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# TENSOR EXTENSION PROPERTIES OF $C(K)$ -REPRESENTATIONS AND APPLICATIONS TO UNCONDITIONALITY

CHRISTOPH KRIEGLER, CHRISTIAN LE MERDY

ABSTRACT. Let  $K$  be any compact set. The  $C^*$ -algebra  $C(K)$  is nuclear and any bounded homomorphism from  $C(K)$  into  $B(H)$ , the algebra of all bounded operators on some Hilbert space  $H$ , is automatically completely bounded. We prove extensions of these results to the Banach space setting, using the key concept of  $R$ -boundedness. Then we apply these results to operators with a uniformly bounded  $H^\infty$ -calculus, as well as to unconditionality on  $L^p$ . We show that any unconditional basis on  $L^p$  'is' an unconditional basis on  $L^2$  after an appropriate change of density.

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## 1. INTRODUCTION

Throughout the paper, we let  $K$  be a nonempty compact set and we let  $C(K)$  be the algebra of all continuous functions  $f: K \rightarrow \mathbb{C}$ , equipped with the supremum norm. A representation of  $C(K)$  on some Banach space  $X$  is a bounded unital homomorphism  $u: C(K) \rightarrow B(X)$  into the algebra  $B(X)$  of all bounded operators on  $X$ . Such representations appear naturally and play a major role in several fields of Operator Theory, including functional calculi, spectral theory and spectral measures, or classification of  $C^*$ -algebras. Several recent papers, in particular [21, 23, 11, 8], have emphasized the rich and fruitful interplays between the notion of  $R$ -boundedness, unconditionality and various functional calculi. The aim of this paper is to establish new properties of  $C(K)$ -representations involving  $R$ -boundedness, and to give applications to  $H^\infty$ -calculus (in the sense of [6, 21]) and to unconditionality in  $L^p$ -spaces.

We recall the definition of  $R$ -boundedness (see [2, 4]). Let  $(\epsilon_k)_{k \geq 1}$  be a sequence of independent Rademacher variables on some probability space  $\Omega_0$ . That is, the  $\epsilon_k$ 's take values in  $\{-1, 1\}$  and  $\text{Prob}(\{\epsilon_k = 1\}) = \text{Prob}(\{\epsilon_k = -1\}) = \frac{1}{2}$ . For any Banach space  $X$ , we let  $\text{Rad}(X) \subset L^2(\Omega_0; X)$  be the closure of  $\text{Span}\{\epsilon_k \otimes x : k \geq 1, x \in X\}$  in  $L^2(\Omega_0; X)$ . Thus for any finite family  $x_1, \dots, x_n$  in  $X$ , we have

$$\left\| \sum_k \epsilon_k \otimes x_k \right\|_{\text{Rad}(X)} = \left( \int_{\Omega_0} \left\| \sum_k \epsilon_k(\lambda) x_k \right\|_X^2 d\lambda \right)^{\frac{1}{2}}.$$

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By definition, a set  $\tau \subset B(X)$  is  $R$ -bounded if there is a constant  $C \geq 0$  such that for any finite families  $T_1, \dots, T_n$  in  $\tau$ , and  $x_1, \dots, x_n$  in  $X$ , we have

$$\left\| \sum_k \epsilon_k \otimes T_k x_k \right\|_{\text{Rad}(X)} \leq C \left\| \sum_k \epsilon_k \otimes x_k \right\|_{\text{Rad}(X)}.$$

In this case, we let  $R(\tau)$  denote the smallest possible  $C$ . It is called the  $R$ -bound of  $\tau$ . By convention, we write  $R(\tau) = \infty$  if  $\tau$  is not  $R$ -bounded.

It will be convenient to let  $\text{Rad}_n(X)$  denote the subspace of  $\text{Rad}(X)$  of all finite sums  $\sum_{k=1}^n \epsilon_k \otimes x_k$ . If  $X = H$  is a Hilbert space, then  $\text{Rad}_n(H) = \ell_n^2(H)$  isometrically and all bounded subsets of  $B(H)$  are automatically  $R$ -bounded. Conversely if  $X$  is not isomorphic to a Hilbert space, then  $B(X)$  contains bounded subsets which are not  $R$ -bounded [1, Prop. 1.13].

In order to motivate the results in this paper, we recall two well-known properties of  $C(K)$ -representations on Hilbert space  $H$ . First, any bounded homomorphism  $u: C(K) \rightarrow B(H)$  is completely bounded, with  $\|u\|_{cb} \leq \|u\|^2$ . This means that for any integer  $n \geq 1$ , the tensor extension  $I_{M_n} \otimes u: M_n(C(K)) \rightarrow M_n(B(H))$  satisfies  $\|I_{M_n} \otimes u\| \leq \|u\|^2$ , if  $M_n(C(K))$  and  $M_n(B(H))$  are equipped with their natural  $C^*$ -algebra norms. This implies that any bounded homomorphism  $u: C(K) \rightarrow B(H)$  is similar to a  $*$ -representation, a result going back at least to [3]. We refer to [28, 30] and the references therein for some information on completely bounded maps and similarity properties.

Second, let  $u: C(K) \rightarrow B(H)$  be a bounded homomorphism. Then for any  $b_1, \dots, b_n$  lying in the commutant of the range of  $u$  and for any  $f_1, \dots, f_n$  in  $C(K)$ , we have

$$(1.1) \quad \left\| \sum_k u(f_k) b_k \right\| \leq \|u\|^2 \sup_{t \in K} \left\| \sum_k f_k(t) b_k \right\|.$$

This property is essentially a rephrasing of the fact that  $C(K)$  is a nuclear  $C^*$ -algebra. More precisely, nuclearity means that the above property holds true for  $*$ -representations (see e.g. [19, Chap. 11] or [28, Chap. 12]), and its extension to arbitrary bounded homomorphisms easily follows from the similarity property mentioned above (see [25] for more explanations and developments).

Now let  $X$  be a Banach space and let  $u: C(K) \rightarrow B(X)$  be a bounded homomorphism. In Section 2, we will show the following analog of (1.1):

$$(1.2) \quad \left\| \sum_k u(f_k) b_k \right\| \leq \|u\|^2 R \left( \left\{ \sum_k f_k(t) b_k : t \in K \right\} \right),$$

provided that the  $b_k$ 's commute with the range of  $u$ .

Section 3 is devoted to sectorial operators  $A$  which have a uniformly bounded  $H^\infty$ -calculus, in the sense that they satisfy an estimate

$$(1.3) \quad \|f(A)\| \leq C \sup_{t>0} |f(t)|$$

for any bounded analytic function on a sector  $\Sigma_\theta$  surrounding  $(0, \infty)$ . Such operators turn out to have a natural  $C(K)$ -functional calculus. Applying (1.2) to the resulting representation  $u: C(K) \rightarrow B(X)$ , we show that (1.3) can be automatically extended to operator valued

analytic functions  $f$  taking their values in the commutant of  $A$ . This is an analog of a remarkable result of Kalton-Weis [23, Thm. 4.4] saying that if an operator  $A$  has a bounded  $H^\infty$ -calculus and  $f$  is an operator valued analytic function taking its values in an  $R$ -bounded subset of the commutant of  $A$ , then the operator  $f(A)$  arising from ‘generalized  $H^\infty$ -calculus’, is bounded.

In Section 4, we introduce matricially  $R$ -bounded maps  $C(K) \rightarrow B(X)$ , a natural analog of completely bounded maps in the Banach space setting. We show that if  $X$  has property  $(\alpha)$ , then any bounded homomorphism  $C(K) \rightarrow B(X)$  is automatically matricially  $R$ -bounded. This extends both the Hilbert space result mentioned above, and a result of De Pagter-Ricker [8, Cor. 2.19] saying that any bounded homomorphism  $C(K) \rightarrow B(X)$  maps the unit ball of  $C(K)$  into an  $R$ -bounded set, provided that  $X$  has property  $(\alpha)$ .

In Section 5, we give an application of matricial  $R$ -boundedness to the case when  $X = L^p$ . A classical result of Johnson-Jones [18] asserts that any bounded operator  $T: L^p \rightarrow L^p$  acts, after an appropriate change of density, as a bounded operator on  $L^2$ . We show versions of this theorem for bases (more generally, for FDD’s). Indeed we show that any unconditional basis (resp. any  $R$ -basis) on  $L^p$  becomes an unconditional basis (resp. a Schauder basis) on  $L^2$  after an appropriate change of density. These results rely on Simard’s extensions of the Johnson-Jones Theorem established in [32].

We end this introduction with a few preliminaries and notation. For any Banach space  $Z$ , we let  $C(K; Z)$  denote the space of all continuous functions  $f: K \rightarrow Z$ , equipped with the supremum norm

$$\|f\|_\infty = \sup\{\|f(t)\|_Z : t \in K\}.$$

We may regard  $C(K) \otimes Z$  as a subspace of  $C(K; Z)$ , by identifying  $\sum_k f_k \otimes z_k$  with the function  $t \mapsto \sum_k f_k(t)z_k$ , for any finite families  $(f_k)_k$  in  $C(K)$  and  $(z_k)_k$  in  $Z$ . Moreover,  $C(K) \otimes Z$  is dense in  $C(K; Z)$ . Note that for any integer  $n \geq 1$ ,  $C(K; M_n)$  coincides with the  $C^*$ -algebra  $M_n(C(K))$  mentioned above.

We will need the so-called ‘contraction principle’, which says that for any  $x_1, \dots, x_n$  in a Banach space  $X$  and any  $\alpha_1, \dots, \alpha_n$  in  $\mathbb{C}$ , we have

$$(1.4) \quad \left\| \sum_k \epsilon_k \otimes \alpha_k x_k \right\|_{\text{Rad}(X)} \leq 2 \sup_k |\alpha_k| \left\| \sum_k \epsilon_k \otimes x_k \right\|_{\text{Rad}(X)}.$$

We also recall that any unital commutative  $C^*$ -algebra is a  $C(K)$ -space (see e.g. [19, Chap. 4]). Thus our results concerning  $C(K)$ -representations apply as well to all these algebras. For example we will apply them to  $\ell^\infty$  in Section 5.

We let  $I_X$  denote the identity mapping on a Banach space  $X$ , and we let  $\chi_B$  denote the indicator function of a set  $B$ . If  $X$  is a dual Banach space, we let  $w^*B(X) \subset B(X)$  be the subspace of all  $w^*$ -continuous operators on  $X$ .

## 2. THE EXTENSION THEOREM

Let  $X$  be an arbitrary Banach space. For any compact set  $K$  and any bounded homomorphism  $u: C(K) \rightarrow B(X)$ , we let

$$E_u = \{b \in B(X) : bu(f) = u(f)b \text{ for any } f \in C(K)\}$$

denote the commutant of the range of  $u$ .

Our main purpose in this section is to prove (1.1). We start with the case when  $C(K)$  is finite dimensional.

**Proposition 2.1.** *Let  $N \geq 1$  and let  $u: \ell_N^\infty \rightarrow B(X)$  be a bounded homomorphism. Let  $(e_1, \dots, e_N)$  be the canonical basis of  $\ell_N^\infty$  and set  $p_i = u(e_i)$ ,  $i = 1, \dots, N$ . Then for any  $b_1, \dots, b_N \in E_u$ , we have*

$$\left\| \sum_{i=1}^N p_i b_i \right\| \leq \|u\|^2 R(\{b_1, \dots, b_N\}).$$

*Proof.* Since  $u$  is multiplicative, each  $p_i$  is a projection and  $p_i p_j = 0$  when  $i \neq j$ . Hence for any choice of signs  $(\alpha_1, \dots, \alpha_N) \in \{-1, 1\}^N$ , we have

$$\sum_{i=1}^N p_i b_i = \sum_{i,j=1}^N \alpha_i \alpha_j p_i p_j b_j.$$

Furthermore,

$$\left\| \sum_i \alpha_i p_i \right\| = \|u(\alpha_1, \dots, \alpha_N)\| \leq \|u\| \|(\alpha_1, \dots, \alpha_N)\|_{\ell_N^\infty} = \|u\|.$$

Therefore for any  $x \in X$ , we have the following chain of inequalities which prove the desired estimate:

$$\begin{aligned} \left\| \sum_i p_i b_i x \right\|^2 &= \int_{\Omega_0} \left\| \sum_i \epsilon_i(\lambda) p_i \sum_j \epsilon_j(\lambda) p_j b_j x \right\|^2 d\lambda \\ &\leq \int_{\Omega_0} \left\| \sum_i \epsilon_i(\lambda) p_i \right\|^2 \left\| \sum_j \epsilon_j(\lambda) p_j b_j x \right\|^2 d\lambda \\ &\leq \|u\|^2 \int_{\Omega_0} \left\| \sum_j \epsilon_j(\lambda) b_j p_j x \right\|^2 d\lambda \\ &\leq \|u\|^2 R(\{b_1, \dots, b_N\})^2 \int_{\Omega_0} \left\| \sum_j \epsilon_j(\lambda) p_j x \right\|^2 d\lambda \\ &\leq \|u\|^4 R(\{b_1, \dots, b_N\})^2 \|x\|^2. \end{aligned}$$

□

The study of infinite dimensional  $C(K)$ -spaces requires the use of second duals and  $w^*$ -topologies. We recall a few well-known facts that will be used later on in this paper. According to the Riesz representation Theorem, the dual space  $C(K)^*$  can be naturally identified with the space  $M(K)$  of Radon measures on  $K$ . Next, the second dual space  $C(K)^{**}$  is a commutative  $C^*$ -algebra for the so-called Arens product. This product extends the product on  $C(K)$  and is separately  $w^*$ -continuous, which means that for any  $\xi \in C(K)^{**}$ , the two linear maps

$$\nu \in C(K)^{**} \longmapsto \nu\xi \in C(K)^{**} \quad \text{and} \quad \nu \in C(K)^{**} \longmapsto \xi\nu \in C(K)^{**}$$

are  $w^*$ -continuous.

Let

$$\mathcal{B}^\infty(K) = \{f: K \rightarrow \mathbb{C} \mid f \text{ bounded, Borel measurable}\},$$

equipped with the sup norm. According to the duality pairing

$$\langle f, \mu \rangle = \int_K f(t) d\mu(t), \quad \mu \in M(K), f \in \mathcal{B}^\infty(K),$$

one can regard  $\mathcal{B}^\infty(K)$  as a closed subspace of  $C(K)^{**}$ . Moreover the restriction of the Arens product to  $\mathcal{B}^\infty(K)$  coincides with the pointwise product. Thus we have natural  $C^*$ -algebra inclusions

$$(2.1) \quad C(K) \subset \mathcal{B}^\infty(K) \subset C(K)^{**}.$$

See e.g. [7, pp. 366-367] and [5, Sec. 9] for further details.

Let  $\widehat{\otimes}$  denote the projective tensor product on Banach spaces. We recall that for any two Banach spaces  $Y_1, Y_2$ , we have a natural identification

$$(Y_1 \widehat{\otimes} Y_2)^* \simeq B(Y_2, Y_1^*),$$

see e.g. [9, VIII.2]. This implies that when  $X$  is a dual Banach space,  $X = (X_*)^*$  say, then  $B(X) = (X_* \widehat{\otimes} X)^*$  is a dual space. The next two lemmas are elementary.

**Lemma 2.2.** *Let  $X = (X_*)^*$  be a dual space, let  $S \in B(X)$ , and let  $R_S, L_S: B(X) \rightarrow B(X)$  be the right and left multiplication operators defined by  $R_S(T) = TS$  and  $L_S(T) = ST$ . Then  $R_S$  is  $w^*$ -continuous whereas  $L_S$  is  $w^*$ -continuous if (and only if)  $S$  is  $w^*$ -continuous.*

*Proof.* The tensor product mapping  $I_{X_*} \otimes S$  on  $X_* \otimes X$  uniquely extends to a bounded map  $r_S: X_* \widehat{\otimes} X \rightarrow X_* \widehat{\otimes} X$ , and we have  $R_S = r_S^*$ . Thus  $R_S$  is  $w^*$ -continuous. Likewise, if  $S$  is  $w^*$ -continuous and if we let  $S_*: X_* \rightarrow X_*$  be its pre-adjoint map, the tensor product mapping  $S_* \otimes I_X$  on  $X_* \otimes X$  extends to a bounded map  $l_S: X_* \widehat{\otimes} X \rightarrow X_* \widehat{\otimes} X$ , and  $L_S = l_S^*$ . Thus  $L_S$  is  $w^*$ -continuous. The ‘only if’ part (which we will not use) is left to the reader.  $\square$

**Lemma 2.3.** *Let  $u: C(K) \rightarrow B(X)$  be a bounded map. Suppose that  $X$  is a dual space. Then there exists a (necessarily unique)  $w^*$ -continuous linear mapping  $\tilde{u}: C(K)^{**} \rightarrow B(X)$  whose restriction to  $C(K)$  coincides with  $u$ . Moreover  $\|\tilde{u}\| = \|u\|$ .*

*If further  $u$  is a homomorphism, and  $u$  is valued in  $w^*B(X)$ , then  $\tilde{u}$  is a homomorphism as well.*

*Proof.* Let  $j: (X_* \widehat{\otimes} X) \hookrightarrow (X_* \widehat{\otimes} X)^{**}$  be the canonical injection and consider its adjoint  $p = j^*: B(X)^{**} \rightarrow B(X)$ . Then set

$$\tilde{u} = p \circ u^{**}: C(K)^{**} \longrightarrow B(X).$$

By construction,  $\tilde{u}$  is  $w^*$ -continuous and extends  $u$ . The equality  $\|\tilde{u}\| = \|u\|$  is clear.

Assume now that  $u$  is a homomorphism and that  $u$  is valued in  $w^*B(X)$ . Let  $\nu, \xi \in C(K)^{**}$  and let  $(f_\alpha)_\alpha$  and  $(g_\beta)_\beta$  be bounded nets in  $C(K)$   $w^*$ -converging to  $\nu$  and  $\xi$  respectively. By both parts of Lemma 2.2, we have the following equalities, where limits are taken in the  $w^*$ -topology of either  $C(K)^{**}$  or  $B(X)$ :

$$\tilde{u}(\nu\xi) = \tilde{u}(\lim_\alpha \lim_\beta f_\alpha g_\beta) = \lim_\alpha \lim_\beta u(f_\alpha g_\beta) = \lim_\alpha \lim_\beta u(f_\alpha)u(g_\beta) = \lim_\alpha u(f_\alpha)\tilde{u}(\xi) = \tilde{u}(\nu)\tilde{u}(\xi).$$

$\square$

We refer e.g. to [17, Lem. 2.4] for the following fact.

**Lemma 2.4.** *Consider  $\tau \subset B(X)$  and set  $\tau^{**} = \{T^{**} : T \in \tau\} \subset B(X^{**})$ . Then  $\tau$  is  $R$ -bounded if and only if  $\tau^{**}$  is  $R$ -bounded, and in this case,*

$$R(\tau) = R(\tau^{**}).$$

For any  $F \in C(K; B(X))$ , we set

$$R(F) = R(\{F(t) : t \in K\}).$$

Note that  $R(F)$  may be infinite. If  $F = \sum_k f_k \otimes b_k$  belongs to the algebraic tensor product  $C(K) \otimes B(X)$ , we set

$$\left\| \sum_k f_k \otimes b_k \right\|_R = R(F) = R\left(\left\{ \sum_k f_k(t) b_k : t \in K \right\}\right).$$

Note that by (1.4), we have

$$(2.2) \quad \|f \otimes b\|_R \leq 2\|f\|_\infty \|b\|, \quad f \in C(K), \quad b \in B(X).$$

From this it is easy to check that  $\|\cdot\|_R$  is finite and is a norm on  $C(K) \otimes B(X)$ .

Whenever  $E \subset B(X)$  is a closed subspace, we let

$$C(K) \overset{R}{\otimes} E$$

denote the completion of  $C(K) \otimes E$  for the norm  $\|\cdot\|_R$ .

**Remark 2.5.** Since the  $R$ -bound of a set is greater than its uniform bound, we have  $\|\cdot\|_\infty \leq \|\cdot\|_R$  on  $C(K) \otimes B(X)$ . Hence the canonical embedding of  $C(K) \otimes B(X)$  into  $C(K; B(X))$  uniquely extends to a contraction

$$J: C(K) \overset{R}{\otimes} B(X) \longrightarrow C(K; B(X)).$$

Moreover  $J$  is 1-1 and for any  $\varphi \in C(K) \overset{R}{\otimes} B(X)$ , we have  $R(J(\varphi)) = \|\varphi\|_R$ . To see this, let  $(F_n)_{n \geq 1}$  be a sequence in  $C(K) \otimes B(X)$  such that  $\|F_n - \varphi\|_R \rightarrow 0$  and let  $F = J(\varphi)$ . Then  $\|F_n\|_R \rightarrow \|\varphi\|_R$  and  $\|F_n - F\|_\infty \rightarrow 0$ . According to the definition of the  $R$ -bound, the latter property implies that  $\|F_n\|_R \rightarrow \|F\|_R$ , which yields the result.

**Theorem 2.6.** *Let  $u: C(K) \rightarrow B(X)$  be a bounded homomorphism.*

(1) *For any finite families  $(f_k)_k$  in  $C(K)$  and  $(b_k)_k$  in  $E_u$ , we have*

$$\left\| \sum_k u(f_k) b_k \right\| \leq \|u\|^2 \left\| \sum_k f_k \otimes b_k \right\|_R.$$

(2) *There is a (necessarily unique) bounded linear map*

$$\widehat{u}: C(K) \overset{R}{\otimes} E_u \longrightarrow B(X)$$

*such that  $\widehat{u}(f \otimes b) = u(f)b$  for any  $f \in C(K)$  and any  $b \in E_u$ . Moreover,  $\|\widehat{u}\| \leq \|u\|^2$ .*

*Proof.* Part (2) clearly follows from part (1). To prove (1) we introduce

$$w: C(K) \longrightarrow B(X^{**}), \quad w(f) = u(f)^{**}.$$

Then  $w$  is a bounded homomorphism and  $\|w\| = \|u\|$ . We let  $\tilde{w}: C(K)^{**} \rightarrow B(X^{**})$  be its  $w^*$ -continuous extension given by Lemma 2.3. Note that  $w$  is valued in  $w^*B(X^{**})$ , so  $\tilde{w}$  is a homomorphism. We claim that

$$\{b^{**} : b \in E_u\} \subset E_{\tilde{w}}.$$

Indeed, let  $b \in E_u$ . Then for all  $f \in C(K)$ , we have

$$b^{**} w(f) = (bu(f))^{**} = (u(f)b)^{**} = w(f)b^{**}.$$

Next for any  $\nu \in C(K)^{**}$ , let  $(f_\alpha)_\alpha$  be a bounded net in  $C(K)$  which converges to  $\nu$  in the  $w^*$ -topology. Then by Lemma 2.2, we have

$$b^{**} \tilde{w}(\nu) = \lim_\alpha b^{**} w(f_\alpha) = \lim_\alpha w(f_\alpha) b^{**} = \tilde{w}(\nu) b^{**},$$

and the claim follows.

Now fix some  $f_1, \dots, f_n \in C(K)$  and  $b_1, \dots, b_n \in E_u$ . For any  $m \in \mathbb{N}$ , there is a finite family  $(t_1, \dots, t_N)$  of  $K$  and a measurable partition  $(B_1, \dots, B_N)$  of  $K$  such that

$$\left\| f_k - \sum_{l=1}^N f_k(t_l) \chi_{B_l} \right\|_\infty \leq \frac{1}{m}, \quad k = 1, \dots, n.$$

We set  $f_k^{(m)} = \sum_{l=1}^N f_k(t_l) \chi_{B_l}$ . Let  $\psi: \ell_N^\infty \rightarrow \mathcal{B}^\infty(K)$  be defined by

$$\psi(\alpha_1, \dots, \alpha_N) = \sum_{l=1}^N \alpha_l \chi_{B_l}.$$

Then  $\psi$  is a norm 1 homomorphism. According to (2.1), we can consider the bounded homomorphism

$$\tilde{w} \circ \psi: \ell_N^\infty \longrightarrow B(X^{**}).$$

Applying Proposition 2.1 to that homomorphism, together with the above claim and Lemma 2.4, we obtain that

$$\begin{aligned} \left\| \sum_k \tilde{w}(f_k^{(m)}) b_k^{**} \right\| &= \left\| \sum_{k,l} f_k(t_l) \tilde{w} \circ \psi(e_l) b_k^{**} \right\| \\ &\leq \|\tilde{w} \circ \psi\|^2 R\left(\left\{ \sum_k f_k(t_l) b_k^{**} : 1 \leq l \leq N \right\}\right) \\ &\leq \|u\|^2 R\left(\left\{ \sum_k f_k(t) b_k^{**} : t \in K \right\}\right) \\ &\leq \|u\|^2 \left\| \sum_k f_k \otimes b_k \right\|_R. \end{aligned}$$

Since  $\|f_k^{(m)} - f_k\|_\infty \rightarrow 0$  for any  $k$ , we have

$$\left\| \sum_k \tilde{w}(f_k^{(m)}) b_k^{**} \right\| \longrightarrow \left\| \sum_k w(f_k) b_k^{**} \right\| = \left\| \sum_k u(f_k) b_k \right\|,$$



and the result follows at once.  $\square$

The following notion is implicit in several recent papers on functional calculi (see in particular [21, 8]).

**Definition 2.7.** *Let  $Z$  be a Banach space and let  $v: Z \rightarrow B(X)$  be a bounded map. We set*

$$R(v) = R(\{v(z) : z \in Z, \|z\| \leq 1\}),$$

*and we say that  $v$  is  $R$ -bounded if  $R(v) < \infty$ .*

**Corollary 2.8.** *Let  $u: C(K) \rightarrow B(X)$  be a bounded homomorphism and let  $v: Z \rightarrow B(X)$  be an  $R$ -bounded map. Assume further that  $u(f)v(z) = v(z)u(f)$  for any  $f \in C(K)$  and any  $z \in Z$ . Then there exists a (necessarily unique) bounded linear map*

$$u \cdot v: C(K; Z) \longrightarrow B(X)$$

*such that  $u \cdot v(f \otimes z) = u(f)v(z)$  for any  $f \in C(K)$  and any  $z \in Z$ . Moreover we have*

$$\|u \cdot v\| \leq \|u\|^2 R(v).$$

*Proof.* Consider any finite families  $(f_k)_k$  in  $C(K)$  and  $(z_k)_k$  in  $Z$  and observe that

$$\left\| \sum_k f_k \otimes v(z_k) \right\|_R = R\left(\left\{v\left(\sum_k f_k(t)z_k\right) : t \in K\right\}\right) \leq R(v) \left\| \sum_k f_k \otimes z_k \right\|_\infty.$$

Then applying Theorem 2.6 and the assumption that  $v$  is valued in  $E_u$ , we obtain that

$$\left\| \sum_k u(f_k)v(z_k) \right\| \leq \|u\|^2 R(v) \left\| \sum_k f_k \otimes z_k \right\|_\infty,$$

which proves the result.  $\square$

**Remark 2.9.** As a special case of Corollary 2.8, we obtain the following result due to De Pagter and Ricker ([8, Prop. 2.27]): Let  $K_1, K_2$  be two compact sets, let

$$u: C(K_1) \longrightarrow B(X) \quad \text{and} \quad v: C(K_2) \longrightarrow B(X)$$

be two bounded homomorphisms which commute, i.e.  $u(f)v(g) = v(g)u(f)$  for all  $f \in C(K_1)$  and  $g \in C(K_2)$ . Assume further that  $R(v) < \infty$ . Then there exists a bounded homomorphism

$$w: C(K_1 \times K_2) \longrightarrow B(X)$$

such that  $w|_{C(K_1)} = u$  and  $w|_{C(K_2)} = v$ , where  $C(K_j)$  is regarded as a subalgebra of  $C(K_1 \times K_2)$  in the natural way.

### 3. UNIFORMLY BOUNDED $H^\infty$ -CALCULUS

We briefly recall the basic notions on  $H^\infty$ -calculus for sectorial operators. For more information, we refer e.g. to [6, 21, 23, 24].

For any  $\theta \in (0, 2\pi)$ , we define

$$\Sigma_\theta = \{re^{i\phi} : r > 0, |\phi| < \theta\}$$

and

$$H^\infty(\Sigma_\theta) = \{f : \Sigma_\theta \rightarrow \mathbb{C} \mid f \text{ is analytic and bounded}\}.$$

This space is equipped with the norm  $\|f\|_{\infty, \theta} = \sup_{\lambda \in \Sigma_\theta} |f(\lambda)|$  and this is a Banach algebra. We consider the auxiliary space

$$H_0^\infty(\Sigma_\theta) = \{f \in H^\infty(\Sigma_\theta) : \exists \epsilon, C > 0 \mid |f(\lambda)| \leq C \min(|\lambda|^\epsilon, |\lambda|^{-\epsilon})\}.$$

A closed linear operator  $A : D(A) \subset X \rightarrow X$  is called  $\omega$ -sectorial, for some  $\omega \in (0, 2\pi)$ , if its domain  $D(A)$  is dense in  $X$ , its spectrum  $\sigma(A)$  is contained in  $\overline{\Sigma_\omega}$ , and for all  $\theta > \omega$  there is a constant  $C_\theta > 0$  such that

$$\|\lambda(\lambda - A)^{-1}\| \leq C_\theta, \quad \lambda \in \mathbb{C} \setminus \overline{\Sigma_\theta}.$$

In this case, we define

$$\omega(A) = \inf\{\omega : A \text{ is } \omega\text{-sectorial}\}.$$

For any  $\theta \in (\omega(A), \pi)$  and any  $f \in H_0^\infty(\Sigma_\theta)$ , we define

$$(3.1) \quad f(A) = \frac{1}{2\pi i} \int_{\Gamma_\gamma} f(\lambda)(\lambda - A)^{-1} d\lambda,$$

where  $\omega(A) < \gamma < \theta$  and  $\Gamma_\gamma$  is the boundary  $\partial\Sigma_\gamma$  oriented counterclockwise. This definition does not depend on  $\gamma$  and the resulting mapping  $f \mapsto f(A)$  is an algebra homomorphism from  $H_0^\infty(\Sigma_\theta)$  into  $B(X)$ . We say that  $A$  has a bounded  $H^\infty(\Sigma_\theta)$ -calculus if the latter homomorphism is bounded, that is, there exists a constant  $C > 0$  such that  $\|f(A)\| \leq C\|f\|_{\infty, \theta}$  for all  $f \in H_0^\infty(\Sigma_\theta)$ .

We will now focus on sectorial operators  $A$  such that  $\omega(A) = 0$ .

**Definition 3.1.** *We say that a sectorial operator  $A$  with  $\omega(A) = 0$  has a uniformly bounded  $H^\infty$ -calculus, if there exists a constant  $C > 0$  such that  $\|f(A)\| \leq C\|f\|_{\infty, \theta}$  for all  $\theta > 0$  and  $f \in H_0^\infty(\Sigma_\theta)$ .*

The space

$$C_\ell([0, \infty)) = \{f : [0, \infty) \rightarrow \mathbb{C} \mid f \text{ is continuous and } \lim_{\infty} f \text{ exists}\}.$$

is a unital commutative  $C^*$ -algebra when equipped with the natural norm

$$\|f\|_{\infty, 0} = \sup\{|f(t)| : t \geq 0\}$$

and involution. For any  $\theta > 0$ , we can regard  $H_0^\infty(\Sigma_\theta)$  as a subalgebra of  $C_\ell([0, \infty))$ , by identifying any  $f \in H_0^\infty(\Sigma_\theta)$  with its restriction  $f|_{[0, \infty)}$ .

For any  $\lambda \in \mathbb{C} \setminus [0, \infty)$ , we let  $R_\lambda \in C_\ell([0, \infty))$  be defined by  $R_\lambda(t) = (\lambda - t)^{-1}$ . Then we let  $\mathcal{R}$  be the unital algebra generated by the  $R_\lambda$ 's. Equivalently,  $\mathcal{R}$  is the algebra of all rational functions of nonpositive degree, whose poles lie outside the half line  $[0, \infty)$ . We

recall that for any  $f \in H_0^\infty(\Sigma_\theta) \cap \mathcal{R}$ , the definition of  $f(A)$  given by (3.1) coincides with the usual rational functional calculus.

The following lemma is closely related to [22, Cor. 6.9].

**Lemma 3.2.** *Let  $A$  be a sectorial operator on  $X$  with  $\omega(A) = 0$ . The following assertions are equivalent.*

- (a)  *$A$  has a uniformly bounded  $H^\infty$ -calculus.*
- (b) *There exists a (necessarily unique) bounded unital homomorphism*

$$u: C_\ell([0, \infty)) \longrightarrow B(X)$$

*such that  $u(R_\lambda) = (\lambda - A)^{-1}$  for any  $\lambda \in \mathbb{C} \setminus [0, \infty)$ .*

*Proof.* Assume (a). We claim that for any  $\theta > 0$  and any  $f \in H_0^\infty(\Sigma_\theta)$ , we have

$$\|f(A)\| \leq C\|f\|_{\infty,0}.$$

Indeed, if  $0 \neq f \in H_0^\infty(\Sigma_{\theta_0})$  for some  $\theta_0 > 0$ , then there exists some  $t_0 > 0$  such that  $f(t_0) \neq 0$ . Now let  $r < R$  such that  $|f(z)| < |f(t_0)|$  for  $|z| < r$  and  $|z| > R$ . Choose for every  $n \in \mathbb{N}$  a  $t_n \in \Sigma_{\theta_0/n}$  such that  $|f(t_n)| = \|f\|_{\infty, \theta_0/n}$ . Necessarily  $|t_n| \in [r, R]$ , and there exists a convergent subsequence  $t_{n_k}$ , whose limit  $t_\infty$  is real. Then

$$\|f\|_{\infty,0} \geq |f(t_\infty)| \geq \liminf_{\theta \rightarrow 0} \|f\|_{\infty,\theta} \geq C^{-1}\|f(A)\|.$$

This readily implies that the rational functional calculus  $(\mathcal{R}, \|\cdot\|_{\infty,0}) \rightarrow B(X)$  is bounded. By Stone-Weierstrass, this extends continuously to  $C_\ell([0, \infty))$ , which yields (b). The uniqueness property is clear.

Assume (b). Then for any  $\theta \in (0, \pi)$  and for any  $f \in H_0^\infty(\Sigma_\theta) \cap \mathcal{R}$ , we have

$$\|f(A)\| \leq \|u\|\|f\|_{\infty,\theta}.$$

By [24, Prop. 2.10] and its proof, this implies that  $A$  has a bounded  $H^\infty(\Sigma_\theta)$ -calculus, with a boundedness constant uniform in  $\theta$ .  $\square$

**Remark 3.3.** An operator  $A$  which admits a bounded  $H^\infty(\Sigma_\theta)$ -calculus for all  $\theta > 0$  does not necessarily have a uniformly bounded  $H^\infty$ -calculus. To get a simple example, consider

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} : \ell_2^2 \longrightarrow \ell_2^2.$$

Then  $\sigma(A) = \{1\}$  and for any  $\theta > 0$  and any  $f \in H_0^\infty(\Sigma_\theta)$ , we have

$$f(A) = \begin{pmatrix} f(1) & f'(1) \\ 0 & f(1) \end{pmatrix}.$$

Assume that  $\theta < \frac{\pi}{2}$ . Using Cauchy's Formula, it is easy to see that  $|f'(1)| \leq (\sin(\theta))^{-1}\|f\|_{\infty,\theta}$  for any  $f \in H_0^\infty(\Sigma_\theta)$ . Thus  $A$  admits a bounded  $H^\infty(\Sigma_\theta)$ -calculus.

Now let  $h$  be a fixed function in  $H_0^\infty(\Sigma_{\frac{\pi}{2}})$  such that  $h(1) = 1$ , set  $g_s(\lambda) = \lambda^{is}$  for any  $s > 0$ , and let  $f_s = hg_s$ . Then  $\|g_s\|_{\infty,0} = 1$ , hence  $\|f_s\|_{\infty,0} \leq \|h\|_{\infty,0}$  for any  $s > 0$ . Further  $g'_s(\lambda) = is\lambda^{is-1}$  and  $f'_s = h'g_s + hg'_s$ . Hence  $f'_s(1) = h'(1) + is$ . Thus

$$\|f_s(A)\|\|f_s\|_{\infty,0}^{-1} \geq |f'_s(1)|\|h_s\|_{\infty,0}^{-1} \longrightarrow \infty$$

when  $s \rightarrow \infty$ . Hence  $A$  does not have a uniformly bounded  $H^\infty$ -calculus.

The above result can also be deduced from Proposition 3.7 below. In fact we will show in that proposition and in Corollary 3.11 that operators with a uniformly bounded  $H^\infty$ -calculus are ‘rare’.

We now turn to the so-called generalized (or operator valued)  $H^\infty$ -calculus. Throughout we let  $A$  be a sectorial operator. We let  $E_A \subset B(X)$  denote the commutant of  $A$ , defined as the subalgebra of all bounded operators  $T: X \rightarrow X$  such that  $T(\lambda - A)^{-1} = (\lambda - A)^{-1}T$  for any  $\lambda$  belonging to the resolvent set of  $A$ . We let  $H_0^\infty(\Sigma_\theta; B(X))$  be the algebra of all bounded analytic functions  $F: \Sigma_\theta \rightarrow B(X)$  for which there exist  $\epsilon, C > 0$  such that  $\|F(\lambda)\| \leq C \min(|\lambda|^\epsilon, |\lambda|^{-\epsilon})$  for any  $\lambda \in \Sigma_\theta$ . Also, we let  $H_0^\infty(\Sigma_\theta; E_A)$  denote the space of all  $E_A$ -valued functions belonging to  $H_0^\infty(\Sigma_\theta; B(X))$ . The generalized  $H^\infty$ -calculus of  $A$  is an extension of (3.1) to this class of functions. Namely for any  $F \in H_0^\infty(\Sigma_\theta; E_A)$ , we set

$$F(A) = \frac{1}{2\pi i} \int_{\Gamma_\gamma} F(\lambda)(\lambda - A)^{-1} d\lambda,$$

where  $\gamma \in (\omega(A), \pi)$ . Again, this definition does not depend on  $\gamma$  and the mapping  $F \mapsto F(A)$  is an algebra homomorphism. The following fundamental result is due to Kalton and Weis.

**Theorem 3.4.** ([21, Thm. 4.4], [23, Thm. 12.7].) *Let  $\omega_0 \geq \omega(A)$  and assume that  $A$  has a bounded  $H^\infty(\Sigma_\theta)$ -calculus for any  $\theta > \omega_0$ . Then for any  $\theta > \omega_0$ , there exists a constant  $C_\theta > 0$  such that for any  $F \in H_0^\infty(\Sigma_\theta; E_A)$ ,*

$$(3.2) \quad \|F(A)\| \leq C_\theta R(\{F(z) : z \in \Sigma_\theta\}).$$

Our aim is to prove a version of this result in the case when  $A$  has a uniformly bounded  $H^\infty$ -calculus. We will obtain in Theorem 3.6 that in this case, the constant  $C_\theta$  in (3.2) can be taken independent of  $\theta$ .

The algebra  $C_\ell([0, \infty))$  is a  $C(K)$ -space and we will apply the results of Section 2 to the bounded homomorphism  $u$  appearing in Lemma 3.2. We recall Remark 2.5.

**Lemma 3.5.** *Let  $J: C_\ell([0, \infty)) \overset{R}{\otimes} B(X) \rightarrow C_\ell([0, \infty); B(X))$  be the canonical embedding. Let  $\theta \in (0, \pi)$ , let  $F \in H_0^\infty(\Sigma_\theta; B(X))$  and let  $\gamma \in (0, \theta)$ .*

(1) *The integral*

$$(3.3) \quad \varphi_F = \frac{1}{2\pi i} \int_{\Gamma_\gamma} R_\lambda \otimes F(\lambda) d\lambda$$

*is absolutely convergent in  $C_\ell([0, \infty)) \overset{R}{\otimes} B(X)$ , and  $J(\varphi_F)$  is equal to the restriction of  $F$  to  $[0, \infty)$ .*

(2) *The set  $\{F(t) : t > 0\}$  is  $R$ -bounded.*

*Proof.* Part (2) readily follows from part (1) and Remark 2.5. To prove (1), observe that for any  $\lambda \in \partial\Sigma_\gamma$ , we have

$$\|R_\lambda \otimes F(\lambda)\|_R \leq 2\|R_\lambda\|_{\infty,0}\|F(\lambda)\| \leq \frac{2}{\sin(\gamma)|\lambda|} \|F(\lambda)\|$$

by (2.2). Thus for appropriate constants  $\epsilon, C > 0$ , we have

$$\|R_\lambda \otimes F(\lambda)\|_R \leq \frac{2C}{\sin(\gamma)} \min(|\lambda|^{\epsilon-1}, |\lambda|^{-\epsilon-1}).$$

This shows that the integral defining  $\varphi_F$  is absolutely convergent. Next, for any  $t > 0$ , we have

$$[J(\varphi_F)](t) = \frac{1}{2\pi i} \int_{\Gamma_\gamma} (R_\lambda \otimes F(\lambda))(t) d\lambda = \frac{1}{2\pi i} \int_{\Gamma_\gamma} \frac{F(\lambda)}{\lambda - t} d\lambda = F(t)$$

by Cauchy's Theorem.  $\square$

**Theorem 3.6.** *Let  $A$  be a sectorial operator with  $\omega(A) = 0$  and assume that  $A$  has a uniformly bounded  $H^\infty$ -calculus. Then there exists a constant  $C > 0$  such that for any  $\theta > 0$  and any  $F \in H_0^\infty(\Sigma_\theta; E_A)$ ,*

$$\|F(A)\| \leq CR(\{F(t) : t > 0\}).$$

*Proof.* Let  $u: C_\ell([0, \infty)) \rightarrow B(X)$  be the representation given by Lemma 3.2. It is plain that  $E_u = E_A$ . Then we let

$$\widehat{u}: C_\ell([0, \infty)) \otimes^R E_A \longrightarrow B(X)$$

be the associated bounded map provided by Theorem 2.6.

Let  $F \in H_0^\infty(\Sigma_\theta; E_A)$  for some  $\theta > 0$ , and let  $\varphi_F \in C_\ell([0, \infty)) \otimes^R E_A$  be defined by (3.3). We claim that

$$F(A) = \widehat{u}(\varphi_F).$$

Indeed for any  $\lambda \in \partial\Sigma_\gamma$ , we have  $u(R_\lambda) = (\lambda - A)^{-1}$ , hence  $\widehat{u}(R_\lambda \otimes F(\lambda)) = (\lambda - A)^{-1}F(\lambda)$ . Thus according to the definition of  $\varphi_F$  and the continuity of  $\widehat{u}$ , we have

$$\widehat{u}(\varphi_F) = \frac{1}{2\pi i} \int_{\Gamma_\gamma} \widehat{u}(R_\lambda \otimes F(\lambda)) d\lambda = \frac{1}{2\pi i} \int_{\Gamma_\gamma} (\lambda - A)^{-1}F(\lambda) d\lambda = F(A).$$

Consequently,

$$\|F(A)\| \leq \|\widehat{u}\| \|\varphi_F\|_R \leq \|u\|^2 \|\varphi_F\|_R.$$

It follows from Lemma 3.5 and Remark 2.5 that  $\|\varphi_F\|_R = R(\{F(t) : t > 0\})$ , and the result follows at once.  $\square$

In the rest of this section we will further investigate operators with a uniformly bounded  $H^\infty$ -calculus. We start with the case when  $X$  is a Hilbert space.

**Proposition 3.7.** *Let  $H$  be a Hilbert space and let  $A$  be a sectorial operator on  $H$ , with  $\omega(A) = 0$ . Then  $A$  admits a uniformly bounded  $H^\infty$ -calculus if and only if there exists an isomorphism  $S: H \rightarrow H$  such that  $S^{-1}AS$  is selfadjoint.*

*Proof.* Assume that  $A$  admits a uniformly bounded  $H^\infty$ -calculus and let  $u: C_\ell([0, \infty)) \rightarrow B(H)$  be the associated representation. According to [28, Thm. 9.1 and Thm. 9.7], there exists an isomorphism  $S: H \rightarrow H$  such that the unital homomorphism  $u_S: C_\ell([0, \infty)) \rightarrow B(H)$  defined by  $u_S(f) = S^{-1}u(f)S$  satisfies  $\|u_S\| \leq 1$ . We let  $B = S^{-1}AS$ . For any  $s \in \mathbb{R}^*$ , we have  $\|R_{is}\|_{\infty,0} = |s|$  and  $u_S(R_{is}) = S^{-1}(is - A)^{-1}S = (is - B)^{-1}$ . Hence

$$\|(is - B)^{-1}\| \leq |s|, \quad s \in \mathbb{R}^*.$$

By the Hille-Yosida Theorem, this implies that  $iB$  and  $-iB$  both generate contractive  $c_0$ -semigroups on  $H$ . Thus  $iB$  generates a unitary  $c_0$ -group. By Stone's Theorem, this implies that  $B$  is selfadjoint.

The converse implication is clear.  $\square$

In the non Hilbertian setting, we will first show that operators with a uniformly bounded  $H^\infty$ -calculus satisfy a spectral mapping theorem with respect to continuous functions defined on the one-point compactification of  $\sigma(A)$ . Then we will discuss the connections with spectral measures and scalar-type operators. We mainly refer to [13, Chap. 5-7] for this topic.

For any compact set  $K$  and any closed subset  $F \subset K$ , we let

$$I_F = \{f \in C(K) : f|_F = 0\}.$$

We recall that the restriction map  $f \mapsto f|_F$  induces a  $*$ -isomorphism  $C(K)/I_F \rightarrow C(F)$ .

**Lemma 3.8.** *Let  $K \subset \mathbb{C}$  be a compact set and let  $u: C(K) \rightarrow B(X)$  be a representation. Let  $\kappa \in C(K)$  be the function defined by  $\kappa(z) = z$  and put  $T = u(\kappa)$ .*

(1) *Then  $\sigma(T) \subset K$  and  $u$  vanishes on  $I_{\sigma(T)}$ .*

*Let  $v: C(\sigma(T)) \simeq C(K)/I_{\sigma(T)} \rightarrow B(X)$  be the representation induced by  $u$ .*

(2) *For any  $f \in C(\sigma(T))$ , we have  $\sigma(v(f)) = f(\sigma(T))$ .*

(3)  *$v$  is an isomorphism onto its range.*

*Proof.* The inclusion  $\sigma(T) \subset K$  is clear. Indeed, for any  $\lambda \notin K$ ,  $(\lambda - T)^{-1}$  is equal to  $u((\lambda - \cdot)^{-1})$ . We will now show that  $u$  vanishes on  $I_{\sigma(T)}$ .

Let  $w: C(K) \rightarrow B(X^*)$  be defined by  $w(f) = [u(f)]^*$ , and let  $\tilde{w}: C(K)^{**} \rightarrow B(X^*)$  be its  $w^*$ -extension. Since  $w$  is valued in  $w^*B(X^*) \simeq B(X)$ , this is a representation (see Lemma 2.3). Let  $\Delta_K$  be the set of all Borel subsets of  $K$ . It is easy to check that the mapping

$$P: \Delta_K \longrightarrow B(X^*), \quad P(B) = \tilde{w}(\chi_B),$$

is a spectral measure of class  $(\Delta_K, X)$  in the sense of [13, p. 119]. According to [13, Prop. 5.8], the operator  $T^*$  is prespectral of class  $X$  (in the sense of [13, Def. 5.5]) and the above mapping  $P$  is its resolution of the identity. Applying [13, Lem. 5.6] and the equality  $\sigma(T^*) = \sigma(T)$ , we obtain that  $\tilde{w}(\chi_{\sigma(T)}) = P(\sigma(T)) = I_{X^*}$ . Therefore for any  $f \in I_{\sigma(T)}$ , we have

$$u(f)^* = \tilde{w}(f(1 - \chi_{\sigma(T)})) = \tilde{w}(f)\tilde{w}(1 - \chi_{\sigma(T)}) = 0.$$

Hence  $u$  vanishes on  $I_{\sigma(T)}$ .

The proofs of (2) and (3) now follow from [13, Prop. 5.9] and the above proof.  $\square$

In the sequel we consider a sectorial operator  $A$  with  $\omega(A) = 0$ . This assumption implies that  $\sigma(A) \subset [0, \infty)$ . By  $C_\ell(\sigma(A))$ , we denote either the space  $C(\sigma(A))$  if  $A$  is bounded, or the space  $\{f: \sigma(A) \rightarrow \mathbb{C} \mid f \text{ is continuous and } \lim_\infty f \text{ exists}\}$  if  $A$  is unbounded. In this case,  $C_\ell(\sigma(A))$  coincides with the space of continuous functions on the one-point compactification of  $\sigma(A)$ . The following strengthens Lemma 3.2.

**Proposition 3.9.** *Let  $A$  be a sectorial operator on  $X$  with  $\omega(A) = 0$ . The following assertions are equivalent.*

(1)  *$A$  has a uniformly bounded  $H^\infty$ -calculus.*

(2) *There exists a (necessarily unique) bounded unital homomorphism*

$$\Psi: C_\ell(\sigma(A)) \longrightarrow B(X)$$

*such that  $\Psi((\lambda - \cdot)^{-1}) = (\lambda - A)^{-1}$  for any  $\lambda \in \mathbb{C} \setminus \sigma(A)$ .*

*In this case,  $\Psi$  is an isomorphism onto its range and for any  $f \in C_\ell(\sigma(A))$ , we have*

$$(3.4) \quad \sigma(\Psi(f)) = f(\sigma(A)) \cup f_\infty,$$

*where  $f_\infty = \emptyset$  if  $A$  is bounded and  $f_\infty = \{\lim_\infty f\}$  if  $A$  is unbounded.*

*Proof.* Assume (1) and let  $u: C_\ell([0, \infty)) \rightarrow B(X)$  be given by Lemma 3.2. We introduce the particular function  $\phi \in C_\ell([0, \infty))$  defined by  $\phi(t) = (1+t)^{-1}$ . Consider the  $*$ -isomorphism

$$\tau: C([0, 1]) \longrightarrow C_\ell([0, \infty)), \quad \tau(g) = g \circ \phi,$$

and set  $T = (1+A)^{-1}$ . If we let  $\kappa(z) = z$  as in Lemma 3.8, we have  $(u \circ \tau)(\kappa) = T$ . Let  $v: C(\sigma(T)) \rightarrow B(X)$  be the resulting factorisation of  $u \circ \tau$ . The spectral mapping theorem gives  $\sigma(A) = \phi^{-1}(\sigma(T) \setminus \{0\})$  and  $0 \in \sigma(T)$  if and only if  $A$  is unbounded. Thus the mapping

$$\tau_A: C(\sigma(T)) \longrightarrow C_\ell(\sigma(A))$$

defined by  $\tau_A(g) = g \circ \phi$  also is a  $*$ -isomorphism. Put  $\Psi = v \circ \tau_A^{-1}: C_\ell(\sigma(A)) \rightarrow B(X)$ . This is a unital bounded homomorphism. Note that  $\phi^{-1}(z) = \frac{1-z}{z}$  for any  $z \in (0, 1]$ . Then for any  $\lambda \in \mathbb{C} \setminus \sigma(A)$ ,

$$\begin{aligned} \Psi((\lambda - \cdot)^{-1}) &= v((\lambda - \cdot)^{-1} \circ \phi^{-1}) = v\left(z \mapsto \left(\lambda - \frac{1-z}{z}\right)^{-1}\right) \\ &= v\left(z \mapsto \frac{z}{(\lambda+1)z-1}\right) \\ &= T((\lambda+1)T-1)^{-1} = (\lambda - A)^{-1}. \end{aligned}$$

Hence  $\Psi$  satisfies (2). Its uniqueness follows from Lemma 3.2. The fact that  $\Psi$  is an isomorphism onto its range, and the spectral property (3.4) follow from the above construction and Lemma 3.8. Finally the implication ‘(2)  $\Rightarrow$  (1)’ also follows from Lemma 3.2.  $\square$

**Remark 3.10.** Let  $A$  be a sectorial operator with a uniformly bounded  $H^\infty$ -calculus, and let  $T = (1+A)^{-1}$ . It follows from Lemma 3.8 and the proof of Proposition 3.9 that there exists a representation

$$v: C(\sigma(T)) \longrightarrow B(X)$$

satisfying  $v(\kappa) = T$  (where  $\kappa(z) = z$ ), such that  $\sigma(v(f)) = f(\sigma(T))$  for any  $f \in C(\sigma(T))$  and  $v$  is an isomorphism onto its range. Also, it follows from the proof of Lemma 3.8 that  $T^*$  is a scalar-type operator of class  $X$ , in the sense of [13, Def. 5.14].

Next according to [13, Thm. 6.24], the operator  $T$  (and hence  $A$ ) is a scalar-type spectral operator if and only if for any  $x \in X$ , the mapping  $C(\sigma(T)) \rightarrow X$  taking  $f$  to  $v(f)x$  for any  $f \in C(\sigma(T))$  is weakly compact.

**Corollary 3.11.** *Let  $A$  be a sectorial operator on  $X$ , with  $\omega(A) = 0$ , and assume that  $X$  does not contain a copy of  $c_0$ . Then  $A$  admits a uniformly bounded  $H^\infty$ -calculus if and only if it is a scalar-type spectral operator.*

*Proof.* The ‘only if’ part follows from the previous remark. Indeed if  $X$  does not contain a copy of  $c_0$ , then any bounded map  $C(K) \rightarrow X$  is weakly compact [9, VI, Thm. 15]. (See also [31] and [8] for related approaches.) The ‘if’ part follows from [16, Prop. 2.7] and its proof.  $\square$

**Remark 3.12.**

(1) The hypothesis on  $X$  in Corollary 3.11 is necessary. Namely it follows from [12, Thm. 3.2] and its proof that if  $c_0 \subset X$ , then there is a sectorial operator  $A$  with a uniformly bounded  $H^\infty$ -calculus on  $X$  which is not scalar-type spectral.

(2) Scalar-type spectral operators on Hilbert space coincide with operators similar to a normal one (see [13, Chap. 7]). Thus when  $X = H$  is a Hilbert space, the above corollary reduces to Proposition 3.7.

#### 4. MATRICIAL $R$ -BOUNDEDNESS

For any integer  $n \geq 1$  and any vector space  $E$ , we will denote by  $M_n(E)$  the space of all  $n \times n$  matrices with entries in  $E$ . We will be mostly concerned with the cases  $E = C(K)$  or  $E = B(X)$ . As mentioned in the introduction, we identify  $M_n(C(K))$  with the space  $C(K; M_n)$  in the usual way. We now introduce a specific norm on  $M_n(B(X))$ . Namely for any  $[T_{ij}] \in M_n(B(X))$ , we set

$$\|[T_{ij}]\|_R = \sup \left\{ \left\| \sum_{i,j=1}^n \epsilon_i \otimes T_{ij}(x_j) \right\|_{\text{Rad}(X)} : x_1, \dots, x_n \in X, \left\| \sum_{j=1}^n \epsilon_j \otimes x_j \right\|_{\text{Rad}(X)} \leq 1 \right\}.$$

Clearly  $\|\cdot\|_R$  is a norm on  $M_n(B(X))$ . Moreover if we consider any element of  $M_n(B(X))$  as an operator on  $\ell_n^2 \otimes X$  in the natural way, and if we equip the latter tensor product with the norm of  $\text{Rad}_n(X)$ , we obtain an isometric identification

$$(4.1) \quad (M_n(B(X)), \|\cdot\|_R) = B(\text{Rad}_n(X)).$$

**Definition 4.1.** Let  $u: C(K) \rightarrow B(X)$  be a bounded linear mapping. We say that  $u$  is *matricially  $R$ -bounded* if there is a constant  $C \geq 0$  such that for any  $n \geq 1$ , and for any  $[f_{ij}] \in M_n(C(K))$ , we have

$$(4.2) \quad \|[u(f_{ij})]\|_R \leq C \|[f_{ij}]\|_{C(K; M_n)}.$$

**Remark 4.2.** The above definition obviously extends to any bounded map  $E \rightarrow B(X)$  defined on an operator space  $E$ , or more generally on any matricially normed space (see [14, 15]). The basic observations below apply to this general case as well.

(1) In the case that  $X = H$  is a Hilbert space, we have

$$\left\| \sum_{j=1}^n \epsilon_j \otimes x_j \right\|_{\text{Rad}(H)} = \left( \sum_{j=1}^n \|x_j\|^2 \right)^{\frac{1}{2}}$$

for any  $x_1, \dots, x_n \in H$ . Consequently, writing that a mapping  $u: C(K) \rightarrow B(H)$  is matricially  $R$ -bounded is equivalent to writing that  $u$  is completely bounded (see e.g. [28]). See Section 5 for the case when  $X$  is an  $L^p$ -space.



(2) The notation  $\|\cdot\|_R$  introduced above is consistent with the one considered so far in Section 2. Indeed let  $b_1, \dots, b_n$  in  $B(X)$ . Then the diagonal matrix  $\text{Diag}\{b_1, \dots, b_n\} \in M_n(B(X))$  and the tensor element  $\sum_{k=1}^n e_k \otimes b_k \in \ell_n^\infty \otimes B(X)$  satisfy

$$\|\text{Diag}\{b_1, \dots, b_n\}\|_R = R(\{b_1, \dots, b_n\}) = \left\| \sum_{k=1}^n e_k \otimes b_k \right\|_R.$$

(3) If  $u: C(K) \rightarrow B(X)$  is matricially  $R$ -bounded (with the estimate (4.2)), then  $u$  is  $R$ -bounded and  $R(u) \leq C$ . Indeed, consider  $f_1, \dots, f_n$  in the unit ball of  $C(K)$ . Then we have  $\|\text{Diag}\{f_1, \dots, f_n\}\|_{C(K; M_n)} \leq 1$ . Hence for any  $x_1, \dots, x_n$  in  $X$ ,

$$\begin{aligned} \left\| \sum_k \epsilon_k \otimes u(f_k) x_k \right\|_{\text{Rad}(X)} &\leq \|\text{Diag}\{u(f_1), \dots, u(f_n)\}\|_R \left\| \sum_k \epsilon_k \otimes x_k \right\|_{\text{Rad}(X)} \\ &\leq C \left\| \sum_k \epsilon_k \otimes x_k \right\|_{\text{Rad}(X)}. \end{aligned}$$

Let  $(g_k)_{k \geq 1}$  be a sequence of complex valued, independent, standard Gaussian random variables on some probability space  $\Omega_G$ . For any  $x_1, \dots, x_n$  in  $X$  let

$$\left\| \sum_k g_k \otimes x_k \right\|_{G(X)} = \left( \int_{\Omega_G} \left\| \sum_k g_k(\lambda) x_k \right\|_X^2 d\lambda \right)^{\frac{1}{2}}.$$

It is well-known that for any scalar valued matrix  $a = [a_{ij}] \in M_n$ , we have

$$(4.3) \quad \left\| \sum_{i,j=1}^n a_{ij} g_i \otimes x_j \right\|_{G(X)} \leq \|a\|_{M_n} \left\| \sum_{j=1}^n g_j \otimes x_j \right\|_{G(X)},$$

see e.g. [10, Cor. 12.17]. For any  $n \geq 1$ , introduce  $\sigma_{n,X}: M_n \rightarrow B(\text{Rad}_n(X))$  by letting

$$\sigma_{n,X}([a_{ij}]) = [a_{ij} I_X].$$

If  $X$  has finite cotype, then we have a uniform equivalence

$$(4.4) \quad \left\| \sum_k \epsilon_k \otimes x_k \right\|_{\text{Rad}(X)} \asymp \left\| \sum_k g_k \otimes x_k \right\|_{G(X)}$$

between Rademacher and Gaussian averages on  $X$  (see e.g. [10, Thm. 12.27]). Combining with (4.3), this implies that

$$\sup_{n \geq 1} \|\sigma_{n,X}\| < \infty.$$

Following [29] we say that  $X$  has property  $(\alpha)$  if there is a constant  $C \geq 1$  such that for any finite family  $(x_{ij})$  in  $X$  and any finite family  $(t_{ij})$  of complex numbers,

$$(4.5) \quad \left\| \sum_{i,j} \epsilon_i \otimes \epsilon_j \otimes t_{ij} x_{ij} \right\|_{\text{Rad}(\text{Rad}(X))} \leq C \sup_{i,j} |t_{ij}| \left\| \sum_{i,j} \epsilon_i \otimes \epsilon_j \otimes x_{ij} \right\|_{\text{Rad}(\text{Rad}(X))}.$$

Equivalently,  $X$  has property  $(\alpha)$  if and only if we have a uniform equivalence

$$\left\| \sum_{i,j} \epsilon_i \otimes \epsilon_j \otimes x_{ij} \right\|_{\text{Rad}(\text{Rad}(X))} \asymp \left\| \sum_{i,j} \epsilon_{ij} \otimes x_{ij} \right\|_{\text{Rad}(X)},$$

where  $(\epsilon_{ij})_{i,j \geq 1}$  be a doubly indexed family of independent Rademacher variables.

The following is a characterization of property  $(\alpha)$  in terms of the  $R$ -boundedness of  $\sigma_{n,X}$ .

**Lemma 4.3.** *A Banach space  $X$  has property  $(\alpha)$  if and only if*

$$\sup_{n \geq 1} R(\sigma_{n,X}) < \infty.$$

*Proof.* Assume that  $X$  has property  $(\alpha)$ . This implies that  $X$  has finite cotype, hence  $X$  satisfies the equivalence property (4.4). Let  $a(1), \dots, a(N)$  be in  $M_n$  and let  $z_1, \dots, z_N$  be in  $\text{Rad}_n(X)$ . Let  $x_{jk}$  in  $X$  such that  $z_k = \sum_j \epsilon_j \otimes x_{jk}$  for any  $k$ . We consider a doubly indexed family  $(\epsilon_{ik})_{i,k \geq 1}$  as above, as well as a doubly indexed family  $(g_{ik})_{i,k \geq 1}$  of independent standard Gaussian variables. Then

$$(4.6) \quad \sum_k \epsilon_k \otimes \sigma_{n,X}(a(k))z_k = \sum_{k,i,j} \epsilon_k \otimes \epsilon_i \otimes a(k)_{ij}x_{jk}.$$

Hence using the properties reviewed above, we have

$$\begin{aligned} \left\| \sum_k \epsilon_k \otimes \sigma_{n,X}(a(k))z_k \right\|_{\text{Rad}(\text{Rad}(X))} &\asymp \left\| \sum_{k,i,j} \epsilon_{ik} \otimes a(k)_{ij}x_{jk} \right\|_{\text{Rad}(X)} \\ &\asymp \left\| \sum_{k,i,j} g_{ik} \otimes a(k)_{ij}x_{jk} \right\|_{G(X)} \\ &\lesssim \left\| \begin{pmatrix} a(1) & 0 \dots & 0 \\ 0 & \ddots & 0 \\ 0 & \dots & 0 & a(N) \end{pmatrix} \right\|_{M_{Nn}} \left\| \sum_{k,j} g_{jk} \otimes x_{jk} \right\|_{G(X)} \\ &\lesssim \max_k \|a(k)\|_{M_n} \left\| \sum_{k,j} \epsilon_{jk} \otimes x_{jk} \right\|_{\text{Rad}(X)} \\ &\lesssim \max_k \|a(k)\|_{M_n} \left\| \sum_{k,j} \epsilon_k \otimes \epsilon_j \otimes x_{jk} \right\|_{\text{Rad}(\text{Rad}(X))} \\ &\lesssim \max_k \|a(k)\|_{M_n} \left\| \sum_k \epsilon_k \otimes z_k \right\|_{\text{Rad}(\text{Rad}(X))}. \end{aligned}$$

This shows that the  $\sigma_{n,X}$ 's are uniformly  $R$ -bounded.

Conversely, assume that for some constant  $C \geq 1$ , we have  $R(\sigma_{n,X}) \leq C$  for all  $n \geq 1$ . Let  $(t_{jk})_{j,k} \in \mathbb{C}^{n^2}$  with  $|t_{jk}| \leq 1$  and for any  $k = 1, \dots, n$ , let  $a(k) \in M_n$  be the diagonal matrix with entries  $t_{1k}, \dots, t_{nk}$  on the diagonal. Then  $\|a(k)\| \leq 1$  for any  $k$ . Hence applying (4.6), we obtain that for any  $(x_{jk})_{j,k}$  in  $X^{n^2}$ ,

$$\begin{aligned} \left\| \sum_{j,k} \epsilon_k \otimes \epsilon_j \otimes t_{jk}x_{jk} \right\|_{\text{Rad}(\text{Rad}(X))} &\leq R(\{a(1), \dots, a(n)\}) \left\| \sum_{j,k} \epsilon_k \otimes \epsilon_j \otimes x_{jk} \right\|_{\text{Rad}(\text{Rad}(X))} \\ &\leq C \left\| \sum_{j,k} \epsilon_k \otimes \epsilon_j \otimes x_{jk} \right\|_{\text{Rad}(\text{Rad}(X))}. \end{aligned}$$

This means that  $X$  has property  $(\alpha)$ . □

**Proposition 4.4.** *Assume that  $X$  has property  $(\alpha)$ . Then any bounded homomorphism  $u: C(K) \rightarrow B(X)$  is matricially  $R$ -bounded.*

*Proof.* Let  $u: C(K) \rightarrow B(X)$  be a bounded homomorphism and let  $w: C(K) \rightarrow B(\text{Rad}_n(X))$  be defined by

$$w(f) = I_{\text{Rad}_n} \otimes u(f).$$

Clearly  $w$  is also a bounded homomorphism, with  $\|w\| = \|u\|$ . Recall the identification (4.1) and note that  $w(f) = \text{Diag}\{u(f), \dots, u(f)\}$  for any  $f \in C(K)$ . Then for any  $a = [a_{ij}] \in M_n$ , we have

$$w(f)\sigma_{n,X}(a) = [a_{ij}u(f)] = \sigma_{n,X}(a)w(f).$$

By Corollary 2.8 and Lemma 4.3, the resulting mapping  $w \cdot \sigma_{n,X}$  satisfies

$$\|w \cdot \sigma_{n,X}: C(K; M_n) \rightarrow B(\text{Rad}_n(X))\| \leq C\|u\|^2$$

where  $C$  does not depend on  $n$ . Let  $E_{ij}$  denote the canonical matrix units of  $M_n$ , for  $i, j = 1, \dots, n$ . Consider  $[f_{ij}] \in C(K; M_n) \simeq M_n(C(K))$  and write this matrix as  $\sum_{i,j} E_{ij} \otimes f_{ij}$ . Then

$$w \cdot \sigma_{n,X}([f_{ij}]) = \sum_{i,j=1}^n w(f_{ij})\sigma_{n,X}(E_{ij}) = \sum_{i,j=1}^n u(f_{ij}) \otimes E_{ij} = [u(f_{ij})].$$

Hence  $\|[u(f_{ij})]\|_R \leq C\|u\|^2\|[f_{ij}]\|_{C(K; M_n)}$ , which proves that  $u$  is matricially  $R$ -bounded.  $\square$

In the case that  $X = H$  is a Hilbert space, it follows from Remark 4.2 (1) that the above proposition reduces to the fact that any bounded homomorphism  $C(K) \rightarrow B(H)$  is completely bounded.

We also observe that applying the above proposition together with Remark 4.2 (3), we obtain the following corollary originally due to De Pagter and Ricker [8, Cor. 2.19]. Indeed, Proposition 4.4 should be regarded as a strengthening of their result.

**Corollary 4.5.** *Assume that  $X$  has property  $(\alpha)$ . Then any bounded homomorphism  $u: C(K) \rightarrow B(X)$  is  $R$ -bounded.*

**Remark 4.6.** The above corollary is nearly optimal. Indeed we claim that if  $X$  does not have property  $(\alpha)$  and if  $K$  is any infinite compact set, then there exists a unital bounded homomorphism

$$u: C(K) \rightarrow B(\text{Rad}(X))$$

which is not  $R$ -bounded.

To prove this, let  $(z_n)_{n \geq 1}$  be an infinite sequence of distinct points in  $K$  and let  $u$  be defined by

$$u(f) \left( \sum_{k \geq 1} \epsilon_k \otimes x_k \right) = \sum_{k \geq 1} f(z_k) \epsilon_k \otimes x_k.$$

According to (1.4), this is a bounded unital homomorphism satisfying  $\|u\| \leq 2$ . Assume now that  $u$  is  $R$ -bounded. Let  $n \geq 1$  be an integer and consider families  $(t_{ij})_{i,j}$  in  $\mathbb{C}^{n^2}$  and

$(x_{ij})_{i,j}$  in  $X^{n^2}$ . For any  $i = 1, \dots, n$ , there exists  $f_i \in C(K)$  such that  $\|f_i\| = \sup_j |t_{ij}|$  and  $f_i(z_j) = t_{ij}$  for any  $j = 1, \dots, n$ . Then

$$\sum_i \epsilon_i \otimes u(f_i) \left( \sum_j \epsilon_j \otimes x_{ij} \right) = \sum_{i,j} t_{ij} \epsilon_i \otimes \epsilon_j \otimes x_{ij},$$

hence

$$\begin{aligned} \left\| \sum_{i,j} t_{ij} \epsilon_i \otimes \epsilon_j \otimes x_{ij} \right\|_{\text{Rad}(\text{Rad}(X))} &\leq R(u) \sup_i \|f_i\| \left\| \sum_{i,j} \epsilon_i \otimes \epsilon_j \otimes x_{ij} \right\|_{\text{Rad}(\text{Rad}(X))} \\ &\leq R(u) \sup_{i,j} |t_{ij}| \left\| \sum_{i,j} \epsilon_i \otimes \epsilon_j \otimes x_{ij} \right\|_{\text{Rad}(\text{Rad}(X))}. \end{aligned}$$

This shows (4.5).

## 5. APPLICATION TO $L^p$ -SPACES AND UNCONDITIONAL BASES

Let  $X$  be a Banach lattice with finite cotype. A classical theorem of Maurey asserts that in addition to (4.4), we have a uniform equivalence

$$\left\| \sum_k \epsilon_k \otimes x_k \right\|_{\text{Rad}(X)} \asymp \left\| \left( \sum_k |x_k|^2 \right)^{\frac{1}{2}} \right\|$$

for finite families  $(x_k)_k$  of  $X$  (see e.g. [10, Thm. 16.18]). Thus a bounded linear mapping  $u: C(K) \rightarrow B(X)$  is matricially  $R$ -bounded if there is a constant  $C \geq 0$  such that for any  $n \geq 1$ , for any matrix  $[f_{ij}] \in M_n(C(K))$  and for any  $x_1, \dots, x_n \in X$ , we have

$$\left\| \left( \sum_i \left| \sum_j u(f_{ij}) x_j \right|^2 \right)^{\frac{1}{2}} \right\| \leq C \| [f_{ij}] \|_{C(K; M_n)} \left\| \left( \sum_j |x_j|^2 \right)^{\frac{1}{2}} \right\|.$$

Mappings satisfying this property were introduced by Simard in [32] under the name of  $\ell^2$ -cb maps. In this section we will apply a factorization property of  $\ell^2$ -cb maps established in [32], in the case when  $X$  is merely an  $L^p$ -space.

Throughout this section, we let  $(\Omega, \mu)$  be a  $\sigma$ -finite measure space. By definition, a density on that space is a measurable function  $g: \Omega \rightarrow (0, \infty)$  such that  $\|g\|_1 = 1$ . For any such function and any  $1 \leq p < \infty$ , we consider the linear mapping

$$\phi_{p,g}: L^p(\Omega, \mu) \longrightarrow L^p(\Omega, g d\mu), \quad \phi_{p,g}(h) = g^{-1/p} h,$$

which is an isometric isomorphism. Note that  $(\Omega, g d\mu)$  is a probability space. Passing from  $(\Omega, \mu)$  to  $(\Omega, g d\mu)$  by means of the maps  $\phi_{p,g}$  is usually called a change of density. A classical theorem of Johnson-Jones [18] asserts that for any bounded operator  $T: L^p(\mu) \rightarrow L^p(\mu)$ , there exists a density  $g$  on  $\Omega$  such that  $\phi_{p,g} \circ T \circ \phi_{p,g}^{-1}: L^p(g d\mu) \rightarrow L^p(g d\mu)$  extends to a bounded operator  $L^2(g d\mu) \rightarrow L^2(g d\mu)$ . The next statement is an analog of that result for  $C(K)$ -representations.

**Proposition 5.1.** *Let  $1 \leq p < \infty$  and let  $u: C(K) \rightarrow B(L^p(\mu))$  be a bounded homomorphism. Then there exists a density  $g: \Omega \rightarrow (0, \infty)$  and a bounded homomorphism  $w: C(K) \rightarrow B(L^2(gd\mu))$  such that*

$$\phi_{p,g} \circ u(f) \circ \phi_{p,g}^{-1} = w(f), \quad f \in C(K),$$

where equality holds on  $L^2(gd\mu) \cap L^p(gd\mu)$ .

*Proof.* Since  $X = L^p(\mu)$  has property  $(\alpha)$ , the mapping  $u$  is matricially  $R$ -bounded by Proposition 4.4. According to the above discussion, this means that  $u$  is  $\ell^2$ -cb in the sense of [32, Def. 2]. The result therefore follows from [32, Thms. 3.4 and 3.6].  $\square$

We will now focus on Schauder bases on separable  $L^p$ -spaces. We refer to [27, Chap. 1] for general information on this topic. We simply recall that a sequence  $(e_k)_{k \geq 1}$  in a Banach space  $X$  is a basis if for every  $x \in X$ , there exists a unique scalar sequence  $(a_k)_{k \geq 1}$  such that  $\sum_k a_k e_k$  converges to  $x$ . A basis  $(e_k)_{k \geq 1}$  is called unconditional if this convergence is unconditional for all  $x \in X$ . We record the following standard characterization.

**Lemma 5.2.** *A sequence  $(e_k)_{k \geq 1} \subset X$  of non-zero vectors is an unconditional basis of  $X$  if and only if  $X = \overline{\text{Span}\{e_k : k \geq 1\}}$  and there exists a constant  $C \geq 1$  such that for any bounded scalar sequence  $(\lambda_k)_{k \geq 1}$  and for any finite scalar sequence  $(a_k)_{k \geq 1}$ ,*

$$(5.1) \quad \left\| \sum_k \lambda_k a_k e_k \right\| \leq C \sup_k |\lambda_k| \left\| \sum_k a_k e_k \right\|.$$

We will need the following elementary lemma.

**Lemma 5.3.** *Let  $(\Omega, \nu)$  be a  $\sigma$ -finite measure space, let  $1 \leq p < \infty$  and let  $Q: L^p(\nu) \rightarrow L^p(\nu)$  be a finite rank bounded operator such that  $Q|_{L^2(\nu) \cap L^p(\nu)}$  extends to a bounded operator  $L^2(\nu) \rightarrow L^2(\nu)$ . Then  $Q(L^p(\nu)) \subset L^2(\nu)$ .*

*Proof.* Let  $E = Q(L^p(\nu) \cap L^2(\nu))$ . By assumption,  $E$  is a finite dimensional subspace of  $L^p(\nu) \cap L^2(\nu)$ . Since  $E$  is automatically closed under the  $L^p$ -norm and  $Q$  is continuous, we obtain that  $Q(L^p(\nu)) = E$ .  $\square$

**Theorem 5.4.** *Let  $1 \leq p < \infty$  and assume that  $(e_k)_{k \geq 1}$  is an unconditional basis of  $L^p(\Omega, \mu)$ . Then there exists a density  $g$  on  $\Omega$  such that  $\phi_{p,g}(e_k) \in L^2(gd\mu)$  for any  $k \geq 1$ , and the sequence  $(\phi_{p,g}(e_k))_{k \geq 1}$  is an unconditional basis of  $L^2(gd\mu)$ .*

*Proof.* Property (5.1) implies that for any  $\lambda = (\lambda_k)_{k \geq 1} \in \ell^\infty$ , there exists a (necessarily unique) bounded operator  $T_\lambda: L^p(\mu) \rightarrow L^p(\mu)$  such that  $T_\lambda(e_k) = \lambda_k e_k$  for any  $k \geq 1$ . Moreover  $\|T_\lambda\| \leq C \|\lambda\|_\infty$ . We can therefore consider the mapping

$$u: \ell^\infty \longrightarrow B(L^p(\mu)), \quad u(\lambda) = T_\lambda,$$

and  $u$  is a bounded homomorphism. By Proposition 5.1, there is a constant  $C_1 > 0$ , and a density  $g$  on  $\Omega$  such that with  $\phi = \phi_{p,g}$ , the mapping

$$\phi T_\lambda \phi^{-1}: L^p(gd\mu) \longrightarrow L^p(gd\mu)$$

extends to a bounded operator

$$S_\lambda: L^2(gd\mu) \longrightarrow L^2(gd\mu)$$

for any  $\lambda \in \ell^\infty$ , with  $\|S_\lambda\| \leq C_1 \|\lambda\|_\infty$ .

Assume first that  $p \geq 2$ , so that  $L^p(gd\mu) \subset L^2(gd\mu)$ . Let  $\lambda = (\lambda_k)_{k \geq 1}$  in  $\ell^\infty$  and let  $(a_k)_{k \geq 1}$  be a finite scalar sequence. Then  $S_\lambda(\phi(e_k)) = \phi T_\lambda \phi^{-1}(\phi(e_k)) = \lambda_k \phi(e_k)$  for any  $k \geq 1$ , hence

$$\left\| \sum_k \lambda_k a_k \phi(e_k) \right\|_{L^2(gd\mu)} = \left\| S_\lambda \left( \sum_k a_k \phi(e_k) \right) \right\|_{L^2(gd\mu)} \leq C_1 \|\lambda\|_\infty \left\| \sum_k a_k \phi(e_k) \right\|_{L^2(gd\mu)}.$$

Moreover the linear span of the  $\phi(e_k)$ 's is dense in  $L^p(gd\mu)$ , hence in  $L^2(gd\mu)$ . By Lemma 5.2, this shows that  $(\phi(e_k))_{k \geq 1}$  is an unconditional basis of  $L^2(gd\mu)$ .

Assume now that  $1 \leq p < 2$ . For any  $n \geq 1$ , let  $f_n \in \ell^\infty$  be defined by  $(f_n)_k = \delta_{n,k}$  for any  $k \geq 1$ , and let  $Q_n: L^p(gd\mu) \rightarrow L^p(gd\mu)$  be the projection defined by

$$Q_n \left( \sum_k a_k \phi(e_k) \right) = a_n \phi(e_n).$$

Then  $Q_n = \phi T_{f_n} \phi^{-1}$  hence  $Q_n$  extends to an  $L^2$  operator. Therefore,  $\phi(e_n)$  belongs to  $L^2(gd\mu)$  by Lemma 5.3.

Let  $p' = p/(p-1)$  be the conjugate number of  $p$ , let  $(e_k^*)_{k \geq 1}$  be the biorthogonal system of  $(e_k)_{k \geq 1}$ , and let  $\phi' = \phi^{*-1}$ . (It is easy to check that  $\phi' = \phi_{p',g}$ , but we will not use this point.) The linear span of the  $e_k^*$ 's is  $w^*$ -dense in  $L^{p'}(\mu)$ . Equivalently, the linear span of the  $\phi'(e_k^*)$ 's is  $w^*$ -dense in  $L^{p'}(gd\mu)$ , hence it is dense in  $L^2(gd\mu)$ . Moreover for any  $\lambda \in \ell^\infty$  and for any  $k \geq 1$ , we have  $T_\lambda^*(e_k^*) = \lambda_k e_k^*$ . Thus for any finite scalar sequence  $(a_k)_{k \geq 1}$ , we have

$$\sum_k \lambda_k a_k \phi'(e_k^*) = (\phi T_\lambda \phi^{-1})^* \left( \sum_k a_k \phi'(e_k^*) \right) = S_\lambda^* \left( \sum_k a_k \phi'(e_k^*) \right).$$

Hence

$$\left\| \sum_k \lambda_k a_k \phi'(e_k^*) \right\|_{L^2(gd\mu)} \leq C_1 \left\| \sum_k a_k \phi'(e_k^*) \right\|_{L^2(gd\mu)}.$$

According to Lemma 5.2, this shows that  $(\phi'(e_k^*))_{k \geq 1}$  is an unconditional basis of  $L^2(gd\mu)$ . It is plain that  $(\phi(e_k))_{k \geq 1} \subset L^2(gd\mu)$  is the biorthogonal system of  $(\phi'(e_k^*))_{k \geq 1} \subset L^2(gd\mu)$ . This shows that in turn,  $(\phi(e_k))_{k \geq 1}$  is an unconditional basis of  $L^2(gd\mu)$ .  $\square$

We will now establish a variant of Theorem 5.4 for conditional bases. Recall that if  $(e_k)_{k \geq 1}$  is a basis on some Banach space  $X$ , the projections  $P_N: X \rightarrow X$  defined by

$$P_N \left( \sum_k a_k e_k \right) = \sum_{k=1}^N a_k e_k$$

are uniformly bounded. We will say that  $(e_k)_{k \geq 1}$  is an  $R$ -basis if the set  $\{P_N : N \geq 1\}$  is actually  $R$ -bounded. It follows from [4, Cor. 3.15] that any unconditional basis on  $L^p$  is an  $R$ -basis. See Remark 5.6 (2) for more on this.

**Proposition 5.5.** *Let  $1 \leq p < \infty$  and let  $(e_k)_{k \geq 1}$  be an  $R$ -basis of  $L^p(\Omega, \mu)$ . Then there exists a density  $g$  on  $\Omega$  such that  $\phi_{p,g}(e_k) \in L^2(gd\mu)$  for any  $k \geq 1$ , and the sequence  $(\phi_{p,g}(e_k))_{k \geq 1}$  is a basis of  $L^2(gd\mu)$ .*

*Proof.* According to [26, Thm. 2.1], there exists a constant  $C \geq 1$  and a density  $g$  on  $\Omega$  such that with  $\phi = \phi_{p,g}$ , we have

$$\|\phi P_N \phi^{-1} h\|_2 \leq C \|h\|_2, \quad N \geq 1, \quad h \in L^2(gd\mu) \cap L^p(gd\mu).$$

Then the proof is similar to the one of Theorem 5.4, using [27, Prop. 1.a.3] instead of Lemma 5.2. We skip the details.  $\square$

**Remark 5.6.**

(1) Theorem 5.4 and Proposition 5.5 can be easily extended to finite dimensional Schauder decompositions. We refer to [27, Sect. 1.g] for general information on this notion. Given a Schauder decomposition  $(X_k)_{k \geq 1}$  of a Banach space  $X$ , let  $P_N$  be the associated projections, namely for any  $N \geq 1$ ,  $P_N: X \rightarrow X$  is the bounded projection onto  $X_1 \oplus \cdots \oplus X_N$  vanishing on  $X_k$  for any  $k \geq N + 1$ . We say that  $(X_k)_{k \geq 1}$  is an  $R$ -Schauder decomposition if the set  $\{P_N : N \geq 1\}$  is  $R$ -bounded. Then we obtain that for any  $1 < p < \infty$  and for any finite dimensional  $R$ -Schauder (resp. unconditional) decomposition  $(X_k)_{k \geq 1}$  of  $L^p(\mu)$ , there exists a density  $g$  on  $\Omega$  such that  $\phi_{p,g}(X_k) \subset L^2(gd\mu)$  for any  $k \geq 1$ , and  $(\phi_{p,g}(X_k))_{k \geq 1}$  is a Schauder (resp. unconditional) decomposition of  $L^2(gd\mu)$ .

(2) The concept of  $R$ -Schauder decompositions can be tracked down to [2], and played a key role in [4] and then in various works on  $L^p$ -maximal regularity and  $H^\infty$ -calculus, see in particular [20, 21]. Let  $C_p$  denote the Schatten spaces. For  $1 < p \neq 2 < \infty$ , an explicit example of a Schauder decomposition on  $L^2([0, 1]; C_p)$  which is not  $R$ -Schauder is given in [4, Sect. 5]. More generally, it follows from [20] that whenever a reflexive Banach space  $X$  has an unconditional basis and is not isomorphic to  $\ell^2$ , then  $X$  admits a finite dimensional Schauder decomposition which is not  $R$ -Schauder. This applies in particular to  $X = L^p([0, 1])$ , for any  $1 < p \neq 2 < \infty$ . However the question whether  $L^p([0, 1])$  admits a Schauder basis which is not  $R$ -Schauder, is apparently an open question.

We finally mention that according to [21, Thm. 3.3], any unconditional decomposition on a Banach space  $X$  with property  $(\Delta)$  is an  $R$ -Schauder decomposition.

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