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► **To cite this version:**

Abdallah Badra. COMPUTING THE FROBENIUS NUMBER. International Journal of Pure and Applied Mathematics, 2010, 60 (1), pp.89-105. hal-00477408

**HAL Id: hal-00477408**

**<https://hal.science/hal-00477408>**

Submitted on 30 Apr 2010

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# COMPUTING THE FROBENIUS NUMBER

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**Abstract.** The Frobenius number  $g(A)$  of a finite subset  $A \subset \mathbb{N}$  such that  $\gcd(A) = 1$  is the largest integer which cannot be expressed as  $\sum_{a \in A} ax_a$  with non-negative integers  $x_a$ . We present an algorithm for the computation of  $g(A)$ . Without loss of generality we suppose that there exist  $a, b \in A$  such that  $\gcd(a, b) = 1$ . We give a formula for  $g(A)$  in the particular case that for all  $c, d \in A$ ,  $c + d$  can be written in the form  $c + d = xa + yb$  with  $x, y \geq 0$  (e.g.  $c + d > ab - a - b$ ). Using Euler polynomials we give a formula for  $g(A)$  in the case that  $A = \{a, b, c\}$ .

**AMS subj. Classification:** 11D04

**Key Words:** Frobenius number, linear diophantine equation.

## 1. Introduction and Statement of Results

Throughout this paper, small letters denote integers. We will set  $\mathbb{N} = \{1, 2, 3, \dots\}$  and  $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ . We consider a finite subset  $A \subset \mathbb{N}$  such that  $\gcd(A) = 1$ . We define  $S(A)$  the additive semi-group of  $\mathbb{N}_0$  generated by  $A$  i.e.  $S(A) = \{\sum_{a \in A} x_a a \mid x_a \in \mathbb{N}_0\}$ . The Frobenius number  $g(A)$  is defined as the largest integer which does not belong to  $S(A)$ . We are interested in computing  $g(A)$  and equivalently  $f(A) = g(A) + \sum_{a \in A} a$  the largest integer which cannot be expressed as  $\sum_{a \in A} ax_a$  with positive integers  $x_a$ . It is well known that  $g(a, b) = ab - a - b$  if  $\gcd(a, b) = 1$ , and  $g(A) = -1$  if

and only if  $1 \in A$ .

For  $\text{card}(A) \geq 3$ , no general formula for  $g(A)$  is known, except in particular cases, see [6] and [8]. Algorithms are developed in [4] and [7] in the case  $A = \{a, b, c\}$ .

Without loss of generality, we can consider only sets  $A$  containing two coprime integers  $a, b$  (see section 2). We fix two integers  $a > 1$  and  $b > 1$  such that  $\text{gcd}(a, b) = 1$ , and  $\mathbf{c} = (c_1, \dots, c_n) \in \mathbb{N}^n$ . Let  $A = \{a, b, c_1, \dots, c_n\}$ .

For all  $\mathbf{t} = (t_1, \dots, t_n) \in \mathbb{N}_0^n$ , we set  $\mathbf{t} \cdot \mathbf{c} = \sum_1^n t_i c_i$ .

For every  $m \in \mathbb{Z}$ , there exist unique integers  $\hat{m}$  and  $\check{m}$  such that  $m = \hat{m}a - \check{m}b$  with  $0 \leq \hat{m} < b$ . We write  $\mathbf{t} \cdot \mathbf{c} = b(\mathbf{t})a - a(\mathbf{t})b$  where  $b(\mathbf{t}) = \hat{\mathbf{t} \cdot \mathbf{c}}$  and  $a(\mathbf{t}) = \check{\mathbf{t} \cdot \mathbf{c}}$ .

A finite subset  $T$  of  $\mathbb{N}_0^n$  is said to be *appropriate* if

$$(1) \quad S(A) = \bigcup_{\mathbf{t} \in T} (S(a, b) + \mathbf{t} \cdot \mathbf{c}).$$

It is said to be *pruned* if  $\mathbf{0} \in T$  and for  $\mathbf{t} \in T \setminus \{\mathbf{0}\}$ , we have  $b(\mathbf{t}) > 0$  and  $a(\mathbf{t}) > 0$ .

Let  $T$  be an *appropriate* and *pruned* subset of  $\mathbb{N}_0^n$ . We can write  $T = \{\mathbf{0} = \mathbf{t}_0, \mathbf{t}_1, \dots, \mathbf{t}_l\}$  with  $0 = b(\mathbf{t}_0) < b(\mathbf{t}_1) \leq b(\mathbf{t}_2) \leq \dots \leq b(\mathbf{t}_l) < b$ . We set  $b_i = b(\mathbf{t}_i)$ ,  $a_i = a(\mathbf{t}_i)$  for  $0 \leq i \leq l$ ,  $b_{l+1} = b$  and  $a_{l+1} = a$ .

We obtain the sequences

$$0 = b_0 < b_1 \leq b_2 \leq \dots \leq b_l < b_{l+1} = b$$

and

$$0 = a_0 < a_1, a_2, \dots, a_l < a_{l+1} = a.$$

For  $0 \leq i \leq l$ , we set  $m_i = \max\{a_j \mid 0 \leq j \leq i\}$  and  $g_i = (b_{i+1} - 1)a - (m_i + 1)b$ .

Our main result is the following

**Theorem 1.**

$$(2) \quad g(A) = \max\{g_i \mid 0 \leq i \leq l\}.$$

A subset  $M \subset \mathbb{N}_0$  is said to be *trimmed* if for all  $m \in M \setminus \{0\}$ ,  $\check{m} > 0$  and for all  $m, d \in M$ ,

$$(3) \quad \hat{m} < \hat{d} \Leftrightarrow \check{m} < \check{d}.$$

We show that there exists an *appropriate* set  $T$  such that

$T.\mathbf{c} = \{\mathbf{t}.\mathbf{c} \mid \mathbf{t} \in T\}$  is *trimmed*.

**Theorem 2.** *Let  $T$  be an appropriate set. If  $T.\mathbf{c}$  is trimmed then*

$$(4) \quad g(A) = \max\{a(b_{i+1} - b_i) + \mathbf{t}_i.\mathbf{c} \mid 0 \leq i \leq l\} - (a + b)$$

where  $\mathbf{t}_0 = \mathbf{0}$ .

We give an algorithm for computing  $g(A)$ . We use Theorem 1 if we start with an arbitrary *appropriate* set  $T$ . And we use Theorem 2 if we start with an *appropriate* set  $T$  such that  $T.\mathbf{c}$  is *trimmed*.

In a particular case we derive a formula for  $g(A)$  from Theorem 2:

**Theorem 3.** *If  $\{c_1, \dots, c_n\}$  is trimmed and for all  $1 \leq i, j \leq n$ , we have  $c_i + c_j \in S(a, b)$  (e.g.  $c_i + c_j > ab - a - b$ ) then*

$$(5) \quad g(A) = \max\{a(b_{i+1} - b_i) + c_i \mid 0 \leq i \leq n\} - (a + b)$$

where  $b_{n+1} = b, b_0 = 0, c_0 = 0$  and for all  $1 \leq i \leq n, b_i = \hat{c}_i$ .

In the case  $n = 1$ , i.e.  $A = \{a, b, c\}$ , we take  $c \notin S(a, b)$  and we write  $c = au_0 - bv_0$  with  $0 < u_0 < b$  and  $0 < v_0 < a$ .

By successive Euclidean divisions we get:

$$(6) \quad \begin{cases} a = u_{-1} = p_0v_0 + u_1, \dots, u_{n-1} = p_nv_n + u_{n+1}, \dots \\ b = v_{-1} = q_0u_0 + v_1, \dots, v_{n-1} = q_nu_n + v_{n+1}, \dots \end{cases}$$

The triplet  $(a, b, c)$  is said to be of level  $n = n(a, b, c)$  if

$$p_0 = q_0, p_1 = q_1, \dots, p_{n-1} = q_{n-1}$$

and  $(p_n \neq q_n \text{ or } v_{n+1} = 0)$ .

Let  $(a, b, c)$  a triplet of level  $n$ . For all  $0 \leq i \leq n$ , we set  $w_{i+1} = u_{i-1} - q_i v_i$  so for  $i \leq n$ ,  $w_i = u_i$  and  $w_{n+1} = (p_n - q_n)v_n + u_{n+1}$ .

We denote by

$$L(a, b, c) = (L_0, L_1) = \begin{cases} (bv_n, bw_{n+1}) & \text{if } n \text{ is even} \\ (av_n, aw_{n+1}) & \text{if } n \text{ is odd} \end{cases},$$

$$l(a, b, c) = (l_0, l_1) = \begin{cases} (aw_n, av_{n+1}) & \text{if } n \text{ is even} \\ (bw_n, bv_{n+1}) & \text{if } n \text{ is odd} \end{cases}.$$

For an integer  $n$  we denote by  $k(n) = k = \lfloor \frac{n}{2} \rfloor$  and  $h(n) = h = \lfloor \frac{n+1}{2} \rfloor$  where  $\lfloor x \rfloor$  is the greatest integer such that  $\lfloor x \rfloor \leq x$ .

**Theorem 4.**

$$(7) \quad \begin{aligned} f(a, b, c) &= aw_{2h} + bw_{2k+1} - \min\{bv_{2h}, av_{2k+1}\} \\ &= l_0 + L_1 - \min\{l_1, L_0\}. \end{aligned}$$

In particular, when  $p_n = q_n$  and  $v_{n+1} = 0$ ,

$$(8) \quad f(a, b, c) = \begin{cases} a \gcd(b, c) + \text{lcm}(b, c) & \text{if } n \text{ is even} \\ b \gcd(a, c) + \text{lcm}(a, c) & \text{if } n \text{ is odd.} \end{cases}$$

**Theorem 5.** Let  $(x_{-1}, x_0, \dots, x_{l+1}), (y_{-1}, y_0, \dots, y_{m+1}), (d_0, d_1, \dots, d_l)$  and  $(e_0, e_1, \dots, e_m)$  be sequences such that the following conditions

$$(9) \quad \begin{cases} x_{-1} = d_0 x_0 - x_1, \dots, x_{l-1} = d_l x_l - x_{l+1}, \\ y_{-1} = e_0 y_0 - y_1, \dots, y_{m-1} = e_m y_m - y_{m+1}, \end{cases}$$

$$(10) \quad \begin{cases} a = x_{-1} > x_0 > x_1 > \dots > x_l > x_{l+1} = 0, \\ b = y_{-1} > y_0 > y_1 > \dots > y_m > y_{m+1} = 0 \end{cases}$$

hold. Let  $n$  be an integer such that

$$(11) \quad \begin{cases} d_0 = e_0, d_1 = e_1, \dots, d_{n-1} = e_{n-1} \\ \text{and } (d_n \neq e_n \text{ or } x_{n+1} = 0). \end{cases}$$

Then,

$$(12) \quad f(a, b, c) = ay_{n+1} + b(x_{n-1} - e_n x_n) + \max\{a(y_n - y_{n+1}), bx_n\}.$$

As a consequence of Theorem 5 we prove the following theorem see [7].

**Theorem 6(Rodseth).** *Let  $R_i$  polynomials defined by induction as follows:  $R_{-1} = 0, R_0 = 1$  and for all  $i > 0$ ,*

$$(13) \quad R_{i+1} = e_i R_i - R_{i-1}.$$

Let  $n$  be the unique integer such that

$$(14) \quad \frac{y_{n+1}}{R_{n+1}} \leq \frac{c}{a} < \frac{y_i}{R_i} \text{ for all } 0 \leq i \leq n.$$

Then,

$$(15) \quad f(a, b, c) = cR_{n+1} + ay_n - \min\{ay_{n+1}, cR_n\}.$$

## 2. Reduction to the case $\gcd(a, b) = 1$

Let  $A = \{b, c_1, \dots, c_n\}$  be a subset of  $\mathbb{N}$  such that  $\gcd(A) = 1$ . We recall the following classical result: It is always possible to choose integers  $x, x_1, \dots, x_n$  such that  $1 = xb + x_1 c_1 + \dots + x_n c_n$ . Then, for  $k$  an integer sufficiently large, we get

$$a = 1 + [k(c_1 + \dots + c_n) - x]b = (x_1 + kb)c_1 + \dots + (x_n + kb)c_n \in S(c_1, \dots, c_n).$$

Hence,  $g(A) = g(A \cup \{a\})$  and  $\gcd(a, b) = 1$ .

In the particular case that

$A = \{a, b, c_1, \dots, c_n\}$  such that  $\gcd(a, b) = \gcd(a, b, c_1, \dots, c_{n-1}) = d \geq 1$  we observe that  $\gcd(\frac{a}{d}, \frac{b}{d}) = 1$ . Therefore, to compute  $g(A)$  we can use Brauer's formula

$$(16) \quad f(A) = df\left(\frac{a}{d}, \frac{b}{d}, \frac{c_1}{d}, \dots, \frac{c_{n-1}}{d}, c_n\right),$$

see [3].

From now on we suppose that  $\gcd(a, b) = 1$ .

### 3. Case $n \geq 1$

We need some remarks.

**R.** If  $m \geq 0$  then  $a > \check{m}$ .

**Proof.** Since  $m = \hat{m}a - \check{m}b \geq 0$  we have  $ba > \hat{m}a \geq \check{m}b$  hence  $a > \check{m}$   $\square$

**R2.** Let  $m = xa - yb$ . Then,

$$0 \leq x < b \Leftrightarrow \hat{m} = x \Leftrightarrow \check{m} = y$$

and

$$-b \leq x < 0 \Leftrightarrow \hat{m} = x + b \Leftrightarrow \check{m} = y + a.$$

**Proof.** To prove the second claim we write  $m = xa - yb = (x + b)a - (y + a)b$  and we use the uniqueness of  $\hat{m}$  and  $\check{m}$   $\square$

**R.** Let  $m = xa - yb$ . Then, there exists a unique integer  $p \in \mathbb{Z}$  such that  $x = pb + \hat{m}$  and  $y = pa + \check{m}$ .

**Proof.** We write  $m = xa - yb = \hat{m}a - \check{m}b$  thus  $(x - \hat{m})a = (y - \check{m})b$ . Since  $\gcd(a, b) = 1$ ,  $p = \frac{x - \hat{m}}{b} = \frac{y - \check{m}}{a}$  is an integer  $\square$

**R4.** We have  $m \in S(a, b)$  if and only if  $\check{m} \leq 0$ .

**Proof.** Clearly  $\check{m} \leq 0 \Rightarrow m = \hat{m}a - \check{m}b \in S(a, b)$ . Conversely, if  $m \in S(a, b)$  then  $m = xa + yb$  with  $x \geq 0$  and  $y \geq 0$ . By **R3** there exists  $p \in \mathbb{Z}$  such that  $x = pb + \hat{m} \geq 0$  thus  $p \geq 0$ . We also have

$-y = pa + \check{m} \leq 0$  and so  $\check{m} = -y - pa \leq 0 \square$

**R5.** For  $d \geq 0$  we set  $G(d) = S(a, b) \cup (S(a, b) + d)$ .

Then, we have

$$m \notin G(d) \Leftrightarrow \check{m} > 0 \text{ and } (\hat{m} < \hat{d} \text{ or } \check{m} > \check{d}).$$

**Proof.** Let  $n = m - d$ . Since  $d \geq 0$ , **R1** shows that  $a > \check{d}$ . Hence,  $\check{m} - \check{d} + a > \check{m}$ . Moreover, we have  $n = (\hat{m} - \hat{d})a - (\check{m} - \check{d})b$  and  $-b < \hat{m} - \hat{d} < b$ . It follows from **R2** that  $\check{n} = \check{m} - \check{d}$  if  $\hat{m} - \hat{d} \geq 0$  and  $\check{n} = \check{m} - \check{d} + a > \check{m}$  if  $\hat{m} - \hat{d} < 0$ . We deduce that

$$\begin{aligned} & \check{m} > 0 \text{ and } (\hat{m} < \hat{d} \text{ or } \check{m} > \check{d}) \\ & \Leftrightarrow \\ & \check{m} > 0 \text{ and } (\hat{m} - \hat{d} \geq 0 \Rightarrow \check{m} - \check{d} > 0) \\ & \Leftrightarrow \\ & \check{m} > 0 \text{ and } \check{n} > 0 \\ & \Leftrightarrow \\ & m \notin S(a, b) \text{ and } n = m - d \notin S(a, b) \\ & \Leftrightarrow \\ & m \notin G(d) \square \end{aligned}$$

As a consequence of **R5** we obtain

**R6.** If  $T$  is *appropriate* then

$$m \notin S(A) \Leftrightarrow \check{m} > 0 \text{ and } \forall \mathbf{t} \in T, \hat{m} < b(\mathbf{t}) \text{ or } \check{m} > a(\mathbf{t}).$$

**Proof of Theorem 1.** For  $0 \leq i \leq l$ , we set

$$F(i) = \{m \in \mathbb{Z} \mid \hat{m} < b_{i+1} \text{ and } \check{m} > m_i\}.$$

**Step 1.**

$$(17) \quad \mathbb{Z} \setminus S(A) = \bigcup_{i=0}^l F(i).$$



Let  $m \in F(i)$ . For every  $0 \leq j \leq l$ ,  $\hat{m} < b_{i+1} \leq b_j$  if  $j \geq i+1$  and  $\check{m} > m_i \geq a_j$  if  $j \leq i$ . Hence, **R6** shows that  $m \notin S(A)$ .

Conversely, let  $m \notin S(A)$  then  $\check{m} > 0$  by **R4**. Since  $b_0 = 0 \leq \hat{m} < b = b_{l+1}$  and  $0 = b_0 < b_1 \leq b_1 \leq \dots \leq b_l \leq b_{l+1} = b$ , there exists  $0 \leq j \leq l$  such that  $\hat{m} < b_{j+1}$ . We put  $i = \min\{j \mid \hat{m} < b_{j+1}\}$ . We thus get for  $0 \leq j \leq i$ ,  $b_j \leq b_i \leq \hat{m} < b_{i+1}$  and  $\check{m} > a_j$  by **R6**. Hence,  $\check{m} > m_i$ . We conclude that  $m \in F(i)$ .

**Step 2.** For  $0 \leq i \leq l$ ,  $g_i = \max F(i)$ .

Since  $0 \leq b_{i+1} - 1 < b$ , we have  $\hat{g}_i = b_{i+1} - 1$  and  $\check{g}_i = m_i + 1 > m_i$ . We thus get  $g_i \in F(i)$ . Moreover, for all  $m \in F(i)$ ,  $m = \hat{m}a - \check{m}b \leq (b_{i+1} - 1)a - (m_i + 1)b$  then  $g_i = \max F(i)$   $\square$

In particular, when  $T = \{\mathbf{0}\}$  i.e. when all  $c_i \in S(a, b)$  we have  $b_0 = 0 < b_1 = b, a_0 = 0$  and  $m_0 = 0$ . Therefore,  $g(A) = g_0 = (b_1 - 1)a - (m_0 + 1)b = ab - a - b = g(a, b)$ .

**Proof of Theorem 2.** If  $T.c$  is *trimmed* then  $0 = b_0 < b_1 < b_1 < \dots < b_l < b_{l+1} = b$  and thus  $0 = a_0 < a_1 < a_1 < \dots < a_l < a_{l+1} = a$ . In particular,  $T$  is *pruned* and for all  $0 \leq i \leq l$ ,  $m_i = a_i$ . We can write  $g_i = a(b_{i+1} - b_i) + ab_i - ba_i - (a + b) = a(b_{i+1} - b_i) + \mathbf{t}_i.c - (a + b)$   $\square$

### Algorithm.

1. For every  $i$ , we choose  $\lambda_i > 0$  such that  $\lambda_i c_i \in S(a, b)$ . Numbers  $\lambda_i$  exist. Indeed, it is sufficient to take  $\lambda_i > \frac{g(a, b)}{c_i} = \frac{ab - a - b}{c_i}$ . The following set

$$U = \{\mathbf{t} = (t_1, \dots, t_n) \in \mathbb{N}_0^n \mid t_i < \lambda_i, 1 \leq i \leq n\}$$

is thus *appropriate*.

We remove from  $U$  all elements  $\mathbf{t} \neq \mathbf{0}$  such that  $b(\mathbf{t}) = 0$  or  $a(\mathbf{t}) \leq 0$ . The set  $T$  of all remaining elements is *pruned* and still *appropriate*.

We number the elements of  $T = \{\mathbf{0} = \mathbf{t}_0, \mathbf{t}_1, \dots, \mathbf{t}_l\}$  in such a

way that  $0 = b(\mathbf{t}_0) < b(\mathbf{t}_1) \leq b(\mathbf{t}_2) \leq \dots \leq b(\mathbf{t}_l)$ . We compute  $m_i = \max\{a_j \mid 0 \leq j \leq i\}$ ,  $g_i = (b_{i+1} - 1)a - (m_i + 1)b$  and  $g(A) = \max\{g_i \mid 0 \leq i \leq l\}$  where  $b_i = b(\mathbf{t}_i)$ ,  $a_i = a(\mathbf{t}_i)$  for  $0 \leq i \leq l$ ,  $b_{l+1} = b$  and  $a_{l+1} = a$ .

2. The algorithm can be modified as follows: For every  $1 \leq i \leq n$ , we start removing from  $A$  all elements  $c_j$  such that  $\hat{c}_i \leq \hat{c}_j$  and  $\check{c}_j \leq \check{c}_i$ . We choose an *appropriate* and *pruned* set  $T = \{\mathbf{0} = \mathbf{t}_0, \mathbf{t}_1, \dots, \mathbf{t}_l\}$ . We can suppose that  $0 = b(\mathbf{t}_0) < b(\mathbf{t}_1) \leq b(\mathbf{t}_2) \leq \dots \leq b(\mathbf{t}_l) < b$ . For all  $0 \leq i \leq l$ , we remove from  $T$  all  $\mathbf{t}_j$  such that  $(b(\mathbf{t}_i) \leq b(\mathbf{t}_j)$  and  $a(\mathbf{t}_j) \leq a(\mathbf{t}_i))$ . Considering the set of the remaining elements we can suppose that  $T.\mathbf{c}$  is trimmed. The Frobenius number can therefore be computed using Theorem 2.

**Proof.**

1. For all  $\mathbf{t} \in \mathbb{N}_0^n$ , we have  $G(\mathbf{t}.\mathbf{c}) = S(a, b) \cup (S(a, b) + \mathbf{t}.\mathbf{c}) \subset S(A)$  thus  $\bigcup_{\mathbf{t} \in U} G(\mathbf{t}.\mathbf{c}) \subset S(A)$ . Conversely, let  $m \in S(A)$ , then  $m = xa + yb + \sum_{i=1}^n x_i c_i$  with  $x \geq 0, y \geq 0$  and  $x_i \geq 0$ . By Euclidean division we write  $x_i = q_i \lambda_i + t_i$ . We thus get  $\mathbf{t} = (t_1, \dots, t_n) \in U$  and  $m = xa + yb + \sum_{i=1}^n q_i \lambda_i c_i + \mathbf{t}.\mathbf{c}$ . Since  $\lambda_i c_i \in S(a, b)$ , we have  $m \in G(\mathbf{t}.\mathbf{c})$ . The equality

$$(18) \quad S(A) = \bigcup_{\mathbf{t} \in U} G(\mathbf{t}.\mathbf{c})$$

follows. Therefore,  $U$  is *appropriate*.

By construction  $T$  is *pruned* and it is still *appropriate*. Indeed, for  $\mathbf{t} \neq \mathbf{0}$ ,  $\mathbf{t}.\mathbf{c} \notin S(a, b)$  if and only if  $a(\mathbf{t}) > 0$  by **R4**. Therefore,

$$(19) \quad T = \{\mathbf{t} \in U \mid \mathbf{t} = \mathbf{0} \text{ or } \mathbf{t}.\mathbf{c} \notin S(a, b)\}$$

and

$$(20) \quad S(A) = \bigcup_{\mathbf{t} \in T} G(\mathbf{t}.\mathbf{c}).$$

2. If  $\hat{c}_i \leq \hat{c}_j$  and  $\check{c}_j \leq \check{c}_i$  then  $c_j = c_i + (\hat{c}_j - \hat{c}_i)a + (\check{c}_i - \check{c}_j)b \in S(c_i, a, b)$ . Therefore,  $g(A) = g(A \setminus \{c_j\})$ . If  $b(\mathbf{t}_i) \leq b(\mathbf{t}_j)$  and  $a(\mathbf{t}_j) \leq a(\mathbf{t}_i)$  then  $\mathbf{t}_j.\mathbf{c} = \mathbf{t}_i.\mathbf{c} + (b(\mathbf{t}_j) - b(\mathbf{t}_i))a + (a(\mathbf{t}_i) - a(\mathbf{t}_j))b \in G(\mathbf{t}_i.\mathbf{c})$  thus  $G(\mathbf{t}_j.\mathbf{c}) \subset G(\mathbf{t}_i.\mathbf{c})$ . We see that (20) is not altered by removing  $\mathbf{t}_j$  from  $T$   $\square$

**Proof of Theorem 3.** Since  $2c_i \in S(a, b)$  for all  $1 \leq i \leq n$ ,  $U = \{\mathbf{t} = (t_1, \dots, t_n) \in \mathbb{N}_0^n \mid 0 \leq t_i \leq 1, 1 \leq i \leq n\}$  is *appropriate*. Furthermore, for  $\mathbf{t} = (t_1, \dots, t_n) \in U$ , if  $\mathbf{t}.\mathbf{c} \notin S(a, b)$  then there exists  $c_i$  and  $y \in S(a, b)$  such that  $\mathbf{t}.\mathbf{c} = c_i + y \in G(c_i)$ . Therefore, the set  $T = \{\mathbf{0} = \mathbf{t}_0, \mathbf{t}_1, \dots, \mathbf{t}_n\}$  where  $\mathbf{t}_i = (0, \dots, t_i, \dots, 0)$  and  $t_i = 1$  is also *appropriate*. Moreover,  $T.\mathbf{c} = \{0, c_1, \dots, c_n\}$  is *trimmed* and  $b(\mathbf{t}_i) = \hat{c}_i$  for all  $1 \leq i \leq n$   $\square$

#### 4. Case $n = 1$

Let  $E$  be a totally ordered set and  $x \in E$ . We call *successor* of  $x$  in  $E$ , and we denote  $x^+$ , the smallest element of  $E$  (if there exists any) such that  $x < x^+$ .

For an *appropriate* set  $T$ , we put  $b(T) = \{b(\mathbf{t}) \mid \mathbf{t} \in T\}$  and  $\mathcal{B} = b(T) \cup \{b\}$ . We equip  $\mathcal{B}$  with the natural order  $\leq$ .

To apply Theorem 2, in the case that  $n = 1$ , it is convenient to formulate it as follows:

**Theorem 2'.** *Let  $T$  be an appropriate set such that  $T.\mathbf{c}$  is trimmed. Then,*

$$(21) \quad g(A) = \max\{a(b(\mathbf{t})^+ - b(\mathbf{t})) + \mathbf{t}.\mathbf{c} \mid \mathbf{t} \in T\} - (a + b)$$

where  $b(\mathbf{t})^+$  is the successor of  $b(\mathbf{t})$  in  $\mathcal{B}$ .

Suppose that there exists integers  $\alpha_1, \dots, \alpha_q$  and a partition  $E_1, \dots, E_q$  of  $T$  such that  $b(\mathbf{t})^+ - b(\mathbf{t}) = \alpha_i$  for all  $\mathbf{t} \in E_i$  and for all  $1 \leq i \leq q$ .

Then,

$$(22) \quad g(A) = \max\{a\alpha_i + \beta_i \mid 1 \leq i \leq q\} - (a + b)$$

where we put  $\beta_i = \max_{E_i} \mathbf{c}$ .

To compute such a partition, we are led to introduce what we call Euler order on  $T$ . We will use Euler polynomials.

### **Euler polynomials**

Let  $(q_0, q_1, \dots, q_i, \dots)$  be a sequence of positive integers. We define Euler polynomials  $Q_i$  by induction as follows:  $Q_{-1} = 0, Q_0 = 1$  and for  $i \geq 0$ ,

$$(23) \quad Q_{i+1}(q_0, \dots, q_i) = q_i Q_i(q_0, \dots, q_{i-1}) + Q_{i-1}(q_0, \dots, q_{i-2}).$$

We set  $Q_{i+1} = Q_{i+1}(q_0, \dots, q_i), Q_i^1 = Q_i(q_1, \dots, q_i), P_{i+1} = Q_i + Q_{i+1}$  and  $P_{i+1}^1 = Q_i^1 + Q_{i+1}^1$ .

We deduce immediately that

$$(24) \quad Q_{n+1} = q_n Q_n + q_{n-2} Q_{n-2} + \dots + q_{n-2i} Q_{n-2i} + Q_{n-2i-1}$$

for  $0 \leq 2i \leq n$ .

### **Euler order $\leq_e$**

**Proposition 1.** *Every integer  $t \in \mathbb{N}_0$  can be written uniquely in the form*

$$(25) \quad t = t_0 Q_0 + t_1 Q_1 + \dots + t_n Q_n$$

where

$$(26) \quad \min\{i \mid t_i > 0\} \text{ is even,}$$

$$(27) \quad 0 \leq t_i \leq q_i \text{ for } 0 \leq i \leq n$$

and

$$(28) \quad t_i = q_i \Rightarrow t_{i-1} = 0 \text{ for } 1 \leq i \leq n.$$

Equality (25) is called Euler expansion of  $t$ .

**Proof.** There exists  $n \in \mathbb{N}_0$  such that  $t < Q_{n+1}$ . By successive Euclidean divisions we can write

$$t = t_n Q_n + s_n \text{ with } 0 \leq s_n < Q_n,$$

$$s_n = t_{n-1} Q_{n-1} + s_{n-1} \text{ with } 0 \leq s_{n-1} < Q_{n-1},$$

$\vdots$

$$s_2 = t_1 Q_1 + s_1 \text{ with } 0 \leq s_1 < Q_1,$$

$$s_1 = t_0 Q_0 \text{ with } t_0 = s_1.$$

We put  $i = \min\{j \mid t_j > 0\}$ . If  $i = 2e$  then  $t = t_{2e} Q_{2e} + t_{2e+1} Q_{2e+1} + \cdots + t_n Q_n$  is Euler expansion of  $t$ . If  $i = 2e + 1$ , using (24) we take  $t = q_0 Q_0 + \cdots + q_{2e} Q_{2e} + (t_{2e+1} - 1) Q_{2e+1} + \cdots + t_n Q_n$  as Euler expansion of  $t$ . Conditions (27) and (28) follow from  $s_{i+1} < Q_{i+1} = Q_{i-1} + q_i Q_i$ . The uniqueness follows from the uniqueness of the Euclidean division and the fact that  $\sum_{i=0}^{2e} t_i Q_i < Q_{2e+1}$  if and only if there exists  $0 \leq j \leq e$  such that  $t_{2j} < q_{2j}$   $\square$

For  $t, x \in \mathbb{N}_0$ , let  $n \in \mathbb{N}_0$  such that  $t, x \leq Q_{n+1}$ . We consider Euler expansions of  $t$  and  $x$  respectively

$$t = t_0 Q_0 + t_1 Q_1 + \cdots + t_n Q_n \text{ and } x = x_0 Q_0 + x_1 Q_1 + \cdots + x_n Q_n.$$

We define Euler order  $\leq_e$  as follows:  $t \leq_e x$  if

$$(t_0, -t_1, \dots, (-1)^i t_i, \dots, (-1)^n t_n) \leq_l (x_0, -x_1, \dots, (-1)^i x_i, \dots, (-1)^n x_n)$$

where  $\leq_l$  is the lexicographic order on  $\mathbb{Z}^n$ .

**Lemma.** Let  $n \in \mathbb{N}_0$ . We consider  $U_n = \{0, 1, \dots, Q_{n+1} - 1\}$  and  $V_n = \{0, 1, \dots, P_n - 1\}$  equipped with induced Euler order  $\leq_e$ .

1. We define a partition of  $U_n = F_1 \cup F_2$  as follows:

$$F_1 = \begin{cases} \{0, 1, \dots, Q_{2k+1} - Q_{2k} - 1\} & \text{if } n = 2k \\ \{Q_{2k+1}, \dots, Q_{2k+2} - 1\} & \text{if } n = 2k + 1 \end{cases}$$

$$F_2 = \begin{cases} \{Q_{2k+1} - Q_{2k}, \dots, Q_{2k+1} - 1\} & \text{if } n = 2k \\ \{0, 1, \dots, Q_{2k+1} - 1\} & \text{if } n = 2k + 1. \end{cases}$$

Then, the successor, in  $U_n$ , of all  $t \in F_1$  (resp.  $t \in F_2$ ) is  $t^+ = t + (-1)^n Q_n$  (resp.  $t^+ = t + (-1)^n [Q_n - Q_{n+1}]$ ). In particular, if  $q_n = 1$  then for all  $t \in F_2$ ,  $t^+ = t + (-1)^{n-1} Q_{n-1}$ .

2. We define a partition of  $V_n = E_1 \cup E_2$  as follows:  $E_1 = \{0, \dots, Q_{2h-1} - 1\}$ ,  $E_2 = \{Q_{2h-1}, \dots, P_n - 1\}$ . Then, the successor, in  $V_n$ , of all  $t \in E_1$  (resp.  $t \in E_2$ ) is  $t^+ = t + Q_{2k}$  (resp.  $t^+ = t - Q_{2h-1}$ ).

**Proof.** Let  $t = t_0 Q_0 + t_1 Q_1 + \dots + t_n Q_n$  be Euler expansion of  $t$ .

1. (a) Suppose that  $t \in F_1$ . It is easily seen that  $t_n > 0$  if  $n = 2k + 1$  and  $t_n < q_n - 1$  or ( $t_n = q_n - 1$  and  $t_{n-1} = 0$ ) if  $n = 2k$ . Therefore,  $t^+ = t + (-1)^n Q_n$ .

(b) Suppose that  $t \in F_2$ , then  $t = t_0 Q_0 + t_1 Q_1 + \dots + t_{2k} Q_{2k}$ . Since  $t < Q_{2k+1}$ , there exists  $j \leq k$  such that  $t_{2j} < q_{2j}$ . Taking  $i = \max\{j \mid t_{2j} < q_{2j}\}$  we can write  $t = t_0 Q_0 + t_1 Q_1 + \dots + t_{2i} Q_{2i} + [q_{2i+2} Q_{2i+2} + \dots + q_{2k} Q_{2k}] = t_0 Q_0 + t_1 Q_1 + \dots + t_{2i} Q_{2i} - Q_{2i+1} + Q_{2k+1}$ .

If  $t_{2i} < q_{2i} - 1$  or ( $t_{2i} = q_{2i} - 1$  and  $t_{2i-1} = 0$ ) then  $t^+ = t_0 Q_0 + t_1 Q_1 + \dots + (t_{2i} + 1) Q_{2i} + (q_{2i+1} - 1) Q_{2i+1} + \dots + q_{2h-1} Q_{2h-1} = t + Q_{2h} - Q_{2k+1}$ .

If  $t_{2i} = q_{2i} - 1$  and  $t_{2i-1} > 0$  then  $t^+ = t_0 Q_0 + t_1 Q_1 + \dots + (t_{2i-1} - 1) Q_{2i-1} + q_{2i+1} Q_{2i+1} + \dots + q_{2h-1} Q_{2h-1} = t + Q_{2h} - Q_{2k+1}$ .

2. It is a particular case: Taking  $q_n = 1$  we get  $P_n = Q_{n-1} + Q_n = Q_{n+1}$  and  $V_n = U_n$ . Moreover, in this case we have  $E_1 = F_2$  and  $E_2 = F_1$  if  $n = 2k$  and  $E_1 = F_1$  and  $E_2 = F_2$  if  $n = 2k+1$   
 $\square$

Let  $(r = r_{-1}, r_0, \dots, r_n)$  and  $(q_0, q_1, \dots, q_n)$  be sequences of positive integers and  $r_{n+1} \geq 0$ . We suppose that

$$(29) \quad r_{i-1} = q_i r_i + r_{i+1}$$

for  $0 \leq i \leq n$ .

We thus have  $r > r_0 > r_1 > \dots > r_n > 0$  and  $r_{n-1} > r_{n+1} \geq 0$ .

We prove by induction the following identities:

$$(30) \quad r = r_i Q_{i+1} + r_{i+1} Q_i$$

and

$$(31) \quad r_0 Q_i = (-1)^i r_i + r Q_{i-1}^1.$$

It follows from (31) that

$$(32) \quad r_0 P_{i+1} = (-1)^i (r_i - r_{i+1}) + r P_i^1.$$

Let  $t \in \mathbb{N}_0$ . Given  $t = t_0 Q_0 + t_1 Q_1 + \dots + t_n Q_n$  its Euler expansion, we associate with  $t$  the following numbers:  $r(t) = \sum_{i=0}^n (-1)^i t_i r_i$  and  $E(t) = \sum_{i=0}^n t_i Q_{i-1}^1$ . It follows from (31) that

$$(33) \quad t r_0 = r(t) + r E(t).$$

Moreover, if  $0 < t < Q_{n+1}$  then

$$(34) \quad r_n \leq r(t) \leq r - r_n$$

and if  $0 < t < P_n$  then

$$(35) \quad r_{2k} \leq r(t) \leq r - r_{2h-1}$$

Indeed, for  $0 < t < Q_{n+1}$ , let  $t = t_0Q_0 + \dots + t_nQ_n$  be Euler expansion of  $t$ . We can write  $t = \sum_{i=e}^k t_{2i}Q_{2i} + \sum_{i=e}^{h-1} t_{2i+1}Q_{2i+1}$  with  $t_{2e} > 0$ . Hence,  $r(t) = \sum_{i=e}^k t_{2i}r_{2i} - \sum_{i=e}^{h-1} t_{2i+1}r_{2i+1}$ .

Using  $r_{2e} = \sum_{i=e}^{h-1} q_{2i+1}r_{2i+1} + r_{2h}$  we get

$$\begin{aligned} r(t) &= (t_{2e} - 1)r_{2e} + \sum_{i=e+1}^k t_{2i}r_{2i} \\ &\quad + \sum_{i=e}^{h-1} (q_{2i+1} - t_{2i+1})r_{2i+1} + r_{2h}. \end{aligned}$$

Now if  $n = 2k + 1$  we get

(36)

$$r_{2h} + r_{2k+1} \leq r(t) \leq \sum_{i=0}^k q_{2i}r_{2i} = r - r_{2k+1} = r - r_n.$$

If  $n = 2k$  there exists  $j \leq k$  such that  $t_{2j} < q_{2j}$  since otherwise  $t = Q_{n+1}$ . We then get

(37)

$$r_n \leq r(e) \leq r(t) \leq r - r_{2j} - r_{2k+1} \leq r - r_{2k} - r_{2k+1}.$$

Hence, assumption (34) follows. In the particular case that  $q_n = 1$  we have  $P_n = Q_{n+1}$  and  $r_{n-1} = r_n + r_{n+1}$ . In this case if  $n = 2k$  we get  $r_{n-1} \leq r(t) \leq r - r_n$  by (36) and if  $n = 2k + 1$  we get  $r_n \leq r(t) \leq r - r_{n-1}$  by (37) thus assumption (35) follows  $\square$

**Proposition 2.** *We suppose that  $r_{n+1} = 0$ . We equip  $\{0, 1, \dots, r-1\}$  with the natural order  $\leq$  and  $U_n = \{0, 1, \dots, Q_{n+1}-1\}$  with Euler order  $\leq_e$ . Then, the mapping  $(U_n, \leq_e) \rightarrow (\{0, 1, \dots, r-1\}, \leq)$ ,  $t \mapsto r(t)$  is strictly increasing.*

**Proof.** Given  $t \leq_e x$  in  $U_n$  we have

$$(t_0, -t_1, \dots, (-1)^j t_j, \dots, (-1)^n t_n) \leq_l (x_0, -x_1, \dots, (-1)^j x_j, \dots, (-1)^n x_n)$$

$$\text{and } r(x) - r(t) = (-1)^j (x_j - t_j) r_j + \sum_{i=j+1}^n (-1)^i x_i r_i - \sum_{i=j+1}^n (-1)^i t_i r_i$$

where  $j$  is the smallest integer such that  $t_j \neq x_j$ .



1. When  $j = 2i$  we get  $x_{2i} > t_{2i} \geq 0$  and  $x_{2i+1} \leq q_{2i+1} - 1$ . Then,
$$r(x) - r(t) = (x_{2i} - t_{2i})r_{2i} + [-x_{2i+1}r_{2i+1} + \cdots + (-1)^n x_n r_n] - [-t_{2i+1}r_{2i+1} + \cdots + (-1)^n t_n r_n] \geq (x_{2i} - t_{2i})r_{2i} - [(q_{2i+1} - 1)r_{2i+1} + \cdots + q_{2h-1}r_{2h-1}] - [q_{2i+2}r_{2i+2} + \cdots + q_{2k}r_{2k}] \geq (x_{2i} - t_{2i})r_{2i} - r_{2i} + r_{2h} + r_{2k+1} > 0.$$
2. When  $j = 2i - 1$  we get  $t_{2i-1} > x_{2i-1} \geq 0$  and  $t_{2i} \leq q_{2i} - 1$ . Then,
$$r(x) - r(t) = (t_{2i-1} - x_{2i-1})r_{2i-1} + [x_{2i}r_{2i} + \cdots + (-1)^n x_n r_n] - [t_{2i}r_{2i} + \cdots + (-1)^n t_n r_n] \geq (t_{2i-1} - x_{2i-1})r_{2i-1} - [q_{2i+1}r_{2i+1} + \cdots + q_{2h-1}r_{2h-1}] - [(q_{2i} - 1)r_{2i} + \cdots + q_{2k}r_{2k}] \geq (t_{2i-1} - x_{2i-1})r_{2i-1} + r_{2h} - r_{2i-1} + r_{2k+1} > 0 \quad \square$$

Now we consider another sequence of positive integers  $(s = s_{-1}, s_0, s_1, \dots, s_n)$  and  $s_{n+1} \geq 0$  such that  $s_{i-1} = q_i s_i + s_{i+1}$  for  $0 \leq i \leq n$ .

We also define  $s(t) = \sum_{i=0}^n (-1)^i t_i s_i$ .

Using (31) we prove the following identity

$$(38) \quad (rs_0 - sr_0)Q_i = (-1)^i (rs_i - sr_i).$$

We derive

$$(39) \quad (rs_0 - sr_0)P_{i+1} = (-1)^i [r(s_i - s_{i+1}) + s(r_{i+1} - r_i)]$$

from (32) and

$$(40) \quad t(rs_0 - sr_0) = rs(t) - sr(t)$$

from (33).

#### **Proof of Theorem 4.**

**First step :** Reduction to the case  $n = 2k$ .

Suppose that  $n = n(a, b, c) = 2k + 1$ .

Since  $g(a, b, c) = g(b, a, c)$ , it suffices to show that  $n(b, a, c)$  is even,

$L(a, b, c) = L(b, a, c)$  and  $l(a, b, c) = l(b, a, c)$ .

We write  $c = b(a - v_0) - a(b - u_0)$ . We consider two cases:

If  $p_0 = q_0 > 1$  then we can write  $a = (a - v_0) + v_0$ ,  $a - v_0 = (q_0 - 1)v_0 + v_1$ ,  $b = (b - u_0) + u_0$ ,  $b - u_0 = (q_0 - 1)u_0 + u_1$ . Therefore,  $n(b, a, c) = n(a, b, c) + 1$ ,  $L(b, a, c) = L(a, b, c)$  and  $l(b, a, c) = l(a, b, c)$ .

If  $p_0 = q_0 = 1$  we get  $a - v_0 = u_1$  and  $b - u_0 = v_1$ . We therefore get  $a = (q_1 + 1)(a - v_0) + v_2$  and  $b = (p_1 + 1)(b - u_0) + u_2$ . It follows that  $n(b, a, c) = n(a, b, c) - 1$ . Furthermore, we observe that if  $n(a, b, c) > 1$  then we have obviously  $L(a, b, c) = L(b, a, c)$  and  $l(a, b, c) = l(b, a, c)$ . If  $n(a, b, c) = 1$  we have  $L(b, a, c) = (av_1, a(b - (q_1 + 1)v_1)) = (av_1, a(u_0 - q_1v_1)) = L(a, b, c)$  and  $l(b, a, c) = (bu_1, bv_2) = l(a, b, c)$ . In both cases,  $n(b, a, c)$  is even. Therefore, the assumption follows.

Moreover, we can write

$$(41) \quad c = w_n w_{n+1} - v_n v_{n+1}.$$

Since  $c > 0$ , we get  $w_{n+1} = (p_n - q_n)v_n + u_{n+1} > 0$  thus  $p_n > q_n$  or  $(p_n = q_n \text{ and } v_{n+1} = 0)$ . When  $p_n = q_n$  and  $v_{n+1} = 0$  we then have

$$(42) \quad c = w_n w_{n+1} = u_n u_{n+1}.$$

In the following steps we suppose that  $n = 2k$ .

**Second step:** Case  $p_n > q_n$ .

Taking  $a = r$ ,  $v_0 = r_0$ ,  $b = s$  and  $u_0 = w_0 = s_0$  it follows from (39) that  $cP_{n+1} = a(w_n - v_{n+1}) + b(w_{n+1} - v_n) \in S(a, b)$  because  $w_{n+1} > v_n$  and  $w_n > v_{n+1}$ . Moreover, it follows from (40) and (35) that  $s(t) = b(t)$  and  $r(t) = a(t)$ . Hence, for all  $0 < t < P_{n+1}$ ,  $tc = ab(t) - ba(t) \notin S(a, b)$  by **R4**. We conclude that

$$(43) \quad P_{n+1} = \min\{t > 0 \mid tc \in S(a, b)\}$$

and by (19)-(20), that  $V_n = \{0, 1, \dots, P_{n+1} - 1\}$  is *appropriate*. Let us show that  $V_n.c$  is *trimmed*. Indeed, for  $0 < t < x < P_{n+1}$ , since

$x-t < P_{n+1}$ , we have  $0 < (x-t)c = (b(x)-b(t))a - (a(x)-a(t))b \notin S(a, b)$ . Then,  $b(x) < b(t)$  if and only if  $a(x) < a(t)$  by **R4**.

Furthermore, it follows from proposition 2 that  $b(t)^+ - b(t) = b(t^+) - b(t) = b(t^+ - t)$ . Writing  $V_n = E_1 \cup E_2$  with the notation of the lemma, Theorem 2 and (22) show that  $g(a, b, c) = \max\{a\alpha_1 + \beta_1, a\alpha_2 + \beta_2\} - (a + b)$  where  $\alpha_1 = b(t^+) - b(t) = b(Q_n)$  for all  $t \in E_1$ ,  $\alpha_2 = b(t^+) - b(t) = -b(Q_{n+1})$  for all  $t \in E_2$ , and  $\beta_i = \max E_i.c$ .

It follows from (31) that  $\alpha_1 = w_n$ , and  $\alpha_2 = v_{n+1}$ . Moreover, we see that  $\beta_2 = c(P_{n+1} - 1)$ ,  $\beta_1 = c(Q_{n+1} - 1) = -cQ_n + cP_{n+1} - c = bv_n - aw_n + cP_{n+1} - c$  by (38). We therefore obtain the formula

$$(44) \quad g(a, b, c) = cP_{n+1} + \max\{av_{n+1}, bv_n\} - (a + b + c)$$

which can be written in the form

$$(45) \quad f(a, b, c) = a(w_n - v_{n+1}) + b(w_{n+1} - v_n) + \max\{av_{n+1}, bv_n\}$$

by (39). Finally,

$$(46) \quad f(a, b, c) = aw_n + bw_{n+1} - \min\{av_{n+1}, bv_n\}.$$

**Third step:** Case  $p_n = q_n$  and  $v_{n+1} = 0$ .

We have  $cQ_{n+1} = bw_{n+1} \in S(a, b)$  by (38) and for all  $0 < t < Q_{n+1}$ ,  $tc = ab(t) - ba(t) \notin S(a, b)$  by (34). We deduce that

$$(47) \quad Q_{n+1} = \min\{t > 0 \mid tc \in S(a, b)\}.$$

Using (19)-(20) we show by a similar argument that  $U_n = \{0, 1, \dots, Q_{n+1} - 1\}$  is *appropriate* and  $U_n.c$  is *trimmed*. Furthermore, since  $v_{n+1} = 0$  the lemma show that  $b(t^+) - b(t) = w_n$  for all  $t \in U_n$ . Therefore, by (21)  $f(a, b, c) = aw_n + cQ_{n+1} = aw_n + bw_{n+1}$ .

Moreover, since  $v_{n+1} = 0$ , we get  $w_n = \gcd(b, w_0) = \gcd(b, c)$  and

$b = w_n Q_{n+1}$  by (30). Using (42) we deduce that  $\text{lcm}(b, c) = bw_{n+1}$

□

**Remark.** The case that ( $p_n = q_n$  and  $v_{n+1} = 0$ ) can be deduced from Brauer's formula (16): We put  $d = \text{gcd}(b, c) = \text{gcd}(b, w_0) = w_n$ ,  $c' = \frac{c}{d} = w_{n+1}$  and  $b' = \frac{b}{d} = Q_{n+1}$ . Using (30) we get  $a = v_n Q_{n+1} + w_{n+1} Q_n > w_{n+1} Q_{n+1} = b'c'$ . We thus have  $f(a, b', c') = g(a, b', c') + a + b' + c' = b'c' + a$ . Hence, (16) show that  $f(a, b, c) = df(a, b', c') = b'c' + da = \text{lcm}(b, c) + a \text{gcd}(b, c)$  □

**Proof of Theorem 5.** We first prove by induction that polynomials  $R_i$  satisfy the following properties: For  $i \leq n$ ,

$$(48) \quad cR_i = ay_i - bx_i,$$

$$(49) \quad cR_{n+1} = ay_{n+1} - b(e_n x_n - x_{n-1}),$$

$$(50) \quad y_{-1} = y_i R_{i+1}(e_0, \dots, e_i) - y_{i+1} R_i(e_0, \dots, e_{i-1})$$

and for  $j \leq i$ ,

$$(51) \quad y_{j-1} = y_i R_{i-j+1}(e_j, \dots, e_i) - y_{i+1} R_{i-j}(e_j, \dots, e_{i-1}).$$

In the particular case that  $e_j = e_{j+1} = \dots = e_i = 2$ , we get

$$(52) \quad R_{i-j+1}(e_j, \dots, e_i) = (i - j + 2)$$

and

$$(53) \quad y_{j-1} - y_j = y_j - y_{j+1} = \dots = (y_i - y_{i+1})$$

so in this case (51) can be written in the form

$$(54) \quad y_{j-1} = (i - j + 2)(y_i - y_{i+1}) + y_{i+1}.$$

Now we consider the set

$$K = \{e_i \mid i = 0 \text{ or } (0 < i < m \text{ and } e_i > 2)\} = \{e_0 = e_{k_0}, e_{k_1}, \dots, e_{k_{s-1}}\}$$

and we set  $k_s = m$ . We can suppose that  $0 = k_0 < k_1 < \dots < k_s = m$ .

We have

$$(55) \quad y_{-1} = (e_0 - 1)y_0 + (y_0 - y_1)$$

and for  $0 < i \leq s$ ,

$$(56) \quad y_{k_{i-1}} - y_{k_i} = (e_{k_i} - 2)y_{k_i} + (y_{k_i} - y_{k_{i+1}}).$$

Furthermore, (54) show that

$$(57) \quad y_{k_{(i-1)}} = (k_i - k_{(i-1)})(y_{k_{i-1}} - y_{k_i}) + y_{k_i}$$

and (53) that for  $0 < i < s$ ,

$$(58) \quad y_{k_{(i-1)}} - y_{k_{(i-1)+1}} = \dots = y_{k_{i-1}} - y_{k_i} > y_{k_i}.$$

To apply Theorem 4 we set

$$v_{-1} = y_{-1}, u_0 = y_0, v_1 = (y_0 - y_1), q_0 = (e_0 - 1)$$

and for  $0 < i < s$ ,

$$v_{2i-1} = y_{k_{i-1}} - y_{k_i}, u_{2i} = y_{k_i}, q_{2i} = (e_{k_i} - 2), p_{2i-1} = (k_i - k_{(i-1)}).$$

Furthermore, if  $e_m > 2$  we set

$$v_{2s-1} = (y_{m-1} - y_m), v_{2s+1} = y_{m+1} = 0, p_{2s-1} = (m - k_{(s-1)}), u_{2s} = y_m, q_{2s} = (e_m - 1)$$

and if  $e_m = 2$  we set

$$v_{2s-1} = y_m, p_{2s-1} = (m + 1 - k_{(s-1)}) \text{ and } u_{2s} = y_{m+1} = 0.$$

We thus get

$$v_{-1} > u_0 > \dots > v_{2i-1} > u_{2i} > v_{2i+1} > \dots$$

and

$$v_{2i-1} = q_{2i}u_{2i} + v_{2i+1}; u_{2i} = p_{2i+1}v_{2i+1} + u_{2i+2}.$$

Using (11) we get  $c = ay_0 - bx_0 = x_n y_n (d_n - e_n) + x_n y_{n+1} - y_n x_{n+1} > 0$ . Then,  $d_n > e_n$  or  $(d_n = e_n \text{ and } x_{n+1} = 0)$ .

To prove (12) we consider two cases:

Case  $n = 0$ .

We can write  $a = (d_0 - 1)x_0 + (x_0 - x_1)$  and  $b = (e_0 - 1)y_0 + (y_0 - y_1)$  with  $d_0 - 1 > e_0 - 1$  or  $d_0 - 1 = e_0 - 1$  and  $x_1 = 0$ . Then,  $L(a, b, c) = (bx_0, b(a - (e_0 - 1)x_0))$ ,  $l(a, b, c) = (ay_0, a(y_0 - y_1))$ . We conclude that  $f(a, b, c) = ay_0 + b(a - (e_0 - 1)x_0) - \min\{a(y_0 - y_1), bx_0\} = ay_1 + b(a - e_0 x_0) + \max\{a(y_0 - y_1), bx_0\}$  by (7).

Case  $n > 0$ .

Let  $r = \max\{i \mid k_i < n\}$ .

1. Suppose that  $d_n \geq e_n > 2$ . We can write  $x_{n-1} - x_n = (d_n - 2)x_n + (x_n - x_{n+1})$  and  $y_{n-1} - y_n = (e_n - 2)y_n + (y_n - y_{n+1})$  with  $d_n - 2 > e_n - 2$  or  $d_n - 2 = e_n - 2$  and  $x_{n+1} = 0$ . It follows that the level of  $(a, b, c)$  is even,  $L(a, b, c) = (bx_n, b(x_{n-1} - (e_n - 1)x_n))$  and  $l(a, b, c) = (ay_n, a(y_n - y_{n+1}))$ . Therefore,  $f(a, b, c) = ay_{n+1} + b(x_{n-1} - e_n x_n) + \max\{a(y_n - y_{n+1}), bx_n\}$  by (7).
2. Suppose that  $d_n > e_n = 2$ . We thus have  $k_{(r+1)} > n$ . we can write  $y_{k_r} = (k_{(r+1)} - k_r)(y_{k_{(r+1)}-1} - y_{k_{(r+1)}}) + y_{k_{(r+1)}}$  and  $x_{k_r} = (n - k_r)(x_{n-1} - x_n) + x_n$  by (57). Moreover, we have  $y_{k_{(r+1)}-1} - y_{k_{(r+1)}} = y_{n-1} - y_n = y_n - y_{n+1}$  by (53) and  $y_{k_r} - (n - k_r)(y_{n-1} - y_n) = y_n$  by (54). Therefore, the level of  $(a, b, c)$  is odd,  $L = (a(y_n - y_{n+1}), ay_n)$  and  $l = (b(x_{n-1} - x_n), bx_n)$ . We deduce that  $f(a, b, c) = ay_{n+1} + b(x_{n-1} - 2x_n) + \max\{a(y_n - y_{n+1}), bx_n\}$  by (7).

3. Suppose that  $d_n = e_n = 2$  and  $x_{n+1} = 0$  then  $k_{(r+1)} \geq n + 1$ . Using (57) we can write  $y_{k_r} = (k_{(r+1)} - k_r)(y_{k_{(r+1)}-1} - y_{k_{(r+1)}}) + y_{k_{(r+1)}} = (n + 1 - k_r)(y_n - y_{n+1}) + y_{n+1}$  and  $x_{k_r} = (n + 1 - k_r)x_n$ . Hence, the level of  $(a, b, c)$  is odd. Since  $y_{k_r} - (n + 1 - k_r)(y_n - y_{n+1}) = y_{n+1}$ , we get  $L = (a(y_n - y_{n+1}), ay_{n+1})$  and  $l = (bx_n, 0)$ . Therefore,  $f(a, b, c) = ay_{n+1} + bx_n = ay_{n+1} + \max\{a(y_n - y_{n+1}), bx_n\}$  because, by (48), we have  $bx_n - ay_n + ay_{n+1} = -cR_n + cR_{n+1}$  and it is easily seen that  $-cR_n + cR_{n+1} < 0$ .

Finally we have proved that

(59)

$$f(a, b, c) = ay_{n+1} + b[x_{n-1} - e_n x_n] + \max\{a(y_n - y_{n+1}), bx_n\} \square$$

To prove Theorem 6 we observe, using (48)-(49), that (11) and the following condition

$$(60) \quad \frac{y_{n+1}}{R_{n+1}} \leq \frac{c}{a} < \frac{y_i}{R_i} \text{ for all } 0 \leq i \leq n.$$

are equivalent. Taking account of (48)-(49) we obtain

$$(61) \quad f(a, b, c) = cR_{n+1} + ay_n - \min\{ay_{n+1}, cR_n\} \square$$

## 5. Examples

1.  $A = \{31, 44, 462, 674, 402, 932, 1214\}$ .

We take  $a = 31, b = 44$ . We obtain

$$(\overset{\wedge}{462}, \overset{\wedge}{674}, \overset{\wedge}{402}, \overset{\wedge}{932}, \overset{\wedge}{1214}) = (22, 26, 30, 40, 42)$$

$$\text{and } (\underset{\vee}{462}, \underset{\vee}{674}, \underset{\vee}{402}, \underset{\vee}{932}, \underset{\vee}{1214}) = (5, 3, 12, 7, 2).$$

We remove 674, 932, 1214 from  $A$  without altering  $g(A)$ . We consider  $(a, b, c_1, c_2) = (31, 44, 462, 402)$ . Applying Theorem 3 we obtain  $g(A) = 761$ .

2.  $A = \{57, 83, 367, 543, 605\}$ .

We take  $a = 57, b = 83$ . We have  $(\hat{367}, \hat{543}, \hat{605}) = (21, 27, 31)$ ,  
 $(\check{367}, \check{543}, \check{605}) = (10, 12, 14)$ ,

$\mathcal{B} = b(T) \cup \{b\} = \{21, 27, 31, 42, 48, 52, 58, 62, 63, 69, 73, 79, 83\}$ ,

$\mathcal{A} = a(T) \cup \{a\} = \{10, 12, 14, 20, 22, 24, 26, 28, 30, 32, 34, 36, 57\}$ .

We obtain  $g(A) = 1603$ .

3.  $A = \{a, b, c\} = \{137, 250, 337\}$ . We have  $(\hat{c}, \check{c}) = (101, 54), l = 4$ ,

$\mathcal{B} = \{53, 101, 154, 202, 250\}, \mathcal{A} = \{25, 54, 79, 108, 137\}$ .

Using Theorem 2 we obtain

$g(137, 250, 337) = \max\{g_1, g_2, g_3\} = 7537$ .

Let us compute  $g(A)$  by Theorem 5. We get  $n(a, b, c) = 1$ ,

$L(a, b, c) = (6576, 7261), l(a, b, c) = (7250, 6250)$ . We obtain

$g(a, b, c) = 7250 + 7261 - 6250 - 137 - 250 - 337 = 7537$ .

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