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Computing the Frobenius Number

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Abstract. The Frobenius number $g(A)$ of a finite subset $A \subset \mathbb{N}$ such that $\gcd(A) = 1$ is the largest integer which cannot be expressed as $\sum_{a \in A} ax_a$ with non-negative integers $x_a$. We present an algorithm for the computation of $g(A)$. Without loss of generality we suppose that there exist $a, b \in A$ such that $\gcd(a, b) = 1$. We give a formula for $g(A)$ in the particular case that for all $c, d \in A$, $c + d$ can be written in the form $c + d = xa + yb$ with $x, y \geq 0$ (e.g. $c + d > ab - a - b$). Using Euler polynomials we give a formula for $g(A)$ in the case that $A = \{a, b, c\}$.

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1. Introduction and Statement of Results
Throughout this paper, small letters denote integers. We will set $\mathbb{N} = \{1, 2, 3 \ldots \}$ and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. We consider a finite subset $A \subset \mathbb{N}$ such that $\gcd(A) = 1$. We define $S(A)$ the additive semigroup of $\mathbb{N}_0$ generated by $A$ i.e. $S(A) = \{\sum_{a \in A} x_a a \mid x_a \in \mathbb{N}_0\}$. The Frobenius number $g(A)$ is defined as the largest integer which does not belong to $S(A)$. We are interested in computing $g(A)$ and equivalently $f(A) = g(A) + \sum_{a \in A} a$ the largest integer which cannot be expressed as $\sum_{a \in A} ax_a$ with positive integers $x_a$. It is well known that $g(a, b) = ab - a - b$ if $\gcd(a, b) = 1$, and $g(A) = -1$ if
and only if $1 \in A$.
For $\text{card}(A) \geq 3$, no general formula for $g(A)$ is known, except in particular cases, see [6] and [8]. Algorithms are developed in [4] and [7] in the case $A = \{a, b, c\}$.
Without loss of generality, we can consider only sets $A$ containing two coprime integers $a, b$ (see section 2). We fix two integers $a > 1$ and $b > 1$ such that $\gcd(a, b) = 1$, and $c = (c_1, \ldots, c_n) \in \mathbb{N}^n$. Let $A = \{a, b, c_1, \ldots, c_n\}$.
For all $t = (t_1, \ldots, t_n) \in \mathbb{N}_0^n$, we set $t \cdot c = \sum_{i=1}^n t_i c_i$.
For every $m \in \mathbb{Z}$, there exist unique integers $\hat{m}$ and $\check{m}$ such that $m = \hat{m}a - \check{m}b$ with $0 \leq \hat{m} < b$. We write $t \cdot c = b(t) a - a(t) b$ where $b(t) = t \cdot c$ and $a(t) = t \cdot c$.
A finite subset $T$ of $\mathbb{N}_0^n$ is said to be appropriate if

$$S(A) = \bigcup_{t \in T} (S(a, b) + t \cdot c).$$

It is said to be pruned if $0 \in T$ and for $t \in T \setminus \{0\}$, we have $b(t) > 0$ and $a(t) > 0$.
Let $T$ be an appropriate and pruned subset of $\mathbb{N}_0^n$. We can write $T = \{0 = t_0, t_1, \ldots, t_l\}$ with $0 = b(t_0) < b(t_1) \leq b(t_2) \leq \cdots \leq b(t_l) < b$. We set $b_i = b(t_i), a_i = a(t_i)$ for $0 \leq i \leq l$, $b_{l+1} = b$ and $a_{l+1} = a$.
We obtain the sequences

$$0 = b_0 < b_1 \leq b_2 \leq \cdots \leq b_l < b_{l+1} = b$$

and

$$0 = a_0 < a_1, a_2, \ldots, a_l < a_{l+1} = a.$$
For $0 \leq i \leq l$, we set $m_i = \max\{a_j \mid 0 \leq j \leq i\}$ and $g_i = (b_{i+1} - 1)a - (m_i + 1)b$.
Our main result is the following
Theorem 1.

\[ g(A) = \max\{ g_i \mid 0 \leq i \leq l \}. \]

A subset \( M \subset \mathbb{N}_0 \) is said to be \textit{trimmed} if for all \( m \in M \setminus \{0\} \), \( \hat{m} > 0 \) and for all \( m, d \in M \),

\[ \hat{m} < \hat{d} \Leftrightarrow \hat{m} < \hat{d}. \]

We show that there exists an \textit{appropriate} set \( T \) such that \( T.c = \{ t.c \mid t \in T \} \) is \textit{trimmed}.

Theorem 2. Let \( T \) be an \textit{appropriate} set. If \( T.c \) is \textit{trimmed} then

\[ g(A) = \max\{ a(b_{i+1} - b_i) + t_i.c \mid 0 \leq i \leq l \} - (a + b) \]

where \( t_0 = 0 \).

We give an algorithm for computing \( g(A) \). We use Theorem 1 if we start with an arbitrary \textit{appropriate} set \( T \). And we use Theorem 2 if we start with an \textit{appropriate} set \( T \) such that \( T.c \) is \textit{trimmed}.

In a particular case we derive a formula for \( g(A) \) from Theorem 2:

Theorem 3. If \( \{c_1, \ldots, c_n\} \) is \textit{trimmed} and for all \( 1 \leq i, j \leq n \), we have \( c_i + c_j \in S(a, b) \) (e.g. \( c_i + c_j > ab - a - b \)) then

\[ g(A) = \max\{ a(b_{i+1} - b_i) + c_i \mid 0 \leq i \leq n \} - (a + b) \]

where \( b_{n+1} = b, b_0 = 0 \) and for all \( 1 \leq i \leq n \), \( b_i = \hat{c}_i \).

In the case \( n = 1 \), i.e. \( A = \{a, b, c\} \), we take \( c \not\in S(a, b) \) and we write \( c = au_0 - bv_0 \) with \( 0 < u_0 < b \) and \( 0 < v_0 < a \).

By successive Euclidean divisions we get:

\[ \begin{align*}
    a &= u_{-1} = p_0v_0 + u_1, \ldots, u_{n-1} = p_nv_n + u_{n+1}, \ldots \\
    b &= v_{-1} = q_0u_0 + v_1, \ldots, v_{n-1} = q_nu_n + v_{n+1}, \ldots
\end{align*} \]

(6)
The triplet \((a, b, c)\) is said to be of level \(n = n(a, b, c)\) if
\[
p_0 = q_0, p_1 = q_1, \ldots, p_{n-1} = q_{n-1}
\]
and \((p_n \neq q_n \text{ or } v_{n+1} = 0)\).

Let \((a, b, c)\) a triplet of level \(n\). For all \(0 \leq i \leq n\), we set \(w_{i+1} = u_i - q_i v_i\) so for \(i \leq n\), \(w_i = u_i\) and \(w_{n+1} = (p_n - q_n)v_n + u_{n+1}\).

We denote by
\[
L(a, b, c) = (L_0, L_1) = \begin{cases} (bv_n, bw_{n+1}) & \text{if } n \text{ is even} \\ (av_n, aw_{n+1}) & \text{if } n \text{ is odd} \end{cases},
\]
\[
l(a, b, c) = (l_0, l_1) = \begin{cases} (aw_n, av_{n+1}) & \text{if } n \text{ is even} \\ (bw_n, bv_{n+1}) & \text{if } n \text{ is odd} \end{cases}.
\]

For an integer \(n\) we denote by \(k(n) = k = \lceil \frac{n}{2} \rceil\) and \(h(n) = h = \left\lfloor \frac{n+1}{2} \right\rfloor\) where \(\lfloor x \rfloor\) is the greatest integer such that \(\lfloor x \rfloor \leq x\).

**Theorem 4.**
\[
f(a, b, c) = aw_{2h} + bw_{2k+1} - \min\{bv_{2h}, av_{2k+1}\} = l_0 + L_1 - \min\{l_1, L_0\}.
\]

In particular, when \(p_n = q_n\) and \(v_{n+1} = 0\),
\[
f(a, b, c) = \begin{cases} a \gcd(b, c) + \operatorname{lcm}(b, c) & \text{if } n \text{ is even} \\ b \gcd(a, c) + \operatorname{lcm}(a, c) & \text{if } n \text{ is odd} \end{cases}.
\]

**Theorem 5.** Let \((x_{-1}, x_0, \ldots, x_{l+1}), (y_{-1}, y_0, \ldots, y_{m+1}), (d_0, d_1, \ldots, d_l)\) and \((e_0, e_1, \ldots, e_m)\) be sequences such that the following conditions
\[
\begin{align*}
\{x_{-1} &= d_0x_0 - x_1, \ldots, x_{l-1} = d_lx_l - x_{l+1}, \\
y_{-1} &= e_0y_0 - y_1, \ldots, y_{m-1} = e_my_m - y_{m+1},
\end{align*}
\]

and \(a = x_{-1} > x_0 > x_1 > \cdots > x_l > x_{l+1} = 0,\)
\(b = y_{-1} > y_0 > y_1 > \cdots > y_m > y_{m+1} = 0\).
hold. Let $n$ be an integer such that

$$
\begin{align*}
&d_0 = e_0, d_1 = e_1, \ldots, d_{n-1} = e_{n-1} \\
&\text{and } (d_n \neq e_n \text{ or } x_{n+1} = 0).
\end{align*}
$$

Then,

$$
\begin{align*}
f(a, b, c) &= ay_{n+1} + b(x_{n-1} - e_n x_n) + \max\{a(y_n - y_{n+1}), bx_n\}.
\end{align*}
$$

As a consequence of Theorem 5 we prove the following theorem see [7].

**Theorem 6(Rodseth).** Let $R_i$ polynomials defined by induction as follows: $R_{-1} = 0, R_0 = 1$ and for all $i > 0$,

$$
R_{i+1} = e_i R_i - R_{i-1}.
$$

Let $n$ be the unique integer such that

$$
\frac{y_{n+1}}{R_{n+1}} \leq \frac{c}{a} < \frac{y_i}{R_i} \text{ for all } 0 \leq i \leq n.
$$

Then,

$$
f(a, b, c) = cR_{n+1} + ay_n - \min\{ay_{n+1}, cR_n\}.
$$

### 2. Reduction to the case $\gcd(a, b) = 1$

Let $A = \{b, c_1, \ldots, c_n\}$ be a subset of $\mathbb{N}$ such that $\gcd(A) = 1$. We recall the following classical result: It is always possible to choose integers $x, x_1, \ldots, x_n$ such that $1 = xb + x_1 c_1 + \cdots + x_n c_n$. Then, for $k$ an integer sufficiently large, we get

$$
a = 1 + [k(c_1 + \cdots + c_n) - x]b = (x_1 + kb)c_1 + \cdots + (x_n + kb)c_n \in S(c_1, \ldots, c_n).$$
Hence, \( g(A) = g(A \cup \{a\}) \) and \( \gcd(a, b) = 1 \).

In the particular case that 

\[ A = \{a, b, c_1, \ldots, c_n\} \] 

such that \( \gcd(a, b) = \gcd(a, b, c_1, \ldots, c_{n-1}) = d \geq 1 \) we observe that \( \gcd\left(\frac{a}{d}, \frac{b}{d}\right) = 1 \). Therefore, to compute \( g(A) \) we can use Brauer’s formula

\[
(16) \quad f(A) = df\left(\frac{a}{d}, \frac{b}{d}, \frac{c_1}{d}, \ldots, \frac{c_{n-1}}{d}, c_n\right),
\]

see [3].

From now on we suppose that \( \gcd(a, b) = 1 \).

3. Case \( n \geq 1 \)

We need some remarks.

**R.** If \( m \geq 0 \) then \( a > \hat{m} \).

**Proof.** Since \( m = \hat{m}a - \tilde{m}b \geq 0 \) we have \( ba > \hat{m}a \geq \tilde{m}b \) hence

\( a > \tilde{m} \) □

**R2.** Let \( m = xa - yb \). Then,

\[ 0 \leq x < b \iff \hat{m} = x \iff \tilde{m} = y \]

and

\[ -b \leq x < 0 \iff \hat{m} = x + b \iff \tilde{m} = y + a. \]

**Proof.** To prove the second claim we write \( m = xa - yb = (x + b)a - (y + a)b \) and we use the uniqueness of \( \hat{m} \) and \( \tilde{m} \) □

**R.** Let \( m = xa - yb \). Then, there exists a unique integer \( p \in \mathbb{Z} \) such that \( x = pb + \hat{m} \) and \( y = pa + \tilde{m} \).

**Proof.** We write \( m = xa - yb = \hat{m}a - \tilde{m}b \) thus \( (x - \hat{m})a = (y - \tilde{m})b \).

Since \( \gcd(a, b) = 1 \), \( p = \frac{x-\hat{m}}{b} = \frac{y-\tilde{m}}{a} \) is an integer □

**R4.** We have \( m \in S(a, b) \) if and only if \( \tilde{m} \leq 0 \).

**Proof.** Clearly \( \tilde{m} \leq 0 \Rightarrow m = \hat{m}a - \tilde{m}b \in S(a, b) \). Conversely, if \( m \in S(a, b) \) then \( m = xa + yb \) with \( x \geq 0 \) and \( y \geq 0 \). By R3 there exists \( p \in \mathbb{Z} \) such that \( x = pb + \hat{m} \geq 0 \) thus \( p \geq 0 \). We also have
\(-y = pa + \hat{m} \leq 0\) and so \(\hat{m} = -y - pa \leq 0\)

**R5.** For \(d \geq 0\) we set \(G(d) = S(a, b) \cup (S(a, b) + d)\).

Then, we have

\[
m \notin G(d) \iff \hat{m} > 0 \text{ and } (\hat{m} < \hat{d} \text{ or } \hat{m} > \hat{d})
\]

**Proof.** Let \(n = m - d\). Since \(d \geq 0\), **R1** shows that \(a > \hat{d}\). Hence, \(\hat{m} - \hat{d} + a > m\). Moreover, we have \(n = (\hat{m} - \hat{d})a - (\hat{m} - \hat{d})b\) and \(-b < \hat{m} - \hat{d} < b\). It follows from **R2** that \(\check{n} = \hat{m} - \hat{d}\) if \(\hat{m} - \hat{d} \geq 0\) and \(\check{n} = \hat{m} - \hat{d} + a > m\) if \(\hat{m} - \hat{d} < 0\). We deduce that

\[
\begin{align*}
\hat{m} > 0 \text{ and } (\check{m} < \check{d} \text{ or } \check{m} > \check{d}) \\
\iff \\
\hat{m} > 0 \text{ and } (\hat{m} - \hat{d} \geq 0 \Rightarrow \hat{m} - \hat{d} > 0) \\
\iff \\
\hat{m} > 0 \text{ and } \check{m} > 0 \\
\iff \\
m \notin S(a, b) \text{ and } n = m - d \notin S(a, b) \\
\iff \\
m \notin G(d)
\end{align*}
\]

As a consequence of **R5** we obtain

**R6.** If \(T\) is appropriate then

\[
m \notin S(A) \iff \hat{m} > 0 \text{ and } \forall t \in T, \check{m} < b(t) \text{ or } \check{m} > a(t).
\]

**Proof of Theorem 1.** For \(0 \leq i \leq l\), we set

\[
F(i) = \{m \in \mathbb{Z} \mid \check{m} < b_{i+1} \text{ and } \check{m} > \check{m}_i\}.
\]

**Step 1.**

\[
\mathbb{Z} \setminus S(A) = \bigcup_{i=0}^{l} F(i).
\]
Let $m \in F(i)$. For every $0 \leq j \leq l$, $\hat{m} < b_{j+1} \leq b_j$ if $j \geq i + 1$ and $\hat{m} > m_i \geq a_j$ if $j \leq i$. Hence, $\textbf{R6}$ shows that $m \not\in S(A)$.

Conversely, let $m \not\in S(A)$ then $\hat{m} > 0$ by $\textbf{R4}$. Since $b_0 = 0 \leq \hat{m} < b = b_{l+1}$ and $0 = b_0 < b_1 \leq \cdots \leq b_l \leq b_{l+1} = b$, there exists $0 \leq j \leq l$ such that $\hat{m} < b_{j+1}$. We put $i = \min\{j \mid \hat{m} < b_{j+1}\}$. We thus get for $0 \leq j \leq i$, $b_j \leq b_i \leq \hat{m} < b_{i+1}$ and $\hat{m} > a_j$ by $\textbf{R6}$.

Hence, $\hat{m} > m_i$. We conclude that $m \in F(i)$.

**Step 2.** For $0 \leq i \leq l$, $g_i = \max F(i)$.

Since $0 \leq b_{i+1} - 1 < b$, we have $\hat{g}_i = b_{i+1} - 1$ and $\hat{g}_i = m_i + 1 > m_i$.

We thus get $g_i \in F(i)$. Moreover, for all $m \in F(i)$, $m = \hat{m}a - \hat{m}b \leq (b_{i+1} - 1)a - (m_i + 1)b$ then $g_i = \max F(i)$ $\Box$

In particular, when $T = \{0\}$ i.e. when all $c_i \in S(a, b)$ we have $b_0 = 0 < b_1 = b, a_0 = 0$ and $m_0 = 0$. Therefore, $g(A) = g_0 = (b_1 - 1)a - (m_0 + 1)b = ab - a - b = g(a, b)$.

**Proof of Theorem 2.** If $T.c$ is trimmed then $0 = b_0 < b_1 < b_1 < \cdots < b_l < b_{l+1} = b$ and thus $0 = a_0 < a_1 < a_1 < \cdots < a_l < a_{l+1} = a$. In particular, $T$ is pruned and for all $0 \leq i \leq l$, $m_i = a_i$. We can write $g_i = a(b_{i+1} - b_i) + ab_i - ba_i - (a+b) = a(b_{i+1} - b_i) + t_i.c - (a+b)$ $\Box$

**Algorithm.**

1. For every $i$, we choose $\lambda_i > 0$ such that $\lambda_i c_i \in S(a, b)$. Numbers $\lambda_i$ exist. Indeed, it is sufficient to take $\lambda_i > \frac{g(a, b)}{c_i} = \frac{ab - a - b}{c_i}$. The following set

$$U = \{ t = (t_1, \ldots, t_n) \in \mathbb{N}_0^n \mid t_i < \lambda_i, 1 \leq i \leq n \}$$

is thus appropriate.

We remove from $U$ all elements $t \neq 0$ such that $b(t) = 0$ or $a(t) \leq 0$. The set $T$ of all remaining elements is pruned and still appropriate.

We number the elements of $T = \{0 = t_0, t_1, \ldots, t_l\}$ in such a
way that \(0 = b(t_0) < b(t_1) \leq b(t_2) \leq \cdots \leq b(t_l)\). We compute 
\[m_i = \max\{a_j \mid 0 \leq j \leq i\}, \quad g_i = (b_{i+1} - 1)a - (m_i + 1)b\] 
and 
\[g(A) = \max\{g_i \mid 0 \leq i \leq l\}\] 
where \(b_i = b(t_i), a_i = a(t_i)\) for 
\(0 \leq i \leq l, b_{l+1} = b\) and \(a_{l+1} = a\).

2. The algorithm can be modified as follows: For every \(1 \leq i \leq n\), we start removing from \(A\) all elements \(c_j\) such that \(\hat{c}_i \leq \hat{c}_j\) and \(\check{c}_j \leq \check{c}_i\). We choose an appropriate and pruned set \(T = \{0 = t_0, t_1, \ldots, t_l\}\). We can suppose that \(0 = b(t_0) < b(t_1) \leq b(t_2) \leq \cdots \leq b(t_l) < b\). For all \(0 \leq i \leq l\), we remove from \(T\) all \(t_j\) such that \((b(t_i) \leq b(t_j)\) and \(a(t_j) \leq a(t_i)\)). Considering the set of the remaining elements we can suppose that \(T.c\) is trimmed. The Frobenius number can therefore be computed using Theorem 2.

**Proof.**

1. For all \(t \in \mathbb{N}_0^n\), we have 
\[G(t.c) = S(a,b) \cup (S(a,b) + t.c) \subset S(A)\] 
thus \(\bigcup_{t \in U} G(t.c) \subset S(A)\). Conversely, let \(m \in S(A)\), then 
\[m = xa + yb + \sum_{i=1}^n x_i c_i\] 
with \(x \geq 0, y \geq 0\) and \(x_i \geq 0\). By Euclidean division we write 
\[x_i = q_i \lambda_i + t_i\] 
We thus get 
\[t = (t_1, \ldots, t_n) \in U\] and 
\[m = xa + yb + \sum_{i=1}^n q_i \lambda_i c_i + t.c\] 
Since \(\lambda_i c_i \in S(a,b)\), we have \(m \in G(t.c)\). The equality
\[S(A) = \bigcup_{t \in U} G(t.c)\] (18)
follows. Therefore, \(U\) is appropriate.

By construction \(T\) is pruned and it is still appropriate. Indeed, for 
\(t \neq 0, t.c \not\in S(a,b)\) if and only if \(a(t) > 0\) by \(\text{R4}\).

Therefore,
\[T = \{t \in U \mid t = 0 \text{ or } t.c \not\in S(a,b)\}\] (19)
and

\[(20)\quad S(A) = \bigcup_{t \in T} G(t, c).\]

2. If \(\hat{c}_i \leq \hat{c}_j\) and \(\check{c}_j \leq \check{c}_i\) then \(c_j = c_i + (\hat{c}_j - \hat{c}_i)a + (\check{c}_i - \check{c}_j)b \in S(c_i, a, b)\). Therefore, \(g(A) = g(A \setminus \{c_j\})\). If \(b(t_i) \leq b(t_j)\) and \(a(t_j) \leq a(t_i)\) then \(t_j, c = t_i, c + (b(t_j) - b(t_i))a + (a(t_i) - a(t_j))b \in G(t_i, c)\) thus \(G(t_j, c) \subset G(t_i, c)\). We see that (20) is not altered by removing \(t_j\) from \(T\). \(\square\)

**Proof of Theorem 3.** Since \(2c_i \in S(a, b)\) for all \(1 \leq i \leq n\), \(U = \{t = (t_1, \ldots , t_n) \in \mathbb{N}_0^n \mid 0 \leq t_i \leq 1, 1 \leq i \leq n\}\) is appropriate. Furthermore, for \(t = (t_1, \ldots , t_n) \in U\), if \(t, c \notin S(a, b)\) then there exists \(c_i\) and \(y \in S(a, b)\) such that \(t, c = c_i + y \in G(c_i)\). Therefore, the set \(T = \{0 = t_0, t_1, \ldots , t_n\}\) where \(t_i = (0, \ldots , t_i, \ldots , 0)\) and \(t_i = 1\) is also appropriate. Moreover, \(T, c = \{0, c_1, \ldots , c_n\}\) is trimmed and \(b(t_i) = \hat{c}_i\) for all \(1 \leq i \leq n\). \(\square\)

4. Case \(n = 1\)

Let \(E\) be a totally ordered set and \(x \in E\). We call successor of \(x\) in \(E\), and we denote \(x^+\), the smallest element of \(E\) (if there exists any) such that \(x < x^+\).

For an appropriate set \(T\), we put \(b(T) = \{b(t) \mid t \in T\}\) and \(B = b(T) \cup \{b\}\). We equip \(B\) with the natural order \(\leq\).

To apply Theorem 2, in the case that \(n = 1\), it is convenient to formulate it as follows:

**Theorem 2’.** Let \(T\) be an appropriate set such that \(T, c\) is trimmed. Then,

\[(21)\quad g(A) = \max\{a(b(t)^+ - b(t)) + t, c \mid t \in T\} - (a + b)\]
where \( b(t)^+ \) is the successor of \( b(t) \) in \( B \).

Suppose that there exists integers \( \alpha_1, \ldots, \alpha_q \) and a partition \( E_1, \ldots, E_q \) of \( T \) such that \( b(t)^+ - b(t) = \alpha_i \) for all \( t \in E_i \) and for all \( 1 \leq i \leq q \).

Then,

\[
(22) \quad g(A) = \max\{a\alpha_i + \beta_i \mid 1 \leq i \leq q\} - (a + b)
\]

where we put \( \beta_i = \max E_i \cdot c \).

To compute such a partition, we are led to introduce what we call Euler order on \( T \). We will use Euler polynomials.

**Euler polynomials**

Let \((q_0, q_1, \ldots, q_i, \ldots)\) be a sequence of positive integers. We define Euler polynomials \( Q_i \) by induction as follows: \( Q_{-1} = 0, Q_0 = 1 \) and for \( i \geq 0, \)

\[
(23) \quad Q_{i+1}(q_0, \ldots, q_i) = q_i Q_i(q_0, \ldots, q_{i-1}) + Q_{i-1}(q_0, \ldots, q_{i-2}).
\]

We set \( Q_{i+1} = Q_{i+1}(q_0, \ldots, q_i), Q_i^1 = Q_i(q_1, \ldots, q_i), P_{i+1} = Q_i + Q_{i+1} \) and \( P_i^1 = Q_i^1 + Q_{i+1}^1 \).

We deduce immediately that

\[
(24) \quad Q_{n+1} = q_n Q_n + q_{n-2} Q_{n-2} + \ldots + q_{n-2i} Q_{n-2i} + Q_{n-2i-1}
\]

for \( 0 \leq 2i \leq n \).

**Euler order** \( \leq_e \)

**Proposition 1.** Every integer \( t \in \mathbb{N}_0 \) can be written uniquely in the form

\[
(25) \quad t = t_0 Q_0 + t_1 Q_1 + \cdots + t_n Q_n
\]

where

\[
(26) \quad \min\{i \mid t_i > 0\} \text{ is even ,}
\]
\[(27) \quad 0 \leq t_i \leq q_i \text{ for } 0 \leq i \leq n\]

and

\[(28) \quad t_i = q_i \Rightarrow t_{i-1} = 0 \text{ for } 1 \leq i \leq n.\]

Equality (25) is called Euler expansion of \(t\).

**Proof.** There exists \(n \in \mathbb{N}_0\) such that \(t < Q_{n+1}\). By successive Euclidean divisions we can write
\[
t = t_n Q_n + s_n \text{ with } 0 \leq s_n < Q_n,
\]
\[
s_n = t_{n-1} Q_{n-1} + s_{n-1} \text{ with } 0 \leq s_{n-1} < Q_{n-1},
\]
\[\vdots\]
\[
s_2 = t_1 Q_1 + s_1 \text{ with } 0 \leq s_1 < Q_1,
\]
\[
s_1 = t_0 Q_0 \text{ with } t_0 = s_1.
\]

We put \(i = \min\{j \mid t_j > 0\}\). If \(i = 2e\) then \(t = t_2e Q_{2e} + t_{2e+1} Q_{2e+1} + \cdots + t_n Q_n\) is Euler expansion of \(t\). If \(i = 2e + 1\), using (24) we take \(t = q_0 Q_0 + \cdots + q_{2e} Q_{2e} + (t_{2e+1} - 1) Q_{2e+1} + \cdots + t_n Q_n\) as Euler expansion of \(t\). Conditions (27) and (28) follow from \(s_{i+1} < Q_{i+1} = Q_{i-1} + q_i Q_i\). The uniqueness follows from the uniqueness of the Euclidean division and the fact that \(\sum_{i=0}^{2e} t_i Q_i < Q_{2e+1}\) if and only if there exists \(0 \leq j \leq e\) such that \(t_{2j} < q_{2j}\) \(\Box\)

For \(t, x \in \mathbb{N}_0\), let \(n \in \mathbb{N}_0\) such that \(t, x \leq Q_{n+1}\). We consider Euler expansions of \(t\) and \(x\) respectively
\[
t = t_0 Q_0 + t_1 Q_1 + \cdots + t_n Q_n \quad \text{and} \quad x = x_0 Q_0 + x_1 Q_1 + \cdots + x_n Q_n.
\]

We define Euler order \(\leq_e\) as follows: \(t \leq_e x\) if
\[(t_0, -t_1, \ldots, (-1)^i t_i, \ldots, (-1)^n t_n) \leq_l (x_0, -x_1, \ldots, (-1)^i x_i, \ldots, (-1)^n x_n)\]
where \(\leq_l\) is the lexicographic order on \(\mathbb{Z}^n\).

**Lemma.** Let \(n \in \mathbb{N}_0\). We consider \(U_n = \{0, 1, \ldots, Q_{n+1} - 1\}\) and \(V_n = \{0, 1, \ldots, P_n - 1\}\) equipped with induced Euler order \(\leq_e\).
1. We define a partition of $U_n = F_1 \cup F_2$ as follows:

\[
F_1 = \begin{cases} 
\{0, 1, \ldots, Q_{2k+1} - Q_{2k} - 1\} & \text{if } n = 2k \\
\{Q_{2k+1}, \ldots, Q_{2k+2} - 1\} & \text{if } n = 2k + 1 
\end{cases}
\]

\[
F_2 = \begin{cases} 
\{Q_{2k+1} - Q_{2k}, \ldots, Q_{2k+1} - 1\} & \text{if } n = 2k \\
\{0, 1, \ldots, Q_{2k+1} - 1\} & \text{if } n = 2k + 1 
\end{cases}
\]

Then, the successor, in $U_n$, of all $t \in F_1$ (resp. $t \in F_2$) is $t^+ = t + (-1)^n Q_n$ (resp. $t^+ = t + (-1)^n[Q_n - Q_{n+1}]$). In particular, if $q_n = 1$ then for all $t \in F_2$, $t^+ = t + (-1)^{n-1} Q_{n-1}$.

2. We define a partition of $V_n = E_1 \cup E_2$ as follows: $E_1 = \{0, \ldots, Q_{2h-1} - 1\}$, $E_2 = \{Q_{2h-1}, \ldots, P_n - 1\}$. Then, the successor, in $V_n$, of all $t \in E_1$ (resp. $t \in E_2$) is $t^+ = t + Q_{2k}$ (resp. $t^+ = t - Q_{2h-1}$).

**Proof.** Let $t = t_0 Q_0 + t_1 Q_1 + \cdots + t_n Q_n$ be Euler expansion of $t$.

1. (a) Suppose that $t \in F_1$. It is easily seen that $t_n > 0$ if $n = 2k + 1$ and $t_n < q_{n-1}$ or ($t_n = q_{n-1}$ and $t_{n-1} = 0$) if $n = 2k$. Therefore, $t^+ = t + (-1)^n Q_n$.

(b) Suppose that $t \in F_2$, then $t = t_0 Q_0 + t_1 Q_1 + \cdots + t_{2k} Q_{2k}$.

Since $t < Q_{2k+1}$, there exists $j \leq k$ such that $t_{2j} < q_{2j}$. Taking $i = \max\{j \mid t_{2j} < q_{2j}\}$ we can write $t = t_0 Q_0 + t_1 Q_1 + \cdots + t_{2i} Q_{2i} + [q_{2i+2} Q_{2i+2} + \cdots + q_{2k} Q_{2k}] = t_0 Q_0 + t_1 Q_1 + \cdots + t_{2i} Q_{2i} - Q_{2i+1} + Q_{2k+1}$.

If $t_{2i} < q_{2i} - 1$ or ($t_{2i} = q_{2i} - 1$ and $t_{2i-1} = 0$) then $t^+ = t_0 Q_0 + t_1 Q_1 + \cdots + (t_{2i} + 1) Q_{2i} + (q_{2i+1} - 1) Q_{2i+1} + \cdots + q_{2n-1} Q_{2n-1} = t + Q_{2h} - Q_{2k+1}$.

If $t_{2i} = q_{2i} - 1$ and $t_{2i-1} > 0$ then $t^+ = t_0 Q_0 + t_1 Q_1 + \cdots + (t_{2i-1} - 1) Q_{2i-1} + q_{2i+1} Q_{2i+1} + \cdots + q_{2h-1} Q_{2h-1} = t + Q_{2h} - Q_{2k+1}$.
2. It is a particular case: Taking $q_n = 1$ we get $P_n = Q_{n-1} + Q_n = Q_{n+1}$ and $V_n = U_n$. Moreover, in this case we have $E_1 = F_2$ and $E_2 = F_1$ if $n = 2k$ and $E_1 = F_1$ and $E_2 = F_2$ if $n = 2k + 1$ □

Let $(r = r_{-1}, r_{0}, \ldots, r_{n})$ and $(q_{0}, q_{1}, \ldots, q_{n})$ be sequences of positive integers and $r_{n+1} \geq 0$. We suppose that

\begin{equation}
 r_{i-1} = q_i r_i + r_{i+1} \quad (29)
\end{equation}

for $0 \leq i \leq n$.

We thus have $r > r_0 > r_1 > \cdots > r_n > 0$ and $r_{n-1} > r_{n+1} \geq 0$.

We prove by induction the following identities:

\begin{equation}
 r = r_i Q_{i+1} + r_{i+1}Q_i \quad (30)
\end{equation}

and

\begin{equation}
 r_0 Q_i = (-1)^i r_i + r P_{i-1} \quad (31)
\end{equation}

It follows from (31) that

\begin{equation}
 r_0 P_{i+1} = (-1)^i (r_i - r_{i+1}) + r P_{i+1} \quad (32)
\end{equation}

Let $t \in \mathbb{N}_0$. Given $t = t_0 Q_0 + t_1 Q_1 + \cdots + t_{n} Q_{n}$ its Euler expansion, we associate with $t$ the following numbers: $r(t) = \sum_{i=0}^{n} (-1)^i t_i r_i$ and $E(t) = \sum_{i=0}^{n} t_i Q_{i-1}$. It follows from (31) that

\begin{equation}
 tr_0 = r(t) + r E(t) \quad (33)
\end{equation}

Moreover, if $0 < t < Q_{n+1}$ then

\begin{equation}
 r_n \leq r(t) \leq r - r_n \quad (34)
\end{equation}

and if $0 < t < P_{n}$ then

\begin{equation}
 r_{2k} \leq r(t) \leq r - r_{2h-1} \quad (35)
\end{equation}
Indeed, for \(0 < t < Q_{n+1}\), let \(t = t_0Q_0 + \cdots + t_nQ_n\) be Euler expansion of \(t\). We can write \(t = \sum_{i=0}^{k} t_iQ_{2i} + \sum_{i=e}^{h-1} t_{2i+1}Q_{2i+1}\) with \(t_{2e} > 0\). Hence, \(r(t) = \sum_{i=e}^{k} t_{2i}r_{2i} - \sum_{i=e}^{h-1} t_{2i+1}r_{2i+1}\).

Using \(r_{2e} = \sum_{i=e}^{h-1} q_{2i+1}r_{2i+1} + r_{2h}\) we get

\[
r(t) = (t_{2e} - 1)r_{2e} + \sum_{i=e+1}^{k} t_{2i}r_{2i} + \sum_{i=e}^{h-1} (q_{2i+1} - t_{2i+1})r_{2i+1} + r_{2h}.
\]

Now if \(n = 2k + 1\) we get

\(r(t) \leq \sum_{i=0}^{k} q_{2i}r_{2i} = r - r_{2k+1} = r - r_n.
\)

If \(n = 2k\) there exists \(j \leq k\) such that \(t_{2j} < q_{2j}\) since otherwise \(t = Q_{n+1}\). We then get

\(r(t) \leq r(t) \leq r - r_{2j} - r_{2k+1} \leq r - r_{2k} - r_{2k+1}.
\)

Hence, assumption (34) follows. In the particular case that \(q_n = 1\) we have \(P_n = Q_{n+1}\) and \(r_{n-1} = r_n + r_{n+1}\). In this case if \(n = 2k\) we get \(r_{n-1} \leq r(t) \leq r - r_n\) by (36) and if \(n = 2k + 1\) we get \(r_n \leq r(t) \leq r - r_{n-1}\) by (37) thus assumption (35) follows.

**Proposition 2.** We suppose that \(r_{n+1} = 0\). We equip \(\{0, 1, \ldots, r - 1\}\) with the natural order \(\leq\) and \(U_n = \{0, 1, \ldots, Q_{n+1} - 1\}\) with Euler order \(\leq_e\). Then, the mapping \((U_n, \leq_e) \rightarrow (\{0, 1, \ldots, r - 1\}, \leq_e), t \mapsto r(t)\) is strictly increasing.

**Proof.** Given \(t \leq_e x\) in \(U_n\) we have

\((t_0, -t_1, \ldots, (-1)^jt_j, \ldots, (-1)^n t_n) \leq_t (x_0, -x_1, \ldots, (-1)^j x_j, \ldots, (-1)^n x_n)\)

and \(r(x) - r(t) = (-1)^j(x_j - t_j)r_j + \sum_{i=j+1}^{n} (-1)^i x_i r_i - \sum_{i=j+1}^{n} (-1)^i t_i r_i\)

where \(j\) is the smallest integer such that \(t_j \neq x_j\).
1. When $j = 2i$ we get $x_{2i} > t_{2i} \geq 0$ and $x_{2i+1} \leq q_{2i+1} - 1$. Then,
\[ r(x) - r(t) = (x_{2i} - t_{2i})r_{2i} + [-x_{2i+1}r_{2i+1} + \cdots + (-1)^{n}x_{n}r_{n}] - \] 
\[ [-t_{2i+1}r_{2i+1} + \cdots + (-1)^{n}t_{n}r_{n}] \geq (x_{2i} - t_{2i})r_{2i} - [(q_{2i+1} - 1)r_{2i+1} + \cdots + q_{2h-1}r_{2h-1}] - [q_{2i+2}r_{2i+2} + \cdots + q_{2k}r_{2k}] \geq (x_{2i} - t_{2i})r_{2i} - r_{2i} + r_{2h} + r_{2k+1} > 0.\]

2. When $j = 2i - 1$ we get $t_{2i-1} > x_{2i-1} \geq 0$ and $t_{2i} \leq q_{2i} - 1$. Then,
\[ r(x) - r(t) = (t_{2i-1} - x_{2i-1})r_{2i-1} + [x_{2i}r_{2i} + \cdots + (-1)^{n}x_{n}r_{n}] - [t_{2i}r_{2i} + \cdots + (-1)^{n}t_{n}r_{n}] \geq (t_{2i-1} - x_{2i-1})r_{2i-1} - [q_{2i+1}r_{2i+1} + \cdots + q_{2h-1}r_{2h-1}] - [(q_{2i} - 1)r_{2i} + \cdots + q_{2k}r_{2k}] \geq (t_{2i-1} - x_{2i-1})r_{2i-1} + r_{2h} - r_{2i-1} + r_{2k+1} > 0 \]

Now we consider another sequence of positive integers $(s = s_{-1}, s_{0}, s_{1}, \ldots, s_{n})$ and $s_{n+1} \geq 0$ such that $s_{i-1} = q_{i}s_{i} + s_{i+1}$ for $0 \leq i \leq n$.
We also define $s(t) = \sum_{i=0}^{n}(-1)^{i}t_{i}s_{i}$.
Using (31) we prove the following identity
\[
(38) \quad (rs_{0} - sr_{0})Q_{i} = (-1)^{i}(rs_{i} - sr_{i}).
\]
We derive
\[
(39) \quad (rs_{0} - sr_{0})P_{i+1} = (-1)^{i}[r(s_{i} - s_{i+1}) + s(r_{i+1} - r_{i})]
\]
from (32) and
\[
(40) \quad t(rs_{0} - sr_{0}) = rs(t) - sr(t)
\]
from (33).

**Proof of Theorem 4.**

**First step**: Reduction to the case $n = 2k$.

Suppose that $n = n(a, b, c) = 2k + 1$.

Since $g(a, b, c) = g(b, a, c)$, it suffices to show that $n(b, a, c)$ is even,
$L(a, b, c) = L(b, a, c)$ and $l(a, b, c) = l(b, a, c)$.
We write \( c = b(a - v_0) - a(b - u_0) \). We consider two cases:

If \( p_0 = q_0 > 1 \) then we can write \( a = (a - v_0) + v_0, a - v_0 = (q_0 - 1)v_0 + v_1, b = (b - u_0) + u_0, b - u_0 = (q_0 - 1)u_0 + u_1 \). Therefore, \( n(b, a, c) = n(a, b, c) + 1 \), \( L(b, a, c) = L(a, b, c) \) and \( l(b, a, c) = l(a, b, c) \).

If \( p_0 = q_0 = 1 \) we get \( a - v_0 = u_1 \) and \( b - u_0 = v_1 \). We therefore get \( a(b, a, c) = n(a, b, c) - 1 \). Furthermore, we observe that if \( n(a, b, c) > 1 \) then we have obviously \( L(a, b, c) = L(b, a, c) \) and \( l(b, a, c) = l(a, b, c) \). If \( n(a, b, c) = 1 \) we have \( L(b, a, c) = (a_1, a(b - (q_0 + 1)v_1)) = (a_1, a(u_0 - q_1v_1)) = L(a, b, c) \) and \( l(b, a, c) = (b_1, b(v_2)) = l(a, b, c) \). In both cases, \( n(b, a, c) \) is even. Therefore, the assumption follows.

Moreover, we can write

\[
(41) \quad c = w_n w_{n+1} - v_n v_{n+1}.
\]

Since \( c > 0 \), we get \( w_{n+1} = (p_n - q_n)v_n + u_{n+1} > 0 \) thus \( p_n > q_n \) or \( (p_n = q_n \) and \( v_{n+1} = 0 \)). When \( p_n = q_n \) and \( v_{n+1} = 0 \) we then have

\[
(42) \quad c = w_n w_{n+1} = u_n u_{n+1}.
\]

In the following steps we suppose that \( n = 2k \).

**Second step:** Case \( p_n > q_n \).

Taking \( a = r, v_0 = r_0, b = s \) and \( u_0 = w_0 = s_0 \) it follows from (39) that \( cP_{n+1} = a(w_n - v_{n+1}) + b(w_{n+1} - v_n) \in S(a, b) \) because \( w_{n+1} > v_n \) and \( w_n > v_{n+1} \). Moreover, it follows from (40) and (35) that \( s(t) = b(t) \) and \( r(t) = a(t) \). Hence, for all \( 0 < t < P_{n+1} \),

\[
(43) \quad P_{n+1} = \min\{t > 0 \mid tc \notin S(a, b)\}
\]

and by (19)-(20), that \( V_n = \{0, 1, \ldots, P_{n+1} - 1\} \) is appropriate. Let us show that \( V_n.c \) is trimmed. Indeed, for \( 0 < t < x < P_{n+1} \), since
x−t < P_{n+1}, we have 0 < (x−t)c = (b(x)−b(t))(a−(a(x)−a(t))b \not\in S(a,b). Then, b(x) < b(t) if and only if a(x) < a(t) by R4.

Furthermore, it follows from proposition 2 that \( b(t^+) − b(t) = b(t^+ − t). \) Writing \( V_n = E_1 \cup E_2 \) with the notation of the lemma, Theorem 2 and (22) show that \( g(a,b,c) = \max\{a\alpha_1 + \beta_1, a\alpha_2 + \beta_2\} − (a + b) \) where \( \alpha_1 = b(t^+) − b(t) = b(Q_n) \) for all \( t \in E_1, \alpha_2 = b(t^+) − b(t) = −b(Q_{n+1}) \) for all \( t \in E_2, \) and \( \beta_i = \max E_i.\)

It follows from (31) that \( \alpha_1 = w_n, \) and \( \alpha_2 = v_{n+1}. \) Moreover, we see that \( \beta_2 = c(P_{n+1} − 1), \beta_1 = c(Q_{n+1} − 1) = −cQ_n + cP_{n+1} − c = bv_n − aw_n + cP_{n+1} − c \) by (38). We therefore obtain the formula

\[
\begin{align*}
(44) \quad g(a, b, c) &= cP_{n+1} + \max\{av_{n+1}, bv_n\} − (a + b + c) \\
(45) \quad f(a, b, c) &= a(w_n − v_{n+1}) + b(w_{n+1} − v_n) + \max\{av_{n+1}, bv_n\} \\
(46) \quad f(a, b, c) &= aw_n + bw_{n+1} − \min\{av_{n+1}, bv_n\}.
\end{align*}
\]

Third step: Case \( p_n = q_n \) and \( v_{n+1} = 0. \)

We have \( cQ_{n+1} = bw_{n+1} \in S(a,b) \) by (38) and for all \( 0 < t < Q_{n+1}, \)
\( tc = ab(t) − ba(t) \not\in S(a,b) \) by (34). We deduce that

\[
(47) \quad Q_{n+1} = \min\{t > 0 \mid tc \in S(a,b)\}.
\]

Using (19)-(20) we show by a similar argument that \( U_n = \{0, 1, \ldots , Q_{n+1} − 1\} \) is appropriate and \( U_n.c \) is trimmed. Furthermore, since \( v_{n+1} = 0 \) the lemma show that \( b(t^+) − b(t) = w_n \) for all \( t \in U_n. \) Therefore, by (21) \( f(a, b, c) = aw_n + cQ_{n+1} = aw_n + bw_{n+1}. \)

Moreover, since \( v_{n+1} = 0, \) we get \( w_n = \gcd(b, w_0) = \gcd(b, c) \) and
\[ b = w_n Q_{n+1} \] by (30). Using (42) we deduce that \( \text{lcm}(b, c) = bw_{n+1} \)

**Remark.** The case that \( p_n = q_n \) and \( v_{n+1} = 0 \) can be deduced from Brauer’s formula (16): We put \( d = \gcd(b, c) = \gcd(b, w_0) = w_n, \ c' = \frac{c}{d} = w_{n+1} \) and \( b' = \frac{b}{d} = Q_{n+1} \). Using (30) we get \( a = v_n Q_{n+1} + w_{n+1} Q_n > w_{n+1} Q_{n+1} = b' c' \). We thus have \( f(a, b', c') = g(a, b', c') + a + b' + c' = b' c' + a \). Hence, (16) show that \( f(a, b, c) = df(a, b', c') = bc' + da = \text{lcm}(b, c) + a \gcd(b, c) \)

**Proof of Theorem 5.** We first prove by induction that polynomials \( R_i \) satisfy the following properties: For \( i \leq n \),

\[ c R_i = ay_i - bx_i, \tag{48} \]

\[ c R_{n+1} = ay_{n+1} - b(e_n x_n - x_{n-1}), \tag{49} \]

\[ y_{-1} = y_i R_{i+1}(e_0, \ldots, e_i) - y_{i+1} R_i(e_0, \ldots, e_{i-1}) \tag{50} \]

and for \( j \leq i \),

\[ y_{j-1} = y_i R_{i-j+1}(e_j, \ldots, e_i) - y_{i+1} R_{i-j}(e_j, \ldots, e_{i-1}). \tag{51} \]

In the particular case that \( e_j = e_{j+1} = \cdots = e_i = 2 \), we get

\[ R_{i-j+1}(e_j, \ldots, e_i) = (i - j + 2) \tag{52} \]

and

\[ y_{j-1} - y_j = y_j - y_{j+1} = \cdots = (y_i - y_{i+1}) \tag{53} \]

so in this case (51) can be written in the form

\[ y_{j-1} = (i - j + 2)(y_i - y_{i+1}) + y_{i+1}. \tag{54} \]
Now we consider the set
\[ K = \{ e_i \mid i = 0 \text{ or } (0 < i < m \text{ and } e_i > 2) \} = \{ e_0 = e_{k_0}, e_{k_1}, \ldots, e_{k_{s-1}} \} \]
and we set \( k_s = m \). We can suppose that \( 0 = k_0 < k_1 < \cdots < k_s = m \).
We have
\[ y_{-1} = (e_0 - 1)y_0 + (y_0 - y_1) \quad (55) \]
and for \( 0 < i \leq s \),
\[ y_{k_{i-1}} - y_{k_i} = (e_{k_i} - 2)y_{k_i} + (y_{k_i} - y_{k_i+1}). \quad (56) \]
Furthermore, (54) show that
\[ y_{k_{(i-1)}} = (k_i - k_{(i-1)})(y_{k_{i-1}} - y_{k_i}) + y_{k_i} \quad (57) \]
and (53) that for \( 0 < i < s \),
\[ y_{k_{(i-1)}} - y_{k_{(i-1)}+1} = \cdots = y_{k_{i-1}} - y_{k_i} > y_{k_i}. \quad (58) \]
To apply Theorem 4 we set
\[ v_{-1} = y_{-1}, u_0 = y_0, v_1 = (y_0 - y_1), q_0 = (e_0 - 1) \]
and for \( 0 < i < s \),
\[ v_{2i-1} = y_{k_{i-1}} - y_{k_i}, u_{2i} = y_{k_i}, q_{2i} = (e_{k_i} - 2), p_{2i-1} = (k_i - k_{(i-1)}). \quad (59) \]
Furthermore, if \( e_m > 2 \) we set
\[ v_{2s-1} = (y_m - y_m), v_{2s+1} = y_{m+1} = 0, p_{2s-1} = (m - k_{(s-1)}), u_{2s} = y_m, q_{2s} = (e_m - 1) \]
and if \( e_m = 2 \) we set
\[ v_{2s-1} = y_m, p_{2s-1} = (m + 1 - k_{(s-1)}) \text{ and } u_{2s} = y_{m+1} = 0. \]
We thus get
\[ v_{-1} > u_0 > \cdots > v_{2i-1} > u_{2i} > v_{2i+1} > \cdots \]
and
\[ v_{2i-1} = q_{2i} v_{2i} + v_{2i+1}; \quad u_{2i} = p_{2i+1} v_{2i+1} + u_{2i+2}. \]

Using (11) we get
\[ c = ay_0 - bx_0 = x_n y_n (d_n - e_n) + x_n y_{n+1} - y_n x_{n+1} > 0. \]
Then, \( d_n > e_n \) or \( (d_n = e_n \text{ and } x_{n+1} = 0) \).

To prove (12) we consider two cases:

Case \( n = 0 \).

We can write \( a = (d_0 - 1)x_0 + (x_0 - x_1) \) and \( b = (e_0 - 1)y_0 + (y_0 - y_1) \)
with \( d_0 - 1 > e_0 - 1 \) or \( d_0 - 1 = e_0 - 1 \) and \( x_1 = 0 \). Then, \( L(a, b, c) = (bx_0, b(a - (e_0 - 1)x_0)), l(a, b, c) = (ay_0, a(y_0 - y_1)) \).

We conclude that \( f(a, b, c) = ay_0 + b(a - (e_0 - 1)x_0) - \min\{a(y_0 - y_1), bx_0\} = ay_1 + b(a - e_0 x_0) + \max\{a(y_0 - y_1), bx_0\} \) by (7).

Case \( n > 0 \).

Let \( r = \max\{i \mid k_i < n\} \).

1. Suppose that \( d_n \geq e_n > 2 \). We can write \( x_{n-1} - x_n = (d_n - 2)x_n + (x_n - x_{n+1}) \) and \( y_{n-1} - y_n = (e_n - 2)y_n + (y_n - y_{n+1}) \) with \( d_n - 2 > e_n - 2 \) or \( d_n - 2 = e_n - 2 \) and \( x_{n+1} = 0 \). It follows that the level of \( (a, b, c) \) is even, \( L(a, b, c) = (bx_n, b(x_{n-1} - (e_n - 1)x_n)) \) and \( l(a, b, c) = (ay_n, a(y_n - y_{n+1})) \). Therefore, \( f(a, b, c) = ay_{n+1} + b(x_{n-1} - e_n x_n) + \max\{a(y_n - y_{n+1}), bx_n\} \) by (7).

2. Suppose that \( d_n > e_n = 2 \). We thus have \( k_{(r+1)} > n \). We can write \( y_{k_r} = (k_{(r+1)} - k_r)(y_{k_{(r+1)}-1} - y_{k_{(r+1)}}) + y_{k_{(r+1)}} \) and \( x_{k_r} = (n - k_r)(x_{n-1} - x_n) + x_n \) by (57). Moreover, we have \( y_{k_{(r+1)}-1} - y_{k_{(r+1)}} = y_{n-1} - y_n = y_n - y_{n+1} \) by (53) and \( y_{k_r} - (n - k_r)(y_{n-1} - y_n) = y_n \) by (54). Therefore, the level of \( (a, b, c) \) is odd, \( L = (a(y_n - y_{n+1}), ay_n) \) and \( l = (b(x_{n-1} - x_n), bx_n) \). We deduce that \( f(a, b, c) = ay_{n+1} + b(x_{n-1} - 2x_n) + \max\{a(y_n - y_{n+1}), bx_n\} \) by (7).
3. Suppose that \( d_n = e_n = 2 \) and \( x_{n+1} = 0 \) then \( k_{(r+1)} \geq n + 1 \).

1. Using (57) we can write \( y_{k_r} = (k_{(r+1)} - k_r)(y_{k_{(r+1)}-1} - y_{k_{(r+1)}}) + y_{k_{(r+1)}} = (n + 1 - k_r)(y_n - y_{n+1}) + y_{n+1} \) and \( x_{k_r} = (n + 1 - k_r)x_n \). Hence, the level of \((a, b, c)\) is odd. Since \( y_{k_r} - (n + 1 - k_r)(y_n - y_{n+1}) = y_{n+1} \), we get \( L = (a(y_n - y_{n+1}), ay_{n+1}) \) and \( l = (bx_n, 0) \). Therefore, \( f(a, b, c) = ay_{n+1} + bx_n = ay_{n+1} + \max\{a(y_n - y_{n+1}), bx_n\} \) because, by (48), we have \( bx_n - ay_n + ay_{n+1} = -cR_n + cR_{n+1} \) and it is easily seen that \(-cR_n + cR_{n+1} < 0\).

Finally we have proved that

\[(59)\]
\[f(a, b, c) = ay_{n+1} + b[x_{n-1} - e_nx_n] + \max\{a(y_n - y_{n+1}), bx_n\}\]

To prove Theorem 6 we observe, using (48)-(49), that (11) and the following condition

\[(60)\]
\[\frac{y_{n+1}}{R_{n+1}} \leq \frac{c}{a} < \frac{y_i}{R_i} \text{ for all } 0 \leq i \leq n.\]

are equivalent. Taking account of (48)-(49) we obtain

\[(61)\]
\[f(a, b, c) = cR_{n+1} + ay_n - \min\{ay_{n+1}, cR_n\}\]

5. Examples

1. \( A = \{31, 44, 462, 674, 402, 932, 1214\} \).

We take \( a = 31, b = 44 \). We obtain
\[(462, 674, 402, 932, 1214) = (22, 26, 30, 40, 42)\]
and \((462, 674, 402, 932, 1214) = (5, 3, 12, 7, 2)\).

We remove 674, 932, 1214 from \( A \) without altering \( g(A) \). We consider \((a, b, c_1, c_2) = (31, 44, 462, 402)\). Applying Theorem 3 we obtain \( g(A) = 761 \).
2. \( A = \{57, 83, 367, 543, 605\} \).

We take \( a = 57, b = 83 \). We have \((367, 543, 605) = (21, 27, 31), (367, 543, 605) = (10, 12, 14)\).

\( \mathcal{B} = b(T) \cup \{b\} = \{21, 27, 31, 42, 48, 52, 58, 62, 63, 69, 73, 79, 83\} \),

\( \mathcal{A} = a(T) \cup \{a\} = \{10, 12, 14, 20, 22, 24, 26, 28, 30, 32, 34, 36, 57\} \).

We obtain \( g(A) = 1603 \).

3. \( A = \{a, b, c\} = \{137, 250, 337\} \). We have \((\hat{c}, \hat{c}) = (101, 54), l = 4\),

\( \mathcal{B} = \{53, 101, 154, 202, 250\}, \mathcal{A} = \{25, 54, 79, 108, 137\} \).

Using Theorem 2 we obtain

\[ g(137, 250, 337) = \max\{g_1, g_2, g_3\} = 7537 \].

Let us compute \( g(A) \) by Theorem 5. We get \( n(a, b, c) = 1, L(a, b, c) = (6576, 7261), l(a, b, c) = (7250, 6250) \). We obtain

\[ g(a, b, c) = 7250 + 7261 - 6250 - 137 - 250 - 337 = 7537 \].

References


[7] O. J. Rodseth, On a linear diophantine problem of Frobenius,