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Finite Projective Spaces, Geometric Spreads of Lines and Multi-Qubits

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Abstract

Given a $(2N - 1)$ -dimensional projective space over $\text{GF}(2)$, $\text{PG}(2N - 1, 2)$, and its geometric spread of lines, there exists a remarkable mapping of this space onto $\text{PG}(N - 1, 4)$ where the lines of the spread correspond to the points and subspaces spanned by pairs of lines to the lines of $\text{PG}(N - 1, 4)$. Under such mapping, a non-degenerate quadric surface of the former space has for its image a non-singular Hermitian variety in the latter space, this quadric being *hyperbolic* or *elliptic* in dependence on N being *even* or *odd*, respectively. We employ this property to show that generalized Pauli groups of N -qubits also form two distinct families according to the parity of N and to put the role of symmetric operators into a new perspective. The $N = 4$ case is taken to illustrate the issue.

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Let $\text{PG}(d, q)$ be a d -dimensional projective space over $\text{GF}(q)$, q being a power of a prime. A t -spread \mathcal{S} of $\text{PG}(d, q)$ is a set of t -dimensional subspaces of $\text{PG}(d, q)$ which partitions its point-set [1]. If the elements of \mathcal{S} in a subspace V form a t -spread on V , one says that \mathcal{S} induces a t -spread on V . A t -spread \mathcal{S} is called *geometric* (or *normal*) if it induces a t -spread on each $(2t + 1)$ -dimensional subspaces of $\text{PG}(d, q)$ spanned by a pair of its elements. It is a well-known fact that $\text{PG}(d, q)$ possesses a t -spread iff $(t + 1)|(d + 1)$; moreover, this condition is also sufficient for $\text{PG}(d, q)$ to have a geometric t -spread. B. Segre showed [2] that a geometric t -spread of $\text{PG}(N(t + 1) - 1, q)$, $N \geq 2$, gives rise to a projective space $\text{PG}(N - 1, q^{t+1})$ as follows: the points of this space are the elements of \mathcal{S} and its lines are the $(2t + 1)$ -dimensional subspaces spanned by any two distinct elements of \mathcal{S} , with incidence inherited from $\text{PG}(N(t + 1) - 1, q)$. For a particular case of $t = 1$ (i. e., a spread of lines), Dye [3] demonstrated that a non-degenerate quadric surface of $\text{PG}(2N - 1, q)$ has always a (geometric) spread of lines, its image is a non-singular Hermitian variety $\text{H}(N - 1, q^2)$ of $\text{PG}(N - 1, q^2)$, and — most importantly — this quadric is *hyperbolic* or *elliptic* according as N is *even* or *odd*, respectively. We shall now show that this last property has for $q = 2$ a very interesting physical implication.

It is already a firmly established fact [4, 5] that the commutation relations between the elements of the generalized Pauli group of N -qubits, $N \geq 2$, can be completely reformulated in the geometrical language of symplectic polar space of rank N and order two, $\text{W}(2N - 1, 2)$; the generalized Pauli operators (discarding the identity) answer to the points of $\text{W}(2N - 1, 2)$, a maximally commuting subset has its representative in a maximal totally isotropic subspace of

$W(2N - 1, 2)$ and commuting translates into collinear. One of the most natural representations of $W(2N - 1, 2)$ is that in terms of the points and the set of totally isotropic subspaces of $PG(2N - 1, 2)$ endowed with a symplectic polarity. Employing this representation, it has been found in [5] that in the real case the *symmetric* elements/operators of the N -qubit Pauli group *always* lie on a *hyperbolic* quadric in the ambient space $PG(2N - 1, 2)$. Combining this fact with Dye’s result, we arrive at our main observation: *it is only for N even when all symmetric generalized Pauli operators of $W(2N - 1, 2)$ can be mapped to the points of an Hermitian variety of the space $PG(N - 1, 4)$ associated through a geometric spread of lines with the ambient space $PG(2N - 1, 2)$.*¹ Hence, in this regard, when it comes to generalized Pauli groups ‘even-numbered’ multi-qubits are found to stand on a slightly different footing than ‘odd-numbered’ ones.

We shall finish this letter by briefly mentioning an especially interesting even case, $N = 4$. Here, a hyperbolic quadric $Q^+(7, 2)$ formed by the symmetric operators is well known for its puzzling triality swapping points and two systems of generators and has for its spread-induced image an Hermitian surface $H(3, 4)$ of $PG(3, 4)$. This Hermitian surface is, in turn, nothing but the generalized quadrangle $GQ(4, 2)$ in disguise (see, e. g., [6]), the dual of which — $GQ(2, 4)$ — was found to play a prominent role in the so-called black-hole-qubit correspondence, by fully encoding the $E_{6(6)}$ symmetric entropy formula describing black holes and black strings in $D = 5$ [7]. Our finding thus, *inter alia*, not only opens up an unexpected window through which also four-qubit Pauli group, like its lower rank cousins, could find its way into some black hole entropy formula(s), but also puts the role of symmetric operators into a new perspective. It is also important to keep in mind this remarkable *three-to-one* correspondence, i. e., that it is always a triple of operators of $PG(2N - 1, 2)$ which comprises a single point of $PG(N - 1, 4)$.

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¹Note that in this case also $PG(N - 1, 4)$ is *odd*-dimensional.

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