



HAL
open science

Tensor extension properties of $C(K)$ -representations and applications to unconditionality

Christoph Kriegler, Christian Le Merdy

► To cite this version:

Christoph Kriegler, Christian Le Merdy. Tensor extension properties of $C(K)$ -representations and applications to unconditionality. *Journal of the Australian Mathematical Society*, 2010, 88, pp.205-230. hal-00477656

HAL Id: hal-00477656

<https://hal.science/hal-00477656>

Submitted on 29 Jan 2021

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

TENSOR EXTENSION PROPERTIES OF $C(K)$ -REPRESENTATIONS AND APPLICATIONS TO UNCONDITIONALITY

CHRISTOPH KRIEGLER, CHRISTIAN LE MERDY

ABSTRACT. Let K be any compact set. The C^* -algebra $C(K)$ is nuclear and any bounded homomorphism from $C(K)$ into $B(H)$, the algebra of all bounded operators on some Hilbert space H , is automatically completely bounded. We prove extensions of these results to the Banach space setting, using the key concept of R -boundedness. Then we apply these results to operators with a uniformly bounded H^∞ -calculus, as well as to unconditionality on L^p . We show that any unconditional basis on L^p 'is' an unconditional basis on L^2 after an appropriate change of density.

2000 Mathematics Subject Classification : 47A60, 46B28.

1. INTRODUCTION

Throughout the paper, we let K be a nonempty compact set and we let $C(K)$ be the algebra of all continuous functions $f: K \rightarrow \mathbb{C}$, equipped with the supremum norm. A representation of $C(K)$ on some Banach space X is a bounded unital homomorphism $u: C(K) \rightarrow B(X)$ into the algebra $B(X)$ of all bounded operators on X . Such representations appear naturally and play a major role in several fields of Operator Theory, including functional calculi, spectral theory and spectral measures, or classification of C^* -algebras. Several recent papers, in particular [21, 23, 11, 8], have emphasized the rich and fruitful interplays between the notion of R -boundedness, unconditionality and various functional calculi. The aim of this paper is to establish new properties of $C(K)$ -representations involving R -boundedness, and to give applications to H^∞ -calculus (in the sense of [6, 21]) and to unconditionality in L^p -spaces.

We recall the definition of R -boundedness (see [2, 4]). Let $(\epsilon_k)_{k \geq 1}$ be a sequence of independent Rademacher variables on some probability space Ω_0 . That is, the ϵ_k 's take values in $\{-1, 1\}$ and $\text{Prob}(\{\epsilon_k = 1\}) = \text{Prob}(\{\epsilon_k = -1\}) = \frac{1}{2}$. For any Banach space X , we let $\text{Rad}(X) \subset L^2(\Omega_0; X)$ be the closure of $\text{Span}\{\epsilon_k \otimes x : k \geq 1, x \in X\}$ in $L^2(\Omega_0; X)$. Thus for any finite family x_1, \dots, x_n in X , we have

$$\left\| \sum_k \epsilon_k \otimes x_k \right\|_{\text{Rad}(X)} = \left(\int_{\Omega_0} \left\| \sum_k \epsilon_k(\lambda) x_k \right\|_X^2 d\lambda \right)^{\frac{1}{2}}.$$

Date: August 28, 2009.

The first author is supported by the Karlsruhe House of Young Scientists and the Franco-German University DFH-UFA, the second author is supported by the research program ANR-06-BLAN-0015.

By definition, a set $\tau \subset B(X)$ is R -bounded if there is a constant $C \geq 0$ such that for any finite families T_1, \dots, T_n in τ , and x_1, \dots, x_n in X , we have

$$\left\| \sum_k \epsilon_k \otimes T_k x_k \right\|_{\text{Rad}(X)} \leq C \left\| \sum_k \epsilon_k \otimes x_k \right\|_{\text{Rad}(X)}.$$

In this case, we let $R(\tau)$ denote the smallest possible C . It is called the R -bound of τ . By convention, we write $R(\tau) = \infty$ if τ is not R -bounded.

It will be convenient to let $\text{Rad}_n(X)$ denote the subspace of $\text{Rad}(X)$ of all finite sums $\sum_{k=1}^n \epsilon_k \otimes x_k$. If $X = H$ is a Hilbert space, then $\text{Rad}_n(H) = \ell_n^2(H)$ isometrically and all bounded subsets of $B(H)$ are automatically R -bounded. Conversely if X is not isomorphic to a Hilbert space, then $B(X)$ contains bounded subsets which are not R -bounded [1, Prop. 1.13].

In order to motivate the results in this paper, we recall two well-known properties of $C(K)$ -representations on Hilbert space H . First, any bounded homomorphism $u: C(K) \rightarrow B(H)$ is completely bounded, with $\|u\|_{cb} \leq \|u\|^2$. This means that for any integer $n \geq 1$, the tensor extension $I_{M_n} \otimes u: M_n(C(K)) \rightarrow M_n(B(H))$ satisfies $\|I_{M_n} \otimes u\| \leq \|u\|^2$, if $M_n(C(K))$ and $M_n(B(H))$ are equipped with their natural C^* -algebra norms. This implies that any bounded homomorphism $u: C(K) \rightarrow B(H)$ is similar to a $*$ -representation, a result going back at least to [3]. We refer to [28, 30] and the references therein for some information on completely bounded maps and similarity properties.

Second, let $u: C(K) \rightarrow B(H)$ be a bounded homomorphism. Then for any b_1, \dots, b_n lying in the commutant of the range of u and for any f_1, \dots, f_n in $C(K)$, we have

$$(1.1) \quad \left\| \sum_k u(f_k) b_k \right\| \leq \|u\|^2 \sup_{t \in K} \left\| \sum_k f_k(t) b_k \right\|.$$

This property is essentially a rephrasing of the fact that $C(K)$ is a nuclear C^* -algebra. More precisely, nuclearity means that the above property holds true for $*$ -representations (see e.g. [19, Chap. 11] or [28, Chap. 12]), and its extension to arbitrary bounded homomorphisms easily follows from the similarity property mentioned above (see [25] for more explanations and developments).

Now let X be a Banach space and let $u: C(K) \rightarrow B(X)$ be a bounded homomorphism. In Section 2, we will show the following analog of (1.1):

$$(1.2) \quad \left\| \sum_k u(f_k) b_k \right\| \leq \|u\|^2 R \left(\left\{ \sum_k f_k(t) b_k : t \in K \right\} \right),$$

provided that the b_k 's commute with the range of u .

Section 3 is devoted to sectorial operators A which have a uniformly bounded H^∞ -calculus, in the sense that they satisfy an estimate

$$(1.3) \quad \|f(A)\| \leq C \sup_{t>0} |f(t)|$$

for any bounded analytic function on a sector Σ_θ surrounding $(0, \infty)$. Such operators turn out to have a natural $C(K)$ -functional calculus. Applying (1.2) to the resulting representation $u: C(K) \rightarrow B(X)$, we show that (1.3) can be automatically extended to operator valued

analytic functions f taking their values in the commutant of A . This is an analog of a remarkable result of Kalton-Weis [23, Thm. 4.4] saying that if an operator A has a bounded H^∞ -calculus and f is an operator valued analytic function taking its values in an R -bounded subset of the commutant of A , then the operator $f(A)$ arising from ‘generalized H^∞ -calculus’, is bounded.

In Section 4, we introduce matricially R -bounded maps $C(K) \rightarrow B(X)$, a natural analog of completely bounded maps in the Banach space setting. We show that if X has property (α) , then any bounded homomorphism $C(K) \rightarrow B(X)$ is automatically matricially R -bounded. This extends both the Hilbert space result mentioned above, and a result of De Pagter-Ricker [8, Cor. 2.19] saying that any bounded homomorphism $C(K) \rightarrow B(X)$ maps the unit ball of $C(K)$ into an R -bounded set, provided that X has property (α) .

In Section 5, we give an application of matricial R -boundedness to the case when $X = L^p$. A classical result of Johnson-Jones [18] asserts that any bounded operator $T: L^p \rightarrow L^p$ acts, after an appropriate change of density, as a bounded operator on L^2 . We show versions of this theorem for bases (more generally, for FDD’s). Indeed we show that any unconditional basis (resp. any R -basis) on L^p becomes an unconditional basis (resp. a Schauder basis) on L^2 after an appropriate change of density. These results rely on Simard’s extensions of the Johnson-Jones Theorem established in [32].

We end this introduction with a few preliminaries and notation. For any Banach space Z , we let $C(K; Z)$ denote the space of all continuous functions $f: K \rightarrow Z$, equipped with the supremum norm

$$\|f\|_\infty = \sup\{\|f(t)\|_Z : t \in K\}.$$

We may regard $C(K) \otimes Z$ as a subspace of $C(K; Z)$, by identifying $\sum_k f_k \otimes z_k$ with the function $t \mapsto \sum_k f_k(t)z_k$, for any finite families $(f_k)_k$ in $C(K)$ and $(z_k)_k$ in Z . Moreover, $C(K) \otimes Z$ is dense in $C(K; Z)$. Note that for any integer $n \geq 1$, $C(K; M_n)$ coincides with the C^* -algebra $M_n(C(K))$ mentioned above.

We will need the so-called ‘contraction principle’, which says that for any x_1, \dots, x_n in a Banach space X and any $\alpha_1, \dots, \alpha_n$ in \mathbb{C} , we have

$$(1.4) \quad \left\| \sum_k \epsilon_k \otimes \alpha_k x_k \right\|_{\text{Rad}(X)} \leq 2 \sup_k |\alpha_k| \left\| \sum_k \epsilon_k \otimes x_k \right\|_{\text{Rad}(X)}.$$

We also recall that any unital commutative C^* -algebra is a $C(K)$ -space (see e.g. [19, Chap. 4]). Thus our results concerning $C(K)$ -representations apply as well to all these algebras. For example we will apply them to ℓ^∞ in Section 5.

We let I_X denote the identity mapping on a Banach space X , and we let χ_B denote the indicator function of a set B . If X is a dual Banach space, we let $w^*B(X) \subset B(X)$ be the subspace of all w^* -continuous operators on X .

2. THE EXTENSION THEOREM

Let X be an arbitrary Banach space. For any compact set K and any bounded homomorphism $u: C(K) \rightarrow B(X)$, we let

$$E_u = \{b \in B(X) : bu(f) = u(f)b \text{ for any } f \in C(K)\}$$

denote the commutant of the range of u .

Our main purpose in this section is to prove (1.1). We start with the case when $C(K)$ is finite dimensional.

Proposition 2.1. *Let $N \geq 1$ and let $u: \ell_N^\infty \rightarrow B(X)$ be a bounded homomorphism. Let (e_1, \dots, e_N) be the canonical basis of ℓ_N^∞ and set $p_i = u(e_i)$, $i = 1, \dots, N$. Then for any $b_1, \dots, b_N \in E_u$, we have*

$$\left\| \sum_{i=1}^N p_i b_i \right\| \leq \|u\|^2 R(\{b_1, \dots, b_N\}).$$

Proof. Since u is multiplicative, each p_i is a projection and $p_i p_j = 0$ when $i \neq j$. Hence for any choice of signs $(\alpha_1, \dots, \alpha_N) \in \{-1, 1\}^N$, we have

$$\sum_{i=1}^N p_i b_i = \sum_{i,j=1}^N \alpha_i \alpha_j p_i p_j b_j.$$

Furthermore,

$$\left\| \sum_i \alpha_i p_i \right\| = \|u(\alpha_1, \dots, \alpha_N)\| \leq \|u\| \|(\alpha_1, \dots, \alpha_N)\|_{\ell_N^\infty} = \|u\|.$$

Therefore for any $x \in X$, we have the following chain of inequalities which prove the desired estimate:

$$\begin{aligned} \left\| \sum_i p_i b_i x \right\|^2 &= \int_{\Omega_0} \left\| \sum_i \epsilon_i(\lambda) p_i \sum_j \epsilon_j(\lambda) p_j b_j x \right\|^2 d\lambda \\ &\leq \int_{\Omega_0} \left\| \sum_i \epsilon_i(\lambda) p_i \right\|^2 \left\| \sum_j \epsilon_j(\lambda) p_j b_j x \right\|^2 d\lambda \\ &\leq \|u\|^2 \int_{\Omega_0} \left\| \sum_j \epsilon_j(\lambda) b_j p_j x \right\|^2 d\lambda \\ &\leq \|u\|^2 R(\{b_1, \dots, b_N\})^2 \int_{\Omega_0} \left\| \sum_j \epsilon_j(\lambda) p_j x \right\|^2 d\lambda \\ &\leq \|u\|^4 R(\{b_1, \dots, b_N\})^2 \|x\|^2. \end{aligned}$$

□

The study of infinite dimensional $C(K)$ -spaces requires the use of second duals and w^* -topologies. We recall a few well-known facts that will be used later on in this paper. According to the Riesz representation Theorem, the dual space $C(K)^*$ can be naturally identified with the space $M(K)$ of Radon measures on K . Next, the second dual space $C(K)^{**}$ is a commutative C^* -algebra for the so-called Arens product. This product extends the product on $C(K)$ and is separately w^* -continuous, which means that for any $\xi \in C(K)^{**}$, the two linear maps

$$\nu \in C(K)^{**} \longmapsto \nu\xi \in C(K)^{**} \quad \text{and} \quad \nu \in C(K)^{**} \longmapsto \xi\nu \in C(K)^{**}$$

are w^* -continuous.

Let

$$\mathcal{B}^\infty(K) = \{f: K \rightarrow \mathbb{C} \mid f \text{ bounded, Borel measurable}\},$$

equipped with the sup norm. According to the duality pairing

$$\langle f, \mu \rangle = \int_K f(t) d\mu(t), \quad \mu \in M(K), f \in \mathcal{B}^\infty(K),$$

one can regard $\mathcal{B}^\infty(K)$ as a closed subspace of $C(K)^{**}$. Moreover the restriction of the Arens product to $\mathcal{B}^\infty(K)$ coincides with the pointwise product. Thus we have natural C^* -algebra inclusions

$$(2.1) \quad C(K) \subset \mathcal{B}^\infty(K) \subset C(K)^{**}.$$

See e.g. [7, pp. 366-367] and [5, Sec. 9] for further details.

Let $\widehat{\otimes}$ denote the projective tensor product on Banach spaces. We recall that for any two Banach spaces Y_1, Y_2 , we have a natural identification

$$(Y_1 \widehat{\otimes} Y_2)^* \simeq B(Y_2, Y_1^*),$$

see e.g. [9, VIII.2]. This implies that when X is a dual Banach space, $X = (X_*)^*$ say, then $B(X) = (X_* \widehat{\otimes} X)^*$ is a dual space. The next two lemmas are elementary.

Lemma 2.2. *Let $X = (X_*)^*$ be a dual space, let $S \in B(X)$, and let $R_S, L_S: B(X) \rightarrow B(X)$ be the right and left multiplication operators defined by $R_S(T) = TS$ and $L_S(T) = ST$. Then R_S is w^* -continuous whereas L_S is w^* -continuous if (and only if) S is w^* -continuous.*

Proof. The tensor product mapping $I_{X_*} \otimes S$ on $X_* \otimes X$ uniquely extends to a bounded map $r_S: X_* \widehat{\otimes} X \rightarrow X_* \widehat{\otimes} X$, and we have $R_S = r_S^*$. Thus R_S is w^* -continuous. Likewise, if S is w^* -continuous and if we let $S_*: X_* \rightarrow X_*$ be its pre-adjoint map, the tensor product mapping $S_* \otimes I_X$ on $X_* \otimes X$ extends to a bounded map $l_S: X_* \widehat{\otimes} X \rightarrow X_* \widehat{\otimes} X$, and $L_S = l_S^*$. Thus L_S is w^* -continuous. The ‘only if’ part (which we will not use) is left to the reader. \square

Lemma 2.3. *Let $u: C(K) \rightarrow B(X)$ be a bounded map. Suppose that X is a dual space. Then there exists a (necessarily unique) w^* -continuous linear mapping $\tilde{u}: C(K)^{**} \rightarrow B(X)$ whose restriction to $C(K)$ coincides with u . Moreover $\|\tilde{u}\| = \|u\|$.*

*If further u is a homomorphism, and u is valued in $w^*B(X)$, then \tilde{u} is a homomorphism as well.*

Proof. Let $j: (X_* \widehat{\otimes} X) \hookrightarrow (X_* \widehat{\otimes} X)^{**}$ be the canonical injection and consider its adjoint $p = j^*: B(X)^{**} \rightarrow B(X)$. Then set

$$\tilde{u} = p \circ u^{**}: C(K)^{**} \longrightarrow B(X).$$

By construction, \tilde{u} is w^* -continuous and extends u . The equality $\|\tilde{u}\| = \|u\|$ is clear.

Assume now that u is a homomorphism and that u is valued in $w^*B(X)$. Let $\nu, \xi \in C(K)^{**}$ and let $(f_\alpha)_\alpha$ and $(g_\beta)_\beta$ be bounded nets in $C(K)$ w^* -converging to ν and ξ respectively. By both parts of Lemma 2.2, we have the following equalities, where limits are taken in the w^* -topology of either $C(K)^{**}$ or $B(X)$:

$$\tilde{u}(\nu\xi) = \tilde{u}(\lim_\alpha \lim_\beta f_\alpha g_\beta) = \lim_\alpha \lim_\beta u(f_\alpha g_\beta) = \lim_\alpha \lim_\beta u(f_\alpha)u(g_\beta) = \lim_\alpha u(f_\alpha)\tilde{u}(\xi) = \tilde{u}(\nu)\tilde{u}(\xi).$$

\square

We refer e.g. to [17, Lem. 2.4] for the following fact.

Lemma 2.4. *Consider $\tau \subset B(X)$ and set $\tau^{**} = \{T^{**} : T \in \tau\} \subset B(X^{**})$. Then τ is R -bounded if and only if τ^{**} is R -bounded, and in this case,*

$$R(\tau) = R(\tau^{**}).$$

For any $F \in C(K; B(X))$, we set

$$R(F) = R(\{F(t) : t \in K\}).$$

Note that $R(F)$ may be infinite. If $F = \sum_k f_k \otimes b_k$ belongs to the algebraic tensor product $C(K) \otimes B(X)$, we set

$$\left\| \sum_k f_k \otimes b_k \right\|_R = R(F) = R\left(\left\{ \sum_k f_k(t) b_k : t \in K \right\}\right).$$

Note that by (1.4), we have

$$(2.2) \quad \|f \otimes b\|_R \leq 2\|f\|_\infty \|b\|, \quad f \in C(K), \quad b \in B(X).$$

From this it is easy to check that $\|\cdot\|_R$ is finite and is a norm on $C(K) \otimes B(X)$.

Whenever $E \subset B(X)$ is a closed subspace, we let

$$C(K) \overset{R}{\otimes} E$$

denote the completion of $C(K) \otimes E$ for the norm $\|\cdot\|_R$.

Remark 2.5. Since the R -bound of a set is greater than its uniform bound, we have $\|\cdot\|_\infty \leq \|\cdot\|_R$ on $C(K) \otimes B(X)$. Hence the canonical embedding of $C(K) \otimes B(X)$ into $C(K; B(X))$ uniquely extends to a contraction

$$J: C(K) \overset{R}{\otimes} B(X) \longrightarrow C(K; B(X)).$$

Moreover J is 1-1 and for any $\varphi \in C(K) \overset{R}{\otimes} B(X)$, we have $R(J(\varphi)) = \|\varphi\|_R$. To see this, let $(F_n)_{n \geq 1}$ be a sequence in $C(K) \otimes B(X)$ such that $\|F_n - \varphi\|_R \rightarrow 0$ and let $F = J(\varphi)$. Then $\|F_n\|_R \rightarrow \|\varphi\|_R$ and $\|F_n - F\|_\infty \rightarrow 0$. According to the definition of the R -bound, the latter property implies that $\|F_n\|_R \rightarrow \|F\|_R$, which yields the result.

Theorem 2.6. *Let $u: C(K) \rightarrow B(X)$ be a bounded homomorphism.*

(1) *For any finite families $(f_k)_k$ in $C(K)$ and $(b_k)_k$ in E_u , we have*

$$\left\| \sum_k u(f_k) b_k \right\| \leq \|u\|^2 \left\| \sum_k f_k \otimes b_k \right\|_R.$$

(2) *There is a (necessarily unique) bounded linear map*

$$\widehat{u}: C(K) \overset{R}{\otimes} E_u \longrightarrow B(X)$$

such that $\widehat{u}(f \otimes b) = u(f)b$ for any $f \in C(K)$ and any $b \in E_u$. Moreover, $\|\widehat{u}\| \leq \|u\|^2$.

Proof. Part (2) clearly follows from part (1). To prove (1) we introduce

$$w: C(K) \longrightarrow B(X^{**}), \quad w(f) = u(f)^{**}.$$

Then w is a bounded homomorphism and $\|w\| = \|u\|$. We let $\tilde{w}: C(K)^{**} \rightarrow B(X^{**})$ be its w^* -continuous extension given by Lemma 2.3. Note that w is valued in $w^*B(X^{**})$, so \tilde{w} is a homomorphism. We claim that

$$\{b^{**} : b \in E_u\} \subset E_{\tilde{w}}.$$

Indeed, let $b \in E_u$. Then for all $f \in C(K)$, we have

$$b^{**} w(f) = (bu(f))^{**} = (u(f)b)^{**} = w(f)b^{**}.$$

Next for any $\nu \in C(K)^{**}$, let $(f_\alpha)_\alpha$ be a bounded net in $C(K)$ which converges to ν in the w^* -topology. Then by Lemma 2.2, we have

$$b^{**} \tilde{w}(\nu) = \lim_\alpha b^{**} w(f_\alpha) = \lim_\alpha w(f_\alpha) b^{**} = \tilde{w}(\nu) b^{**},$$

and the claim follows.

Now fix some $f_1, \dots, f_n \in C(K)$ and $b_1, \dots, b_n \in E_u$. For any $m \in \mathbb{N}$, there is a finite family (t_1, \dots, t_N) of K and a measurable partition (B_1, \dots, B_N) of K such that

$$\left\| f_k - \sum_{l=1}^N f_k(t_l) \chi_{B_l} \right\|_\infty \leq \frac{1}{m}, \quad k = 1, \dots, n.$$

We set $f_k^{(m)} = \sum_{l=1}^N f_k(t_l) \chi_{B_l}$. Let $\psi: \ell_N^\infty \rightarrow \mathcal{B}^\infty(K)$ be defined by

$$\psi(\alpha_1, \dots, \alpha_N) = \sum_{l=1}^N \alpha_l \chi_{B_l}.$$

Then ψ is a norm 1 homomorphism. According to (2.1), we can consider the bounded homomorphism

$$\tilde{w} \circ \psi: \ell_N^\infty \longrightarrow B(X^{**}).$$

Applying Proposition 2.1 to that homomorphism, together with the above claim and Lemma 2.4, we obtain that

$$\begin{aligned} \left\| \sum_k \tilde{w}(f_k^{(m)}) b_k^{**} \right\| &= \left\| \sum_{k,l} f_k(t_l) \tilde{w} \circ \psi(e_l) b_k^{**} \right\| \\ &\leq \|\tilde{w} \circ \psi\|^2 R\left(\left\{ \sum_k f_k(t_l) b_k^{**} : 1 \leq l \leq N \right\}\right) \\ &\leq \|u\|^2 R\left(\left\{ \sum_k f_k(t) b_k^{**} : t \in K \right\}\right) \\ &\leq \|u\|^2 \left\| \sum_k f_k \otimes b_k \right\|_R. \end{aligned}$$

Since $\|f_k^{(m)} - f_k\|_\infty \rightarrow 0$ for any k , we have

$$\left\| \sum_k \tilde{w}(f_k^{(m)}) b_k^{**} \right\| \longrightarrow \left\| \sum_k w(f_k) b_k^{**} \right\| = \left\| \sum_k u(f_k) b_k \right\|,$$

and the result follows at once. \square

The following notion is implicit in several recent papers on functional calculi (see in particular [21, 8]).

Definition 2.7. *Let Z be a Banach space and let $v: Z \rightarrow B(X)$ be a bounded map. We set*

$$R(v) = R(\{v(z) : z \in Z, \|z\| \leq 1\}),$$

and we say that v is R -bounded if $R(v) < \infty$.

Corollary 2.8. *Let $u: C(K) \rightarrow B(X)$ be a bounded homomorphism and let $v: Z \rightarrow B(X)$ be an R -bounded map. Assume further that $u(f)v(z) = v(z)u(f)$ for any $f \in C(K)$ and any $z \in Z$. Then there exists a (necessarily unique) bounded linear map*

$$u \cdot v: C(K; Z) \longrightarrow B(X)$$

such that $u \cdot v(f \otimes z) = u(f)v(z)$ for any $f \in C(K)$ and any $z \in Z$. Moreover we have

$$\|u \cdot v\| \leq \|u\|^2 R(v).$$

Proof. Consider any finite families $(f_k)_k$ in $C(K)$ and $(z_k)_k$ in Z and observe that

$$\left\| \sum_k f_k \otimes v(z_k) \right\|_R = R\left(\left\{v\left(\sum_k f_k(t)z_k\right) : t \in K\right\}\right) \leq R(v) \left\| \sum_k f_k \otimes z_k \right\|_\infty.$$

Then applying Theorem 2.6 and the assumption that v is valued in E_u , we obtain that

$$\left\| \sum_k u(f_k)v(z_k) \right\| \leq \|u\|^2 R(v) \left\| \sum_k f_k \otimes z_k \right\|_\infty,$$

which proves the result. \square

Remark 2.9. As a special case of Corollary 2.8, we obtain the following result due to De Pagter and Ricker ([8, Prop. 2.27]): Let K_1, K_2 be two compact sets, let

$$u: C(K_1) \longrightarrow B(X) \quad \text{and} \quad v: C(K_2) \longrightarrow B(X)$$

be two bounded homomorphisms which commute, i.e. $u(f)v(g) = v(g)u(f)$ for all $f \in C(K_1)$ and $g \in C(K_2)$. Assume further that $R(v) < \infty$. Then there exists a bounded homomorphism

$$w: C(K_1 \times K_2) \longrightarrow B(X)$$

such that $w|_{C(K_1)} = u$ and $w|_{C(K_2)} = v$, where $C(K_j)$ is regarded as a subalgebra of $C(K_1 \times K_2)$ in the natural way.

3. UNIFORMLY BOUNDED H^∞ -CALCULUS

We briefly recall the basic notions on H^∞ -calculus for sectorial operators. For more information, we refer e.g. to [6, 21, 23, 24].

For any $\theta \in (0, 2\pi)$, we define

$$\Sigma_\theta = \{re^{i\phi} : r > 0, |\phi| < \theta\}$$

and

$$H^\infty(\Sigma_\theta) = \{f : \Sigma_\theta \rightarrow \mathbb{C} \mid f \text{ is analytic and bounded}\}.$$

This space is equipped with the norm $\|f\|_{\infty, \theta} = \sup_{\lambda \in \Sigma_\theta} |f(\lambda)|$ and this is a Banach algebra. We consider the auxiliary space

$$H_0^\infty(\Sigma_\theta) = \{f \in H^\infty(\Sigma_\theta) : \exists \epsilon, C > 0 \mid |f(\lambda)| \leq C \min(|\lambda|^\epsilon, |\lambda|^{-\epsilon})\}.$$

A closed linear operator $A : D(A) \subset X \rightarrow X$ is called ω -sectorial, for some $\omega \in (0, 2\pi)$, if its domain $D(A)$ is dense in X , its spectrum $\sigma(A)$ is contained in $\overline{\Sigma_\omega}$, and for all $\theta > \omega$ there is a constant $C_\theta > 0$ such that

$$\|\lambda(\lambda - A)^{-1}\| \leq C_\theta, \quad \lambda \in \mathbb{C} \setminus \overline{\Sigma_\theta}.$$

In this case, we define

$$\omega(A) = \inf\{\omega : A \text{ is } \omega\text{-sectorial}\}.$$

For any $\theta \in (\omega(A), \pi)$ and any $f \in H_0^\infty(\Sigma_\theta)$, we define

$$(3.1) \quad f(A) = \frac{1}{2\pi i} \int_{\Gamma_\gamma} f(\lambda)(\lambda - A)^{-1} d\lambda,$$

where $\omega(A) < \gamma < \theta$ and Γ_γ is the boundary $\partial\Sigma_\gamma$ oriented counterclockwise. This definition does not depend on γ and the resulting mapping $f \mapsto f(A)$ is an algebra homomorphism from $H_0^\infty(\Sigma_\theta)$ into $B(X)$. We say that A has a bounded $H^\infty(\Sigma_\theta)$ -calculus if the latter homomorphism is bounded, that is, there exists a constant $C > 0$ such that $\|f(A)\| \leq C\|f\|_{\infty, \theta}$ for all $f \in H_0^\infty(\Sigma_\theta)$.

We will now focus on sectorial operators A such that $\omega(A) = 0$.

Definition 3.1. *We say that a sectorial operator A with $\omega(A) = 0$ has a uniformly bounded H^∞ -calculus, if there exists a constant $C > 0$ such that $\|f(A)\| \leq C\|f\|_{\infty, \theta}$ for all $\theta > 0$ and $f \in H_0^\infty(\Sigma_\theta)$.*

The space

$$C_\ell([0, \infty)) = \{f : [0, \infty) \rightarrow \mathbb{C} \mid f \text{ is continuous and } \lim_{\infty} f \text{ exists}\}.$$

is a unital commutative C^* -algebra when equipped with the natural norm

$$\|f\|_{\infty, 0} = \sup\{|f(t)| : t \geq 0\}$$

and involution. For any $\theta > 0$, we can regard $H_0^\infty(\Sigma_\theta)$ as a subalgebra of $C_\ell([0, \infty))$, by identifying any $f \in H_0^\infty(\Sigma_\theta)$ with its restriction $f|_{[0, \infty)}$.

For any $\lambda \in \mathbb{C} \setminus [0, \infty)$, we let $R_\lambda \in C_\ell([0, \infty))$ be defined by $R_\lambda(t) = (\lambda - t)^{-1}$. Then we let \mathcal{R} be the unital algebra generated by the R_λ 's. Equivalently, \mathcal{R} is the algebra of all rational functions of nonpositive degree, whose poles lie outside the half line $[0, \infty)$. We

recall that for any $f \in H_0^\infty(\Sigma_\theta) \cap \mathcal{R}$, the definition of $f(A)$ given by (3.1) coincides with the usual rational functional calculus.

The following lemma is closely related to [22, Cor. 6.9].

Lemma 3.2. *Let A be a sectorial operator on X with $\omega(A) = 0$. The following assertions are equivalent.*

- (a) *A has a uniformly bounded H^∞ -calculus.*
- (b) *There exists a (necessarily unique) bounded unital homomorphism*

$$u: C_\ell([0, \infty)) \longrightarrow B(X)$$

such that $u(R_\lambda) = (\lambda - A)^{-1}$ for any $\lambda \in \mathbb{C} \setminus [0, \infty)$.

Proof. Assume (a). We claim that for any $\theta > 0$ and any $f \in H_0^\infty(\Sigma_\theta)$, we have

$$\|f(A)\| \leq C\|f\|_{\infty,0}.$$

Indeed, if $0 \neq f \in H_0^\infty(\Sigma_{\theta_0})$ for some $\theta_0 > 0$, then there exists some $t_0 > 0$ such that $f(t_0) \neq 0$. Now let $r < R$ such that $|f(z)| < |f(t_0)|$ for $|z| < r$ and $|z| > R$. Choose for every $n \in \mathbb{N}$ a $t_n \in \Sigma_{\theta_0/n}$ such that $|f(t_n)| = \|f\|_{\infty, \theta_0/n}$. Necessarily $|t_n| \in [r, R]$, and there exists a convergent subsequence t_{n_k} , whose limit t_∞ is real. Then

$$\|f\|_{\infty,0} \geq |f(t_\infty)| \geq \liminf_{\theta \rightarrow 0} \|f\|_{\infty,\theta} \geq C^{-1}\|f(A)\|.$$

This readily implies that the rational functional calculus $(\mathcal{R}, \|\cdot\|_{\infty,0}) \rightarrow B(X)$ is bounded. By Stone-Weierstrass, this extends continuously to $C_\ell([0, \infty))$, which yields (b). The uniqueness property is clear.

Assume (b). Then for any $\theta \in (0, \pi)$ and for any $f \in H_0^\infty(\Sigma_\theta) \cap \mathcal{R}$, we have

$$\|f(A)\| \leq \|u\|\|f\|_{\infty,\theta}.$$

By [24, Prop. 2.10] and its proof, this implies that A has a bounded $H^\infty(\Sigma_\theta)$ -calculus, with a boundedness constant uniform in θ . \square

Remark 3.3. An operator A which admits a bounded $H^\infty(\Sigma_\theta)$ -calculus for all $\theta > 0$ does not necessarily have a uniformly bounded H^∞ -calculus. To get a simple example, consider

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} : \ell_2^2 \longrightarrow \ell_2^2.$$

Then $\sigma(A) = \{1\}$ and for any $\theta > 0$ and any $f \in H_0^\infty(\Sigma_\theta)$, we have

$$f(A) = \begin{pmatrix} f(1) & f'(1) \\ 0 & f(1) \end{pmatrix}.$$

Assume that $\theta < \frac{\pi}{2}$. Using Cauchy's Formula, it is easy to see that $|f'(1)| \leq (\sin(\theta))^{-1}\|f\|_{\infty,\theta}$ for any $f \in H_0^\infty(\Sigma_\theta)$. Thus A admits a bounded $H^\infty(\Sigma_\theta)$ -calculus.

Now let h be a fixed function in $H_0^\infty(\Sigma_{\frac{\pi}{2}})$ such that $h(1) = 1$, set $g_s(\lambda) = \lambda^{is}$ for any $s > 0$, and let $f_s = hg_s$. Then $\|g_s\|_{\infty,0} = 1$, hence $\|f_s\|_{\infty,0} \leq \|h\|_{\infty,0}$ for any $s > 0$. Further $g'_s(\lambda) = is\lambda^{is-1}$ and $f'_s = h'g_s + hg'_s$. Hence $f'_s(1) = h'(1) + is$. Thus

$$\|f_s(A)\|\|f_s\|_{\infty,0}^{-1} \geq |f'_s(1)|\|h_s\|_{\infty,0}^{-1} \longrightarrow \infty$$

when $s \rightarrow \infty$. Hence A does not have a uniformly bounded H^∞ -calculus.

The above result can also be deduced from Proposition 3.7 below. In fact we will show in that proposition and in Corollary 3.11 that operators with a uniformly bounded H^∞ -calculus are ‘rare’.

We now turn to the so-called generalized (or operator valued) H^∞ -calculus. Throughout we let A be a sectorial operator. We let $E_A \subset B(X)$ denote the commutant of A , defined as the subalgebra of all bounded operators $T: X \rightarrow X$ such that $T(\lambda - A)^{-1} = (\lambda - A)^{-1}T$ for any λ belonging to the resolvent set of A . We let $H_0^\infty(\Sigma_\theta; B(X))$ be the algebra of all bounded analytic functions $F: \Sigma_\theta \rightarrow B(X)$ for which there exist $\epsilon, C > 0$ such that $\|F(\lambda)\| \leq C \min(|\lambda|^\epsilon, |\lambda|^{-\epsilon})$ for any $\lambda \in \Sigma_\theta$. Also, we let $H_0^\infty(\Sigma_\theta; E_A)$ denote the space of all E_A -valued functions belonging to $H_0^\infty(\Sigma_\theta; B(X))$. The generalized H^∞ -calculus of A is an extension of (3.1) to this class of functions. Namely for any $F \in H_0^\infty(\Sigma_\theta; E_A)$, we set

$$F(A) = \frac{1}{2\pi i} \int_{\Gamma_\gamma} F(\lambda)(\lambda - A)^{-1} d\lambda,$$

where $\gamma \in (\omega(A), \pi)$. Again, this definition does not depend on γ and the mapping $F \mapsto F(A)$ is an algebra homomorphism. The following fundamental result is due to Kalton and Weis.

Theorem 3.4. ([21, Thm. 4.4], [23, Thm. 12.7].) *Let $\omega_0 \geq \omega(A)$ and assume that A has a bounded $H^\infty(\Sigma_\theta)$ -calculus for any $\theta > \omega_0$. Then for any $\theta > \omega_0$, there exists a constant $C_\theta > 0$ such that for any $F \in H_0^\infty(\Sigma_\theta; E_A)$,*

$$(3.2) \quad \|F(A)\| \leq C_\theta R(\{F(z) : z \in \Sigma_\theta\}).$$

Our aim is to prove a version of this result in the case when A has a uniformly bounded H^∞ -calculus. We will obtain in Theorem 3.6 that in this case, the constant C_θ in (3.2) can be taken independent of θ .

The algebra $C_\ell([0, \infty))$ is a $C(K)$ -space and we will apply the results of Section 2 to the bounded homomorphism u appearing in Lemma 3.2. We recall Remark 2.5.

Lemma 3.5. *Let $J: C_\ell([0, \infty)) \overset{R}{\otimes} B(X) \rightarrow C_\ell([0, \infty); B(X))$ be the canonical embedding. Let $\theta \in (0, \pi)$, let $F \in H_0^\infty(\Sigma_\theta; B(X))$ and let $\gamma \in (0, \theta)$.*

(1) *The integral*

$$(3.3) \quad \varphi_F = \frac{1}{2\pi i} \int_{\Gamma_\gamma} R_\lambda \otimes F(\lambda) d\lambda$$

is absolutely convergent in $C_\ell([0, \infty)) \overset{R}{\otimes} B(X)$, and $J(\varphi_F)$ is equal to the restriction of F to $[0, \infty)$.

(2) *The set $\{F(t) : t > 0\}$ is R -bounded.*

Proof. Part (2) readily follows from part (1) and Remark 2.5. To prove (1), observe that for any $\lambda \in \partial\Sigma_\gamma$, we have

$$\|R_\lambda \otimes F(\lambda)\|_R \leq 2\|R_\lambda\|_{\infty,0}\|F(\lambda)\| \leq \frac{2}{\sin(\gamma)|\lambda|} \|F(\lambda)\|$$

by (2.2). Thus for appropriate constants $\epsilon, C > 0$, we have

$$\|R_\lambda \otimes F(\lambda)\|_R \leq \frac{2C}{\sin(\gamma)} \min(|\lambda|^{\epsilon-1}, |\lambda|^{-\epsilon-1}).$$

This shows that the integral defining φ_F is absolutely convergent. Next, for any $t > 0$, we have

$$[J(\varphi_F)](t) = \frac{1}{2\pi i} \int_{\Gamma_\gamma} (R_\lambda \otimes F(\lambda))(t) d\lambda = \frac{1}{2\pi i} \int_{\Gamma_\gamma} \frac{F(\lambda)}{\lambda - t} d\lambda = F(t)$$

by Cauchy's Theorem. \square

Theorem 3.6. *Let A be a sectorial operator with $\omega(A) = 0$ and assume that A has a uniformly bounded H^∞ -calculus. Then there exists a constant $C > 0$ such that for any $\theta > 0$ and any $F \in H_0^\infty(\Sigma_\theta; E_A)$,*

$$\|F(A)\| \leq CR(\{F(t) : t > 0\}).$$

Proof. Let $u: C_\ell([0, \infty)) \rightarrow B(X)$ be the representation given by Lemma 3.2. It is plain that $E_u = E_A$. Then we let

$$\widehat{u}: C_\ell([0, \infty)) \otimes^R E_A \longrightarrow B(X)$$

be the associated bounded map provided by Theorem 2.6.

Let $F \in H_0^\infty(\Sigma_\theta; E_A)$ for some $\theta > 0$, and let $\varphi_F \in C_\ell([0, \infty)) \otimes^R E_A$ be defined by (3.3). We claim that

$$F(A) = \widehat{u}(\varphi_F).$$

Indeed for any $\lambda \in \partial\Sigma_\gamma$, we have $u(R_\lambda) = (\lambda - A)^{-1}$, hence $\widehat{u}(R_\lambda \otimes F(\lambda)) = (\lambda - A)^{-1}F(\lambda)$. Thus according to the definition of φ_F and the continuity of \widehat{u} , we have

$$\widehat{u}(\varphi_F) = \frac{1}{2\pi i} \int_{\Gamma_\gamma} \widehat{u}(R_\lambda \otimes F(\lambda)) d\lambda = \frac{1}{2\pi i} \int_{\Gamma_\gamma} (\lambda - A)^{-1}F(\lambda) d\lambda = F(A).$$

Consequently,

$$\|F(A)\| \leq \|\widehat{u}\| \|\varphi_F\|_R \leq \|u\|^2 \|\varphi_F\|_R.$$

It follows from Lemma 3.5 and Remark 2.5 that $\|\varphi_F\|_R = R(\{F(t) : t > 0\})$, and the result follows at once. \square

In the rest of this section we will further investigate operators with a uniformly bounded H^∞ -calculus. We start with the case when X is a Hilbert space.

Proposition 3.7. *Let H be a Hilbert space and let A be a sectorial operator on H , with $\omega(A) = 0$. Then A admits a uniformly bounded H^∞ -calculus if and only if there exists an isomorphism $S: H \rightarrow H$ such that $S^{-1}AS$ is selfadjoint.*

Proof. Assume that A admits a uniformly bounded H^∞ -calculus and let $u: C_\ell([0, \infty)) \rightarrow B(H)$ be the associated representation. According to [28, Thm. 9.1 and Thm. 9.7], there exists an isomorphism $S: H \rightarrow H$ such that the unital homomorphism $u_S: C_\ell([0, \infty)) \rightarrow B(H)$ defined by $u_S(f) = S^{-1}u(f)S$ satisfies $\|u_S\| \leq 1$. We let $B = S^{-1}AS$. For any $s \in \mathbb{R}^*$, we have $\|R_{is}\|_{\infty,0} = |s|$ and $u_S(R_{is}) = S^{-1}(is - A)^{-1}S = (is - B)^{-1}$. Hence

$$\|(is - B)^{-1}\| \leq |s|, \quad s \in \mathbb{R}^*.$$

By the Hille-Yosida Theorem, this implies that iB and $-iB$ both generate contractive c_0 -semigroups on H . Thus iB generates a unitary c_0 -group. By Stone's Theorem, this implies that B is selfadjoint.

The converse implication is clear. \square

In the non Hilbertian setting, we will first show that operators with a uniformly bounded H^∞ -calculus satisfy a spectral mapping theorem with respect to continuous functions defined on the one-point compactification of $\sigma(A)$. Then we will discuss the connections with spectral measures and scalar-type operators. We mainly refer to [13, Chap. 5-7] for this topic.

For any compact set K and any closed subset $F \subset K$, we let

$$I_F = \{f \in C(K) : f|_F = 0\}.$$

We recall that the restriction map $f \mapsto f|_F$ induces a $*$ -isomorphism $C(K)/I_F \rightarrow C(F)$.

Lemma 3.8. *Let $K \subset \mathbb{C}$ be a compact set and let $u: C(K) \rightarrow B(X)$ be a representation. Let $\kappa \in C(K)$ be the function defined by $\kappa(z) = z$ and put $T = u(\kappa)$.*

(1) *Then $\sigma(T) \subset K$ and u vanishes on $I_{\sigma(T)}$.*

Let $v: C(\sigma(T)) \simeq C(K)/I_{\sigma(T)} \rightarrow B(X)$ be the representation induced by u .

(2) *For any $f \in C(\sigma(T))$, we have $\sigma(v(f)) = f(\sigma(T))$.*

(3) *v is an isomorphism onto its range.*

Proof. The inclusion $\sigma(T) \subset K$ is clear. Indeed, for any $\lambda \notin K$, $(\lambda - T)^{-1}$ is equal to $u((\lambda - \cdot)^{-1})$. We will now show that u vanishes on $I_{\sigma(T)}$.

Let $w: C(K) \rightarrow B(X^*)$ be defined by $w(f) = [u(f)]^*$, and let $\tilde{w}: C(K)^{**} \rightarrow B(X^*)$ be its w^* -extension. Since w is valued in $w^*B(X^*) \simeq B(X)$, this is a representation (see Lemma 2.3). Let Δ_K be the set of all Borel subsets of K . It is easy to check that the mapping

$$P: \Delta_K \longrightarrow B(X^*), \quad P(B) = \tilde{w}(\chi_B),$$

is a spectral measure of class (Δ_K, X) in the sense of [13, p. 119]. According to [13, Prop. 5.8], the operator T^* is prespectral of class X (in the sense of [13, Def. 5.5]) and the above mapping P is its resolution of the identity. Applying [13, Lem. 5.6] and the equality $\sigma(T^*) = \sigma(T)$, we obtain that $\tilde{w}(\chi_{\sigma(T)}) = P(\sigma(T)) = I_{X^*}$. Therefore for any $f \in I_{\sigma(T)}$, we have

$$u(f)^* = \tilde{w}(f(1 - \chi_{\sigma(T)})) = \tilde{w}(f)\tilde{w}(1 - \chi_{\sigma(T)}) = 0.$$

Hence u vanishes on $I_{\sigma(T)}$.

The proofs of (2) and (3) now follow from [13, Prop. 5.9] and the above proof. \square

In the sequel we consider a sectorial operator A with $\omega(A) = 0$. This assumption implies that $\sigma(A) \subset [0, \infty)$. By $C_\ell(\sigma(A))$, we denote either the space $C(\sigma(A))$ if A is bounded, or the space $\{f: \sigma(A) \rightarrow \mathbb{C} \mid f \text{ is continuous and } \lim_\infty f \text{ exists}\}$ if A is unbounded. In this case, $C_\ell(\sigma(A))$ coincides with the space of continuous functions on the one-point compactification of $\sigma(A)$. The following strengthens Lemma 3.2.

Proposition 3.9. *Let A be a sectorial operator on X with $\omega(A) = 0$. The following assertions are equivalent.*

(1) *A has a uniformly bounded H^∞ -calculus.*

(2) *There exists a (necessarily unique) bounded unital homomorphism*

$$\Psi: C_\ell(\sigma(A)) \longrightarrow B(X)$$

such that $\Psi((\lambda - \cdot)^{-1}) = (\lambda - A)^{-1}$ for any $\lambda \in \mathbb{C} \setminus \sigma(A)$.

In this case, Ψ is an isomorphism onto its range and for any $f \in C_\ell(\sigma(A))$, we have

$$(3.4) \quad \sigma(\Psi(f)) = f(\sigma(A)) \cup f_\infty,$$

where $f_\infty = \emptyset$ if A is bounded and $f_\infty = \{\lim_\infty f\}$ if A is unbounded.

Proof. Assume (1) and let $u: C_\ell([0, \infty)) \rightarrow B(X)$ be given by Lemma 3.2. We introduce the particular function $\phi \in C_\ell([0, \infty))$ defined by $\phi(t) = (1+t)^{-1}$. Consider the $*$ -isomorphism

$$\tau: C([0, 1]) \longrightarrow C_\ell([0, \infty)), \quad \tau(g) = g \circ \phi,$$

and set $T = (1+A)^{-1}$. If we let $\kappa(z) = z$ as in Lemma 3.8, we have $(u \circ \tau)(\kappa) = T$. Let $v: C(\sigma(T)) \rightarrow B(X)$ be the resulting factorisation of $u \circ \tau$. The spectral mapping theorem gives $\sigma(A) = \phi^{-1}(\sigma(T) \setminus \{0\})$ and $0 \in \sigma(T)$ if and only if A is unbounded. Thus the mapping

$$\tau_A: C(\sigma(T)) \longrightarrow C_\ell(\sigma(A))$$

defined by $\tau_A(g) = g \circ \phi$ also is a $*$ -isomorphism. Put $\Psi = v \circ \tau_A^{-1}: C_\ell(\sigma(A)) \rightarrow B(X)$. This is a unital bounded homomorphism. Note that $\phi^{-1}(z) = \frac{1-z}{z}$ for any $z \in (0, 1]$. Then for any $\lambda \in \mathbb{C} \setminus \sigma(A)$,

$$\begin{aligned} \Psi((\lambda - \cdot)^{-1}) &= v((\lambda - \cdot)^{-1} \circ \phi^{-1}) = v\left(z \mapsto \left(\lambda - \frac{1-z}{z}\right)^{-1}\right) \\ &= v\left(z \mapsto \frac{z}{(\lambda+1)z-1}\right) \\ &= T((\lambda+1)T-1)^{-1} = (\lambda - A)^{-1}. \end{aligned}$$

Hence Ψ satisfies (2). Its uniqueness follows from Lemma 3.2. The fact that Ψ is an isomorphism onto its range, and the spectral property (3.4) follow from the above construction and Lemma 3.8. Finally the implication ‘(2) \Rightarrow (1)’ also follows from Lemma 3.2. \square

Remark 3.10. Let A be a sectorial operator with a uniformly bounded H^∞ -calculus, and let $T = (1+A)^{-1}$. It follows from Lemma 3.8 and the proof of Proposition 3.9 that there exists a representation

$$v: C(\sigma(T)) \longrightarrow B(X)$$

satisfying $v(\kappa) = T$ (where $\kappa(z) = z$), such that $\sigma(v(f)) = f(\sigma(T))$ for any $f \in C(\sigma(T))$ and v is an isomorphism onto its range. Also, it follows from the proof of Lemma 3.8 that T^* is a scalar-type operator of class X , in the sense of [13, Def. 5.14].

Next according to [13, Thm. 6.24], the operator T (and hence A) is a scalar-type spectral operator if and only if for any $x \in X$, the mapping $C(\sigma(T)) \rightarrow X$ taking f to $v(f)x$ for any $f \in C(\sigma(T))$ is weakly compact.

Corollary 3.11. *Let A be a sectorial operator on X , with $\omega(A) = 0$, and assume that X does not contain a copy of c_0 . Then A admits a uniformly bounded H^∞ -calculus if and only if it is a scalar-type spectral operator.*

Proof. The ‘only if’ part follows from the previous remark. Indeed if X does not contain a copy of c_0 , then any bounded map $C(K) \rightarrow X$ is weakly compact [9, VI, Thm. 15]. (See also [31] and [8] for related approaches.) The ‘if’ part follows from [16, Prop. 2.7] and its proof. \square

Remark 3.12.

(1) The hypothesis on X in Corollary 3.11 is necessary. Namely it follows from [12, Thm. 3.2] and its proof that if $c_0 \subset X$, then there is a sectorial operator A with a uniformly bounded H^∞ -calculus on X which is not scalar-type spectral.

(2) Scalar-type spectral operators on Hilbert space coincide with operators similar to a normal one (see [13, Chap. 7]). Thus when $X = H$ is a Hilbert space, the above corollary reduces to Proposition 3.7.

4. MATRICIAL R -BOUNDEDNESS

For any integer $n \geq 1$ and any vector space E , we will denote by $M_n(E)$ the space of all $n \times n$ matrices with entries in E . We will be mostly concerned with the cases $E = C(K)$ or $E = B(X)$. As mentioned in the introduction, we identify $M_n(C(K))$ with the space $C(K; M_n)$ in the usual way. We now introduce a specific norm on $M_n(B(X))$. Namely for any $[T_{ij}] \in M_n(B(X))$, we set

$$\|[T_{ij}]\|_R = \sup \left\{ \left\| \sum_{i,j=1}^n \epsilon_i \otimes T_{ij}(x_j) \right\|_{\text{Rad}(X)} : x_1, \dots, x_n \in X, \left\| \sum_{j=1}^n \epsilon_j \otimes x_j \right\|_{\text{Rad}(X)} \leq 1 \right\}.$$

Clearly $\|\cdot\|_R$ is a norm on $M_n(B(X))$. Moreover if we consider any element of $M_n(B(X))$ as an operator on $\ell_n^2 \otimes X$ in the natural way, and if we equip the latter tensor product with the norm of $\text{Rad}_n(X)$, we obtain an isometric identification

$$(4.1) \quad (M_n(B(X)), \|\cdot\|_R) = B(\text{Rad}_n(X)).$$

Definition 4.1. Let $u: C(K) \rightarrow B(X)$ be a bounded linear mapping. We say that u is *matricially R -bounded* if there is a constant $C \geq 0$ such that for any $n \geq 1$, and for any $[f_{ij}] \in M_n(C(K))$, we have

$$(4.2) \quad \|[u(f_{ij})]\|_R \leq C \|[f_{ij}]\|_{C(K; M_n)}.$$

Remark 4.2. The above definition obviously extends to any bounded map $E \rightarrow B(X)$ defined on an operator space E , or more generally on any matricially normed space (see [14, 15]). The basic observations below apply to this general case as well.

(1) In the case that $X = H$ is a Hilbert space, we have

$$\left\| \sum_{j=1}^n \epsilon_j \otimes x_j \right\|_{\text{Rad}(H)} = \left(\sum_{j=1}^n \|x_j\|^2 \right)^{\frac{1}{2}}$$

for any $x_1, \dots, x_n \in H$. Consequently, writing that a mapping $u: C(K) \rightarrow B(H)$ is matricially R -bounded is equivalent to writing that u is completely bounded (see e.g. [28]). See Section 5 for the case when X is an L^p -space.

(2) The notation $\|\cdot\|_R$ introduced above is consistent with the one considered so far in Section 2. Indeed let b_1, \dots, b_n in $B(X)$. Then the diagonal matrix $\text{Diag}\{b_1, \dots, b_n\} \in M_n(B(X))$ and the tensor element $\sum_{k=1}^n e_k \otimes b_k \in \ell_n^\infty \otimes B(X)$ satisfy

$$\|\text{Diag}\{b_1, \dots, b_n\}\|_R = R(\{b_1, \dots, b_n\}) = \left\| \sum_{k=1}^n e_k \otimes b_k \right\|_R.$$

(3) If $u: C(K) \rightarrow B(X)$ is matricially R -bounded (with the estimate (4.2)), then u is R -bounded and $R(u) \leq C$. Indeed, consider f_1, \dots, f_n in the unit ball of $C(K)$. Then we have $\|\text{Diag}\{f_1, \dots, f_n\}\|_{C(K; M_n)} \leq 1$. Hence for any x_1, \dots, x_n in X ,

$$\begin{aligned} \left\| \sum_k \epsilon_k \otimes u(f_k) x_k \right\|_{\text{Rad}(X)} &\leq \|\text{Diag}\{u(f_1), \dots, u(f_n)\}\|_R \left\| \sum_k \epsilon_k \otimes x_k \right\|_{\text{Rad}(X)} \\ &\leq C \left\| \sum_k \epsilon_k \otimes x_k \right\|_{\text{Rad}(X)}. \end{aligned}$$

Let $(g_k)_{k \geq 1}$ be a sequence of complex valued, independent, standard Gaussian random variables on some probability space Ω_G . For any x_1, \dots, x_n in X let

$$\left\| \sum_k g_k \otimes x_k \right\|_{G(X)} = \left(\int_{\Omega_G} \left\| \sum_k g_k(\lambda) x_k \right\|_X^2 d\lambda \right)^{\frac{1}{2}}.$$

It is well-known that for any scalar valued matrix $a = [a_{ij}] \in M_n$, we have

$$(4.3) \quad \left\| \sum_{i,j=1}^n a_{ij} g_i \otimes x_j \right\|_{G(X)} \leq \|a\|_{M_n} \left\| \sum_{j=1}^n g_j \otimes x_j \right\|_{G(X)},$$

see e.g. [10, Cor. 12.17]. For any $n \geq 1$, introduce $\sigma_{n,X}: M_n \rightarrow B(\text{Rad}_n(X))$ by letting

$$\sigma_{n,X}([a_{ij}]) = [a_{ij} I_X].$$

If X has finite cotype, then we have a uniform equivalence

$$(4.4) \quad \left\| \sum_k \epsilon_k \otimes x_k \right\|_{\text{Rad}(X)} \asymp \left\| \sum_k g_k \otimes x_k \right\|_{G(X)}$$

between Rademacher and Gaussian averages on X (see e.g. [10, Thm. 12.27]). Combining with (4.3), this implies that

$$\sup_{n \geq 1} \|\sigma_{n,X}\| < \infty.$$

Following [29] we say that X has property (α) if there is a constant $C \geq 1$ such that for any finite family (x_{ij}) in X and any finite family (t_{ij}) of complex numbers,

$$(4.5) \quad \left\| \sum_{i,j} \epsilon_i \otimes \epsilon_j \otimes t_{ij} x_{ij} \right\|_{\text{Rad}(\text{Rad}(X))} \leq C \sup_{i,j} |t_{ij}| \left\| \sum_{i,j} \epsilon_i \otimes \epsilon_j \otimes x_{ij} \right\|_{\text{Rad}(\text{Rad}(X))}.$$

Equivalently, X has property (α) if and only if we have a uniform equivalence

$$\left\| \sum_{i,j} \epsilon_i \otimes \epsilon_j \otimes x_{ij} \right\|_{\text{Rad}(\text{Rad}(X))} \asymp \left\| \sum_{i,j} \epsilon_{ij} \otimes x_{ij} \right\|_{\text{Rad}(X)},$$

where $(\epsilon_{ij})_{i,j \geq 1}$ be a doubly indexed family of independent Rademacher variables.

The following is a characterization of property (α) in terms of the R -boundedness of $\sigma_{n,X}$.

Lemma 4.3. *A Banach space X has property (α) if and only if*

$$\sup_{n \geq 1} R(\sigma_{n,X}) < \infty.$$

Proof. Assume that X has property (α) . This implies that X has finite cotype, hence X satisfies the equivalence property (4.4). Let $a(1), \dots, a(N)$ be in M_n and let z_1, \dots, z_N be in $\text{Rad}_n(X)$. Let x_{jk} in X such that $z_k = \sum_j \epsilon_j \otimes x_{jk}$ for any k . We consider a doubly indexed family $(\epsilon_{ik})_{i,k \geq 1}$ as above, as well as a doubly indexed family $(g_{ik})_{i,k \geq 1}$ of independent standard Gaussian variables. Then

$$(4.6) \quad \sum_k \epsilon_k \otimes \sigma_{n,X}(a(k))z_k = \sum_{k,i,j} \epsilon_k \otimes \epsilon_i \otimes a(k)_{ij}x_{jk}.$$

Hence using the properties reviewed above, we have

$$\begin{aligned} \left\| \sum_k \epsilon_k \otimes \sigma_{n,X}(a(k))z_k \right\|_{\text{Rad}(\text{Rad}(X))} &\asymp \left\| \sum_{k,i,j} \epsilon_{ik} \otimes a(k)_{ij}x_{jk} \right\|_{\text{Rad}(X)} \\ &\asymp \left\| \sum_{k,i,j} g_{ik} \otimes a(k)_{ij}x_{jk} \right\|_{G(X)} \\ &\lesssim \left\| \begin{pmatrix} a(1) & 0 \dots & 0 \\ 0 & \ddots & 0 \\ 0 & \dots & 0 & a(N) \end{pmatrix} \right\|_{M_{Nn}} \left\| \sum_{k,j} g_{jk} \otimes x_{jk} \right\|_{G(X)} \\ &\lesssim \max_k \|a(k)\|_{M_n} \left\| \sum_{k,j} \epsilon_{jk} \otimes x_{jk} \right\|_{\text{Rad}(X)} \\ &\lesssim \max_k \|a(k)\|_{M_n} \left\| \sum_{k,j} \epsilon_k \otimes \epsilon_j \otimes x_{jk} \right\|_{\text{Rad}(\text{Rad}(X))} \\ &\lesssim \max_k \|a(k)\|_{M_n} \left\| \sum_k \epsilon_k \otimes z_k \right\|_{\text{Rad}(\text{Rad}(X))}. \end{aligned}$$

This shows that the $\sigma_{n,X}$'s are uniformly R -bounded.

Conversely, assume that for some constant $C \geq 1$, we have $R(\sigma_{n,X}) \leq C$ for all $n \geq 1$. Let $(t_{jk})_{j,k} \in \mathbb{C}^{n^2}$ with $|t_{jk}| \leq 1$ and for any $k = 1, \dots, n$, let $a(k) \in M_n$ be the diagonal matrix with entries t_{1k}, \dots, t_{nk} on the diagonal. Then $\|a(k)\| \leq 1$ for any k . Hence applying (4.6), we obtain that for any $(x_{jk})_{j,k}$ in X^{n^2} ,

$$\begin{aligned} \left\| \sum_{j,k} \epsilon_k \otimes \epsilon_j \otimes t_{jk}x_{jk} \right\|_{\text{Rad}(\text{Rad}(X))} &\leq R(\{a(1), \dots, a(n)\}) \left\| \sum_{j,k} \epsilon_k \otimes \epsilon_j \otimes x_{jk} \right\|_{\text{Rad}(\text{Rad}(X))} \\ &\leq C \left\| \sum_{j,k} \epsilon_k \otimes \epsilon_j \otimes x_{jk} \right\|_{\text{Rad}(\text{Rad}(X))}. \end{aligned}$$

This means that X has property (α) . □

Proposition 4.4. *Assume that X has property (α) . Then any bounded homomorphism $u: C(K) \rightarrow B(X)$ is matricially R -bounded.*

Proof. Let $u: C(K) \rightarrow B(X)$ be a bounded homomorphism and let $w: C(K) \rightarrow B(\text{Rad}_n(X))$ be defined by

$$w(f) = I_{\text{Rad}_n} \otimes u(f).$$

Clearly w is also a bounded homomorphism, with $\|w\| = \|u\|$. Recall the identification (4.1) and note that $w(f) = \text{Diag}\{u(f), \dots, u(f)\}$ for any $f \in C(K)$. Then for any $a = [a_{ij}] \in M_n$, we have

$$w(f)\sigma_{n,X}(a) = [a_{ij}u(f)] = \sigma_{n,X}(a)w(f).$$

By Corollary 2.8 and Lemma 4.3, the resulting mapping $w \cdot \sigma_{n,X}$ satisfies

$$\|w \cdot \sigma_{n,X}: C(K; M_n) \rightarrow B(\text{Rad}_n(X))\| \leq C\|u\|^2$$

where C does not depend on n . Let E_{ij} denote the canonical matrix units of M_n , for $i, j = 1, \dots, n$. Consider $[f_{ij}] \in C(K; M_n) \simeq M_n(C(K))$ and write this matrix as $\sum_{i,j} E_{ij} \otimes f_{ij}$. Then

$$w \cdot \sigma_{n,X}([f_{ij}]) = \sum_{i,j=1}^n w(f_{ij})\sigma_{n,X}(E_{ij}) = \sum_{i,j=1}^n u(f_{ij}) \otimes E_{ij} = [u(f_{ij})].$$

Hence $\|[u(f_{ij})]\|_R \leq C\|u\|^2\|[f_{ij}]\|_{C(K; M_n)}$, which proves that u is matricially R -bounded. \square

In the case that $X = H$ is a Hilbert space, it follows from Remark 4.2 (1) that the above proposition reduces to the fact that any bounded homomorphism $C(K) \rightarrow B(H)$ is completely bounded.

We also observe that applying the above proposition together with Remark 4.2 (3), we obtain the following corollary originally due to De Pagter and Ricker [8, Cor. 2.19]. Indeed, Proposition 4.4 should be regarded as a strengthening of their result.

Corollary 4.5. *Assume that X has property (α) . Then any bounded homomorphism $u: C(K) \rightarrow B(X)$ is R -bounded.*

Remark 4.6. The above corollary is nearly optimal. Indeed we claim that if X does not have property (α) and if K is any infinite compact set, then there exists a unital bounded homomorphism

$$u: C(K) \rightarrow B(\text{Rad}(X))$$

which is not R -bounded.

To prove this, let $(z_n)_{n \geq 1}$ be an infinite sequence of distinct points in K and let u be defined by

$$u(f) \left(\sum_{k \geq 1} \epsilon_k \otimes x_k \right) = \sum_{k \geq 1} f(z_k) \epsilon_k \otimes x_k.$$

According to (1.4), this is a bounded unital homomorphism satisfying $\|u\| \leq 2$. Assume now that u is R -bounded. Let $n \geq 1$ be an integer and consider families $(t_{ij})_{i,j}$ in \mathbb{C}^{n^2} and

$(x_{ij})_{i,j}$ in X^{n^2} . For any $i = 1, \dots, n$, there exists $f_i \in C(K)$ such that $\|f_i\| = \sup_j |t_{ij}|$ and $f_i(z_j) = t_{ij}$ for any $j = 1, \dots, n$. Then

$$\sum_i \epsilon_i \otimes u(f_i) \left(\sum_j \epsilon_j \otimes x_{ij} \right) = \sum_{i,j} t_{ij} \epsilon_i \otimes \epsilon_j \otimes x_{ij},$$

hence

$$\begin{aligned} \left\| \sum_{i,j} t_{ij} \epsilon_i \otimes \epsilon_j \otimes x_{ij} \right\|_{\text{Rad}(\text{Rad}(X))} &\leq R(u) \sup_i \|f_i\| \left\| \sum_{i,j} \epsilon_i \otimes \epsilon_j \otimes x_{ij} \right\|_{\text{Rad}(\text{Rad}(X))} \\ &\leq R(u) \sup_{i,j} |t_{ij}| \left\| \sum_{i,j} \epsilon_i \otimes \epsilon_j \otimes x_{ij} \right\|_{\text{Rad}(\text{Rad}(X))}. \end{aligned}$$

This shows (4.5).

5. APPLICATION TO L^p -SPACES AND UNCONDITIONAL BASES

Let X be a Banach lattice with finite cotype. A classical theorem of Maurey asserts that in addition to (4.4), we have a uniform equivalence

$$\left\| \sum_k \epsilon_k \otimes x_k \right\|_{\text{Rad}(X)} \asymp \left\| \left(\sum_k |x_k|^2 \right)^{\frac{1}{2}} \right\|$$

for finite families $(x_k)_k$ of X (see e.g. [10, Thm. 16.18]). Thus a bounded linear mapping $u: C(K) \rightarrow B(X)$ is matricially R -bounded if there is a constant $C \geq 0$ such that for any $n \geq 1$, for any matrix $[f_{ij}] \in M_n(C(K))$ and for any $x_1, \dots, x_n \in X$, we have

$$\left\| \left(\sum_i \left| \sum_j u(f_{ij}) x_j \right|^2 \right)^{\frac{1}{2}} \right\| \leq C \| [f_{ij}] \|_{C(K; M_n)} \left\| \left(\sum_j |x_j|^2 \right)^{\frac{1}{2}} \right\|.$$

Mappings satisfying this property were introduced by Simard in [32] under the name of ℓ^2 -cb maps. In this section we will apply a factorization property of ℓ^2 -cb maps established in [32], in the case when X is merely an L^p -space.

Throughout this section, we let (Ω, μ) be a σ -finite measure space. By definition, a density on that space is a measurable function $g: \Omega \rightarrow (0, \infty)$ such that $\|g\|_1 = 1$. For any such function and any $1 \leq p < \infty$, we consider the linear mapping

$$\phi_{p,g}: L^p(\Omega, \mu) \longrightarrow L^p(\Omega, g d\mu), \quad \phi_{p,g}(h) = g^{-1/p} h,$$

which is an isometric isomorphism. Note that $(\Omega, g d\mu)$ is a probability space. Passing from (Ω, μ) to $(\Omega, g d\mu)$ by means of the maps $\phi_{p,g}$ is usually called a change of density. A classical theorem of Johnson-Jones [18] asserts that for any bounded operator $T: L^p(\mu) \rightarrow L^p(\mu)$, there exists a density g on Ω such that $\phi_{p,g} \circ T \circ \phi_{p,g}^{-1}: L^p(g d\mu) \rightarrow L^p(g d\mu)$ extends to a bounded operator $L^2(g d\mu) \rightarrow L^2(g d\mu)$. The next statement is an analog of that result for $C(K)$ -representations.

Proposition 5.1. *Let $1 \leq p < \infty$ and let $u: C(K) \rightarrow B(L^p(\mu))$ be a bounded homomorphism. Then there exists a density $g: \Omega \rightarrow (0, \infty)$ and a bounded homomorphism $w: C(K) \rightarrow B(L^2(gd\mu))$ such that*

$$\phi_{p,g} \circ u(f) \circ \phi_{p,g}^{-1} = w(f), \quad f \in C(K),$$

where equality holds on $L^2(gd\mu) \cap L^p(gd\mu)$.

Proof. Since $X = L^p(\mu)$ has property (α) , the mapping u is matricially R -bounded by Proposition 4.4. According to the above discussion, this means that u is ℓ^2 -cb in the sense of [32, Def. 2]. The result therefore follows from [32, Thms. 3.4 and 3.6]. \square

We will now focus on Schauder bases on separable L^p -spaces. We refer to [27, Chap. 1] for general information on this topic. We simply recall that a sequence $(e_k)_{k \geq 1}$ in a Banach space X is a basis if for every $x \in X$, there exists a unique scalar sequence $(a_k)_{k \geq 1}$ such that $\sum_k a_k e_k$ converges to x . A basis $(e_k)_{k \geq 1}$ is called unconditional if this convergence is unconditional for all $x \in X$. We record the following standard characterization.

Lemma 5.2. *A sequence $(e_k)_{k \geq 1} \subset X$ of non-zero vectors is an unconditional basis of X if and only if $X = \overline{\text{Span}\{e_k : k \geq 1\}}$ and there exists a constant $C \geq 1$ such that for any bounded scalar sequence $(\lambda_k)_{k \geq 1}$ and for any finite scalar sequence $(a_k)_{k \geq 1}$,*

$$(5.1) \quad \left\| \sum_k \lambda_k a_k e_k \right\| \leq C \sup_k |\lambda_k| \left\| \sum_k a_k e_k \right\|.$$

We will need the following elementary lemma.

Lemma 5.3. *Let (Ω, ν) be a σ -finite measure space, let $1 \leq p < \infty$ and let $Q: L^p(\nu) \rightarrow L^p(\nu)$ be a finite rank bounded operator such that $Q|_{L^2(\nu) \cap L^p(\nu)}$ extends to a bounded operator $L^2(\nu) \rightarrow L^2(\nu)$. Then $Q(L^p(\nu)) \subset L^2(\nu)$.*

Proof. Let $E = Q(L^p(\nu) \cap L^2(\nu))$. By assumption, E is a finite dimensional subspace of $L^p(\nu) \cap L^2(\nu)$. Since E is automatically closed under the L^p -norm and Q is continuous, we obtain that $Q(L^p(\nu)) = E$. \square

Theorem 5.4. *Let $1 \leq p < \infty$ and assume that $(e_k)_{k \geq 1}$ is an unconditional basis of $L^p(\Omega, \mu)$. Then there exists a density g on Ω such that $\phi_{p,g}(e_k) \in L^2(gd\mu)$ for any $k \geq 1$, and the sequence $(\phi_{p,g}(e_k))_{k \geq 1}$ is an unconditional basis of $L^2(gd\mu)$.*

Proof. Property (5.1) implies that for any $\lambda = (\lambda_k)_{k \geq 1} \in \ell^\infty$, there exists a (necessarily unique) bounded operator $T_\lambda: L^p(\mu) \rightarrow L^p(\mu)$ such that $T_\lambda(e_k) = \lambda_k e_k$ for any $k \geq 1$. Moreover $\|T_\lambda\| \leq C \|\lambda\|_\infty$. We can therefore consider the mapping

$$u: \ell^\infty \longrightarrow B(L^p(\mu)), \quad u(\lambda) = T_\lambda,$$

and u is a bounded homomorphism. By Proposition 5.1, there is a constant $C_1 > 0$, and a density g on Ω such that with $\phi = \phi_{p,g}$, the mapping

$$\phi T_\lambda \phi^{-1}: L^p(gd\mu) \longrightarrow L^p(gd\mu)$$

extends to a bounded operator

$$S_\lambda: L^2(gd\mu) \longrightarrow L^2(gd\mu)$$

for any $\lambda \in \ell^\infty$, with $\|S_\lambda\| \leq C_1 \|\lambda\|_\infty$.

Assume first that $p \geq 2$, so that $L^p(gd\mu) \subset L^2(gd\mu)$. Let $\lambda = (\lambda_k)_{k \geq 1}$ in ℓ^∞ and let $(a_k)_{k \geq 1}$ be a finite scalar sequence. Then $S_\lambda(\phi(e_k)) = \phi T_\lambda \phi^{-1}(\phi(e_k)) = \lambda_k \phi(e_k)$ for any $k \geq 1$, hence

$$\left\| \sum_k \lambda_k a_k \phi(e_k) \right\|_{L^2(gd\mu)} = \left\| S_\lambda \left(\sum_k a_k \phi(e_k) \right) \right\|_{L^2(gd\mu)} \leq C_1 \|\lambda\|_\infty \left\| \sum_k a_k \phi(e_k) \right\|_{L^2(gd\mu)}.$$

Moreover the linear span of the $\phi(e_k)$'s is dense in $L^p(gd\mu)$, hence in $L^2(gd\mu)$. By Lemma 5.2, this shows that $(\phi(e_k))_{k \geq 1}$ is an unconditional basis of $L^2(gd\mu)$.

Assume now that $1 \leq p < 2$. For any $n \geq 1$, let $f_n \in \ell^\infty$ be defined by $(f_n)_k = \delta_{n,k}$ for any $k \geq 1$, and let $Q_n: L^p(gd\mu) \rightarrow L^p(gd\mu)$ be the projection defined by

$$Q_n \left(\sum_k a_k \phi(e_k) \right) = a_n \phi(e_n).$$

Then $Q_n = \phi T_{f_n} \phi^{-1}$ hence Q_n extends to an L^2 operator. Therefore, $\phi(e_n)$ belongs to $L^2(gd\mu)$ by Lemma 5.3.

Let $p' = p/(p-1)$ be the conjugate number of p , let $(e_k^*)_{k \geq 1}$ be the biorthogonal system of $(e_k)_{k \geq 1}$, and let $\phi' = \phi^{*-1}$. (It is easy to check that $\phi' = \phi_{p',g}$, but we will not use this point.) The linear span of the e_k^* 's is w^* -dense in $L^{p'}(\mu)$. Equivalently, the linear span of the $\phi'(e_k^*)$'s is w^* -dense in $L^{p'}(gd\mu)$, hence it is dense in $L^2(gd\mu)$. Moreover for any $\lambda \in \ell^\infty$ and for any $k \geq 1$, we have $T_\lambda^*(e_k^*) = \lambda_k e_k^*$. Thus for any finite scalar sequence $(a_k)_{k \geq 1}$, we have

$$\sum_k \lambda_k a_k \phi'(e_k^*) = (\phi T_\lambda \phi^{-1})^* \left(\sum_k a_k \phi'(e_k^*) \right) = S_\lambda^* \left(\sum_k a_k \phi'(e_k^*) \right).$$

Hence

$$\left\| \sum_k \lambda_k a_k \phi'(e_k^*) \right\|_{L^2(gd\mu)} \leq C_1 \left\| \sum_k a_k \phi'(e_k^*) \right\|_{L^2(gd\mu)}.$$

According to Lemma 5.2, this shows that $(\phi'(e_k^*))_{k \geq 1}$ is an unconditional basis of $L^2(gd\mu)$. It is plain that $(\phi(e_k))_{k \geq 1} \subset L^2(gd\mu)$ is the biorthogonal system of $(\phi'(e_k^*))_{k \geq 1} \subset L^2(gd\mu)$. This shows that in turn, $(\phi(e_k))_{k \geq 1}$ is an unconditional basis of $L^2(gd\mu)$. \square

We will now establish a variant of Theorem 5.4 for conditional bases. Recall that if $(e_k)_{k \geq 1}$ is a basis on some Banach space X , the projections $P_N: X \rightarrow X$ defined by

$$P_N \left(\sum_k a_k e_k \right) = \sum_{k=1}^N a_k e_k$$

are uniformly bounded. We will say that $(e_k)_{k \geq 1}$ is an R -basis if the set $\{P_N : N \geq 1\}$ is actually R -bounded. It follows from [4, Cor. 3.15] that any unconditional basis on L^p is an R -basis. See Remark 5.6 (2) for more on this.

Proposition 5.5. *Let $1 \leq p < \infty$ and let $(e_k)_{k \geq 1}$ be an R -basis of $L^p(\Omega, \mu)$. Then there exists a density g on Ω such that $\phi_{p,g}(e_k) \in L^2(gd\mu)$ for any $k \geq 1$, and the sequence $(\phi_{p,g}(e_k))_{k \geq 1}$ is a basis of $L^2(gd\mu)$.*

Proof. According to [26, Thm. 2.1], there exists a constant $C \geq 1$ and a density g on Ω such that with $\phi = \phi_{p,g}$, we have

$$\|\phi P_N \phi^{-1} h\|_2 \leq C \|h\|_2, \quad N \geq 1, \quad h \in L^2(gd\mu) \cap L^p(gd\mu).$$

Then the proof is similar to the one of Theorem 5.4, using [27, Prop. 1.a.3] instead of Lemma 5.2. We skip the details. \square

Remark 5.6.

(1) Theorem 5.4 and Proposition 5.5 can be easily extended to finite dimensional Schauder decompositions. We refer to [27, Sect. 1.g] for general information on this notion. Given a Schauder decomposition $(X_k)_{k \geq 1}$ of a Banach space X , let P_N be the associated projections, namely for any $N \geq 1$, $P_N: X \rightarrow X$ is the bounded projection onto $X_1 \oplus \cdots \oplus X_N$ vanishing on X_k for any $k \geq N + 1$. We say that $(X_k)_{k \geq 1}$ is an R -Schauder decomposition if the set $\{P_N : N \geq 1\}$ is R -bounded. Then we obtain that for any $1 < p < \infty$ and for any finite dimensional R -Schauder (resp. unconditional) decomposition $(X_k)_{k \geq 1}$ of $L^p(\mu)$, there exists a density g on Ω such that $\phi_{p,g}(X_k) \subset L^2(gd\mu)$ for any $k \geq 1$, and $(\phi_{p,g}(X_k))_{k \geq 1}$ is a Schauder (resp. unconditional) decomposition of $L^2(gd\mu)$.

(2) The concept of R -Schauder decompositions can be tracked down to [2], and played a key role in [4] and then in various works on L^p -maximal regularity and H^∞ -calculus, see in particular [20, 21]. Let C_p denote the Schatten spaces. For $1 < p \neq 2 < \infty$, an explicit example of a Schauder decomposition on $L^2([0, 1]; C_p)$ which is not R -Schauder is given in [4, Sect. 5]. More generally, it follows from [20] that whenever a reflexive Banach space X has an unconditional basis and is not isomorphic to ℓ^2 , then X admits a finite dimensional Schauder decomposition which is not R -Schauder. This applies in particular to $X = L^p([0, 1])$, for any $1 < p \neq 2 < \infty$. However the question whether $L^p([0, 1])$ admits a Schauder basis which is not R -Schauder, is apparently an open question.

We finally mention that according to [21, Thm. 3.3], any unconditional decomposition on a Banach space X with property (Δ) is an R -Schauder decomposition.

Acknowledgements. We are grateful to the referee for various remarks and references which improved the presentation of the paper.

REFERENCES

- [1] W. Arendt, and S. Bu, *The operator-valued Marcinkiewicz multiplier theorem and maximal regularity*, Math. Z. 240 (2002), 311-343.
- [2] E. Berkson, and T. A. Gillespie, *Spectral decompositions and harmonic analysis on UMD Banach spaces*, Studia Math. 112 (1994), 13-49.
- [3] J. W. Bunce, *Representations of strongly amenable C^* -algebras*, Proc. Amer. Math. Soc. 32 (1972), 241-246.
- [4] P. Clément, B. de Pagter, F. A. Sukochev, and H. Witvliet, *Schauder decompositions and multiplier theorems*, Studia Math. 138 (2000), 135-163.
- [5] J. B. Conway, *A course in operator theory*, Graduate studies in Math. 21, Amer. Math. Soc., Providence, RI, 2000.

- [6] M. Cowling, I. Doust, A. McIntosh, and A. Yagi, *Banach space operators with a bounded H^∞ functional calculus*, J. Aust. Math. Soc., Ser. A 60 (1996), 51-89.
- [7] G. Dales, *Banach algebras and automatic continuity*, London Math. Soc. Monographs, New series 24, Oxford University Press, 2000.
- [8] B. de Pagter, and W. J. Ricker, *$C(K)$ -representations and R -boundedness*, J. Lond. Math. Soc. 76 (2007), 498-512 .
- [9] J. Diestel, and J. J. Uhl, *Vector measures*, Mathematical Surveys and Monographs 15, Amer. Math. Soc., Providence, RI, 1977.
- [10] J. Diestel, H. Jarchow, and A. Tonge, *Absolutely summing operators*, Cambridge Studies in Advanced Mathematics 43, Cambridge University Press, 1995.
- [11] I. Doust, and T. A. Gillespie, *Well-boundedness of sums and products of operators*, J. London Math. Soc. 68 (2003), 183-192.
- [12] I. Doust, and R. deLaubenfels, *Functional calculus, integral representations, and Banach space geometry*, Quaestiones Math. 17 (1994), 161-171.
- [13] H. R. Dowson, *Spectral theory of linear operators*, London Math. Soc. Monographs 12, 1978.
- [14] E. Effros, and Z.-J. Ruan, *On matricially normed spaces*, Pacific J. Math. 132 (1988), 243-264.
- [15] E. Effros, and Z.-J. Ruan, *Operator spaces*, London Math. Soc. Monographs, New series 23, Oxford University Press, 2000.
- [16] A. M. Fröhlich, and L. Weis, *H^∞ calculus and dilations*, Bull. Soc. Math. France 134 (2006), 487-508.
- [17] M. Hoffmann, N. J. Kalton, and T. Kucherenko, *R -bounded approximating sequences and applications to semigroups*, J. Math. Anal. Appl. 294 (2004), 373-386.
- [18] W. B. Johnson, and L. Jones, *Every L^p operator is an L^2 operator*, Proc. Amer. Math. Soc. 72 (1978), 309-312.
- [19] R. V. Kadison, and J. R. Ringrose, *Fundamentals of the theory of operator algebras, I and II*, Graduate studies in Math. 16, Amer. Math. Soc., Providence, RI, 1997.
- [20] N. J. Kalton, and G. Lancien, *A solution to the problem of L^p -maximal regularity*, Math. Z. 235 (2000), 559-568.
- [21] N. J. Kalton, and L. Weis, *The H^∞ -calculus and sums of closed operators*, Math. Ann. 321 (2001), 319-345.
- [22] N. J. Kalton, and L. Weis, *The H^∞ functional calculus and square function estimates*, preprint.
- [23] P. C. Kunstmann, and L. Weis, *Maximal L_p -regularity for parabolic equations, Fourier multiplier theorems and H^∞ -functional calculus*, pp. 65-311 in "Functional analytic methods for evolution equations", Lect. Notes in Math. 1855, Springer, 2004.
- [24] C. Le Merdy, *H^∞ -functional calculus and applications to maximal regularity*, Publ. Math. Besançon 16 (1998), 41-77.
- [25] C. Le Merdy, *A strong similarity property of nuclear C^* -algebras*, Rocky Mountain J. Math. 30 (2000), 279-292.
- [26] C. Le Merdy, and A. Simard, *A Factorization property of R -bounded sets of operators on L^p -spaces*, Math. Nachr. 243 (2002), 146-155.
- [27] J. Lindenstrauss, and L. Tzafriri, *Classical Banach spaces I*, Springer-Verlag, 1977.
- [28] V. I. Paulsen, *Completely bounded maps and operator algebras*, Cambridge Studies in Advanced Mathematics 78, Cambridge University Press, 2002.
- [29] G. Pisier, *Some results on Banach spaces without local unconditional structure*, Compos. Math. 37 (1978), 3-19.
- [30] G. Pisier, *Similarity problems and completely bounded maps (Second, expanded version)*, Lect. Notes in Math. 1618, Springer, 2001.
- [31] W. J. Ricker, *Operator algebras generated by commuting projections: a vector measure approach*, Lect. Notes in Math. 1711, Springer, 1999.
- [32] A. Simard, *Factorization of sectorial operators with bounded H^∞ -functional calculus*, Houston J. Math. 25 (1999), 351-370.

INSTITUT FÜR ANALYSIS, KAISERSTRASSE 89, 76133 KARLSRUHE, GERMANY

LABORATOIRE DE MATHÉMATIQUES, UNIVERSITÉ DE FRANCHE-COMTÉ, 25030 BESANÇON CEDEX,
FRANCE

E-mail address: christoph.kriegler@univ-fcomte.fr

LABORATOIRE DE MATHÉMATIQUES, UNIVERSITÉ DE FRANCHE-COMTÉ, 25030 BESANÇON CEDEX,
FRANCE

E-mail address: clemerdy@univ-fcomte.fr