long. To conclude using the remark, we just need to check that  $y_0$  is the only index in  $[y_0, y_0 + \chi - 1]$  whose corresponding moving point belongs to  $C_0$ . Due to property 2 and theorem 1, a moving point of  $C_0$  appears exactly every  $w(G^A)$  indices on any route. The index  $y_1$  of the next moving point in  $C_0$  is hence equal to  $y_0 + w(G^A)$ , which is larger than  $y_0 + \chi$  by hypothesis.

As can be clearly seen in the proof, we can pick any licit  $\chi$ -coloring of  $C_0$  to color the other connected components by cyclic permutation, and then pick any of the  $\chi$  colors to define a valid assignment of density  $\frac{1}{2\chi-1}$ . One may think that an astute choice of the initial coloration or an astute selection of the color to pick could lead to an assignment of larger density.

## 7 Upper bounds on the maximum density of a graph

In this section we provide negative results on the density of graphs. An important parameter, as underlined in section 6, is the chromatic number of the conflicts graph  $G^C$ . We have the following property:

**Theorem 4** *If there exists a  $\frac{1}{k}$ -dense assignment, then the conflicts graph is  $k$ -colorable.*

*Proof.* Let  $N_0$  be the set of allowed moving points in a  $\frac{1}{k}$ -dense assignment. This set is an independent one, as no two moving points can be in conflict if they both have the allowed status. Let  $N_l$ ,  $l \in [1, k-1]$  be the sets of moving points defined as follows:

$$N_l = \{p(i, x) \mid p(i, x-l) \in N_0 \text{ and } p(i, x) \notin N_0 \cup \dots \cup N_{l-1}\}$$

Due to their construction, whatever  $l$ ,  $N_l$  is isomorphic to a subset of  $N_0$ , which implies that  $N_l$  is an independent set of the conflicts graph. We can note that  $N_p$  and  $N_q$  are disjoint if  $p \neq q$ , as a necessary condition for a moving point to be in  $N_l$  is precisely that it does not belong to any set  $N_0$  to  $N_{l-1}$ . To conclude that  $N_0, \dots, N_{k-1}$  define a partition of the conflicts graph into  $k$  disjoint independent sets, which implies that the conflicts graph is  $k$ -colorable, it remains to prove that the union of the  $N_l$ 's is the set of all moving points. Let  $\Omega$  be this union:

$$\Omega = \bigcup_{l=0}^{k-1} N_l$$

Let us take any moving point  $p(j, t)$  and prove that it is in  $\Omega$ . As the assignment is  $\frac{1}{k}$ -dense, there is a moving point  $p(j, s)$  in  $N_0$  such that distance  $t - s < k$ . In short, we denote by  $d$  the distance  $t - s$ . Since  $d$  is an integer of  $[0, k-1]$ , we can consider set  $N_d$ . By construction, moving point  $p(j, t)$  is either in one of the sets  $N_0, \dots, N_{d-1}$ , or in set  $N_d$ . Thus  $p(j, t) \in \Omega$ .

As a direct corollary, we have the following upper bound on the density:

**Corollary 1** *Let  $\chi$  be the chromatic number of the conflicts graph  $G^C$ . Then the density of any valid assignment is smaller than or equal to  $\frac{1}{\chi}$ .*

This corollary gives an upper bound on the maximum density of an assignment. For instance, if the conflicts graph is not bipartite, there is no hope of finding a solution with a density larger than  $\frac{1}{3}$ .

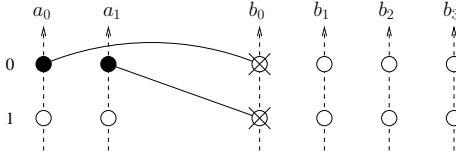
In the previous section we prove that if  $w(G^I) \geq \chi$ , then there exists a  $\frac{1}{2\chi-1}$ -dense solution, and we conjecture that a  $\frac{1}{\chi+1}$ -dense solution exists, which would reduce to 1 the gap between the upper and lower bounds. Nevertheless one may wonder what one can expect if the assumption  $w(G^I) \geq \chi$  is not verified. The following result shows that things get considerably worse, as the maximum density for a conflicts graph can be arbitrarily small:

**Theorem 5** *For any integer  $k \geq 2$  there exists a routes representation for which no  $\frac{1}{k}$ -dense solution exists, even if the associated conflicts graph is bipartite.*

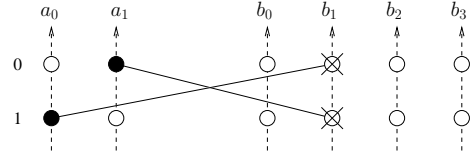
*Proof.* We first construct a bipartite conflicts graph for which no  $\frac{1}{k}$ -dense solution exists, and then we expose a routes representation corresponding to this conflicts graph.

Our bipartite conflicts graph has  $k$  routes on the left-hand side, denoted  $a_0, \dots, a_{k-1}$  and  $K = k^k$  routes on the right-hand side, denoted  $b_0, \dots, b_{K-1}$ . The basic idea of the construction is to block every  $\frac{1}{k}$ -dense solution on the left-hand side routes using the right-hand side routes. More precisely any  $\frac{1}{k}$ -dense assignment on the left part will forbid  $k$  consecutive moving points on one right-hand side route.

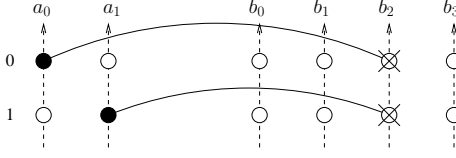
Consider the index interval  $I = [0, \dots, k-1]$ . Any  $\frac{1}{k}$ -dense solution has at least one allowed moving point in this interval on every route. Thus we can associate to a



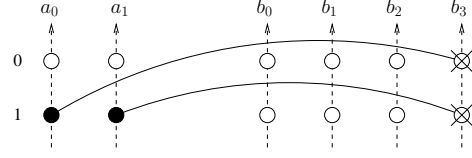
**Fig. 12.** Construction for  $x = (0, 0)$



**Fig. 13.** Construction for  $x = (1, 0)$



**Fig. 14.** Construction for  $x = (0, 1)$



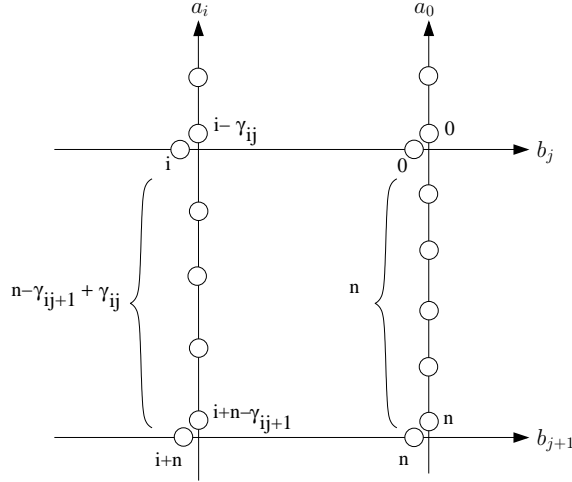
**Fig. 15.** Construction for  $x = (1, 1)$

$\frac{1}{k}$ -dense solution a  $k$ -uple  $(x_0, \dots, x_{k-1})$  of  $I^k$  representing the smallest index in  $I$  of allowed moving points on route  $a_0, \dots, a_{k-1}$ , respectively. This  $k$ -uple can also be seen as the writing in basis  $k$  of the number  $x = x_0 + x_1k + \dots x_{k-1}k^{k-1}$ . The number of distinct  $k$ -uple of  $I^k$  is clearly  $k^k$ . Now, for each  $k$ -uple  $(x_0, \dots, x_{k-1})$  of  $I^k$ , we join each left-hand side moving point  $p(a_i, x_i)$  with moving point  $p(b_x, l)$ . In this way we block  $k$  consecutive moving points on right-hand side route  $b_x$ . Figures 12 to 15 show the construction for  $k = 2$ . To obtain an infinite (periodic) graph, we then shift every indices by one and repeat the construction, and so on. The resulting graph for  $k = 2$  is represented figure 17. This graph is bipartite (edges appear only between a left-hand side route and a right-hand side route) and fulfills the conditions that a moving point of a route  $i$  is in conflict with at most one moving point of another route  $j$ . Indeed a number  $x$  has a unique representation in basis  $k$ .

Clearly no  $\frac{1}{k}$ -dense solution exists for this conflicts graph, since any solution that would be  $\frac{1}{k}$ -dense on the left-hand side necessarily has  $k$  consecutive forbidden moving points on a route of the right-hand side. Notice that  $w(G^C)$  for the conflicts graph is equal to 1. This can be easily shown by exhibiting a path from  $p(a_0, 0)$  to  $p(a_0, 1)$ , using the characterization of  $w(G^C)$  as the number of connected components. For instance we can use path  $(p(a_0, 0), p(b_1, 0), p(a_1, 0), p(b_0, 1), p(a_0, 1))$  which exists in the construction for all  $k$ .

Once a non  $\frac{1}{k}$ -dense conflicts graph is identified, we have to find a corresponding routes representation to complete the proof. This representation can be obtained as follows: conflicts graph left-hand side routes are represented vertically, and right-hand side routes are represented horizontally. A moving point is present at every intersection.

The number of moving points between consecutive intersections is chosen according to the conflicts graph. On routes  $b$ 's, there is no moving point between intersections with routes  $a$ 's. Recall that in the construction of the conflicts graph, moving points are indexed in such a way that  $p(a_0, 0)$  is in conflict with  $p(b_j, 0)$  for all  $j = 0, \dots, K-1$ . Let  $\gamma_{ij}$  be the index in our construction of the conflicts graph such that  $p(a_i, 0)$  is in conflict with  $p(b_j, \gamma_{ij})$ . We have to construct our routes representation in such a way that  $\Delta_{a_i, b_j} = \gamma_{ij}$  for all the routes. Assume that the routes representation has been obtained up to route  $b_j$ . We represent horizontally route  $b_{j+1}$  and  $n$  moving points between routes  $b_j$  and  $b_{j+1}$  on route  $a_0$ . We make use of the possibility of varying instantaneous speed on a route to space a different number of moving points between  $b_j$  and  $b_{j+1}$  on other routes  $a$ 's. Precisely we place  $n + \gamma_{i,j} - \gamma_{i,j+1}$  moving points between the 2 consecutive intersections on route  $a_i$ . Figure 16 depicts the situation at



**Fig. 16.** Construction of the route representation, adding route  $b_{j+1}$

the instant when a conflict occurs between  $p(a_0, 0)$  and  $p(b_j, 0)$ . Since the representation is assumed to correspond to the conflicts graph up to route  $b_j$ , we must have a conflict between  $p(a_i, i)$  and  $p(b_j, i + \gamma_{ij})$ . It is then easy to see that at the intersection of routes  $a_i$  and  $b_{j+1}$ , the conflict involves at this time moving points  $p(b_{j+1}, i + n)$  and  $p(a_i, (i - \gamma_{ij}) + (n + \gamma_{ij} - \gamma_{ij+1}))$ , that is  $p(a_i, i + n - \gamma_{ij+1})$ . We obtain the required result.

For this construction to be valid, we simply need  $n + \gamma_{i,j} - \gamma_{i,j+1}$  to be positive, i.e.  $n \geq \gamma_{i,j+1} - \gamma_{i,j}$ . Notice that this number  $n$  of moving points separating intersections on route  $a_0$  is not required to be the same between any 2 consecutive intersections. Since  $|\gamma|$  is uniformly bounded by  $k^k - 1$ , any value of  $n$  larger than  $2(k^k - 1)$  is suitable for all intersections. A routes graph representation for  $k = 2$  is given in figure 18. The values  $\gamma_{i,j+1} - \gamma_{i,j}$  have been represented between consecutive intersections of route  $a_1$  with the  $b_j$ 's.

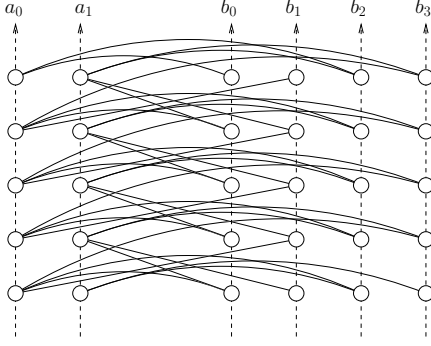
## 8 Density of bipartite conflicts graphs

In the previous two sections, we give positive and negative results on the maximum density of a graph depending on its chromatic number  $\chi$ . In this section, we study the special case of bipartite graphs, that is to say when  $\chi = 2$ . We give in particular a characterization of graphs admitting a  $\frac{1}{2}$ -dense solution.

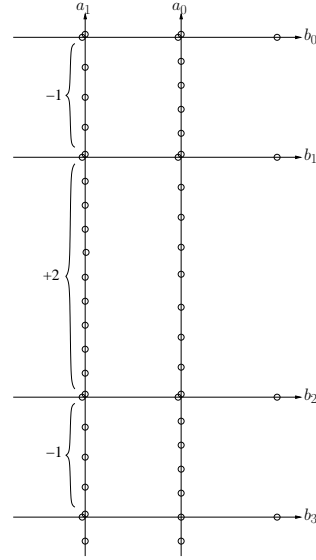
Thanks to previous sections, we already have some results on bipartite conflicts graphs. Applying corollary 1 implies that we cannot hope for a density greater than  $\frac{1}{2}$ . Rather worse, theorem 5 shows that the maximum density of a bipartite conflicts graph can be arbitrarily close to 0. Now looking at some good news, theorem 3 proves that if  $w(G^A) \geq 2$ , then there exists a  $\frac{1}{3}$ -dense solution. It implies that if  $w(G^A) \neq 1$ , the maximum density is equal either to  $\frac{1}{2}$  or to  $\frac{1}{3}$ . In this section, we precisely characterize all the  $\frac{1}{2}$ -dense solutions. The following lemma is useful for this purpose:

**Lemma 2** *There exists a  $\frac{1}{2}$ -dense assignment if and only if there exists an exactly  $\frac{1}{2}$ -dense assignment.*

*Proof.* It is obvious that the set of exactly  $\frac{1}{2}$ -dense solutions is included in the set of  $\frac{1}{2}$ -dense solutions. Let us hence prove the converse. We suppose that there exists a



**Fig. 17.** Conflicts graph for  $k = 2$



**Fig. 18.** A routes representation for conflicts graph of figure 17

$\frac{1}{2}$ -dense solution. Clearly the forbidden status cannot be assigned to two successive moving points on a route  $i$ . But neither can the allowed status, since otherwise there would be two successive moving points with the forbidden status on every route intersecting route  $i$ . As we cannot have two successive moving points with the allowed status nor the forbidden status, we have an exactly  $\frac{1}{2}$ -dense solution.

**Remark 3** An exactly  $\frac{1}{2}$ -dense assignment defines a proper 2-coloring of the conflicts graph.

*Proof.* The set  $A$  of allowed moving points in an assignment is by definition an independent set. If the assignment is exactly  $\frac{1}{2}$ -dense, the reverse assignment, swapping the allowed and forbidden status, is also a  $\frac{1}{2}$ -dense solution. Hence complementary set  $\overline{A}$  of forbidden moving points is an independent set. Therefore an exactly  $\frac{1}{2}$ -dense assignment defines a partition of  $V^C$  into two independent sets  $(A, \overline{A})$ , in other words a proper bicoloring of  $G^C$ .

A  $\frac{1}{2}$ -dense assignment corresponds to a 2-coloring of the bipartite conflicts graph. Symmetrically to any 2-coloring  $\lambda : V^C \mapsto \{0, 1\}$  of the conflicts graph, we can associate a valid assignment  $A(\lambda)$ , where allowed moving points correspond by convention to color 1. Thus by a slight abuse of language, we speak about the colors of an assignment or the density of a coloring. Clearly there is no reason for an arbitrary 2-coloring to be  $\frac{1}{2}$ -dense. The following lemma gives a necessary and sufficient condition for this to happen. As in section 5.2, we assume that there exists a cycle of minimum positive weight passing through an intersection of route 1.

**Lemma 3** An assignment is  $\frac{1}{2}$ -dense if and only if it is exactly  $\frac{1}{2}$ -dense on route 1.

*Proof.* The remark is based on the fact that a bicoloring on a connected graph is entirely defined by the color assigned to one single vertex. Indeed as colors necessarily alternate along a path, two vertices are of the same color if and only if the length of a path between them is even. Consider a coloring  $\lambda$  which is exactly  $\frac{1}{2}$ -dense on route 1. Due to lemma 2 we have to prove that on any route  $i$  colors of moving points

do alternate. Let  $p(i, x)$  and  $p(i, x+1)$  be 2 successive moving points of route  $i$ , and  $C_\alpha$  be the connected component of  $p(i, x)$ . By definition there exists a path  $P$  from  $p(1, \alpha)$  to  $p(i, x)$ . Due to remark 1, there exists a path  $P'$ , isomorphic to  $P$ , between  $p(1, \alpha+1)$  and  $p(i, x+1)$ . If  $l$  is the length of  $P$ , the result is that  $\lambda(p(i, x)) = \lambda(p(1, \alpha)) + l \bmod 2$  and  $\lambda(p(i, x+1)) = \lambda(p(1, \alpha+1)) + l \bmod 2$ . Since colors alternate on route 1,  $p(i, x)$  and  $p(i, x+1)$  are also of different colors.

We now introduce a new definition necessary for the characterization of theorem 6. Recall that  $w(G^A)$  is the smallest strictly positive cycle weight in the routes graph. Due to property 2, the conflicts graph has  $w(G^A)$  (isomorphic) connected components, and there exists a path between  $p(1, 0)$  and  $p(1, w(G^A))$ .

**Definition 2.** We define  $L(G^C)$  as the length of a shortest path between  $p(1, 0)$  and  $p(1, w(G^A))$  in conflicts graph  $G^C$ .

The length of a path in  $G^C$  is, as usual, the number of its edges. Notice that since  $G^C$  is bipartite, the length of any path between two moving points is of the same parity. It implies that in any 2-coloring  $\lambda$ , we have  $\lambda(p(1, w(G^A))) = \lambda(p(1, 0)) + L(G^C) \bmod 2$ . We derive the following characterization of  $\frac{1}{2}$ -dense assignments:

**Theorem 6 (Characterization of  $\frac{1}{2}$ -dense assignments)** A conflicts graph  $G^C$  admits a  $\frac{1}{2}$ -dense assignment if and only if  $G^C$  is bipartite and  $w(G^A) + L(G^C)$  is even.

*Proof.* Suppose that  $w(G^A) + L(G^C) = 0 \bmod 2$ . Since the conflicts graph is bipartite, each connected component  $C_\alpha$  can be partitioned into two independent sets. Consider such a partition  $(A_0, \overline{A_0})$  of  $C_0$ . For  $\alpha \in [1, w(G^A) - 1]$ , let  $(A_\alpha, \overline{A_\alpha})$  be the partition isomorphic to  $(A_0, \overline{A_0})$  by the canonical mapping  $\varphi_\alpha : p(i, x) \mapsto p(i, x + \alpha)$ . For each connected component, we choose an independent set  $X_\alpha \in \{A_\alpha, \overline{A_\alpha}\}$ . Then both  $X = \bigcup_{\alpha=0}^{w(G^A)-1} X_\alpha$  and its complementary set  $\overline{X} = V^C \setminus X$  are independent sets of the conflicts graph. Hence whatever the choice of the  $X_\alpha$ 's, we obtain a bicoloring  $(X, \overline{X})$  of the conflicts graph. We claim that the following coloring is  $\frac{1}{2}$ -dense :

$$\begin{cases} X_{2\beta} = A_{2\beta} \\ X_{2\beta+1} = \overline{A_{2\beta+1}} \end{cases} \quad (1)$$

By construction, colors alternate on moving points  $p(1, 0), \dots, p(1, w(G^C))$ . To conclude that the density of the assignment is equal to  $\frac{1}{2}$ , we need to check that  $p(1, w(G^A) - 1)$  and  $p(1, w(G^A))$  have different colors. On one hand, by construction of  $X$ , color of  $p(1, w(G^A) - 1)$  is equal to  $\lambda(p(1, 0)) + w(G^A) \bmod 2$ . On the other hand, since  $L(G^C)$  is the length of a path from  $p(1, 0)$  to  $p(1, w(G^A))$ , we have  $\lambda(p(1, w(G^A))) = \lambda(p(1, 0)) + L(G^C) \bmod 2$ . Due to the hypothesis  $w(G^A) + L(G^C) = 0 \bmod 2$ , the result is that  $\lambda(p(1, w(G^A))) = \lambda(p(1, w(G^A) - 1) + 1 \bmod 2$ . Using lemma 3, we can conclude that the assignment is  $\frac{1}{2}$ -dense.

Conversely, suppose that there exists a  $\frac{1}{2}$ -dense assignment. Thanks to lemma 2 we can assume that this assignment is exactly  $\frac{1}{2}$ -dense. Let  $\lambda$  be its associated 2-coloring. Then we must have  $\lambda(p(1, x)) = \lambda(p(1, 0)) + x \bmod 2$  as colors alternate on a route. Since  $\lambda$  is a proper coloring, it must also verify  $\lambda(p(1, w(G^A))) = \lambda(p(1, 0)) + w(G^A) \bmod 2$ . We get  $w(G^A) = L(G^C) \bmod 2$ .

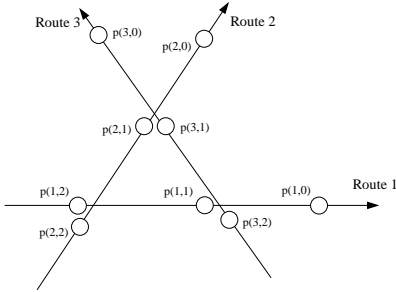
Thanks to theorems 3, 5 and 6, we know precisely the maximum density of solutions according to  $w(G^A)$  and  $L(G^C)$  for bipartite conflicts graphs. These results are

summarized in table 1. The only case where the maximum density is not fixed by parameters  $w(G^A)$  and  $L(G^C)$  corresponds to  $w(G^A) = 1$  and  $L(G^C)$  even. On the one hand theorem 6 asserts that the maximum density is at most  $\frac{1}{3}$ . For instance, for routes representation given figure 5, we have  $w(G^A) = 1$  and  $L(G^C) = 4$  (see conflicts graph figure 7). Hence no  $\frac{1}{2}$ -dense assignment exists, but a  $\frac{1}{3}$ -dense assignment can be easily found, selecting one moving point every three points along the single path of  $G^C$ . On the other hand proof of theorem 5 exhibits bipartite conflicts graphs with maximum density arbitrarily close to 0.

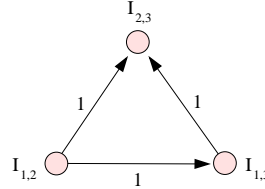
	$w(G^A)$ even	$w(G^A) \neq 1$ odd	$w(G^A) = 1$
$L(G^C)$ even	$1/2$	$1/3$	$\in (0, \frac{1}{3}]$
$L(G^C)$ odd	$1/3$	$1/2$	$1/2$

**Table 1.** Maximum density value for bipartite conflicts graphs

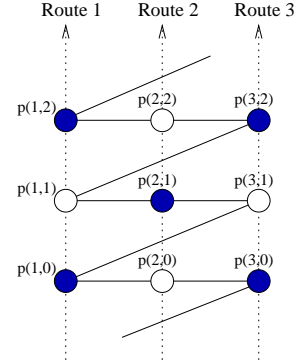
Figure 19 illustrates the results of theorem 6 on a routes representation composed of 3 routes. Routes graph  $G^A$  is represented in figure 20. It is an elementary cycle of weight  $w(G^A) = 1$ . Conflicts graph is represented in figure 21. Clearly it is a bipartite graph, with  $L(G^C) = 3$ . Therefore the solution is  $\frac{1}{2}$ -dense.



**Fig. 19.** A routes representation with three routes



**Fig. 20.** Routes graph of figure 19



**Fig. 21.** Conflicts graph of figure 19 with a  $\frac{1}{2}$ -dense coloring

## 9 Conclusion

In this paper, we described a new air traffic paradigm, in which aircraft are compelled to follow immaterial periodic moving points generated on a route joining their origin to their destination. More specifically, given a set of  $n$  routes, the problem is to find an allowed/forbidden assignment of the moving points without conflict between allowed moving points, which maximizes their density. In this paper, we gave lower and upper bounds for the maximum density of a solution. We first proved that for  $n$  routes, we always have a  $\frac{1}{n}$ -dense solution. If the conflicts graph is  $\chi$ -colorable and has more than  $\chi$  connected components, we showed that there exists a  $\frac{1}{2\chi-1}$ -dense solution.

Besides, we proposed the following conjecture: for a  $\chi$ -colorable conflicts graph with  $w(G^A) \geq \chi$ , there is always a  $\frac{1}{\chi+1}$ -dense solution. We also showed that the density of any solution is always less than or equal to  $\frac{1}{\chi}$ ,  $\chi$  being the chromatic number of the conflicts graph. We then proved that there exists routes representations for which no  $\frac{1}{k}$ -dense solution exists, whatever  $k \in \mathbb{N}^*$ . We finally characterized instances which admit a  $\frac{1}{2}$ -dense solution.

In this work, we only address a part of the problems whose resolution is necessary to deal with operational constraints. Firstly, one has to find one or several 3D-profiles for each origin-destination. To solve this problem, we have to deal with meteorological conditions like winds, and space constraints, consisting for example in avoiding military areas. Besides, one must determine the pace of the moving points and the optimal phase shifts between all routes. This optimization is really relevant as it could lead to find intersections for which there is no conflict between any two moving points. The problem studied in this paper can also be slightly modified. For example, we could accept solutions having a low conflict rate. The remaining conflicts per allowed moving point could then be limited to one. This method would lead to deal with a bigger part of the traffic, and still limit the number of conflicts. A complementary approach could consist in limiting the space concentration of the remaining conflicting intersections by trying to dispatch them among the airspace. We could also try to solve a weighted version of our problem, the weights being proportional to the number of flights on each route. Another related problem is to obtain a subset of routes that admits a  $\frac{1}{k}$ -dense solution (or an exactly  $\frac{1}{k}$ -dense solution),  $k$  being given. Future work consists in obtaining more operational solutions, by solving the problems introduced previously, and in studying extensions of the problem presented in this paper.

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