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Finite Projective Spaces, Geometric Spreads of Lines and Multi-Qubits

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Abstract
Given a $(2N-1)$-dimensional projective space over GF(2), $\mathrm{PG}(2N-1,2)$, and its geometric spread of lines, there exists a remarkable mapping of this space onto $\mathrm{PG}(N-1,4)$ where the lines of the spread correspond to the points and subspaces spanned by pairs of lines to the lines of $\mathrm{PG}(N-1,4)$. Under such mapping, a non-degenerate quadric surface of the former space has for its image a non-singular Hermitian variety in the latter space, this quadric being *hyperbolic* or *elliptic* in dependence on N being *even* or *odd*, respectively. We employ this property to show that generalized Pauli groups of N-qubits also form two distinct families according to the parity of N and to put the role of symmetric Pauli operators into a new perspective. The $N=4$ case is taken to illustrate the issue, due to its link with the so-called black-hole/qubit correspondence.

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Multiple qubit states play a key role in various fields of quantum information theory like quantum computing, coding and quantum error-correction (see, e. g., [1]). Recently, and rather surprisingly, they have also been recognized to be of great relevance for getting insights into the nature of entropy formulas of a certain class of stringy black hole solutions (see, e. g., [2]). It is, therefore, important to deepen our understanding of these fundamental buildings blocks of quantum world. In the present note we do so through the geometry of their associated generalized Pauli groups.

Let $\mathrm{PG}(d,q)$ be a d-dimensional projective space over $\mathrm{GF}(q)$, q being a power of a prime.[1] A t-spread $\mathcal{S}$ of $\mathrm{PG}(d,q)$ is a set of t-dimensional subspaces of $\mathrm{PG}(d,q)$ which partitions its point-set [4]. If the elements of $\mathcal{S}$ in a subspace V form a t-spread on V, one says that $\mathcal{S}$ induces a t-spread on V. A t-spread $\mathcal{S}$ is called *geometric* (or normal) if it induces a t-spread on each $(2t+1)$-dimensional subspaces of $\mathrm{PG}(d,q)$ spanned by a pair of its elements. It is a well-known fact that $\mathrm{PG}(d,q)$ possesses a t-spread iff $(t+1)|(d+1)$; moreover, this condition is also sufficient for $\mathrm{PG}(d,q)$ to have a geometric t-spread. B. Segre showed [5] that a geometric t-spread of $\mathrm{PG}(N(t+1)-1,q)$, $N\geq 2$, gives rise to a projective space $\mathrm{PG}(N-1,q^{t+1})$ as follows: the points of this space are the elements of $\mathcal{S}$ and its lines are the $(2t+1)$-dimensional subspaces spanned by any two distinct elements of $\mathcal{S}$, with incidence inherited from $\mathrm{PG}(N(t+1)-1,q)$. For a particular case of $t=1$ (i. e., a spread of lines), Dye [6] demonstrated that a *hyperbolic* or an *elliptic* quadric of $\mathrm{PG}(2N-1,q)$ has an induced (geometric) spread of lines if and only if N is, respectively, *even* or *odd*, in which case it is mapped onto a non-singular Hermitian variety $\mathrm{H}(N-1,q^2)$ of $\mathrm{PG}(N-1,q^2)$. We shall now show that this property has for $q=2$ a very interesting physical implication.

It is already a firmly established fact [7, 8, 9, 10] that the commutation relations between the elements of the generalized Pauli group of N-qubits, $N\geq 2$, can be completely reformulated in the geometrical language of symplectic polar space of rank N and order two, $\mathrm{W}(2N-1,2)$; the generalized Pauli operators (discarding the identity) answer to the

[1] For the standard mathematical nomenclature and notation employed in what follows, see, e. g., [3].

points of $\mathrm{W}(2N-1,2)$, a maximally commuting subset has its representative in a maximal totally isotropic subspace of $\mathrm{W}(2N-1,2)$ and commuting translates into collinear. One of the most natural representations of $\mathrm{W}(2N-1,2)$ is that in terms of the points and the set of totally isotropic subspaces of $\mathrm{PG}(2N-1,2)$ endowed with a symplectic polarity. Employing this representation, it has been found in [8] that in the real case the *symmetric* elements/operators of the N-qubit Pauli group *always* lie on a *hyperbolic* quadric in the ambient space $\mathrm{PG}(2N-1,2)$. Combining this fact with Dye's result, we arrive at our main observation: *it is only for N <u>even</u> when <u>all</u> symmetric generalized Pauli operators of $W(2N-1,2)$ can be mapped to the points of an Hermitian variety of the space $PG(N-1,4)$ associated through a geometric spread of lines with the ambient space $PG(2N-1,2)$*. Hence, in this regard, when it comes to generalized Pauli groups 'even-numbered' multi-qubits are found to stand on a slightly different footing than 'odd-numbered' ones.

We shall finish this communication by briefly mentioning an especially interesting even case, $N=4$. Here, a hyperbolic quadric $\mathrm{Q}^+(7,2)$ of $\mathrm{PG}(7,2)$ formed by the symmetric operators is well known for its puzzling triality swapping points and two systems of generators and has for its spread-induced image an Hermitian surface $\mathrm{H}(3,4)$ of $\mathrm{PG}(3,4)$ (see, e. g., [11]). This Hermitian surface is, in turn, nothing but the generalized quadrangle $\mathrm{GQ}(4,2)$ in disguise (see, e. g., [12]), the dual of which — $\mathrm{GQ}(2,4)$ — was found to play a prominent role in the so-called black-hole-qubit correspondence, by fully encoding the $E_{6(6)}$ symmetric entropy formula describing black holes and black strings in $D=5$ [13]. Our finding thus, *inter alia*, not only opens up an unexpected window through which also four-qubit Pauli group, like its lower rank cousins, could find its way into some black hole entropy formula(s), but also puts the role of symmetric operators into a new perspective. It is also important to keep in mind this remarkable *three-to-one* correspondence, i. e., that it is always a triple of (collinear) operators of the ambient space $\mathrm{PG}(7,2)$ which comprises a single point of $\mathrm{PG}(3,4)$.

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