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Abstract

We address in this paper a nonlinear parabolic system, which is built to retain the main mathematical difficulties of the P1 radiative diffusion physical model. We propose a finite volume fractional-step scheme for this problem enjoying the following properties. First, we show that each discrete solution satisfies *a priori* L^∞-estimates, through a discrete maximum principle; by a topological degree argument, this yields the existence of a solution, which is proven to be unique. Second, we establish uniform (with respect to the size of the meshes and the time step) L^2-bounds for the space and time translates; this proves, by the Kolmogorov theorem, the relative compactness of any sequence of solutions obtained through a sequence of discretizations the time and space steps of which tend to zero; the limits of converging subsequences are then shown to be a solution to the continuous problem. Estimates of time translates of the discrete solutions are obtained through the formalization of a generic argument, interesting for its own sake.

Key words : P1 radiative transfer model, finite volumes, parabolic systems, convergence analysis.

1 Introduction

We address in this paper the following nonlinear parabolic system:

$$\left|\begin{array}{ll} \dfrac{\partial u}{\partial t} - \Delta u + u^4 - \varphi = 0 & \text{for } (x,t) \in \Omega \times (0,T), \\ \varphi - \Delta\varphi - u^4 = 0 & \text{for } (x,t) \in \Omega \times (0,T), \\ u(x,0) = u_0(x) & \text{for } x \in \Omega, \\ u(x,t) = 0 & \text{for } (x,t) \in \partial\Omega \times (0,T), \\ \nabla\varphi \cdot n = 0 & \text{for } (x,t) \in \partial\Omega \times (0,T), \end{array}\right. \tag{1}$$

where Ω is a connected bounded subset of $\mathbb{R}^d$, $d = 2$ or $d = 3$, which is supposed to be polygonal ($d = 2$) or polyhedral ($d = 3$), $T < \infty$ is the final time, u and φ are two real-valued functions defined on $\Omega \times [0, T)$ and $\partial\Omega$ stands for the boundary of Ω of outward normal n. The initial value for u, denoted by u_0, is supposed to lie in $\mathrm{L}^\infty(\Omega) \cap \mathrm{H}^1_0(\Omega)$ and to satisfy $u_0(x) \geq 0$ for almost every $x \in \Omega$.

This system of partial differential equations is inspired from a simplified radiative transfer physical model, the so-called $\mathbf{P}_1$ model, sometimes used in computational fluid dynamics for the simulation of high temperature optically thick flows, as encountered for instance in fire modelling (see e.g. [11] for an exposition of the theory, [10, 2, 9, 12] for recent developments and applications or the documentation of the CFX or FLUENT commercial codes for a synthetic description). In this context, the unknown u stands for the temperature, φ for the radiative intensity and the first equation is the energy balance. System (1) has been derived with the aim of retaining the main mathematical difficulties of the initial physical model; in particular, adding a convection term in the first equation would only require minor changes in the theory developed hereafter.

In this short paper, we give a finite volume scheme for the discretization of (1) and prove the existence and uniqueness of the discrete solution and its convergence to a solution of (1), thus showing that this problem indeed admits a solution, in a weak sense which will be defined.

2 The finite volume scheme

Even though the arguments developed in this paper are valid for any general admissible discretization in the sense of Definition 9.1, p. 762 in [4], we choose, for the sake of simplicity, to restrict the presentation to simplicial meshes. We thus suppose given a triangulation $\mathcal{M}$ of Ω, that is a finite collection of d-simplicial control volumes K, pairwise disjoint, and such that $\bar{\Omega} = \cup_{K\in\mathcal{M}} \bar{K}$; the mesh is supposed to be conforming in the sense that two neighbouring simplices share a whole face (*i.e.* there is no hanging node). In addition, we assume that, for any $K \in \mathcal{M}$, the circumcenter x_K of K lies in K; note that, for each neighbouring control volumes K and L, the segment $[x_K, x_L]$ is orthogonal to the face $K|L$ separating K from L.

For each simplex K, we denote by $\mathcal{E}(K)$ the set of the faces of K and by $|K|$ the measure of K. The set of faces of the mesh $\mathcal{E}$ is split into the set $\mathcal{E}_{\mathrm{int}}$ of internal ones (*i.e.* separating two control volumes) and the set $\mathcal{E}_{\mathrm{ext}}$ of faces included in the domain boundary. For each internal face $\sigma = K|L$, we denote by $|\sigma|$ the (d-1)-dimensional measure of σ and by d_σ the distance $d(x_K, x_L)$; for an external face σ of a control volume K, d_σ stands for distance from x_K to σ. The regularity of the mesh is characterized by the parameter $\theta_\mathcal{M}$ defined by:

$$\theta_\mathcal{M} := \min_{K\in\mathcal{M}} \frac{\rho_K}{h_K} \tag{2}$$

where ρ_K and h_K stands for the diameter of the largest ball included in K and the diameter of K, respectively.

We denote by $\mathrm{H}_{\mathcal{M}}(\Omega) \subset \mathrm{L}^2(\Omega)$ the space of functions which are piecewise constant over each control volume $K \in \mathcal{M}$. For all $u \in \mathrm{H}_{\mathcal{M}}(\Omega)$ and for all $K \in \mathcal{M}$, we denote by u_K the constant value of u in K. The space $\mathrm{H}_{\mathcal{M}}(\Omega)$ is equipped with the following Euclidean structure. For $(u, v) \in (\mathrm{H}_{\mathcal{M}}(\Omega))^2$, we define the following inner product:

$$[u, v]_{\mathcal{M}} := \sum_{\sigma \in \mathcal{E}_{\rm int} \ (\sigma = K|L)} \frac{|\sigma|}{d_\sigma} \ (u_L - u_K)(v_L - v_K) + \sum_{\sigma \in \mathcal{E}_{\rm ext} \ (\sigma \in \mathcal{E}(K))} \frac{|\sigma|}{d_\sigma} \ u_K \ v_K. \quad (3)$$

Thanks to the discrete Poincaré inequality (5) given below, this scalar product defines a norm on $\mathrm{H}_{\mathcal{M}}(\Omega)$:

$$\|u\|_{1,\mathcal{M}} := [u, u]_{\mathcal{M}}^{1/2}. \quad (4)$$

The following discrete Poincaré inequality holds (see lemma 9.1, p. 765, in [4]):

$$\|u\|_{L^2(\Omega)} \leq \mathrm{diam}(\Omega) \ \|u\|_{1,\mathcal{M}} \qquad \forall u \in \mathrm{H}_{\mathcal{M}}(\Omega). \quad (5)$$

We also define the following semi-inner product and semi-norm:

$$< u, v >_{\mathcal{M}} := \sum_{\sigma \in \mathcal{E}_{\rm int} \ (\sigma = K|L)} \frac{|\sigma|}{d_\sigma} \ (u_L - u_K)(v_L - v_K), \qquad |u|_{1,\mathcal{M}} := < u, u >_{\mathcal{M}}^{1/2}.$$

These inner products can be seen as discrete analogues to the standard H^1-inner product, with, in the first one, an implicitly assumed zero boundary condition.
For any function $u \in \mathrm{H}_{\mathcal{M}}(\Omega)$, we also define the following discrete H^{-1}-norm:

$$\|u\|_{-1,\mathcal{M}} := \sup_{v \in \mathrm{H}_{\mathcal{M}}(\Omega), \, v \neq 0} \frac{\displaystyle\int_\Omega u \, v \, \mathrm{d}x}{\|v\|_{1,\mathcal{M}}}.$$

By inequality (5), the $\|\cdot\|_{-1,\mathcal{M}}$-norm is controlled by the $\mathrm{L}^2(\Omega)$-norm.

The discrete Laplace operators associated with homogeneous Dirichlet and Neumann boundary conditions, denoted by $\Delta_{\mathcal{M},\mathrm{D}}(\cdot)$ and $\Delta_{\mathcal{M},\mathrm{N}}(\cdot)$ respectively, are defined as follows:

$$\begin{aligned} &\forall \psi \in \mathrm{H}_{\mathcal{M}}(\Omega), \\ &(\Delta_{\mathcal{M},\mathrm{N}}(\psi))_K = \frac{1}{|K|} \sum_{\sigma = K|L} \frac{|\sigma|}{d_\sigma} \ (\psi_L - \psi_K), \\ &(\Delta_{\mathcal{M},\mathrm{D}}(\psi))_K = (\Delta_{\mathcal{M},\mathrm{N}}(\psi))_K + \frac{1}{|K|} \sum_{\sigma \in \mathcal{E}_{\rm ext} \cap \mathcal{E}(K)} \frac{|\sigma|}{d_\sigma} \ (-\psi_K). \end{aligned} \quad (6)$$

The links between these operators and the above defined inner products is clarified by the following identities:

$$\forall \psi \in \mathrm{H}_{\mathcal{M}}(\Omega), \qquad \sum_{K \in \mathcal{M}} -|K| \ \psi_K \ (\Delta_{\mathcal{M},\mathrm{N}}(\psi))_K = < \psi, \psi >_{\mathcal{M}}$$

$$\text{and} \qquad \sum_{K \in \mathcal{M}} -|K| \ \psi_K \ (\Delta_{\mathcal{M},\mathrm{D}}(\psi))_K = [\psi, \psi]_{\mathcal{M}}.$$

Finally, we suppose given a partition of the interval $(0,T)$, which we assume regular for the sake and simplicity, with $t^0 = 0, \ldots, t^n = n\,\delta t, \ldots t^N = T$.

A each time t_n, an approximation of the solution $(u^n, \varphi^n) \in \mathrm{H}_{\mathcal{M}}(\Omega) \times \mathrm{H}_{\mathcal{M}}(\Omega)$ is given by the following finite volume scheme:

$$\forall K \in \mathcal{M}, \quad \left| \begin{array}{ll} (i) & \dfrac{u_K^{n+1} - u_K^n}{\delta t} - (\Delta_{\mathcal{M},\mathrm{D}}(u^{n+1}))_K + |u_K^{n+1}|\ (u_K^{n+1})^3 - \varphi_K^n = 0, \\ (ii) & \varphi_K^{n+1} - (\Delta_{\mathcal{M},\mathrm{N}}(\varphi^{n+1}))_K - (u_K^{n+1})^4 = 0. \end{array} \right. \tag{7}$$

In the first equation, the term u^4 is discretized as $|u_K^{n+1}|\ (u_K^{n+1})^3$ to ensure positivity (see proof of Proposition 3.1 and Remark 1 below). The scheme is initialized as follows:

$$\forall K \in \mathcal{M}, \quad u_K^0 = \frac{1}{|K|} \int_K u_0(x)\ \mathrm{d}x \tag{8}$$

and φ^0 is given by (7)-(ii), where we set $n+1 = 0$.

3 A priori L^∞ estimates, existence and uniqueness of the discrete solution

We prove in this section the following result.

Proposition 3.1

1. The scheme (7) has a unique solution.
2. For $0 \le n \le N$, the unknown φ^n satisfies the following estimate:
$$\forall K \in \mathcal{M}, \quad 0 \le \varphi_K^n \le \left[\max_{L \in \mathcal{M}} u_L^n \right]^4.$$
3. For $1 \le n \le N$, the unknown u^n satisfies the following estimate:
$$\forall K \in \mathcal{M}, \quad 0 \le u_K^n \le \max_{L \in \mathcal{M}} u_L^{n-1}.$$

Proof Step one: positivity of the unknowns.

We first observe that, from its definition (8) and thanks to the fact that u_0 is non-negative, u^0 is a non-negative function. Let us suppose that this property still holds at time step n. We write the second equation of the scheme (7) as:

$$\varphi_K^n - (\Delta_{\mathcal{M},\mathrm{N}}(\varphi^n))_K = (u_K^n)^4.$$

This set of relations is a linear system for φ^n, the matrix of which is an M-matrix: indeed, from the definition (6) of $\Delta_{\mathcal{M},\mathrm{N}}(\cdot)$, it can be easily checked that its diagonal

is strictly dominant and has only positive entries, and all its off-diagonal entries are non-positive. Since the right-hand side of this equation is non-negative, φ^n is also non-negative. The first equation of the scheme (7) now can be recast as:

$$\left[\frac{1}{\delta t} + |u_K^{n+1}|\ (u_K^{n+1})^2\right] u_K^{n+1} - (\Delta_{\mathcal{M},\mathrm{D}}(u^{n+1}))_K = \frac{1}{\delta t} u_K^n + \varphi_K^n. \tag{9}$$

This set of relations can be viewed as a matrix system for the unknown u^{n+1} the matrix of which depends on u^{n+1} but is also an M-matrix whatever u^{n+1} may be. Since we know that $\varphi^n \geq 0$, the right-hand side of this equation is by assumption non-negative and so is u^{n+1} too.

Step two: upper bounds.

Let $\bar{\varphi}$ be a constant function of $\mathrm{H}_{\mathcal{M}}(\Omega)$. From the definition (6) of $\Delta_{\mathcal{M},\mathrm{N}}(\cdot)$, we see that $\Delta_{\mathcal{M},\mathrm{N}}(\bar{\varphi}) = 0$ and the second relation of the scheme thus implies:

$$(\varphi^{n+1} - \bar{\varphi})_K - (\Delta_{\mathcal{M},\mathrm{N}}(\varphi^{n+1} - \bar{\varphi}))_K = (u_K^{n+1})^4 - \bar{\varphi}_K.$$

Choosing for the constant value of $\bar{\varphi}$ the quantity $(\max_{K\in\mathcal{M}} u_K^{n+1})^4$ yields a non-positive right-hand side, and so, from the above mentionned property of the matrix of this linear system, $(\varphi^{n+1} - \bar{\varphi})_K \leq 0,\ \forall K \in \mathcal{M}$, which equivalently reads:

$$\varphi_K^{n+1} \leq (\max_{L\in\mathcal{M}} u_L^{n+1})^4, \quad \forall K \in \mathcal{M}. \tag{10}$$

Let us now turn to the estimate of the first unknown u^{n+1}. Let K_0 be a control volume where u^{n+1} reaches its maximum value. From the definition (6) of $\Delta_{\mathcal{M},\mathrm{D}}(\cdot)$, it appears that:

$$-(\Delta_{\mathcal{M},\mathrm{D}}(u^{n+1}))_{K_0} \geq 0.$$

The first relation of the scheme reads:

$$\frac{1}{\delta t}(u_{K_0}^{n+1} - u_{K_0}^n) - (\Delta_{\mathcal{M},\mathrm{D}}(u^{n+1}))_{K_0} + \left[|u_{K_0}^{n+1}|\ (u_{K_0}^{n+1})^3 - \varphi_{K_0}^n\right] = 0. \tag{11}$$

By the inequality (10), we see that supposing that $u_{K_0}^{n+1} > \max_{K\in\mathcal{M}} u_K^n$ yields that the first and third term of the preceding relation are positive, while the second one is non-negative, which is in contradiction with the fact that their sum is zero.

Step three: existence of a solution.

Let us suppose that we have obtained a solution to the scheme up to time step n. Let the function $F(\cdot)$ be defined as follows:

$$\left|\begin{array}{rcl} \mathbb{R}^{\mathrm{card}(\mathcal{M})} \times [0,1] & \longrightarrow & \mathbb{R}^{\mathrm{card}(\mathcal{M})}, \\ ((u_K)_{K\in\mathcal{M}},\ \alpha) & \mapsto & (v_K)_{K\in\mathcal{M}} \quad \text{such that:} \\ \qquad \forall K \in \mathcal{M}, & & v_K = \dfrac{1}{\delta t}(u_K - u_K^n) - (\Delta_{\mathcal{M},\mathrm{D}}(u))_K + \alpha\left[|u_K|\,(u_K)^3 - \varphi_K^n\right]. \end{array}\right.$$

The solution u^{n+1} of the first relation of the scheme is the solution to:

$$F((u_K^{n+1})_{K\in\mathcal{M}}, 1) = 0. \tag{12}$$

First, we observe that the function F_0, which maps $\mathbb{R}^{\mathrm{card}(\mathcal{M})}$ onto $\mathbb{R}^{\mathrm{card}(\mathcal{M})}$ and is defined by $F_0((u_K)_{K\in\mathcal{M}}) = F((u_K)_{K\in\mathcal{M}}, 0)$, is affine and one-to-one. Second, we see from their proofs that the estimates on u^{n+1} proven in step one and step two for $\alpha = 1$ in fact holds uniformly for $\alpha \geq 0$. The existence of a solution to (12) then follows by a topological degree argument (see e.g. [3]).
Finally, the existence (and uniqueness) of the solution to the second equation of the scheme, which is a linear system, follows from the above mentionned properties of the associated matrix.

Step four: uniqueness of the solution.

Let us suppose that the solution is unique up to step n and that there exist two solutions u^{n+1} and v^{n+1} to the first equation of the scheme. By the identity $a^4 - b^4 = (a-b)\,(a^3 + a^2 b + ab^2 + +b^3)$, the difference $\delta u = u^{n+1} - v^{n+1}$ satisfies the following system of equations:

$$\forall K \in \mathcal{M}, \quad \left[\frac{1}{\delta t} + \left((u_K^{n+1})^3 + (u_K^{n+1})^2 v_K^{n+1} + u_K^{n+1}(v_K^{n+1})^2 + (v_K^{n+1})^3\right)\right] \delta u_K \\ -(\Delta_{\mathcal{M},\mathrm{D}}(\delta u))_K = 0.$$

Since we know from the precedent analysis that both u_K^{n+1} and v_K^{n+1} are non-negative, this set of relations can be seen as a matrix system for δu the matrix of which is an M-matrix; we thus get $\delta u = 0$, which proves the uniqueness of the solution. □

Remark 1 We see from Relation (9) that the discretization of u^4 as a product of a positive quantity (here $|u^{n+1}|^p$) and $(u^{n+1})^q$ (where $p+q=4$) with q odd (*i.e.* $q=3$ or $q=1$) is essential to prove the non-negativity of u^{n+1}; note that we have indeed observed in practice some (non-physical) negative values when this term is discretized as $(u^{n+1})^4$.

4 Convergence to a solution of the continuous problem

Let $\mathrm{H}_{\mathcal{D}}$ be the space of piecewise constant functions over each $K \times I^n$, for $K \in \mathcal{M}$ and $I^n = [t^n,\ t^{n+1}),\ 0 \leq n \leq N-1$. To each sequence $(u^n)_{n=0,N}$ of functions of $\mathrm{H}_{\mathcal{M}}(\Omega)$, we associate the function $u \in \mathrm{H}_{\mathcal{D}}$ defined by $u(x,t) = u^n(x)$ for $t^n \leq t < t^{n+1},\ 0 \leq n \leq N-1$. In addition, for any $u \in \mathrm{H}_{\mathcal{D}}$, we define $\partial_{t,\mathcal{D}}(u) \in \mathrm{H}_{\mathcal{D}}$ by $\partial_{t,\mathcal{D}}(u)(x,t) = \partial_{t,\mathcal{D}}(u)^n(x)$ for $t^n \leq t < t^{n+1},\ 0 \leq n \leq N-1$ where the function $\partial_{t,\mathcal{D}}(u)^n \in \mathrm{H}_{\mathcal{M}}(\Omega)$ is defined by:

$$\partial_{t,\mathcal{D}}(u)^n(x) := \frac{u^{n+1}(x) - u^n(x)}{\delta t} \qquad \left(i.e.\ \partial_{t,\mathcal{D}}(u)_K^n = \frac{u_K^{n+1} - u_K^n}{\delta t}, \quad \forall K \in \mathcal{M}\right).$$

For any function $u \in \mathrm{H}_{\mathcal{D}}$, we define the following norms and semi-norms:

$$\|u\|^2_{\mathrm{L}^2(0,T;\,\mathrm{H}^1_{\mathcal{M}})} := \delta t \sum_{n=0}^{N} \|u^n\|^2_{1,\mathcal{M}},$$
$$\|u\|^2_{\mathrm{L}^2(0,T;\,\mathrm{H}^{-1}_{\mathcal{M}})} := \delta t \sum_{n=0}^{N-1} \|u^n\|^2_{-1,\mathcal{M}},$$
$$|u|^2_{\mathrm{L}^2(0,T;\,\mathrm{H}^1_{\mathcal{M}})} := \delta t \sum_{n=0}^{N} |u^n|^2_{1,\mathcal{M}}.$$

The norms $\|\cdot\|_{\mathrm{L}^2(0,T;\,\mathrm{H}^1_{\mathcal{M}})}$ and $|\cdot|_{\mathrm{L}^2(0,T;\,\mathrm{H}^1_{\mathcal{M}})}$ can be seen as discrete equivalents of the $\mathrm{L}^2(0,T;\mathrm{H}^1(\Omega))$-norm, and $\|\cdot\|_{\mathrm{L}^2(0,T;\,\mathrm{H}^{-1}_{\mathcal{M}})}$ may be considered as a discrete $\mathrm{L}^2(0,T;\mathrm{H}^{-1}(\Omega))$-norm.

The following result provides estimates of the solution to the considered scheme.

Proposition 4.1 (Estimates in energy norms) Let u and φ be the functions of $\mathrm{H}_{\mathcal{D}}$ associated to $(u^n)_{0\le n\le N} \in \mathrm{H}_{\mathcal{M}}(\Omega)^{N+1}$ and $(\varphi^n)_{0\le n\le N} \in \mathrm{H}_{\mathcal{M}}(\Omega)^{N+1}$ respectively, themselves being given by the scheme (7) and the initial condition (8). Then the following estimate holds:

$$\begin{aligned}\|u\|_{\mathrm{L}^2(0,T;\,\mathrm{H}^1_{\mathcal{M}})} + \|\partial_{t,\mathcal{D}}(u)\|_{\mathrm{L}^2(0,T;\,\mathrm{H}^{-1}_{\mathcal{M}})} + \|\varphi\|_{L^2((0,T)\times\Omega)} \\ + |\varphi|_{\mathrm{L}^2(0,T;\,\mathrm{H}^1_{\mathcal{M}})} + \|\partial_{t,\mathcal{D}}(\varphi)\|_{L^2((0,T)\times\Omega)} \le c_{\mathrm{e}},\end{aligned} \tag{13}$$

where the real number c_{e} only depends on Ω, the initial data $u_0(\cdot)$ and (as a decreasing function) on the parameter $\theta_{\mathcal{M}}$ characterizing the regularity of the mesh, defined by (2).

Proof First, we recall that, by a standard reordering of the summations, we have, for any function $u \in \mathrm{H}_{\mathcal{M}}(\Omega)$:

$$-\sum_{K\in\mathcal{M}} |K|\, u_K\, (\Delta_{\mathcal{M},\mathrm{D}}(u))_K = \|u\|^2_{1,\mathcal{M}}, \qquad -\sum_{K\in\mathcal{M}} |K|\, u_K\, (\Delta_{\mathcal{M},\mathrm{N}}(u))_K = |u|^2_{1,\mathcal{M}}.$$

Multiplying by $2\,\delta t\,|K|\,u_K^{n+1}$ Equation (7)-(i), using the equality $2\,a\,(a-b) = a^2 + (a-b)^2 - b^2$, summing over each control volume of the mesh and using the first of the preceding identities yields, for $0 \le n \le N-1$:

$$\begin{aligned}\|u^{n+1}\|^2_{L^2(\Omega)} + \|u^{n+1}-u^n\|^2_{L^2(\Omega)} - \|u^n\|^2_{L^2(\Omega)} + 2\,\delta t\,\|u^{n+1}\|^2_{1,\mathcal{M}} \\ + 2\,\delta t \int_\Omega (u^{n+1})^5\,\mathrm{d}x = 2\,\delta t \int_\Omega \varphi^n u^{n+1}\,\mathrm{d}x.\end{aligned}$$

By Proposition 3.1, u^{n+1} is non-negative and $\varphi_K^n \le \bar{u}_0^4,\ \forall K \in \mathcal{M}$, where $\bar{u}_0$ stands for $\max_{K\in\mathcal{M}} u_K^0$ and is thus bounded by the L^∞-norm of on the initial data $u_0(\cdot)$. Therefore, we get:

$$\|u^{n+1}\|^2_{L^2(\Omega)} - \|u^n\|^2_{L^2(\Omega)} + 2\,\delta t\,\|u^{n+1}\|^2_{1,\mathcal{M}} \le 2\,\delta t\,\bar{u}_0^4 \int_\Omega u^{n+1}\,\mathrm{d}x.$$

By Cauchy-Schwarz' inequality, Young's inequality and the discrete Poincaré inequality (5), we thus obtain:

$$\|u^{n+1}\|_{L^2(\Omega)}^2 - \|u^n\|_{L^2(\Omega)}^2 + \delta t \, \|u^{n+1}\|_{1,\mathcal{M}}^2 \leq \delta t \, |\Omega| \, \mathrm{diam}(\Omega)^2 \, \bar{u}_0^8.$$

Summing from $n = 0$ to $n = N - 1$, we get:

$$\|u^N\|_{L^2(\Omega)}^2 + \sum_{n=1}^{N} \delta t \, \|u^n\|_{1,\mathcal{M}}^2 \leq T \, |\Omega| \, \mathrm{diam}(\Omega)^2 \, \bar{u}_0^8 + \|u^0\|_{L^2(\Omega)}^2.$$

Since, by assumption, $u_0 \in \mathrm{H}_0^1(\Omega)$, the discrete H^1 norm of u^0, $\|u^0\|_{1,\mathcal{M}}$ is bounded by $c\|u_0\|_{\mathrm{H}^1(\Omega)}$ where the real number c only depends on Ω and, in a decreasing way, on the parameter $\theta_{\mathcal{M}}$ characterizing the regularity of the mesh (see e.g. Lemma 3.3 in [7]). Together with the preceding relation, this provides the control of the first term in (13).

We now turn to the estimate of $\|\partial_{t,\mathcal{D}}(u)\|_{\mathrm{L}^2(0,T;\,\mathrm{H}_{\mathcal{M}}^{-1})}$. Let v be a function of $\mathrm{H}_{\mathcal{M}}(\Omega)$; multiplying by $|K| \, v_K$ the first equation of the scheme (7)-(i) and summing over $K \in \mathcal{M}$, we get for $0 \leq n \leq N - 1$:

$$\int_\Omega \partial_{t,\mathcal{D}}(u)^n \; v \, \mathrm{d}x = -[u^{n+1}, v]_{\mathcal{M}} - \int_\Omega \left[(u^{n+1})^4 - \varphi^n\right] \; v \, \mathrm{d}x.$$

By the fact that, as both $(u^{n+1})^4$ and φ^n are non-negative functions bounded by $\bar{u}_0^4$, the difference is itself bounded by $\bar{u}_0^4$, using the Cauchy-Schwarz inequality and the discrete Poincaré inequality (5), we get:

$$\int_\Omega \partial_{t,\mathcal{D}}(u)^n \; v \, \mathrm{d}x \leq \left[\|u^{n+1}\|_{1,\mathcal{M}} + \bar{u}_0^4 \, |\Omega|^{1/2} \, \mathrm{diam}(\Omega)\right] \|v\|_{1,\mathcal{M}},$$

and so:

$$\|\partial_{t,\mathcal{D}}(u)^n\|_{-1,\mathcal{M}} \leq \|u^{n+1}\|_{1,\mathcal{M}} + \bar{u}_0^4 \, |\Omega|^{1/2} \, \mathrm{diam}(\Omega),$$

which, by the bound of $\|u\|_{\mathrm{L}^2(0,T;\,\mathrm{H}_{\mathcal{M}}^1)}$, yields the control of the second term in (13).

As far as φ is concerned, the second equation of the scheme (7)-(ii) and the initialization (8) yields for $0 \leq n \leq N$:

$$\|\varphi^n\|_{L^2(\Omega)}^2 + |\varphi^n|_{1,\mathcal{M}}^2 \leq \int_\Omega \bar{u}_0^4 \; \varphi^n \, \mathrm{d}x.$$

Thus, by Young's inequality, we get:

$$\frac{1}{2} \, \|\varphi^n\|_{L^2(\Omega)}^2 + |\varphi^n|_{1,\mathcal{M}}^2 \leq \frac{1}{2} \, |\Omega| \, \bar{u}_0^8.$$

Multiplying by δt and summing over the time steps, this provides the estimates of $\|\varphi\|_{L^2((0,T)\times\Omega)}$ and $|\varphi|_{\mathrm{L}^2(0,T;\,\mathrm{H}_{\mathcal{M}}^1)}$ we are searching for.

To obtain a control on $\partial_{t,\mathcal{D}}(\varphi)$, we need a sharper estimate on $\partial_{t,\mathcal{D}}(u)$. Our starting point is once again Equation (7)-(i), which we multiply this time by $|K| \, \partial_{t,\mathcal{D}}(u)^n$

before summing over $K \in \mathcal{M}$, to get for $0 \leq n \leq N-1$, once again by the identity $a^2 - b^2 \leq a^2 + (a-b)^2 - b^2 = 2\,a\,(a-b)$ and invoking the Cauchy-Schwarz inequality:

$$\|\partial_{t,\mathcal{D}}(u)^n\|^2_{L^2(\Omega)} + \frac{1}{2\,\delta t}\,\left(\|u^{n+1}\|^2_{1,\mathcal{M}} - \|u^n\|^2_{1,\mathcal{M}}\right) \leq |\Omega|^{1/2}\,\bar{u}_0^4\,\|\partial_{t,\mathcal{D}}(u)^n\|_{L^2(\Omega)},$$

so, by Young's inequality:

$$\|\partial_{t,\mathcal{D}}(u)^n\|^2_{L^2(\Omega)} + \frac{1}{\delta t}\,\left(\|u^{n+1}\|^2_{1,\mathcal{M}} - \|u^n\|^2_{1,\mathcal{M}}\right) \leq |\Omega|\,\bar{u}_0^8.$$

Multiplying by δt and summing over the time steps yields:

$$\|\partial_{t,\mathcal{D}}(u)\|^2_{L^2((0,T)\times\Omega)} + \|u^N\|^2_{1,\mathcal{M}} \leq |\Omega|\,T\,\bar{u}_0^8 + \|u^0\|^2_{1,\mathcal{M}}, \tag{14}$$

which provides an estimate for $\|\partial_{t,\mathcal{D}}(u)\|_{L^2((0,T)\times\Omega)}$. Taking now the difference of the second equation of the scheme (7)-(ii) at two consecutive time steps and using (8) for the first one, we obtain for $0 \leq n \leq N-1$:

$$\begin{aligned}\forall K \in \mathcal{M}, \qquad \partial_{t,\mathcal{D}}(\varphi)^n_K - (\Delta_{\mathcal{M},\mathrm{N}}(\partial_{t,\mathcal{D}}(\varphi)^n))_K &= \frac{(u_K^{n+1})^4 - (u_K^n)^4}{\delta t} \\ &= \left[(u_K^{n+1})^3 + (u_K^{n+1})^2\,u_K^n + u_K^{n+1}\,(u_K^n)^2 + (u_K^n)^3\right]\partial_{t,\mathcal{D}}(u)^n_K.\end{aligned}$$

Multiplying by $|K|\;\partial_{t,\mathcal{D}}(\varphi)^n_K$ over each control volume of the mesh and summing yields:

$$\|\partial_{t,\mathcal{D}}(\varphi)^n\|^2_{L^2(\Omega)} + |\partial_{t,\mathcal{D}}(\varphi)^n|^2_{1,\mathcal{M}} \leq 4\bar{u}_0^3\;\|\partial_{t,\mathcal{D}}(u)^n\|_{L^2(\Omega)}\;\|\partial_{t,\mathcal{D}}(\varphi)^n\|_{L^2(\Omega)},$$

which, using Young's inequality, multiplying by δt and summing over the time steps yields the desired estimate for $\|\partial_{t,\mathcal{D}}(\varphi)\|_{L^2((0,T)\times\Omega)}$, thanks to (14). □

We are now in position to prove the following existence and convergence result.

THEOREM 4.2 Let $(u^{(m)})_{m\in\mathbb{N}}$ and $(\varphi^{(m)})_{m\in\mathbb{N}}$ be a sequence of solutions to (7) with a sequence of discretizations such that the space and time step, $h^{(m)}$ and $\delta t^{(m)}$ respectively, tends to zero. We suppose that the parameters $\theta_{\mathcal{M}^{(m)}}$ characterizing the regularity of the meshes of this sequence are bounded away from zero, *i.e.* $\theta_{\mathcal{M}^{(m)}} \geq \theta > 0,\ \forall\, m \in \mathbb{N}$. Then there exists a subsequence, still denoted by $(u^{(m)})_{m\in\mathbb{N}}$ and $(\varphi^{(m)})_{m\in\mathbb{N}}$ and two functions $\tilde{u}$ and $\tilde{\varphi}$ such that:

1. $u^{(m)}$ and $\varphi^{(m)}$ tends to $\tilde{u}$ and $\tilde{\varphi}$ respectively in $\mathrm{L}^2((0,T)\times\Omega)$,

2. $\tilde{u}$ and $\tilde{\varphi}$ are solution to the continuous problem (1) in the following weak sense:

$\tilde{u} \in \mathrm{L}^\infty((0,T)\times\Omega) \cap \mathrm{L}^2(0,T;\,\mathrm{H}^1_0(\Omega)),\ \tilde{\varphi} \in \mathrm{L}^\infty((0,T)\times\Omega) \cap \mathrm{L}^2(0,T;\,\mathrm{H}^1(\Omega))$ and:

$$\left|\begin{array}{ll} \displaystyle\int_{0,T}\int_\Omega \left[-\frac{\partial\psi}{\partial t}\,\tilde{u} + \nabla\tilde{u}\cdot\nabla\psi + (\tilde{u}^4 - \tilde{\varphi})\,\psi\right]\,\mathrm{d}x\,\mathrm{d}t = \int_\Omega \psi(x,0)\,u_0(x)\,\mathrm{d}x, & \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \forall\psi \in \mathrm{C}_c^\infty([0,T)\times\Omega), & \\ \displaystyle\int_{0,T}\int_\Omega \left[(\tilde{\varphi} - \tilde{u}^4)\,\psi + \nabla\tilde{\varphi}\cdot\nabla\psi\right]\,\mathrm{d}x\,\mathrm{d}t = 0, & \forall\psi \in \mathrm{C}_c^\infty([0,T)\times\bar{\Omega}). \end{array}\right. \tag{15}$$

Proof The following estimates of the space translates can be found in [4], Lemma 9.3, p. 770 and Lemma 18.3, p. 851:

$$\forall v \in \mathrm{H}_{\mathcal{M}},\ \forall \eta \in \mathbb{R}^d,$$

$$\|\hat{v}(\cdot+\eta)-\hat{v}(\cdot)\|^2_{L^2(\mathbb{R}^d)} \leq \|v\|^2_{1,\mathcal{M}}\ |\eta|\ \left[|\eta| + c(\Omega)\, h\right],$$

$$\|\hat{v}(\cdot+\eta)-\hat{v}(\cdot)\|^2_{L^2(\mathbb{R}^d)} \leq |\eta|\ \left[|v|^2_{1,\mathcal{M}}\ (|\eta| + 2\,h) + 2\,|\partial\Omega|\,\|v\|_{\mathrm{L}^\infty(\Omega)}\right],$$

where $\hat{v}$ stands for the extension by zero of v to $\mathbb{R}^d$ and the real number $c(\Omega)$ only depends on the domain.

For m given, let $\hat{u}^{(m)}$ and $\hat{\varphi}^{(m)}$ be the functions of $\mathrm{L}^\infty(\mathbb{R}^d\times\mathbb{R})$ obtained by extending $u^{(m)}$ and $\varphi^{(m)}$ by 0 from $\Omega\times[0,T)$ to $\mathbb{R}^d\times\mathbb{R}$. The estimates of Proposition 4.1 of $\|u^{(m)}\|_{\mathrm{L}^2(0,T;\,\mathrm{H}^1_{\mathcal{M}})}$ and $|\varphi^{(m)}|_{\mathrm{L}^2(0,T;\,\mathrm{H}^1_{\mathcal{M}})}$, together with the L^∞-bound for $\varphi^{(m)}$, thus allow to bound independently of m the space translates of $\hat{u}^{(m)}$ and $\hat{\varphi}^{(m)}$ in the $\mathrm{L}^2((0,T)\times\Omega)$-norm. In addition, Theorem A.2 applied with $\|\cdot\|_*$ equal to $\|\cdot\|_{1,\mathcal{M}}$ for $u^{(m)}$ and with $\|\cdot\|_*$ equal to $\|\cdot\|_{\mathrm{L}^2(\Omega)}$ for $\varphi^{(m)}$, together with the estimates of $\|u^{(m)}\|_{\mathrm{L}^2(0,T;\,\mathrm{H}^1_{\mathcal{M}})}$, $\|\partial_{t,\mathcal{D}}(u)^{(m)}\|_{\mathrm{L}^2(0,T;\,\mathrm{H}^{-1}_{\mathcal{M}})}$, $\|\varphi\|_{L^2((0,T)\times\Omega)}$ and $\|\partial_{t,\mathcal{D}}(\varphi)\|_{L^2((0,T)\times\Omega)}$ of Proposition 4.1 allows to bound the time translates of $\hat{u}^{(m)}$ and $\hat{\varphi}^{(m)}$, still independently of m and in the $\mathrm{L}^2((0,T)\times\Omega)$-norm. In addition, by Proposition 3.1, the sequences $(\hat{u}^{(m)})_{m\in\mathbb{N}}$ and $(\hat{\varphi}^{(m)})_{m\in\mathbb{N}}$ are uniformly bounded in $\mathrm{L}^\infty((0,T)\times\Omega)$, and thus in $\mathrm{L}^2((0,T)\times\Omega)$. By Kolmogorov theorem (see e.g. [4], Theorem 14.1, p. 833), these sequences are relatively compact and strongly converge in $\mathrm{L}^2((0,T)\times\Omega)$ to, respectively, $\tilde{u}$ and $\tilde{\varphi}$. Moreover, the uniform bounds of $\|u^{(m)}\|_{\mathrm{L}^2(0,T;\,\mathrm{H}^1_{\mathcal{M}})}$ and $|\varphi^{(m)}|_{\mathrm{L}^2(0,T;\,\mathrm{H}^1_{\mathcal{M}})}$ prove, by Theorem 14.2 and Theorem 14.3, p. 833 and p.834, in [4], that $\tilde{u}$ and $\tilde{\varphi}$ lie respectively in $\mathrm{L}^2(0,T;\,\mathrm{H}^1_0(\Omega))$ and $\mathrm{L}^2(0,T;\,\mathrm{H}^1(\Omega))$.

To prove that $\tilde{u}$ and $\tilde{\varphi}$ are solution to the continuous problem, it remains to prove that (15) holds. This proof is rather standard (see e.g. the proof of Theorem 18.1 pp. 858–862 in [4] for a similar, although more complicated, problem) and we only give here the main arguments. Let ψ be a function of $\mathrm{C}^\infty_c([0,T)\times\Omega)$. We define ψ^n_K by $\psi^n_K = \psi(x_K,t^n)$. Multiplying the first equation of (7) by $\delta t\,|K|\ \psi^{n+1}_K$ and summing up over the control volumes and the time steps, we get for any element of the sequence of discrete solutions:

$$\sum_{n=0}^{N-1}\sum_{K\in\mathcal{M}} \delta t\,|K|\ \psi^{n+1}_K \left[\frac{u^{n+1}_K - u^n_K}{\delta t} - (\Delta_{\mathcal{M},\mathrm{D}}(u^{n+1}))_K + (u^{n+1}_K)^4 - \varphi^n_K\right] = T_1 + T_2 + T_3 = 0,$$

where, for enhanced readability, the superscript $^{(m)}$ has been omitted and:

$$\left|\begin{array}{l} T_1 = \displaystyle\sum_{n=0}^{N-1} \sum_{K\in\mathcal{M}} |K|\ \psi_K^{n+1}\ \left[u_K^{n+1} - u_K^n\right], \\ T_2 = \displaystyle\sum_{n=0}^{N-1} \sum_{K\in\mathcal{M}} -\delta t\,|K|\ \psi_K^{n+1}\ (\Delta_{\mathcal{M},\mathrm{D}}(u^{n+1}))_K, \\ T_3 = \displaystyle\sum_{n=0}^{N-1} \sum_{K\in\mathcal{M}} \delta t\,|K|\ \psi_K^{n+1}\ \left[(u_K^{n+1})^4 - \varphi_K^n\right]. \end{array}\right.$$

Reordering the summations and using the fact that $\psi(\cdot,T)=0$, we get for T_1:

$$T_1 = -\sum_{K\in\mathcal{M}} |K|\ \psi_K^1\ u_K^0 + \sum_{n=1}^{N-1}\sum_{K\in\mathcal{M}} |K|\ u_K^n\ \left[\psi_K^n - \psi_K^{n+1}\right].$$

The first term of the right hand side reads:

$$T_{1,1} = -\int_\Omega u_0(x)\,\psi(x,0)\,\mathrm{d}x \quad + \sum_{K\in\mathcal{M}} \int_K (u_0(x) - u_K^0)\,\psi(x,0)\,\mathrm{d}x + \sum_{K\in\mathcal{M}} |K|\ u_K^0\ \underbrace{\left[\frac{1}{|K|}\int_K \psi(x,0)\,\mathrm{d}x - \psi(x_K,\delta t)\right]}_{R_\psi}.$$

On one hand, u^0 converges to u_0 in $\mathrm{L}^1\Omega$ and $\psi(\cdot,0)\in \mathrm{C}_c^\infty(\Omega)$, so the second term of $T_{1,1}$ tends to zero with h; on the other hand, since $u_K^0 \le \bar{u}_0$, $\forall K\in\mathcal{M}$ and, from the regularity of ψ, $|R_\psi| \le c_\psi\ (\delta t + h)$, the third term of $T_{1,1}$ also tends to zero with δt and h. Let us now turn to the second term in the expression of T_1:

$$T_{1,2} = \sum_{n=1}^{N-1}\sum_{K\in\mathcal{M}} |K|\ u_K^n\ \left[\psi_K^n - \psi_K^{n+1}\right] = -\int_{\delta t}^T \int_\Omega u(x,t)\ \frac{\partial\psi}{\partial t}(x,t)\,\mathrm{d}x\,\mathrm{d}t + \sum_{n=1}^{N-1}\sum_{K\in\mathcal{M}} \delta t\,|K|\ u_K^n\ (R_\psi)_K^n$$

with:

$$(R_\psi)_K^n = \frac{1}{\delta t\,|K|}\int_{t^n}^{t^{n+1}} \int_K \frac{\partial\psi}{\partial t}(x,t)\,\mathrm{d}x\,\mathrm{d}t - \frac{\psi(x_K,t^{n+1}) - \psi(x_K,t^n)}{\delta t},$$

and thus $|(R_\psi)_K^n| \le c_\psi\ (\delta t + h)$. Since $u_K^n \le \bar{u}_0$, $\forall K\in\mathcal{M}$ and $0\le n\le N$, we get:

$$T_1 \to -\int_\Omega u_0(x)\,\psi(x,0)\,\mathrm{d}x - \int_0^T\int_\Omega \tilde{u}(x,t)\ \frac{\partial\psi}{\partial t}(x,t)\,\mathrm{d}x\,\mathrm{d}t \qquad \text{as} \quad m\to\infty.$$

Reordering the summations in T_2 and using the fact that $\psi(\cdot,T)=0$, we obtain:

$$\begin{aligned} T_2 &= \sum_{n=0}^{N-2}\sum_{K\in\mathcal{M}} -\delta t\,|K|\ u_K^{n+1}\ (\Delta_{\mathcal{M},\mathrm{D}}(\psi^{n+1}))_K \\ &= -\int_{\delta t}^T\int_\Omega u(x,t)\ \Delta\psi(x,t)\,\mathrm{d}x\,\mathrm{d}t + \sum_{n=0}^{N-2}\sum_{K\in\mathcal{M}} -\delta t\,|K|\ u_K^{n+1} \sum_{\sigma\in\mathcal{E}(K)} |\sigma|\ (R_\psi)_\sigma^{n+1}, \end{aligned}$$

where the residual term $(R_\psi)_\sigma^{n+1}$ is the difference of the mean value of $\nabla\psi\cdot n$ over $\sigma\times(t^{n+1},t^{n+2})$ and its finite volume approximation. The fact that the second term in the right hand side of this relation tends to zero is thus a classical consequence of the control of $\|u\|_{\mathrm{L}^2(0,T;\,\mathrm{H}^1_{\mathcal{M}})}$ and the consistency of the diffusive fluxes $(R_\psi)_\sigma^{n+1}$ (see Theorem 9.1, pp. 772–776, in [4]) and yields, as $\tilde{u}$ is known to belong to $\mathrm{H}^1_0(\Omega)$:

$$T_2 \to \int_0^T \int_\Omega \nabla\tilde{u}(x,t)\cdot\nabla\psi(x,t)\,\mathrm{d}x\,\mathrm{d}t \qquad \text{as} \quad m\to\infty.$$

Finally, T_3 reads:

$$T_3 = \int_{\delta t}^T \int_\Omega \psi(x,t)\ \left[u(x,t)^4-\varphi(x,t-\delta t)\right] - \sum_{n=0}^{N-2}\sum_{K\in\mathcal{M}} \delta t\,|K|\ \left[(u_K^{n+1})^4-\varphi_K^n\right]\ (R_\psi)_K^{n+1}$$

with:

$$(R_\psi)_K^{n+1} = \frac{1}{\delta t\,|K|}\int_{t^{n+1}}^{t^{n+2}}\int_K \psi(x,t)\,\mathrm{d}x\,\mathrm{d}t - \psi(x_K,t^{n+1}).$$

The second term tends to zero by the L^∞-estimates for u and φ and the regularity of ψ. Since $u^{(m)}$ tends to $\tilde{u}$ in $\mathrm{L}^2((0,T)\times\Omega)$ and is bounded in $\mathrm{L}^\infty((0,T)\times\Omega)$, $u^{(m)}$ converges to $\tilde{u}$ in $\mathrm{L}^2(0,T;\,\mathrm{L}^p(\Omega))$, for any $p\in[1,+\infty)$; in addition, from the time translates estimates, $\varphi^{(m)}(\cdot,\cdot-\delta t)$ also converge to $\tilde{\varphi}$. We thus get:

$$T_3 \to \int_0^T\int_\Omega \psi(x,t)\ \left[\tilde{u}(x,t)^4-\tilde{\varphi}(x,t)\right]\,\mathrm{d}x\,\mathrm{d}t \qquad \text{as} \quad m\to\infty.$$

Gathering the results for T_1, T_2 and T_3, we obtain the first relation of (15). The second relation is obtained using the same arguments; the convergence of the diffusion term in case of Neumann boundary conditions poses an additional difficulty which is solved in Theorem 10.3, pp. 810–815, in [4]. □

The uniqueness of the solution to the problem under consideration is left beyond the scope of the present paper. Note however that such a result would imply, by a standard argument, the convergence of the whole sequence to the solution.

5 Conclusion

We propose in this paper a finite volume scheme for a problem capturing the essential difficulties of a simple radiative transfer model (the so-called $\mathbf{P}_1$ model), which enjoys the following properties: the discrete solution exists, is unique, and satisfies a discrete maximum principle; in addition, it converges (possibly up to the extraction of a subsequence) to a solution of the continuous problem, which yields, as a by-product, that such a solution indeed exists. For the proof of this latter result, we state and prove an abstract estimate allowing to bound the time translates of a finite volume discrete function, as a function of (possibly discrete) norms of the function

itself and of its discrete time derivative; although this estimate is underlying in some already available analysis (see chapter IV in [4], [5] or [6]), this formulation is new and should be useful to tackle new problems. Variants of the presented numerical scheme are now successfully running for the modelling of radiative transfer in the ISIS free software [8] developed at IRSN and devoted to the simulation of fires in confined buildings (see [1]), as nuclear power plants.

A Estimation of time translates

The objective of this appendix is to state and prove an abstract result allowing to bound the time translates of a discrete solution. We begin by a technical lemma.

LEMMA A.1 Let $(t^n)_{0\leq n\leq N}$ be such that $t^0 = 0$, $t^n = n\delta t$, $t^N = T$, τ be a positive real number and $\chi_\tau^n : \mathbb{R} \to \mathbb{R}$ be the function defined by $\chi_\tau^n(t) = 1$ if $t < t^n \leq t + \tau$ and $\chi_\tau^n(t) = 0$ otherwise. Then, for any family of real numbers $(\alpha_n)_{n=1,N}$ and, respectively, for any real number t, we have the following identities:

$$(i) \qquad \int_{\mathbb{R}} \left[\sum_{n=1}^{N} \alpha_n \chi_\tau^n(t)\right] \mathrm{d}t = \tau \sum_{n=1}^{N} \alpha_n,$$

$$(ii) \qquad \int_t^{t+\delta t} \left[\sum_{n=1}^{N} \chi_\tau^n(s)\right] \mathrm{d}s \leq \tau.$$

Proof The function $\chi_\tau^n(t)$ is equal to one for $t \in [t^n - \tau,\ t^n)$, so we have:

$$\int_{\mathbb{R}} \left[\sum_{n=1}^{N} \alpha_n \chi_\tau^n(t)\right] \mathrm{d}t = \sum_{n=1}^{N} \alpha_n \int_{t^n-\tau}^{t^n} \mathrm{d}t.$$

To obtain the inequality (ii), we remark that $t \in [t^n - \tau,\ t^n)$ is equivalent to $t - t^n \in [-\tau,\ 0)$ and so $\chi_\tau^n(t) = 1$ is equivalent to $\chi_\tau^0(t - t^n) = 1$ (under the assumption $t^0 = 0$). We thus have:

$$\int_t^{t+\delta t} \left[\sum_{n=1}^{N} \chi_\tau^n(s)\right] \mathrm{d}s = \sum_{n=1}^{N} \int_{t-t^n}^{t-t^n+\delta t} \chi_\tau^0(s)\, \mathrm{d}s \leq \int_{\mathbb{R}} \chi_\tau^0(s)\, \mathrm{d}s = \tau.$$

□

We now introduce some notations. Let $\mathrm{H}_{\mathcal{M}}(\Omega)$ and $\mathrm{H}_{\mathcal{D}}$ be the discrete functional spaces introduced in section 2 and 4 respectively. We suppose given a norm $\|\cdot\|_*$ on $\mathrm{H}_{\mathcal{M}}(\Omega)$, over which we also define the dual norm $\|\cdot\|^*$ with respect to the L^2-inner product:

$$\forall u \in \mathrm{H}_{\mathcal{M}}(\Omega), \qquad \|u\|^* := \sup_{v\in \mathrm{H}_{\mathcal{M}}(\Omega),\, v\neq 0} \frac{\displaystyle\int_\Omega u\, v\, \mathrm{d}x}{\|v\|_*}.$$

These two spatial norms may be associated to a corresponding norm on $\mathrm{H}_{\mathcal{D}}$ as follows:

$$\forall u \in \mathrm{H}_{\mathcal{D}},\ u = (u^n)_{0\leq n\leq N}, \qquad \|u\|^2_{\mathrm{L}^2(0,T;\,\mathrm{H}_{\mathcal{M},*})} = \sum_{n=0}^{N} \delta t\ \|u^n\|^2_*,$$

$$\text{and} \qquad \|u\|^2_{\mathrm{L}^2(0,T;\,\mathrm{H}^*_{\mathcal{M}})} = \sum_{n=0}^{N-1} \delta t\ \|u^n\|^{*2}.$$

We are now in position to state the following result.

THEOREM A.2 Let u be a function of $\mathrm{H}_{\mathcal{D}}$ and τ a real number. We denote by $\hat{u}$ the extension by zero of u to $\mathbb{R}^d \times \mathbb{R}$. Then we have:

$$\|\hat{u}(\cdot,\cdot+\tau) - \hat{u}(\cdot,\cdot)\|^2_{\mathrm{L}^2(\mathbb{R}^d\times\mathbb{R})} \leq \tau\ \Big[2\ \|u\|^2_{\mathrm{L}^2(0,T;\,\mathrm{H}_{\mathcal{M},*})} + \frac{1}{2}\ \|\partial_{t,\mathcal{D}}(u)\|^2_{\mathrm{L}^2(0,T;\,\mathrm{H}^*_{\mathcal{M}})} + 2\ \|u\|^2_{\mathrm{L}^\infty(0,T;\,\mathrm{L}^2(\Omega))}\Big].$$

Proof Let u be a function of $\mathrm{H}_{\mathcal{D}}$ and $t \in \mathbb{R}$. Let τ be a real number that we suppose positive. The following identity holds:

$$\hat{u}(\cdot,t+\tau) - \hat{u}(\cdot,t) = \chi^0_\tau(t)\ u^0 + \sum_{n=1}^{N-1} \chi^n_\tau(t)\left[u^n - u^{n-1}\right] - \chi^N_\tau(t)\ u^{N-1}.$$

For $s \in \mathbb{R}$ we define $n(s)$ by: $n(s) = -1$ if $s < 0$, $n(s)$ is the index such that $t^{n(s)} \leq s < t^{n(s)+1}$ for $0 \leq s < t^N$, $n(s) = N+1$ for $t \geq t^N$. Let $n_0(t)$ and $n_1(t)$ be given by $n_0(t) = n(t)$, $n_1(t) = n(t+\tau)$. We adopt the convention $u^{-1} = u^N = 0$. With this notation, we have for $u(\cdot,t+\tau) - u(\cdot,t)$ the following equivalent expression:

$$u(\cdot,t+\tau) - u(\cdot,t) = u^{n_1(t)} - u^{n_0(t)},$$

and thus:

$$\int_\Omega \left[u(x,t+\tau) - u(x,t)\right]^2\,\mathrm{d}x = \int_\Omega \left[u^{n_1(t)} - u^{n_0(t)}\right]\ \left[\chi^0_\tau(t)\ u^0 + \sum_{n=1}^{N-1} \chi^n_\tau(t)\left[u^n - u^{n-1}\right] - \chi^N_\tau(t)\ u^{N-1}\right]\mathrm{d}x.$$

Developping, we get:

$$\int_\Omega \left[u(x,t+\tau) - u(x,t)\right]^2\,\mathrm{d}x = T_1(t) + T_2(t) + T_3(t)$$

with:

$$\left|\begin{array}{l} T_1(t) = \chi^0_\tau(t) \displaystyle\int_\Omega \left[u^{n_1(t)} - u^{n_0(t)}\right]\ u^0\,\mathrm{d}x, \\ T_2(t) = \displaystyle\sum_{n=1}^{N-1} \chi^n_\tau(t) \int_\Omega \left[u^{n_1(t)} - u^{n_0(t)}\right]\left[u^n - u^{n-1}\right]\,\mathrm{d}x, \\ T_3(t) = -\chi^N_\tau(t) \displaystyle\int_\Omega \left[u^{n_1(t)} - u^{n_0(t)}\right]\ u^{N-1}\,\mathrm{d}x. \end{array}\right.$$

We first estimate the integral of $T_1(t)$ over $\mathbb{R}$. Since $\chi_\tau^0(t)$ is equal to 1 in the interval $[-\tau, 0)$ and 0 elsewhere, and since $u^{n_0(t)} = 0$ for any negative t, we get:

$$\int_{\mathbb{R}} T_1(t)\,\mathrm{d}t = \int_{-\tau}^{0}\int_{\Omega} u^{n_1(t)}\ u^0\,\mathrm{d}x\,\mathrm{d}t \leq \tau\ \|u\|^2_{\mathrm{L}^\infty(0,T;\,\mathrm{L}^2(\Omega))}.$$

By the same arguments, we get the same bound for the integral of $T_3(t)$:

$$\int_{\mathbb{R}} T_3(t)\,\mathrm{d}t \leq \tau\ \|u\|^2_{\mathrm{L}^\infty(0,T;\,\mathrm{L}^2(\Omega))}.$$

From the definition of the $\|\cdot\|^*$ norm, we get:

$$T_2(t) \leq \delta t \sum_{n=1}^{N-1} \chi_\tau^n(t) \quad \|\partial_{t,\mathcal{D}}(u)^{n-1}\|^*\ \|u^{n_1(t)} - u^{n_0(t)}\|_*,$$

and thus, by Young's inequality:

$$T_2(t) \leq \delta t \sum_{n=1}^{N-1} \chi_\tau^n(t) \quad \left[\frac{1}{2}\ \|\partial_{t,\mathcal{D}}(u)^{n-1}\|^{*\,2} + \|u^{n_0(t)}\|_*^2 + \|u^{n_1(t)}\|_*^2\right].$$

Integrating over the time, we get:

$$\int_{\mathbb{R}} T_2(t)\,\mathrm{d}t \leq T_{2,1} + T_{2,2} + T_{3,3},$$

where the term $T_{2,1}$ reads and, by Lemma A.1 (Relation (i)), satisfies:

$$\begin{aligned} T_{2,1} = \frac{\delta t}{2}\int_{\mathbb{R}} \sum_{n=1}^{N-1} \chi_\tau^n(t) \quad \|\partial_{t,\mathcal{D}}(u)^{n-1}\|^{*\,2}\,\mathrm{d}t \ &= \frac{\tau}{2}\sum_{n=0}^{N-2} \delta t\ \|\partial_{t,\mathcal{D}}(u)^{n}\|^{*\,2} \\ &\leq \frac{\tau}{2}\ \|\partial_{t,\mathcal{D}}(u)\|^2_{\mathrm{L}^2(0,T;\,\mathrm{H}^*_{\mathcal{M}})}. \end{aligned}$$

Since $u^{n_0(t)} = u^m$ for $t^m \leq t < t^{m+1}$, the term $T_{2,2}$ reads and satisfies, once again by Lemma A.1 (Relation (ii)):

$$T_{2,2} = \delta t \sum_{m=0}^{N-1}\left[\int_{t^m}^{t^{m+1}} \sum_{n=1}^{N-1} \chi_\tau^n(t)\,\mathrm{d}t\right]\|u^m\|_*^2 \leq \tau \sum_{m=0}^{N-1} \delta t\ \|u^m\|_*^2 = \tau\ \|u\|^2_{\mathrm{L}^2(0,T;\,\mathrm{H}_{\mathcal{M},*})}.$$

Finally, $u^{n_1(t)} = u^m$ for $t^m - \tau \leq t < t^{m+1} - \tau$, and thus, by the same argument:

$$T_{2,3} = \delta t \sum_{m=0}^{N-1}\left[\int_{t^m-\tau}^{t^{m+1}-\tau} \sum_{n=1}^{N-1} \chi_\tau^n(t)\,\mathrm{d}t\right]\|u^m\|_*^2 \leq \tau \sum_{m=0}^{N-1} \delta t\ \|u^m\|_*^2 = \tau\ \|u\|^2_{\mathrm{L}^2(0,T;\,\mathrm{H}_{\mathcal{M},*})}.$$

This concludes the proof for positive τ. The case of negative τ follows by symmetry. □

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