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SEMICLASSICAL MEASURES FOR THE SCHRÖDINGER EQUATION ON THE TORUS

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Abstract. In this article, the structure of semiclassical measures for solutions to the linear Schrödinger equation on the torus is analysed. We show that the disintegration of such a measure on every invariant lagrangian torus is absolutely continuous with respect to the Lebesgue measure. We obtain an expression of the Radon-Nikodym derivative in terms of the sequence of initial data and show that it satisfies an explicit propagation law. As a consequence, we also prove an observability inequality, saying that the L^2-norm of a solution on any open subset of the torus controls the full L^2-norm.

1. Introduction

Consider the torus $\mathbb{T}^d := (\mathbb{R}/2\pi\mathbb{Z})^d$ equipped with the standard flat metric. We denote by Δ the associated Laplacian. We are interested in understanding dynamical properties related to propagation of singularities by the (time-dependent) linear Schrödinger equation

$$i\frac{\partial u}{\partial t}(t,x) = \left(-\frac{1}{2}\Delta + V(t,x)\right) u(t,x), \qquad u\rceil_{t=0} = u_0 \in L^2(\mathbb{T}^d).$$

More precisely, given a sequence of initial conditions $u_n \in L^2(\mathbb{T}^d)$, we shall investigate the regularity properties of the *Wigner distributions* and *semiclassical measures* associated with $u_n(t,x)$. These describe how the L^2-norm is distributed in the cotangent bundle $T^*\mathbb{T}^d = \mathbb{T}^d \times \mathbb{R}^d$ (position $\times$ frequency). Our main results, Theorems 1 and 3 below, provide a description of the regularity properties and, more generally, the global structure of semiclassical measures associated to sequences of solutions to the Schrödinger equation.

These results are aimed to give a description of the high-frequency behavior of the linear Schrödinger flow. This aspect of the dynamics is particularly relevant in the study of the quantum-classical correspondence principle, but is also related to other dynamical properties such as dispersion and unique continuation (see the discussion below and the articles [19, 21, 3] for a more precise account and detailed references on these issues). As a corollary of Theorem 3, we prove an observability inequality on any open subset of the torus, for the Schrödinger equation with a time-independent potential : Theorem 4.

We assume the following regularity condition on the potential $V \in L^\infty\left(\mathbb{R} \times \mathbb{T}^d\right)$:

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(R) For every $T > 0$, for every $\epsilon > 0$, there exists a *compact* set $K_\epsilon \subset [0,T] \times \mathbb{T}^d$, of Lebesgue measure $< \epsilon$, and $V_\epsilon \in C([0,T] \times \mathbb{T}^d)$, such that $|V - V_\epsilon| \leq \epsilon$ on $\big([0,T] \times \mathbb{T}^d\big) \setminus K_\epsilon$.

We believe that this assumption should not be necessary. In any case, assumption (R) already covers a broad class of examples.

We shall focus on the propagator starting at time 0, denoted by $U_V(t)$; *i.e.* $u(t) = U_V(t)u_0$.

Let us define the notion of Wigner distribution. We will use the semiclassical point of view, and denote by (u_h) our family of initial conditions, where $h > 0$ is a real parameter going to 0. The parameter h acts as a scaling factor on the frequencies, and the limit $h \longrightarrow 0^+$ corresponds to the high-frequency regime. We will always assume that the functions u_h are normalized in $L^2(\mathbb{T}^d)$. The *Wigner distribution* associated to u_h (at scale h) is a distribution on the cotangent bundle $T^*\mathbb{T}^d$, defined by

$$\int_{T^*\mathbb{T}^d} a(x,\xi) w^h_{u_h}(dx,d\xi) = \langle u_h, \mathrm{Op}_h(a) u_h \rangle_{L^2(\mathbb{T}^d)}, \qquad \text{for all } a \in C_c^\infty(T^*\mathbb{T}^d),$$

where $\mathrm{Op}_h(a)$ is the operator on $L^2(\mathbb{T}^d)$ associated to a by the Weyl quantization (Section 8). More explicitly, we have

$$\int_{T^*\mathbb{T}^d} a(x,\xi) w^h_{u_h}(dx,d\xi) = \frac{1}{(2\pi)^{d/2}} \sum_{k,j \in \mathbb{Z}^d} \hat{u}_h(k) \overline{\hat{u}_h(j)} \hat{a}_{j-k}\left(\frac{h}{2}(k+j)\right),$$

where $\hat{u}_h(k) := \int_{\mathbb{T}^d} u_h(x) \frac{e^{-ik.x}}{(2\pi)^{d/2}} dx$ and $\hat{a}_k(\xi) := \int_{\mathbb{T}^d} a(x,\xi) \frac{e^{-ik.x}}{(2\pi)^{d/2}} dx$ denote the respective Fourier coefficients of u_h and a, with respect to the variable $x \in \mathbb{T}^d$. We note that, if a is a function on $T^*\mathbb{T}^d = \mathbb{T}^d \times \mathbb{R}^d$ that depends only on the first coordinate, then

$$\int_{T^*\mathbb{T}^d} a(x) w^h_{u_h}(dx,d\xi) = \int_{\mathbb{T}^d} a(x) |u_h(x)|^2 dx. \tag{1}$$

The main object of our study will be the Wigner distributions $w^h_{U_V(t)u_h}$. When no confusion arises, we will more simply denote them by $w_h(t,\cdot)$. By standard estimates on the norm of $\mathrm{Op}_h(a)$ (the Calderón-Vaillancourt theorem, section 8), $t \mapsto w_h(t,\cdot)$ belongs to $L^\infty(\mathbb{R}; \mathcal{D}'\left(T^*\mathbb{T}^d\right))$, and is uniformly bounded in that space as $h \longrightarrow 0^+$. Thus, one can extract subsequences that converge in the weak-$*$ topology on $L^\infty(\mathbb{R}; \mathcal{D}'\left(T^*\mathbb{T}^d\right))$. In other words, after possibly extracting a subsequence, we have

$$\int_{\mathbb{R}} \varphi(t) a(x,\xi) w_h(t,dx,d\xi) dt \underset{h \longrightarrow 0}{\longrightarrow} \int_{\mathbb{R}} \varphi(t) a(x,\xi) \mu(t,dx,d\xi) dt$$

for all $\varphi \in L^1(\mathbb{R})$ and $a \in C_c^\infty(T^*\mathbb{T}^d)$. It also follows from standard properties of the Weyl quantization that the limit μ has the following properties :

- $\mu \in L^\infty(\mathbb{R}; \mathcal{M}_+(T^*\mathbb{T}^d))$, meaning that for almost all t, $\mu(t,\cdot)$ is a positive measure on $T^*\mathbb{T}^d$.
- The unitary character of $U_V(t)$ implies that $\int_{T^*\mathbb{T}^d} \mu(t,dx,d\xi)$ does not depend on t; from the normalization of u_h, we have $\int_{T^*\mathbb{T}^d} \mu(t,dx,d\xi) \leq 1$, the inequality coming

from the fact that $T^*\mathbb{T}^d$ is not compact, and that there may be an escape of mass to infinity.

- Define the geodesic flow $\phi_\tau : T^*\mathbb{T}^d \longrightarrow T^*\mathbb{T}^d$ by $\phi_\tau(x,\xi) := (x+\tau\xi,\xi)$ $(\tau \in \mathbb{R})$. The Weyl quantization enjoys the following property :

$$\left[-\frac{1}{2}\Delta, \mathrm{Op}_h(a)\right] = \frac{1}{ih}\mathrm{Op}_h(\xi\cdot\partial_x a). \tag{2}$$

This implies that $\mu(t,\cdot)$ is invariant under ϕ_τ, for almost all t and all $\tau \in \mathbb{R}$ (the argument is recalled in Lemma 11).

We refer to [19] for details. We can now state our first main result, which deals with the regularity properties of the measures μ.

Theorem 1. *(i) Let μ be a weak-$*$ limit of the family w_h. Then, for almost all t, $\int_{\mathbb{R}^d}\mu(t,\cdot,d\xi)$ is an absolutely continuous measure on $\mathbb{T}^d$.*

(ii) In fact, the following stronger statement holds. Let $\bar\mu$ be the measure on $\mathbb{R}^d$ image of $\mu(t,\cdot)$ under the projection map $(x,\xi)\mapsto\xi$. Then $\bar\mu$ does not depend on t.

For every bounded measurable function f, and every L^1-function $\theta(t)$ write

$$\int_{\mathbb{R}}\int_{\mathbb{T}^d\times\mathbb{R}^d} f(x,\xi)\mu(t,dx,d\xi)\theta(t)dt = \int_{\mathbb{R}}\int_{\mathbb{R}^d}\left(\int_{\mathbb{T}^d} f(x,\xi)\mu_\xi(t,dx)\right)\bar\mu(d\xi)\theta(t)dt,$$

where $\mu_\xi(t,\cdot)$ is the disintegration[1] of $\mu(t,\cdot)$ with respect to the variable ξ. Then for $\bar\mu$-almost every ξ, the measure $\mu_\xi(t,\cdot)$ is absolutely continuous.

The first assertion in Theorem 1 may be restated in a simpler, concise way.

Corollary 2. *Let (u_n) be a sequence in $L^2(\mathbb{T}^d)$, such that $\|u_n\|_{L^2(\mathbb{T}^d)}=1$ for all n. Consider the sequence of probability measures ν_n on $\mathbb{T}^d$, defined by*

$$\nu_n(dx) = \left(\int_0^1 |U_V(t)u_n(x)|^2 dt\right)dx. \tag{3}$$

Let ν be any weak-$*$ limit of the sequence (ν_n) : then ν is absolutely continuous.

Our next result enlightens the structure of the set of semiclassical measures arising as weak-$*$ limits of sequences (w_h). It gives a description of the Radon-Nikodym derivatives of the measures $\int_{\mathbb{R}^d}\mu(t,\cdot,d\xi)$ and clarifies the link between $\mu(0,\cdot)$ and $\mu(t,\cdot)$. It was already noted in [19] (in the case $V=0$) that the dependence of $\mu(t,\cdot)$ on the sequence of initial conditions is a subtle issue : although $w_h(0,\cdot) = w^h_{u_h}$ completely determines $w_h(t,\cdot) = w^h_{U_V(t)u_h}$ for all t, it is not true that the weak-$*$ limits of $w_h(0,\cdot)$ determine $\mu(t,\cdot)$ for all t. In [19], one can find examples of two sequences (u_h) and (v_h) of initial conditions, such that $w^h_{u_h}$ and $w^h_{v_h}$ have the same limit in $\mathcal{D}'(T^*\mathbb{T}^d)$, but $w^h_{U_V(t)u_h}$ and $w^h_{U_V(t)v_h}$ have different limits in $L^\infty(\mathbb{R};\mathcal{D}'(T^*\mathbb{T}^d))$.

In order to state Theorem 3, we must introduce some notation. We call a submodule $\Lambda\subset\mathbb{Z}^d$ primitive if $\langle\Lambda\rangle\cap\mathbb{Z}^d=\Lambda$ (where $\langle\Lambda\rangle$ denotes the linear subspace of $\mathbb{R}^d$ spanned by

[1]When $\mu(t,\cdot)$ is a probability measure, $\mu_\xi(t,\cdot)$ is the conditional law of x knowing ξ, when the pair (x,ξ) is distributed according to $\mu(t,\cdot)$

Λ). If b is a function on $\mathbb{T}^d$, let $\widehat{b}_k$, $k \in \mathbb{Z}^d$, denote the Fourier coefficients of b. If $\widehat{b}_k = 0$ for $k \notin \Lambda$, we will say that b has only Fourier modes in Λ. This means that b is constant in the directions orthogonal to $\langle \Lambda \rangle$. Let $L^p_\Lambda(\mathbb{T}^d)$ denote the subspace of $L^p\left(\mathbb{T}^d\right)$ consisting of functions with Fourier modes in Λ. If $b \in L^2(\mathbb{T}^d)$, we denote by $\langle b \rangle_\Lambda$ its orthogonal projection onto $L^2_\Lambda(\mathbb{T}^d)$, in other words, the average of b along $\Lambda^\perp$:

$$\langle b \rangle_\Lambda (x) := \sum_{k \in \Lambda} \widehat{b}_k (t) \frac{e^{ik\cdot x}}{(2\pi)^{d/2}}.$$

Given $b \in L^\infty_\Lambda\left(\mathbb{T}^d\right)$, we will denote by m_b the multiplication operator by b, acting on $L^2_\Lambda(\mathbb{T}^d)$.

Finally, we denote by $U_{\langle V \rangle_\Lambda}(t)$ the unitary propagator of the equation

$$i\frac{\partial v}{\partial t}(t,x) = \left(-\frac{1}{2}\Delta + \langle V \rangle_\Lambda(t,x)\right) v(t,x), \qquad v\rceil_{t=0} \in L^2_\Lambda(\mathbb{T}^d).$$

Theorem 3. *For any sequence (u_h), we can extract a subsequence such that the following hold :*

- *the subsequence $w_h(t,\cdot)$ converges weakly-$*$ to a limit $\mu(t,\cdot)$;*
- *for each primitive submodule $\Lambda \subset \mathbb{Z}^d$, we can build from the sequence of initial conditions (u_h) a nonnegative trace class operator σ_Λ, acting on $L^2_\Lambda(\mathbb{T}^d)$;*[2]
- *for almost all t, we have*

$$\int_{\mathbb{R}^d} \mu(t,\cdot,d\xi) = \sum_\Lambda \nu_\Lambda(t,\cdot),$$

where $\nu_\Lambda(t,\cdot)$ is the measure on $\mathbb{T}^d$, whose non-vanishing Fourier modes correspond to frequencies in Λ, defined by

$$\int_{\mathbb{T}^d} b(x)\nu_\Lambda(t,dx) = \mathrm{Tr}\left(m_{\langle b \rangle_\Lambda} U_{\langle V \rangle_\Lambda}(t)\, \sigma_\Lambda\, U_{\langle V \rangle_\Lambda}(t)^*\right),$$

if $b \in L^\infty\left(\mathbb{T}^d\right)$.

Theorem 3 tells us more about the dependence of $\mu(t,\cdot)$ with respect to t. If two sequences of initial conditions (u_h) and (v_h) give rise to the same family of operators σ_Λ, then they also give rise to the same limit $\mu(t,\cdot)$. There are cases in which the measures ν_Λ can be determined from the semiclassical measure $\mu(0,\cdot)$ of the sequence of initial data : in Corollary 30 in Section 6 we show that if $\mu(0, \mathbb{T}^d \times \Lambda^\perp) = 0$ then ν_Λ vanishes identically.

Technically speaking, the operators σ_Λ are built in terms of 2-microlocal semiclassical measures, that describe how the sequences (u_h) concentrate along certain coisotropic manifolds in phase-space. The technical construction of σ_Λ will only be achieved at the end of Section 5.

We shall prove, as a consequence of Theorem 3, the following result:

[2]This means that the integral kernel of σ_Λ is constant in the directions orthogonal to Λ.

Theorem 4. *Suppose $V \in L^\infty(\mathbb{T}^d)$ does not depend on time and satisfies condition (R). Then for every open set $\omega \subset \mathbb{T}^d$ and every $T > 0$ there exists a constant $C = C(T, \omega) > 0$ such that:*

$$\|u_0\|^2_{L^2(\mathbb{T}^d)} \leq C \int_0^T \|U_V(t)\, u_0\|^2_{L^2(\omega)}\, dt, \tag{4}$$

for every initial datum $u_0 \in L^2\left(\mathbb{T}^d\right)$.

Note that this result implies the unique continuation property for the Schrödinger propagator U_V from any open set $(0,T) \times \omega$. In other words, if $U_V(t)\, u_0 = 0$ on ω for all $t \in [0,T]$, then $u_0 = 0$. Estimate (4) is usually known as an *observability inequality*; these type of estimates are especially relevant in Control Theory (see [18]).

As a consequence of this result, with the notation of Theorem 1 (ii), we deduce the following :

Corollary 5. *For $\bar{\mu}$-almost every ξ, we have*

$$\int_0^T \mu_\xi(t,\omega)dt \geq \frac{T}{C(T,\omega)}.$$

This lower bound is uniform w.r.t. the initial data u_h and to ξ.

Relations to other work. In the case $V = 0$, Corollary 2 and the first assertion in Theorem 1 have been obtained by Zygmund [28] in the case $d = 1$. In the final remark of [5], Bourgain indicates a proof in arbitrary dimension, using fine properties of the distribution of lattice points on paraboloids. When the sequence (u_n) consists of eigenfunctions of Δ ($\nu_n(dx) = |u_n(x)|^2 dx$, in that case), the conclusion of Corollary 2 was proved by Zygmund ($d = 2$), Bourgain (no restriction on d) and precised in terms of regularity by Jakobson in [17], by studying the distribution of lattice points on ellipsoids. More results on the regularity of μ can be found in [1, 8, 25, 24].

Our methods are very different, and there is no obvious adaptation of the technique of [5, 17] to the case $V \neq 0$. Theorem 3 was proved in dimension $d = 2$ for $V = 0$ in [20] using semiclassical methods, and we develop and refine the ideas therein. We use in a decisive way the dynamics of the geodesic flow (since we are on a flat torus, the geodesic flow is a completely explicit object), and we use the decomposition of the momentum space into resonant vectors of various orders. The other main ingredient is the two-microlocal calculus, in the spirit of the developments by Nier [26] and Fermanian-Kammerer [10, 11], and also [23, 12]. Our proof is written on the "square" torus. More precisely, the property of the lattice $\Gamma = \mathbb{Z}^d \subset \mathbb{R}^d$ and of the scalar product $\langle \cdot, \cdot \rangle$ (principal symbol of the laplacian) that we use is that $[\langle x, y\rangle \in \mathbb{Q}\ \forall y \in \mathbb{Q}\Gamma \Leftrightarrow x \in \mathbb{Q}\Gamma]$. This assumption can be removed and the results can be adapted to more general lattices, but this requires a slightly different presentation, that will appear in the work [2]. Moreover, it seems reasonable to think that Theorems 1 and 3 can be extended to more general completely integrable systems and their quantizations [2]. The generalized statement would be that the disintegration of the limit

measure on regular lagrangian tori is absolutely continuous, with respect to the Lebesgue measure on these tori.

Theorem 4 was first established by Jaffard [16] in the case $V = 0$ using techniques based on the theory of lacunary Fourier series developed by Kahane. Since then, several proofs of this result based on microlocal methods and semiclassical measures (still for $V = 0$) are available [6, 22, 21]. Our proof of Theorem 4 will follow the lines of that given in [21] and is based on the structure and propagation result for semiclassical measures obtained in Theorem 3. At the same time as this paper was being written, Burq and Zworski [7] have given a proof of Theorem 4 in the case $V \in C(\mathbb{T}^2)$, which is an adaptation of their previous work [6]. Here, we exploit our results about the structure of semiclassical measures to avoid the semiclassical normal form argument (Burq and Zworski's Propositions 2.5 and 2.10) and to lower the regularity of the potential.

Corollary 5 implies Corollary 4 of the article by Wunsch [27] (which is expressed in terms of wavefront sets) and holds in arbitrary dimension whereas Wunsch's method is restricted to $d = 2$.

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2. Decomposition of an invariant measure on the torus

Before we start our construction in §3, we recall a few basic facts on the geodesic flow and its invariant measures.

Denote by $\mathcal{L}$ the family of all submodules Λ of $\mathbb{Z}^d$ which are *primitive*, in the sense that $\langle\Lambda\rangle \cap \mathbb{Z}^d = \Lambda$ (where $\langle\Lambda\rangle$ denotes the linear subspace of $\mathbb{R}^d$ spanned by Λ). For each $\Lambda \in \mathcal{L}$, we define

$$\Lambda^\perp := \left\{\xi \in \mathbb{R}^d : \xi \cdot k = 0, \quad \forall k \in \Lambda\right\},$$
$$\mathbb{T}_\Lambda := \langle\Lambda\rangle / 2\pi\Lambda.$$

Note that $\mathbb{T}_\Lambda$ is a submanifold of $\mathbb{T}^d$ diffeomorphic to a torus of dimension $\operatorname{rk}\Lambda$. Its cotangent bundle $T^*\mathbb{T}_\Lambda$ is $\mathbb{T}_\Lambda \times \langle\Lambda\rangle$. We shall use the notation $\mathbb{T}_{\Lambda^\perp}$ to refer to the torus $\Lambda^\perp / \left(2\pi\mathbb{Z}^d \cap \Lambda^\perp\right)$. Denote by $\Omega_j \subset \mathbb{R}^d$, for $j = 0, ..., d$, the set of resonant vectors of order exactly j, that is:

$$\Omega_j := \left\{\xi \in \mathbb{R}^d : \operatorname{rk}\Lambda_\xi = d - j\right\},$$

where $\Lambda_\xi := \left\{k \in \mathbb{Z}^d : k \cdot \xi = 0\right\}$. Note that the sets Ω_j form a partition of $\mathbb{R}^d$, and that $\Omega_0 = \{0\}$; more generally, $\xi \in \Omega_j$ if and only if the geodesic issued from any $x \in \mathbb{T}^d$ in the direction ξ is dense in a subtorus of $\mathbb{T}^d$ of dimension j. The set $\Omega := \bigcup_{j=0}^{d-1} \Omega_j$ is usually called the set of *resonant* directions, whereas $\Omega_d = \mathbb{R}^d \setminus \Omega$ is referred to as the set of *non-resonant* vectors. Finally, write

$$R_\Lambda := \Lambda^\perp \cap \Omega_{d - \operatorname{rk}\Lambda}.$$

The relevance of these definitions to the study of the geodesic flow is explained by the following remark. Saying that $\xi \in R_\Lambda$ is equivalent to saying that (for any $x_0 \in \mathbb{T}^d$) the

time-average $\frac{1}{T}\int_0^T \delta_{x_0+t\xi}(x)\,dt$ converges weakly to the Haar measure on the torus $x_0+\mathbb{T}_{\Lambda^\perp}$, as $T\to\infty$.

By construction, for $\xi\in R_\Lambda$ we have $\Lambda_\xi=\Lambda$; moreover, if $\operatorname{rk}\Lambda=d-1$ then $R_\Lambda=\Lambda^\perp\backslash\{0\}$. Finally,

$$\mathbb{R}^d=\bigsqcup_{\Lambda\in\mathcal{L}} R_\Lambda, \tag{5}$$

that is, the sets R_Λ form a partition of $\mathbb{R}^d$. As a consequence, the following result holds.

Lemma 6. *Let μ be a finite, positive Radon measure[3] on $T^*\mathbb{T}^d$. Then μ decomposes as a sum of positive measures:*

$$\mu=\sum_{\Lambda\in\mathcal{L}}\mu\rceil_{\mathbb{T}^d\times R_\Lambda}. \tag{6}$$

Given any $\mu\in\mathcal{M}_+\left(T^*\mathbb{T}^d\right)$ we define the Fourier coefficients of μ as the complex measures on $\mathbb{R}^d$:

$$\widehat{\mu}(k,\cdot):=\int_{\mathbb{T}^d}\frac{e^{-ik\cdot x}}{(2\pi)^{d/2}}\mu(dx,\cdot),\quad k\in\mathbb{Z}^d.$$

One has, in the sense of distributions,

$$\mu(x,\xi)=\sum_{k\in\mathbb{Z}^d}\widehat{\mu}(k,\xi)\frac{e^{ik\cdot x}}{(2\pi)^{d/2}}.$$

Lemma 7. *Let $\mu\in\mathcal{M}_+\left(T^*\mathbb{T}^d\right)$ and $\Lambda\in\mathcal{L}$. The distribution:*

$$\langle\mu\rangle_\Lambda(x,\xi):=\sum_{k\in\Lambda}\widehat{\mu}(k,\xi)\frac{e^{ik\cdot x}}{(2\pi)^{d/2}}$$

is a finite, positive Radon measure on $T^*\mathbb{T}^d$.

Proof. Let $a\in C_c^\infty\left(T^*\mathbb{T}^d\right)$ and $\{v_1,...,v_n\}$ be a basis of $\Lambda^\perp$. Suppose

$$a(x,\xi)=\sum_{k\in\mathbb{Z}^d}\widehat{a}(k,\xi)\frac{e^{ik\cdot x}}{(2\pi)^{d/2}};$$

then it is not difficult to see that

$$\begin{aligned}\langle a\rangle_\Lambda(x,\xi)&:=\lim_{T_1,...,T_n\to\infty}\frac{1}{T_1...T_n}\int_0^{T_1}...\int_0^{T_n}a\left(x+\sum_{j=1}^n t_jv_j,\xi\right)dt_1...dt_n\\&=\sum_{k\in\Lambda}\widehat{a}(k,\xi)\frac{e^{ik\cdot x}}{(2\pi)^{d/2}},\end{aligned}$$

[3]We denote by $\mathcal{M}_+\left(T^*\mathbb{T}^d\right)$ the set of all such measures.

that $\langle a\rangle_\Lambda$ is non-negative as soon as a is, $\|\langle a\rangle_\Lambda\|_{L^\infty\left(T^*\mathbb{T}^d\right)} \leq \|a\|_{L^\infty\left(T^*\mathbb{T}^d\right)}$, and that $\langle a\rangle_\Lambda \in C_c^\infty\left(T^*\mathbb{T}^d\right)$ as well. Therefore,

$$\left\langle \langle\mu\rangle_\Lambda, a\right\rangle = \int_{T^*\mathbb{T}^d} \langle a\rangle_\Lambda (x,\xi)\, \mu\,(dx, d\xi)$$

defines a positive distribution, which is a positive Radon measure by Schwartz's theorem. □

Recall that a measure $\mu \in \mathcal{M}_+\left(T^*\mathbb{T}^d\right)$ is invariant under the action of the geodesic flow[4] on $T^*\mathbb{T}^d$ whenever:

$$(\phi_\tau)_*\,\mu = \mu, \quad \text{with } \phi_\tau(x,\xi) = (x+\tau\xi, \xi), \tag{7}$$

for all $\tau \in \mathbb{R}$. Let us also introduce, for $v \in \mathbb{R}^d$ the translations $\tau^v : T^*\mathbb{T}^d \to T^*\mathbb{T}^d$ defined by:

$$\tau^v(x,\xi) = (x+v,\xi).$$

Lemma 8. *Let μ be a positive invariant measure on $T^*\mathbb{T}^d$. Then every term in the decomposition (6) is a positive invariant measure, and*

$$\mu\rceil_{\mathbb{T}^d\times R_\Lambda} = \langle\mu\rangle_\Lambda\rceil_{\mathbb{T}^d\times R_\Lambda}. \tag{8}$$

Moreover, this last identity is equivalent to the following invariance property:

$$\tau^v_*\mu\rceil_{\mathbb{T}^d\times R_\Lambda} = \mu\rceil_{\mathbb{T}^d\times R_\Lambda}, \quad \textit{for every } v \in \Lambda^\perp.$$

Proof. The invariance of the measures $\mu\rceil_{\mathbb{T}^d\times R_\Lambda}$ is clearly a consequence of that of μ and of the form of the geodesic flow on $T^*\mathbb{T}^d$. To check (8) is suffices to show that $\widehat{\mu}(k,\cdot)\rceil_{R_\Lambda} = 0$ as soon as $k \notin \Lambda$. Start noticing that (7) is equivalent to the fact that μ solves the equation:

$$\xi\cdot\nabla_x\mu(x,\xi) = 0.$$

This is in turn equivalent to:

$$i\,(k\cdot\xi)\,\widehat{\mu}(k,\xi) = 0, \qquad \text{for every } k \in \mathbb{Z}^d,$$

from which we infer:

$$\operatorname{supp}\widehat{\mu}(k,\cdot) \subset \left\{\xi\in\mathbb{R}^d : k\cdot\xi = 0\right\}. \tag{9}$$

Now remark that $R_\Lambda \cap \left\{\xi\in\mathbb{R}^d : k\cdot\xi = 0\right\} \neq \emptyset$ if and only if $k \in \Lambda$. This concludes the proof of the lemma. □

[4]In what follows, we shall refer to such a measure simply as a *positive invariant measure*.

3. Second microlocalization on a resonant affine subspace

We now start with our main construction. Theorem 1 (i) and Corollary 2 will be proved at the end of §4, and Theorem 3 in §5.

Given $\Lambda \in \mathcal{L}$, we denote by $\mathcal{S}^1_\Lambda$ the class of smooth functions $a(x,\xi,\eta)$ on $T^*\mathbb{T}^d \times \langle\Lambda\rangle$ that are:

(i) compactly supported w.r.t. $(x,\xi) \in T^*\mathbb{T}^d$,

(ii) homogeneous of degree zero at infinity in $\eta \in \langle\Lambda\rangle$. That is, if we denote by $\mathbb{S}_{\langle\Lambda\rangle}$ the unit sphere in $\langle\Lambda\rangle$ (*i.e.* $\mathbb{S}_{\langle\Lambda\rangle} := \langle\Lambda\rangle \cap \mathbb{S}^{d-1}$) there exist $R_0 > 0$ and $a_{\hom} \in C^\infty_c\left(T^*\mathbb{T}^d \times \mathbb{S}_{\langle\Lambda\rangle}\right)$ with

$$a(x,\xi,\eta) = a_{\hom}\left(x,\xi,\frac{\eta}{|\eta|}\right), \quad \text{for } |\eta| > R_0 \text{ and } (x,\xi) \in T^*\mathbb{T}^d;$$

we also write

$$a(x,\xi,\infty\eta) = a_{\hom}\left(x,\xi,\frac{\eta}{|\eta|}\right), \quad \text{for } \eta \neq 0;$$

(iii) such that their non-vanishing Fourier coefficients (in the x variable) correspond to frequencies $k \in \Lambda$:

$$a(x,\xi,\eta) = \sum_{k\in\Lambda} \widehat{a}(k,\xi,\eta)\,\frac{e^{ik\cdot x}}{(2\pi)^{d/2}}.$$

We will also express this fact by saying that *a has only x-Fourier modes in Λ.*

Let (u_h) be a bounded sequence in $L^2\left(\mathbb{T}^d\right)$ and suppose that its Wigner distributions $w_h(t) := w^h_{U_V(t)u_h}$ converge to a semiclassical measure $\mu \in L^\infty\left(\mathbb{R}; \mathcal{M}_+\left(T^*\mathbb{T}^d\right)\right)$ in the weak-$*$ topology of $L^\infty\left(\mathbb{R}; \mathcal{D}'\left(T^*\mathbb{T}^d\right)\right)$.

Our purpose in this section is to analyse the structure of the restriction $\mu\rceil_{\mathbb{T}^d\times R_\Lambda}$. To achieve this we shall introduce a two-microlocal distribution describing the concentration of the sequence $(U_V(t)\,u_h)$ on the resonant subspaces:

$$\Lambda^\perp = \left\{\xi \in \mathbb{R}^d : P_\Lambda(\xi) = 0\right\},$$

where P_Λ denotes the orthogonal projection of $\mathbb{R}^d$ onto $\langle\Lambda\rangle$. Similar objects have been introduced in the local, Euclidean, case by Nier [26] and Fermanian-Kammerer [10, 11] under the name of two-microlocal semiclassical measures. A specific concentration scale may also be specified in the two-microlocal variable, giving rise to the two-scale semiclassical measures studied by Miller [23] and Gérard and Fermanian-Kammerer [12]. We shall follow the approach in [11], although it will be important to take into account the global nature of the objects we shall be dealing with.

By Lemma 8, it suffices to characterize the action of $\mu\rceil_{\mathbb{T}^d\times R_\Lambda}$ on test functions having only x-Fourier modes in Λ. With this in mind, we introduce two auxiliary distributions which describe more precisely how $w_h(t)$ concentrates along $\mathbb{T}^d \times \Lambda^\perp$ and that act on symbols on the class $\mathcal{S}^1_\Lambda$.

Let $\chi \in C_c^\infty(\mathbb{R})$ be a nonnegative cut-off function that is identically equal to one near the origin. Let $R > 0$. For $a \in \mathcal{S}^1_\Lambda$, we define

$$\left\langle w^\Lambda_{h,R}(t), a\right\rangle := \int_{T^*\mathbb{T}^d} \left(1 - \chi\left(\frac{P_\Lambda(\xi)}{Rh}\right)\right) a\left(x, \xi, \frac{P_\Lambda(\xi)}{h}\right) w_h(t)(dx, d\xi),$$

and

$$\langle w_{\Lambda,h,R}(t), a\rangle := \int_{T^*\mathbb{T}^d} \chi\left(\frac{P_\Lambda(\xi)}{Rh}\right) a\left(x, \xi, \frac{P_\Lambda(\xi)}{h}\right) w_h(t)(dx, d\xi). \tag{10}$$

Remark 9. *If $\Lambda = \{0\}$ then $w^\Lambda_{h,R} = 0$ and $w_{\Lambda,h,R}(t) = w_h(t) \otimes \delta_0$.*

Remark 10. *For every $R > 0$ and $a \in \mathcal{S}^1_\Lambda$ the following holds.*

$$\int_{T^*\mathbb{T}^d} a\left(x, \xi, \frac{P_\Lambda(\xi)}{h}\right) w_h(t)(dx, d\xi) = \left\langle w^\Lambda_{h,R}(t), a\right\rangle + \langle w_{\Lambda,h,R}(t), a\rangle.$$

The Calderón-Vaillancourt theorem (see the appendix for a precise statement) ensures that both $w^\Lambda_{h,R}$ and $w_{\Lambda,h,R}$ are bounded in $L^\infty\left(\mathbb{R}; (\mathcal{S}^1_\Lambda)'\right)$. After possibly extracting subsequences, we have the existence of a limit : for every $\varphi \in L^1(\mathbb{R})$ and $a \in \mathcal{S}^1_\Lambda$,

$$\int_{\mathbb{R}} \varphi(t) \left\langle \tilde{\mu}^\Lambda(t, \cdot), a\right\rangle dt := \lim_{R\to\infty} \lim_{h\to 0^+} \int_{\mathbb{R}} \varphi(t) \left\langle w^\Lambda_{h,R}(t), a\right\rangle dt,$$

and

$$\int_{\mathbb{R}} \varphi(t) \langle \tilde{\mu}_\Lambda(t, \cdot), a\rangle dt := \lim_{R\to\infty} \lim_{h\to 0^+} \int_{\mathbb{R}} \varphi(t) \langle w_{\Lambda,h,R}(t), a\rangle dt. \tag{11}$$

Define, for $(x, \xi, \eta) \in T^*\mathbb{T}^d \times \mathbb{R}^d$ and $\tau \in \mathbb{R}$,

$$\phi^0_\tau(x, \xi, \eta) := (x + \tau\xi, \xi, \eta),$$

and, when $\eta \neq 0$,

$$\phi^1_\tau(x, \xi, \eta) := \left(x + \tau\frac{\eta}{|\eta|}, \xi, \eta\right).$$

Since the distributions[5] $w^\Lambda_{h,R}$ and $w_{\Lambda,h,R}$ satisfy a transport equation with respect to the ξ-variable the following result holds.

Lemma 11. *The distributions $\tilde{\mu}_\Lambda(t, \cdot)$ and $\tilde{\mu}^\Lambda(t, \cdot)$ are ϕ^0_τ-invariant for almost every t:*

$$\left(\phi^0_\tau\right)_* \tilde{\mu}_\Lambda(t, \cdot) = \tilde{\mu}_\Lambda(t, \cdot), \quad \left(\phi^0_\tau\right)_* \tilde{\mu}^\Lambda(t, \cdot) = \tilde{\mu}^\Lambda(t, \cdot), \quad \text{for every } \tau \in \mathbb{R}$$

Proof. Let $a \in C_c^\infty\left(T^*\mathbb{T}^d\right)$. Then

$$\frac{d}{dt}\langle w_h(t), a\rangle = i\left\langle u_h(t, \cdot), \left[-\frac{1}{2}\Delta + V(t, \cdot), \mathrm{Op}_h(a)\right] u_h(t, \cdot)\right\rangle. \tag{12}$$

Now, using identity (2) for the Weyl quantization we deduce:

$$\frac{d}{dt}\langle w_h(t), a\rangle = \frac{1}{h}\langle w_h(t), \xi \cdot \partial_x a\rangle + \left\langle \mathcal{L}^h_V(t), a\right\rangle, \tag{13}$$

[5]It is convenient to use the word "distribution", but we actually mean elements of $L^\infty\left(\mathbb{R}; (\mathcal{S}^1_\Lambda)'\right)$.

where

$$\left\langle \mathcal{L}_V^h(t), a\right\rangle := i\left\langle u_h(t,\cdot), \left[V(t,\cdot), \mathrm{Op}_h(a)\right] u_h(t,\cdot)\right\rangle. \tag{14}$$

Note that this quantity is bounded in h for t varying on a compact set. Integration in t against a function $\varphi \in C_c^1(\mathbb{R})$ gives:

$$\int_{\mathbb{R}} \varphi(t)\left\langle w_h(t), \xi\cdot\partial_x a\right\rangle dt = -h\int_{\mathbb{R}} \varphi'(t)\left\langle w_h(t), a\right\rangle dt - h\int_{\mathbb{R}} \varphi(t)\left\langle \mathcal{L}_V^h(t), a\right\rangle dt.$$

Replacing a in the above identity by

$$\chi\left(\frac{P_\Lambda(\xi)}{Rh}\right) a\left(x,\xi,\frac{P_\Lambda(\xi)}{h}\right) \text{ or } \left(1-\chi\left(\frac{P_\Lambda(\xi)}{Rh}\right)\right) a\left(x,\xi,\frac{P_\Lambda(\xi)}{h}\right)$$

and letting $h\to 0^+$ and $R\to\infty$ we obtain:

$$\left\langle \tilde\mu_\Lambda(t,\cdot), \xi\cdot\partial_x a\right\rangle = 0 \text{ and } \left\langle \tilde\mu^\Lambda(t,\cdot), \xi\cdot\partial_x a\right\rangle = 0$$

which is the desired invariance property. □

Positivity and invariance properties of the accumulation points $\tilde\mu_\Lambda(t,\cdot)$ and $\tilde\mu^\Lambda(t,\cdot)$ are described in the next two results.

Theorem 12. *(i) For a.e. $t\in\mathbb{R}$, $\tilde\mu^\Lambda(t,\cdot)$ is positive, 0-homogeneous and supported at infinity in the variable η (i.e., it vanishes when paired with a compactly supported function). As a consequence, $\tilde\mu^\Lambda(t,\cdot)$ may be identified [6] with a positive measure on $T^*\mathbb{T}^d\times\mathbb{S}_{\langle\Lambda\rangle}$.*

For a.e. $t\in\mathbb{R}$, the projection of $\tilde\mu_\Lambda(t,\cdot)$ on $T^*\mathbb{T}^d$ is a positive measure.

(ii) Both $\tilde\mu^\Lambda(t,\cdot)$ and $\tilde\mu_\Lambda(t,\cdot)$ are ϕ_τ^0-invariant.

(iii) Let

$$\mu^\Lambda(t,\cdot) := \int_{\langle\Lambda\rangle} \tilde\mu^\Lambda(t,\cdot,d\eta)\rceil_{(x,\xi)\in\mathbb{T}^d\times R_\Lambda}, \quad \mu_\Lambda(t,\cdot) := \int_{\langle\Lambda\rangle} \tilde\mu_\Lambda(t,\cdot,d\eta)\rceil_{(x,\xi)\in\mathbb{T}^d\times R_\Lambda}.$$

Then both $\mu^\Lambda(t,\cdot)$ and $\mu_\Lambda(t,\cdot)$ are positive measures on $T^*\mathbb{T}^d$, invariant by the geodesic flow, and satisfy:

$$\mu(t,\cdot)\rceil_{\mathbb{T}^d\times R_\Lambda} = \mu^\Lambda(t,\cdot) + \mu_\Lambda(t,\cdot). \tag{15}$$

Note that identity (15) is a consequence of the decomposition property expressed in Remark 10.

The following result is the key step of our proof, it states that both μ^Λ and μ_Λ have some extra regularity in the variable x, for two different reasons :

Theorem 13. *(i) For a.e. $t\in\mathbb{R}$, $\tilde\mu_\Lambda(t,\cdot)$ is concentrated on $\mathbb{T}^d\times\Lambda^\perp\times\langle\Lambda\rangle$ and its projection on $\mathbb{T}^d$ is absolutely continuous with respect to the Lebesgue measure.*

(ii) For a.e. $t\in\mathbb{R}$, the measure $\tilde\mu^\Lambda(t,\cdot)$ satisfies the invariance property:

$$\left(\phi_\tau^1\right)_* \tilde\mu^\Lambda(t,\cdot) = \tilde\mu^\Lambda(t,\cdot), \quad \tau\in\mathbb{R}. \tag{16}$$

[6] More precisely, there exists a positive measure $M^\Lambda(t,\cdot)$ on $T^*\mathbb{T}^d\times\mathbb{S}_{\langle\Lambda\rangle}$ such that $\int_{T^*\mathbb{T}^d\times\langle\Lambda\rangle} a(x,\xi,\eta)\tilde\mu^\Lambda(t,d\xi,d\eta) = \int_{T^*\mathbb{T}^d\times\mathbb{S}_{\langle\Lambda\rangle}} a(x,\xi,\infty\eta) M^\Lambda(t,d\xi,d\eta)$. For simplicity we will identify $M^\Lambda(t,\cdot)$ and $\tilde\mu^\Lambda(t,\cdot)$, and we will write the integrals in the most convenient way according to the context.

Remark 14. *As we shall prove in Section 5, the distributions $\tilde{\mu}_\Lambda(t,\cdot)$ verify a propagation law that is related to unitary propagator generated by the self-adjoint operator $\frac{1}{2}\Delta + \langle V\rangle_\Lambda(t,\cdot)$, where $\langle V\rangle_\Lambda$ denotes the average of V along $\Lambda^\perp$.*

Remark 15. *The invariance property (16) provides $\tilde{\mu}^\Lambda$ with additional regularity. This is clearly seen when $\operatorname{rk}\Lambda = 1$. In that case, (16) implies that, for a.e. $t\in\mathbb{R}$, the measure $\tilde{\mu}^\Lambda(t,\cdot)$ satisfies for every $v\in\mathbb{S}_{\langle\Lambda\rangle}$:*

$$(\tau_s^v)_*\,\tilde{\mu}^\Lambda(t,\cdot)\rceil_{\mathbb{T}^d\times R_\Lambda\times\langle\Lambda\rangle} = \tilde{\mu}^\Lambda(t,\cdot)\rceil_{\mathbb{T}^d\times R_\Lambda\times\langle\Lambda\rangle},\quad s\in\mathbb{R}. \tag{17}$$

On the other hand, Lemma 8 implies that (17) also holds for every $v\in\Lambda^\perp$. Therefore, we conclude that $\tilde{\mu}^\Lambda(t,\cdot)\rceil_{\mathbb{T}^d\times R_\Lambda\times\langle\Lambda\rangle}$ is constant in $x\in\mathbb{T}^d$ in this case.

Remark 16. *Theorems 12 (iii), and 13 (i), together with Lemma 6 imply that, for a.e. $t\in\mathbb{R}$, we have a decomposition:*

$$\mu(t,\cdot) = \sum_{\Lambda\in\mathcal{L}}\mu^\Lambda(t,\cdot) + \sum_{\Lambda\in\mathcal{L}}\mu_\Lambda(t,\cdot),$$

where the second term in the above sum defines a positive measure whose projection on $\mathbb{T}^d$ is absolutely continuous with respect to the Lebesgue measure.

The rest of this section is devoted to the proofs of Theorems 12 and 13.

3.1. **Computation and structure of $\tilde{\mu}_\Lambda$.** We use the linear isomorphism

$$\chi_\Lambda : \Lambda^\perp\times\langle\Lambda\rangle\to\mathbb{R}^d : (s,y)\mapsto s+y$$

and denote by $\tilde{\chi}_\Lambda : T^*\Lambda^\perp\times T^*\langle\Lambda\rangle\to T^*\mathbb{R}^d$ the induced canonical transformation. Explicitly, $\tilde{\chi}_\Lambda$ goes as follows : let $(s,\sigma)\in T^*\Lambda^\perp = \Lambda^\perp\times\left(\Lambda^\perp\right)^*$ and $(y,\eta)\in T^*\langle\Lambda\rangle = \langle\Lambda\rangle\times\langle\Lambda\rangle^*$. Extend σ to a linear form on $\mathbb{R}^d$ vanishing on $\langle\Lambda\rangle$, and η to a linear form on $\mathbb{R}^d$ vanishing on $\Lambda^\perp$. Then $\tilde{\chi}_\Lambda(s,\sigma,y,\eta) = (s+y,\sigma+\eta)\in T^*\mathbb{R}^d = \mathbb{R}^d\times(\mathbb{R}^d)^*$.

The map χ_Λ goes to the quotient and gives a smooth Riemannian covering :

$$\pi_\Lambda : \mathbb{T}_{\Lambda^\perp}\times\mathbb{T}_\Lambda\to\mathbb{T}^d : (s,y)\mapsto s+y;$$

$\tilde{\pi}_\Lambda$ will denote its extension to the cotangent bundles $T^*\mathbb{T}_{\Lambda^\perp}\times T^*\mathbb{T}_\Lambda\to T^*\mathbb{T}^d$. Let p_Λ denote the degree of π_Λ.

There is a linear isomorphism $T_\Lambda : L^2_{\mathrm{loc}}\left(\mathbb{R}^d\right)\to L^2_{\mathrm{loc}}\left(\Lambda^\perp\times\langle\Lambda\rangle\right)$ given by

$$T_\Lambda u := \frac{1}{\sqrt{p_\Lambda}}\,(u\circ\chi_\Lambda).$$

Note that because of the factor $p_\Lambda^{-1/2}$, T_Λ maps $L^2\left(\mathbb{T}^d\right)$ isometrically into a subspace of $L^2(\mathbb{T}_{\Lambda^\perp}\times\mathbb{T}_\Lambda) = L^2(\mathbb{T}_{\Lambda^\perp};L^2(\mathbb{T}_\Lambda))$. Moreover, T_Λ maps $L^2_\Lambda\left(\mathbb{T}^d\right)$ into $L^2(\mathbb{T}_\Lambda)\subset L^2(\mathbb{T}_{\Lambda^\perp}\times\mathbb{T}_\Lambda)$, since if the non-vanishing Fourier modes of u correspond only to frequencies $k\in\Lambda$, then

$$T_\Lambda u(s,y) = \frac{1}{\sqrt{p_\Lambda}}u(y)\quad\text{for every } s\in\mathbb{T}_{\Lambda^\perp}. \tag{18}$$

Since $\tilde{\chi}_\Lambda$ is linear, the following holds for any $a \in C^\infty\left(T^*\mathbb{R}^d\right)$:

$$T_\Lambda \operatorname{Op}_h(a) = \operatorname{Op}_h(a \circ \tilde{\chi}_\Lambda) T_\Lambda.$$

Denote by $\operatorname{Op}_h^{\Lambda^\perp}$ and $\operatorname{Op}_h^{\Lambda}$ the Weyl quantization operators defined on smooth test functions on $T^*\Lambda^\perp \times T^*\langle\Lambda\rangle$ which act on the variables $T^*\Lambda^\perp$ and $T^*\langle\Lambda\rangle$ respectively, leaving the other frozen. The composition $\operatorname{Op}_h^{\Lambda^\perp} \operatorname{Op}_h^{\Lambda}$ gives the whole Weyl quantization Op_h on $T^*\Lambda^\perp \times T^*\langle\Lambda\rangle$. Now, if $a \in \mathcal{S}^1_\Lambda$ we have, in view of (18), that $a \circ \tilde{\pi}_\Lambda$ does not depend on $s \in \mathbb{T}_{\Lambda^\perp}$ and therefore we write $a \circ \tilde{\pi}_\Lambda(\sigma, y, \eta)$ for $a \circ \tilde{\pi}_\Lambda(s, \sigma, y, \eta)$. We have

$$T_\Lambda \operatorname{Op}_h(a) = \operatorname{Op}_h^{\Lambda}(a \circ \tilde{\pi}_\Lambda(hD_s, \cdot)) T_\Lambda. \tag{19}$$

Note that for every $\sigma \in \Lambda^\perp$, the operators $\operatorname{Op}_h^{\Lambda}(a \circ \tilde{\pi}_\Lambda(\sigma, \cdot))$ map $L^2(\mathbb{T}_\Lambda)$ into itself. To be even more precise, it maps the subspace $T_\Lambda(L^2_\Lambda(\mathbb{T}^d))$ into itself.

Remark 17. *Let $a \in \mathcal{S}^1_\Lambda$; set $a_R(x,\xi,\eta) := \chi(\eta/R)\, a(x,\xi,\eta)$ and define $a^h_{R,\Lambda} \in C^\infty_c\left(\Lambda^\perp \times T^*\mathbb{T}_\Lambda\right)$ by*

$$a^h_{R,\Lambda}(\sigma, y, \eta) := a_R(\tilde{\pi}_\Lambda(\sigma, y, h\eta), \eta) = a_R(y, \sigma + h\eta, \eta), \quad (y,\eta) \in T^*\mathbb{T}_\Lambda, \quad \sigma \in \Lambda^\perp.$$

It is simple to check that (19) gives:

$$T_\Lambda \operatorname{Op}_h(a) T^*_\Lambda = \operatorname{Op}_1^{\Lambda}\left(a^h_{R,\Lambda}(hD_s, \cdot)\right),$$

and

$$\langle w_{\Lambda,h,R}(t), a\rangle = \left\langle T_\Lambda u_h(t,\cdot), \operatorname{Op}_1^{\Lambda}\left(a^h_{R,\Lambda}(hD_s,\cdot)\right) T_\Lambda u_h(t,\cdot)\right\rangle_{L^2\left(\mathbb{T}_{\Lambda^\perp}; L^2(\mathbb{T}_\Lambda)\right)}.$$

Note that for every $R > 0$, $t \in \mathbb{R}$ and $(s,\sigma) \in T^*\mathbb{T}_{\Lambda^\perp}$, the operator

$$\operatorname{Op}_1^{\Lambda}\left(a^h_{R,\Lambda}(\sigma,\cdot)\right)$$

is compact on $L^2(\mathbb{T}_\Lambda)$, since $a^h_{R,\Lambda}$ is compactly supported in the variable η.

Given a Hilbert space H, denote respectively by $\mathcal{K}(H)$ and $\mathcal{L}^1(H)$ the spaces of compact and trace class operators on H. A measure on a polish space T, taking values in $\mathcal{L}^1(H)$, is defined as a bounded linear functional ρ from $C_c(T)$ to $\mathcal{L}^1(H)$; ρ is said to be positive if, for every nonnegative $b \in C_c(T)$, $\rho(b)$ is a positive hermitian operator. The set of such measures is denoted by $\mathcal{M}_+(T; \mathcal{L}^1(H))$; they can be identified in a natural way to positive linear functionals on $C_c(T; \mathcal{K}(H))$. Background and further details on operator-valued measures may be found for instance in [14].

In view of Remark 17, it turns out that the limiting object relevant in the computation of $\tilde{\mu}_\Lambda$ is the one presented in the next result. For $K \in C^\infty_c\left(T^*\mathbb{T}_{\Lambda^\perp}; \mathcal{K}\left(L^2(\mathbb{T}_\Lambda)\right)\right)$ denote:

$$\left\langle n^\Lambda_h(t), K\right\rangle := \left\langle T_\Lambda U_V(t) u_h, K(s, hD_s) T_\Lambda U_V(t) u_h\right\rangle_{L^2\left(\mathbb{T}_{\Lambda^\perp}; L^2(\mathbb{T}_\Lambda)\right)}. \tag{20}$$

Proposition 18. *Suppose (u_h) is bounded in $L^2\left(\mathbb{T}^d\right)$. Then, modulo a subsequence, the following convergence takes place:*

$$\lim_{h\to 0^+} \int_{\mathbb{R}} \varphi(t) \left\langle n^\Lambda_h(t), K\right\rangle dt = \int_{\mathbb{R}} \varphi(t) \operatorname{Tr} \int_{T^*\mathbb{T}_{\Lambda^\perp}} K(s,\sigma)\, \tilde{\rho}_\Lambda(t, ds, d\sigma)\, dt, \tag{21}$$

for every $K \in C_c^\infty\left(T^*\mathbb{T}_{\Lambda^\perp}; \mathcal{K}\left(L^2\left(\mathbb{T}_\Lambda\right)\right)\right)$ *and every* $\varphi \in L^1(\mathbb{R})$*; in other words,* $\tilde{\rho}_\Lambda$ *is the limit of* $n_h^\Lambda(t)$ *in the weak-* topology of* $L^\infty\left(\mathbb{R}, \mathcal{D}'\left(T^*\mathbb{T}_{\Lambda^\perp}; \mathcal{L}^1\left(L^2\left(\mathbb{T}_\Lambda\right)\right)\right)\right)$.

Then $\tilde{\rho}_\Lambda$ *is an* L^∞*-function in* t *taking values in the set of positive,* $\mathcal{L}^1\left(L^2\left(\mathbb{T}_\Lambda\right)\right)$*-valued measures on* $T^*\mathbb{T}_{\Lambda^\perp}$*. We have* $\int_{T^*\mathbb{T}_{\Lambda^\perp}} \operatorname{Tr} \tilde{\rho}_\Lambda(t, ds, d\sigma) \leq 1$ *for* a.e. t.

Moreover, for almost every t *the measure* $\tilde{\rho}_\Lambda(t, \cdot)$ *is invariant by the geodesic flow* $\phi_\tau|_{T^*\mathbb{T}_{\Lambda^\perp}} : (s, \sigma) \mapsto (s + \tau\sigma, \sigma)$ $(\tau \in \mathbb{R})$.

This result is the analogue of Theorems 1 and 2 of [19] in the context of operator-valued measures. Its proof follows the lines of those results, after the adaptation of the symbolic calculus to operator valued symbols as developed for instance in [14].

When taking the limits $h \longrightarrow 0$ and $R \longrightarrow +\infty$ one should have in mind the following facts. For any $a \in \mathcal{S}_\Lambda^1$, we have for fixed R

$$\operatorname{Op}_1^\Lambda\left(a_{R,\Lambda}^h(\sigma, \cdot)\right) = \operatorname{Op}_1^\Lambda\left(a_{R,\Lambda}^0(\sigma, \cdot)\right) + \mathcal{O}(h)$$

where the remainder $\mathcal{O}(h)$ is estimated in the operator norm (using the Calderón-Vaillancourt theorem). In addition, the following limit takes place in the strong topology of $C_c^\infty\left(T^*\mathbb{T}_{\Lambda^\perp}; \mathcal{L}\left(L^2\left(\mathbb{T}_\Lambda\right)\right)\right)$:

$$\lim_{R\to\infty} \operatorname{Op}_1^\Lambda\left(a_{R,\Lambda}^0(\sigma, \cdot)\right) = \operatorname{Op}_1^\Lambda\left(a_\Lambda^0(\sigma, \cdot)\right),$$

where a_Λ^0 is defined by setting $h = 0$ and $R = \infty$ in the definition of $a_{R,\Lambda}^h$. In other words, $a_\Lambda^0(\sigma, y, \eta) = a(\tilde{\pi}_\Lambda(\sigma, y, 0), \eta) = a(y, \sigma, \eta)$.

Combining what we have done so far, we find

Corollary 19. *Let* $\tilde{\rho}_\Lambda \in L^\infty\left(\mathbb{R}; \mathcal{M}_+\left(T^*\mathbb{T}_{\Lambda^\perp}; \mathcal{L}^1\left(L^2\left(\mathbb{T}_\Lambda\right)\right)\right)\right)$ *be a weak-* limit of* $\left(n_h^\Lambda\right)$*. Let* $\tilde{\mu}_\Lambda$ *be defined by* (10) *and* (11)*. Then, for every* $a \in \mathcal{S}_\Lambda^1$ *and a.e.* $t \in \mathbb{R}$ *we have:*

$$\begin{aligned}&\int_{T^*\mathbb{T}^d\times\langle\Lambda\rangle} a(x, \xi, \eta)\, \tilde{\mu}_\Lambda(t, dx, d\xi, d\eta)\\ &= \operatorname{Tr}\left(\int_{T^*\mathbb{T}_{\Lambda^\perp}} \operatorname{Op}_1^\Lambda\left(a_\Lambda^0(\sigma, \cdot)\right) \tilde{\rho}_\Lambda(t, ds, d\sigma)\right).\end{aligned}$$

Remark 20. *If* $a \in \mathcal{S}_\Lambda^1$ *does not depend on* $\eta \in \mathbb{R}^d$ *then the above identity can be rewritten as:*

$$\int_{T^*\mathbb{T}^d\times\langle\Lambda\rangle} a(x, \xi)\, \tilde{\mu}_\Lambda(t, dx, d\xi, d\eta) = \operatorname{Tr}_{L^2(\mathbb{T}_\Lambda)}\left(\int_{T^*\mathbb{T}_{\Lambda^\perp}} m_{a\circ\pi_\Lambda}(\sigma)\, \tilde{\rho}_\Lambda(t, ds, d\sigma)\right), \tag{22}$$

where for $\sigma \in \Lambda^\perp$*,* $m_a(\sigma)$ *denotes the operator of multiplication by* $a(., \sigma)$ *in* $L^2\left(\mathbb{T}_\Lambda\right)$.

Since all the arguments above actually hold with $L^2(\mathbb{T}_\Lambda)$ *replaced by the smaller space* $T_\Lambda(L^2_\Lambda(\mathbb{T}^d))$*, and since* $m_{a\circ\pi_\Lambda}(\sigma) = T_\Lambda m_a(\sigma) T_\Lambda^*$ *on this space (where* $m_a(\sigma)$ *is again the multiplication operator by* $a(., \sigma)$*), we can write the above identity as:*

$$\int_{T^*\mathbb{T}^d\times\langle\Lambda\rangle} a(x, \xi)\, \tilde{\mu}_\Lambda(t, dx, d\xi, d\eta) = \operatorname{Tr}_{L^2_\Lambda(\mathbb{T}^d)}\left(\int_{T^*\mathbb{T}_{\Lambda^\perp}} m_a(\sigma) T_\Lambda^* \tilde{\rho}_\Lambda(t, ds, d\sigma)\, T_\Lambda\right). \tag{23}$$

And when $a = a(x)$ does not depend on ξ, this reduces to

$$\int_{T^*\mathbb{T}^d\times\langle\Lambda\rangle} a(x)\,\tilde{\mu}_\Lambda(t,dx,d\xi,d\eta) = \mathrm{Tr}_{L^2_\Lambda(\mathbb{T}^d)}\left(\int_{T^*\mathbb{T}_{\Lambda^\perp}} m_a T^*_\Lambda \tilde{\rho}_\Lambda(t,ds,d\sigma)\,T_\Lambda\right) \tag{24}$$

which proves the absolute continuity of the projection of $\tilde{\mu}_\Lambda$ to $\mathbb{T}^d$.

3.2. **Computation and structure of $\tilde{\mu}^\Lambda$.** The positivity of $\tilde{\mu}^\Lambda(t,\cdot)$ can be deduced following the lines of [12] §2.1, or those of the proof of Theorem 1 in [14]; the idea is recalled in Corollary 35 in the appendix. Given $a \in \mathcal{S}^1_\Lambda$ there exists $R_0 > 0$ and $a_{\hom} \in C^\infty_c\left(T^*\mathbb{T}^d \times \mathbb{S}_{\langle\Lambda\rangle}\right)$ such that

$$a(x,\xi,\eta) = a_{\hom}\left(x,\xi,\frac{\eta}{|\eta|}\right), \quad \text{for } |\eta| \geq R_0.$$

Clearly, for R large enough, the value $\left\langle w^\Lambda_{h,R}(t), a\right\rangle$ only depends on $a_{\hom}$. Therefore, the limiting distribution $\tilde{\mu}^\Lambda(t,\cdot)$ can be viewed as an element of the dual of $C^\infty_c\left(T^*\mathbb{T}^d \times \mathbb{S}_{\langle\Lambda\rangle}\right)$. Let us now check the invariance property (16). Set

$$a^R(x,\xi,\eta) := \left(1-\chi\left(\frac{\eta}{R}\right)\right) a(x,\xi,\eta).$$

Notice that since a has only Fourier modes in Λ:

$$\frac{\xi}{h}\cdot\partial_x a^R\left(x,\xi,\frac{P_\Lambda\xi}{h}\right) = \frac{P_\Lambda\xi}{h}\cdot\partial_x a^R\left(x,\xi,\frac{P_\Lambda\xi}{h}\right).$$

Therefore, by equations (13) and (14), and taking into account that a^R vanishes near $\eta = 0$, we have, for every $\varphi \in C^1_c(\mathbb{R})$:

$$\int_{\mathbb{R}} \varphi(t)\left\langle w^\Lambda_{h,R}(t), \frac{\eta}{|\eta|}\cdot\partial_x a^R\right\rangle dt = -\int_{\mathbb{R}} \varphi'(t)\left\langle w^\Lambda_{h,R}(t), \frac{1}{|\eta|}a^R\right\rangle dt \tag{25}$$

$$+\int_{\mathbb{R}} \varphi(t)\left\langle \mathcal{L}^h_V(t), \frac{1}{|\eta|}a^R\right\rangle dt. \tag{26}$$

Writing $\eta = r\omega$ with $r > 0$ and $\omega \in \mathbb{S}_{\langle\Lambda\rangle}$ we find, for R large enough:

$$b^R(x,\xi,\eta) := \frac{1}{|\eta|}a^R(x,\xi,\eta) = \frac{1}{r}\left(1-\chi\left(\frac{r}{R}\omega\right)\right) a_{\hom}(x,\xi,\omega);$$

moreover, since b^R is homogeneous of degree -1 in the variable η, the Calderón-Vaillancourt theorem implies that the operator:

$$B^\Lambda_{h,R} := \mathrm{Op}_h\left(b^R\left(x,\xi,\frac{P_\Lambda\xi}{h}\right)\right)$$

satisfies:

$$\limsup_{h\to 0^+}\left\|B^\Lambda_{h,R}\right\|_{\mathcal{L}\left(L^2\left(\mathbb{T}^d\right)\right)} \leq \frac{C}{R}.$$

Therefore,

$$\lim_{R\to\infty}\lim_{h\to 0^+}\int_{\mathbb{R}}\varphi'(t)\left\langle w_{h,R}^{\Lambda}(t),\frac{1}{|\eta|}a^R\right\rangle dt=0,$$

and

$$\limsup_{h\to 0^+}\left\langle \mathcal{L}_V^h(t),\frac{1}{|\eta|}a^R\right\rangle \leq C\limsup_{h\to 0^+}\left\|\left[V,B_{h,R}^{\Lambda}\right]\right\|_{\mathcal{L}\left(L^2\left(\mathbb{T}^d\right)\right)}\leq\frac{C'}{R}\left\|V\right\|_{L^\infty\left(\mathbb{T}^d\right)}.$$

After letting $h\to 0^+$ and $R\to\infty$ in (25), (26), we conclude that for almost every $t\in\mathbb{R}$:

$$\omega\cdot\nabla_x\tilde{\mu}^{\Lambda}(t,x,\xi,\omega)=0.$$

This is equivalent to (16).

4. Successive second microlocalizations corresponding to a sequence of lattices

Let us summarize what we have done in the previous section. The semiclassical measure $\mu(t,.)$ has been decomposed as a sum

$$\mu(t,.)=\sum_{\Lambda}\mu_{\Lambda}(t,.)+\sum_{\Lambda}\mu^{\Lambda}(t,.),$$

where Λ runs over the set of primitive submodules of $\mathbb{Z}^d$, and where

$$\mu_{\Lambda}(t,.)=\int_{\langle\Lambda\rangle}\tilde{\mu}_{\Lambda}(t,.,d\eta)\rceil_{\mathbb{T}^d\times R_{\Lambda}},\qquad \mu^{\Lambda}(t,.)=\int_{\langle\Lambda\rangle}\tilde{\mu}^{\Lambda}(t,.,d\eta)\rceil_{\mathbb{T}^d\times R_{\Lambda}}.$$

The "distributions" $\tilde{\mu}_{\Lambda}$ and $\tilde{\mu}^{\Lambda}$ have the following properties :

- $\tilde{\mu}_{\Lambda}(t,dx,d\xi,d\eta)$ is in $L^\infty\left(\mathbb{R};(\mathcal{S}^1_{\Lambda})'\right)$;
- $\int_{\langle\Lambda\rangle}\tilde{\mu}_{\Lambda}(t,.,d\eta)$ is in $L^\infty(\mathbb{R},\mathcal{M}_+(T^*\mathbb{T}^d))$;
- for $a\in\mathcal{S}^1_{\Lambda}$, we have

$$\int_{T^*\mathbb{T}^d\times\langle\Lambda\rangle}a(x,\xi,\eta)\,\tilde{\mu}_{\Lambda}(t,dx,d\xi,d\eta)=\mathrm{Tr}\left(\int_{T^*\mathbb{T}_{\Lambda^\perp}}\mathrm{Op}_1^{\Lambda}\left(a(\cdot,\sigma,\cdot)\right)\tilde{\rho}_{\Lambda}(t,ds,d\sigma)\right)$$

  where $\tilde{\rho}_{\Lambda}(t)$ is a positive measure on $T^*\mathbb{T}_{\Lambda^\perp}$, taking values in $\mathcal{L}^1(T_{\Lambda}(L^2_{\Lambda}(\mathbb{T}^d)))$, invariant under the geodesic flow $(s,\sigma)\mapsto(s+\tau\sigma,\sigma)$ $(\tau\in\mathbb{R})$.

In addition,

- for $a\in\mathcal{S}^1_{\Lambda}$, $\langle\tilde{\mu}^{\Lambda}(t,dx,d\xi,d\eta),a(x,\xi,\eta)\rangle$ is obtained as the limit of

$$\left\langle w_{h,R}^{\Lambda}(t),a\right\rangle:=\int_{T^*\mathbb{T}^d}\left(1-\chi\left(\frac{P_{\Lambda}(\xi)}{Rh}\right)\right)a\left(x,\xi,\frac{P_{\Lambda}(\xi)}{h}\right)w_h(t)\,(dx,d\xi),$$

  where the weak-$*$ limit holds in $L^\infty(\mathbb{R},\mathcal{S}^{1'}_{\Lambda})$, as $h\longrightarrow 0$ then $R\longrightarrow+\infty$ (along subsequences);
- $\tilde{\mu}^{\Lambda}(t,dx,d\xi,d\eta)$ is in $L^\infty(\mathbb{R},\mathcal{M}_+(T^*\mathbb{T}^d\times\mathbb{S}_{\langle\Lambda\rangle}))$;

- $\tilde{\mu}^{\Lambda}$ is invariant by the two flows, $\phi^0_\tau : (x,\xi,\eta) \mapsto (x+\tau\xi,\xi,\eta)$, and $\phi^1_\tau : (x,\xi,\eta) \mapsto (x+\tau\frac{\eta}{|\eta|},\xi,\eta)$ $(\tau\in\mathbb{R})$.

This can be considered as the first step of an induction procedure, the k-th step of which will read as follows :

Step k of the induction : At step k, we have decomposed $\mu(t,.)$ as a sum

$$\mu(t,.) = \sum_{1\le l\le k} \sum_{\Lambda_1\supset\Lambda_2\supset\ldots\supset\Lambda_l} \mu_{\Lambda_l}^{\Lambda_1\Lambda_2\ldots\Lambda_{l-1}}(t,.) + \sum_{\Lambda_1\supset\Lambda_2\supset\ldots\supset\Lambda_k} \mu^{\Lambda_1\Lambda_2\ldots\Lambda_k}(t,.),$$

where the sums run over the *strictly decreasing* sequences of primitive submodules of $\mathbb{Z}^d$ (of lengths $l\le k$ in the first term, of length k in the second term). These measures themselves are obtained as

$$\mu_{\Lambda_l}^{\Lambda_1\Lambda_2\ldots\Lambda_{l-1}}(t,.) = \int_{R_{\Lambda_2}(\Lambda_1)\times\ldots\times R_{\Lambda_l}(\Lambda_{l-1})\times\langle\Lambda_l\rangle} \tilde{\mu}_{\Lambda_l}^{\Lambda_1\Lambda_2\ldots\Lambda_{l-1}}(t,.,d\eta_1,\ldots,d\eta_l)\rceil_{\mathbb{T}^d\times R_{\Lambda_1}},$$

$$\mu^{\Lambda_1\Lambda_2\ldots\Lambda_k}(t,.) = \int_{R_{\Lambda_2}(\Lambda_1)\times\ldots\times R_{\Lambda_k}(\Lambda_{k-1})\times\langle\Lambda_k\rangle} \tilde{\mu}^{\Lambda_1\Lambda_2\ldots\Lambda_k}(t,.,d\eta_1,\ldots,d\eta_k)\rceil_{\mathbb{T}^d\times R_{\Lambda_1}},$$

where we denoted $R_\Lambda(\Lambda') := \Lambda^\perp\cap\langle\Lambda'\rangle\cap\Omega_{\mathrm{rk}\,\Lambda'-\mathrm{rk}\,\Lambda}$, for $\Lambda\subset\Lambda'$.

Let us denote by $\mathcal{S}^k_{\Lambda_1,\ldots,\Lambda_k}$ the class of smooth functions $a(x,\xi,\eta_1,\ldots,\eta_k)$ on $T^*\mathbb{T}^d\times\langle\Lambda_1\rangle\times\ldots\times\langle\Lambda_k\rangle$ that are (i) smooth and compactly supported in $(x,\xi)\in T^*\mathbb{T}^d$; (ii) homogeneous of degree 0 at infinity in each variable $\eta_1,\ldots,\eta_k$; (iii) such that their non-vanishing x-Fourier coefficients correspond to frequencies in Λ_k.

The "distributions" $\tilde{\mu}_{\Lambda_l}^{\Lambda_1\Lambda_2\ldots\Lambda_{l-1}}$ and $\tilde{\mu}^{\Lambda_1\Lambda_2\ldots\Lambda_k}$ have the following properties :

- $\tilde{\mu}_{\Lambda_l}^{\Lambda_1\Lambda_2\ldots\Lambda_{l-1}}$ is in $L^\infty\left(\mathbb{R},(\mathcal{S}^l_{\Lambda_1,\ldots,\Lambda_l})'\right)$. With respect to the variables $\eta_j\in\langle\Lambda_j\rangle$, $j=1,\ldots,l-1$, it is 0-homogeneous and supported at infinity. Thus, (as in footnote 6) we may identify it with a distribution on the unit sphere $\mathbb{S}_{\langle\Lambda_1\rangle}\times\ldots\times\mathbb{S}_{\langle\Lambda_{l-1}\rangle}$;
- $\int_{\langle\Lambda_l\rangle}\tilde{\mu}_{\Lambda_l}^{\Lambda_1\Lambda_2\ldots\Lambda_{l-1}}(t,.,d\eta_l)$ is in $L^\infty(\mathbb{R},\mathcal{M}_+(T^*\mathbb{T}^d\times\mathbb{S}_{\langle\Lambda_1\rangle}\times\ldots\times\mathbb{S}_{\langle\Lambda_{l-1}\rangle}))$;
- for $a\in\mathcal{S}^l_{\Lambda_1,\ldots,\Lambda_l}$, we have

(27)

$$\int_{T^*\mathbb{T}^d\times\langle\Lambda_1\rangle\times\ldots\times\langle\Lambda_{l-1}\rangle} a\left(x,\xi,\eta_1,\ldots,\eta_l\right)\tilde{\mu}_{\Lambda_l}^{\Lambda_1\Lambda_2\ldots\Lambda_{l-1}}\left(t,dx,d\xi,d\eta_1,\ldots,d\eta_l\right)$$

(28)

$$= \mathrm{Tr}\left(\int_{T^*\mathbb{T}_{\Lambda_l^\perp}\times\mathbb{S}_{\langle\Lambda_1\rangle}\times\ldots\times\mathbb{S}_{\langle\Lambda_{l-1}\rangle}} \mathrm{Op}_1^{\Lambda_l}\left(a\left(\cdot,\sigma,\infty\eta_1,\ldots,\infty\eta_{l-1},\cdot\right)\right)\tilde{\rho}_{\Lambda_l}^{\Lambda_1\Lambda_2\ldots\Lambda_{l-1}}\left(t,ds,d\sigma,d\eta_1,\ldots,d\eta_{l-1}\right)\right)$$

where $\tilde{\rho}_{\Lambda_l}^{\Lambda_1\Lambda_2\ldots\Lambda_{l-1}}(t)$ is a positive measure on $T^*\mathbb{T}_{\Lambda_l^\perp}\times\mathbb{S}_{\langle\Lambda_1\rangle}\times\ldots\times\mathbb{S}_{\langle\Lambda_{l-1}\rangle}$, taking values in $\mathcal{L}^1(T_{\Lambda_l}L^2_{\Lambda_l}(\mathbb{T}^d))$. It is invariant under the flows $(s,\sigma,\eta_1,\ldots,\eta_{l-1})\mapsto(s+\tau\sigma,\sigma,\eta_1,\ldots,\eta_{l-1})$ and $(s,\sigma,\eta_1,\ldots,\eta_{l-1})\mapsto(s+\tau\frac{\eta_j}{|\eta_j|},\sigma,\eta_1,\ldots,\eta_{l-1})$ $(\tau\in\mathbb{R},\ j=$

$1, \dots, l-1$). Equation (28) implies that the projection of $\tilde{\mu}^{\Lambda_1\Lambda_2\dots\Lambda_{l-1}}_{\Lambda_l}$ on $\mathbb{T}^d$ is absolutely continuous.

- For $a \in \mathcal{S}^k_{\Lambda_1,\dots,\Lambda_k}$, $\langle \tilde{\mu}^{\Lambda_1\Lambda_2\dots\Lambda_k}(t, dx, d\xi, d\eta_1, \dots, d\eta_k), a(x, \xi, \eta_1, \dots, \eta_k)\rangle$ is obtained as the limit of

$$\left\langle w_h(t, dx, d\xi), a\left(x, \xi, \frac{P_{\Lambda_1}\xi}{h}, \cdots, \frac{P_{\Lambda_k}\xi}{h}\right)\left(1-\chi\left(\frac{P_{\Lambda_1}\xi}{R_1 h}\right)\right)\dots\left(1-\chi\left(\frac{P_{\Lambda_k}\xi}{R_k h}\right)\right)\right\rangle.$$

  The weak limit holds in $L^\infty(\mathbb{R}, (\mathcal{S}^k_{\Lambda_1,\dots,\Lambda_k})')$, as $h \longrightarrow 0$ then $R_1 \longrightarrow +\infty$,..., $R_k \longrightarrow +\infty$ (along subsequences);
- $\tilde{\mu}^{\Lambda_1\Lambda_2\dots\Lambda_k}$ is in $L^\infty(\mathbb{R}, \mathcal{M}_+(T^*\mathbb{T}^d \times \mathbb{S}_{\langle\Lambda_1\rangle} \times \dots \mathbb{S}_{\langle\Lambda_k\rangle}))$;
- $\tilde{\mu}^{\Lambda_1\Lambda_2\dots\Lambda_k}$ is invariant by the $k+1$ flows, $\phi^0_\tau : (x, \xi, \eta) \mapsto (x + \tau\xi, \xi, \eta_1, \dots, \eta_k)$, and $\phi^j_\tau : (x, \xi, \eta_1, \dots, \eta_k) \mapsto (x + \tau\frac{\eta_j}{|\eta_j|}, \xi, \eta_1, \dots, \eta_k)$ (where $j = 1, \dots, k$, $\tau \in \mathbb{R}$).

How to go from step k to step $k+1$.

The term $\sum_{1\le l\le k}\sum_{\Lambda_1\supset\Lambda_2\supset\dots\supset\Lambda_l} \mu^{\Lambda_1\Lambda_2\dots\Lambda_{l-1}}_{\Lambda_l}$ remains untouched after step k.

To decompose further the term $\sum_{\Lambda_1\supset\Lambda_2\supset\dots\supset\Lambda_k} \mu^{\Lambda_1\Lambda_2\dots\Lambda_k}$, we proceed as follows. Using the positivity of $\tilde{\mu}^{\Lambda_1\Lambda_2\dots\Lambda_k}$, we use the procedure described in Section 2 to write

$$\tilde{\mu}^{\Lambda_1\Lambda_2\dots\Lambda_k} = \sum_{\Lambda_{k+1}\subset\Lambda_k} \tilde{\mu}^{\Lambda_1\Lambda_2\dots\Lambda_k}\rceil_{\eta_k\in R_{\Lambda_{k+1}}(\Lambda_k)},$$

where the sum runs over all primitive submodules Λ_{k+1} of Λ_k. Moreover, by the proof of Lemma 8, all the x-Fourier modes of $\tilde{\mu}^{\Lambda_1\Lambda_2\dots\Lambda_k}\rceil_{\eta_k\in R_{\Lambda_{k+1}}(\Lambda_k)}$ are in Λ_{k+1}. To generalize the analysis of Section 3, we consider test functions $a \in \mathcal{S}^{k+1}_{\Lambda_1,\dots\Lambda_{k+1}}$. For such a function a, we let

$$\begin{aligned}\left\langle w^{\Lambda_1\Lambda_2\dots\Lambda_{k+1}}_{h,R_1,\dots,R_k}(t), a\right\rangle &:= \int_{T^*\mathbb{T}^d}\left(1-\chi\left(\frac{P_{\Lambda_1}(\xi)}{R_1 h}\right)\right)\dots\left(1-\chi\left(\frac{P_{\Lambda_k}(\xi)}{R_k h}\right)\right)\left(1-\chi\left(\frac{P_{\Lambda_{k+1}}(\xi)}{R_{k+1} h}\right)\right)\\ &\qquad a\left(x, \xi, \frac{P_{\Lambda_1}(\xi)}{h}, \cdots, \frac{P_{\Lambda_{k+1}}(\xi)}{h}\right) w_h(t)(dx, d\xi),\end{aligned}$$

and

$$\begin{aligned}\left\langle w^{\Lambda_1\Lambda_2\dots\Lambda_k}_{\Lambda_{k+1},h,R_1,\dots,R_k}(t), a\right\rangle &:= \int_{T^*\mathbb{T}^d}\left(1-\chi\left(\frac{P_{\Lambda_1}(\xi)}{R_1 h}\right)\right)\dots\left(1-\chi\left(\frac{P_{\Lambda_k}(\xi)}{R_k h}\right)\right)\chi\left(\frac{P_{\Lambda_{k+1}}(\xi)}{R_{k+1} h}\right)\\ &\qquad a\left(x, \xi, \frac{P_{\Lambda_1}(\xi)}{h}, \cdots, \frac{P_{\Lambda_{k+1}}(\xi)}{h}\right) w_h(t)(dx, d\xi).\end{aligned}$$

By the Calderón-Vaillancourt theorem, both $w^{\Lambda_1\Lambda_2\dots\Lambda_k}_{\Lambda_{k+1},h,R_1,\dots,R_k}$ and $w^{\Lambda_1\Lambda_2\dots\Lambda_{k+1}}_{h,R_1,\dots,R_k}$ are bounded in $L^\infty(\mathbb{R},(\mathcal{S}^{k+1}_{\Lambda_1,\dots,\Lambda_{k+1}})')$. After extracting subsequences, we can take the following limits :

$$\lim_{R_{k+1}\longrightarrow+\infty}\cdots\lim_{R_1\longrightarrow+\infty}\lim_{h\longrightarrow 0}\left\langle w^{\Lambda_1\Lambda_2\dots\Lambda_{k+1}}_{h,R_1,\dots,R_k}(t)\,,a\right\rangle =: \left\langle \tilde{\mu}^{\Lambda_1\Lambda_2\dots\Lambda_{k+1}},a\right\rangle,$$

and

$$\lim_{R_{k+1}\longrightarrow+\infty}\cdots\lim_{R_1\longrightarrow+\infty}\lim_{h\longrightarrow 0}\left\langle w^{\Lambda_1\Lambda_2\dots\Lambda_k}_{\Lambda_{k+1},h,R_1,\dots,R_k}(t)\,,a\right\rangle =: \left\langle \tilde{\mu}^{\Lambda_1\Lambda_2\dots\Lambda_k}_{\Lambda_{k+1}},a\right\rangle.$$

By the arguments of §3, one then shows that $\tilde{\mu}^{\Lambda_1\Lambda_2\dots\Lambda_{k+1}}$ and $\tilde{\mu}^{\Lambda_1\Lambda_2\dots\Lambda_k}_{\Lambda_{k+1}}$ satisfy all of the induction hypotheses at step $k+1$. In particular, we obtain the following analogues of Theorems 12 and 13.

Theorem 21. *(i) $\tilde{\mu}^{\Lambda_1\Lambda_2\dots\Lambda_{k+1}}(t,\cdot)$ is positive, zero-homogeneous in the variables $\eta_1\in\langle\Lambda_1\rangle,\dots,\eta_{k+1}\in\langle\Lambda_{k+1}\rangle$, and supported at infinity. It can thus be identified with a positive measure on $T^*\mathbb{T}^d\times\mathbb{S}_{\langle\Lambda_1\rangle}\times\dots\times\mathbb{S}_{\langle\Lambda_{k+1}\rangle}$.*

$\tilde{\mu}^{\Lambda_1\Lambda_2\dots\Lambda_k}_{\Lambda_{k+1}}(t,\cdot)$ is zero-homogeneous in the variables $\eta_1\in\langle\Lambda_1\rangle,\dots,\eta_k\in\langle\Lambda_k\rangle$, and supported at infinity. It can thus be identified with a distribution on $T^*\mathbb{T}^d\times\mathbb{S}_{\langle\Lambda_1\rangle}\times\dots\times\mathbb{S}_{\langle\Lambda_k\rangle}\times\langle\Lambda_{k+1}\rangle$.

The projection of $\tilde{\mu}^{\Lambda_1\Lambda_2\dots\Lambda_k}_{\Lambda_{k+1}}(t,\cdot)$ on $T^*\mathbb{T}^d\times\mathbb{S}_{\langle\Lambda_1\rangle}\times\dots\times\mathbb{S}_{\langle\Lambda_k\rangle}$ is positive.

(ii) For a.e. $t\in\mathbb{R}$, $\tilde{\mu}^{\Lambda_1\Lambda_2\dots\Lambda_{k+1}}(t,\cdot)$ and $\tilde{\mu}^{\Lambda_1\Lambda_2\dots\Lambda_k}_{\Lambda_{k+1}}(t,.)$ satisfy the invariance properties:

$$\left(\phi^j_\tau\right)_*\tilde{\mu}^{\Lambda_1\Lambda_2\dots\Lambda_k}_{\Lambda_{k+1}}(t,.)=\tilde{\mu}^{\Lambda_1\Lambda_2\dots\Lambda_k}_{\Lambda_{k+1}}(t,.),$$
$$\left(\phi^j_\tau\right)_*\tilde{\mu}^{\Lambda_1\Lambda_2\dots\Lambda_{k+1}}(t,\cdot)=\tilde{\mu}^{\Lambda_1\Lambda_2\dots\Lambda_{k+1}}(t,\cdot),$$

for $j=0,\dots,k$, $\tau\in\mathbb{R}$.

(iii) Let

$$\mu^{\Lambda_1\Lambda_2\dots\Lambda_k}_{\Lambda_{k+1}}(t,.)=\int_{R_{\Lambda_2}(\Lambda_1)\times\dots\times R_{\Lambda_{k+1}}(\Lambda_k)\times\langle\Lambda_{k+1}\rangle}\tilde{\mu}^{\Lambda_1\Lambda_2\dots\Lambda_k}_{\Lambda_{k+1}}(t,.,d\eta_1,\dots,d\eta_{k+1})\rceil_{(x,\xi)\in\mathbb{T}^d\times R_{\Lambda_1}},$$

$$\mu^{\Lambda_1\Lambda_2\dots\Lambda_{k+1}}(t,.)=\int_{R_{\Lambda_2}(\Lambda_1)\times\dots\times R_{\Lambda_{k+1}}(\Lambda_k)\times\langle\Lambda_{k+1}\rangle}\tilde{\mu}^{\Lambda_1\Lambda_2\dots\Lambda_{k+1}}(t,.,d\eta_1,\dots,d\eta_{k+1})\rceil_{(x,\xi)\in\mathbb{T}^d\times R_{\Lambda_1}}.$$

Then both $\mu^{\Lambda_1\Lambda_2\dots\Lambda_k}_{\Lambda_{k+1}}(t,.)$ and $\mu^{\Lambda_1\Lambda_2\dots\Lambda_{k+1}}(t,.)$ are positive measures on $T^*\mathbb{T}^d$, invariant by the geodesic flow, and satisfy:

$$\mu^{\Lambda_1\Lambda_2\dots\Lambda_k}\rceil_{\eta_k\in R_{\Lambda_{k+1}}(\Lambda_k)}(t,.)=\mu^{\Lambda_1\Lambda_2\dots\Lambda_k}_{\Lambda_{k+1}}(t,.)+\mu^{\Lambda_1\Lambda_2\dots\Lambda_{k+1}}(t,.). \tag{29}$$

Theorem 22. *(i) For a.e. $t\in\mathbb{R}$, $\tilde{\mu}^{\Lambda_1\Lambda_2\dots\Lambda_k}_{\Lambda_{k+1}}(t,.)$ is supported on $\mathbb{T}^d\times\Lambda_{k+1}^\perp\times\mathbb{S}_{\langle\Lambda_1\rangle}\times\dots\times\mathbb{S}_{\langle\Lambda_k\rangle}\times\langle\Lambda_{k+1}\rangle$ and its projection on $\mathbb{T}^d$ is absolutely continuous with respect to the Lebesgue measure.*

(ii) The measure $\tilde{\mu}^{\Lambda_1\Lambda_2\dots\Lambda_{k+1}}(t,\cdot)$ satisfies the additional invariance properties:

$$\left(\phi^{k+1}_\tau\right)_*\tilde{\mu}^{\Lambda_1\Lambda_2\dots\Lambda_{k+1}}(t,\cdot)=\tilde{\mu}^{\Lambda_1\Lambda_2\dots\Lambda_{k+1}}(t,\cdot),$$

for $\tau\in\mathbb{R}$.

The ideas are identical to those of Sections 2 and 3, and detailed proofs will be omitted.

Remark 23. *By construction, if* $\Lambda_{k+1} = \{0\}$*, we have* $\tilde{\mu}^{\Lambda_1\Lambda_2\dots\Lambda_{k+1}} = 0$*, and the induction stops. The measure* $\mu^{\Lambda_1\Lambda_2\dots\Lambda_k}_{\Lambda_{k+1}}$ *is then constant in* x*.*

Similarly to Remark 15, one can also see that if $\operatorname{rk}\Lambda_{k+1} = 1$*, the invariance properties of* $\tilde{\mu}^{\Lambda_1\Lambda_2\dots\Lambda_{k+1}}$ *imply that it is constant in* x*.*

Proof of Theorem 1 (i) and of Corollary 2. We write

$$\mu(t,.) = \sum_{1\leq l\leq d+1} \sum_{\Lambda_1\supset\Lambda_2\supset\dots\supset\Lambda_l} \mu^{\Lambda_1\Lambda_2\dots\Lambda_{l-1}}_{\Lambda_l}(t,.),$$

and we know that each term is a positive measure on $T^*\mathbb{T}^d$, whose projection on $\mathbb{T}^d$ is absolutely continuous. This proves Theorem 1 (i).

Corollary 2 is a direct consequence of Theorem 1 (i) and of the identity (1), with one little subtlety. Because $T^*\mathbb{T}^d$ is not compact, if w_h converges weakly-$*$ to μ and $\left(\int_0^1 |U_V(t)u_h(x)|^2 dt\right) dx$ converges weakly-$*$ to a probability measure ν on $\mathbb{T}^d$, it does not follow automatically that

$$\nu = \int_0^1 \int_{\mathbb{R}^d} \mu(t,\cdot,d\xi)dt.$$

This is only true if we know a priori that $\int_{\mathbb{T}^d\times\mathbb{R}^d} \mu(t,dx,d\xi) = 1$ for almost all t, which means that there is no escape of mass to infinity. To check that Theorem 1 implies Corollary 2, we must explain why, for any normalized sequence $(u_n) \in L^2(\mathbb{T}^d)$, we can find a sequence of parameters $h_n \longrightarrow 0$ such that the sequence $w^{h_n}_{u_n}$ does not escape to infinity. Let us choose h_n such that

$$\sum_{k\in\mathbb{Z}^d, \|k\|\leq h_n^{-1}} |\hat{u}_n(k)|^2 \underset{n\longrightarrow+\infty}{\longrightarrow} 1, \tag{30}$$

which is always possible. If we let $\tilde{u}_n(x) = \sum_{k\in\mathbb{Z}^d,\|k\|\leq h_n^{-1}} \hat{u}_n(k)\frac{e^{ik.x}}{(2\pi)^{d/2}}$, equation (30) implies that $w^{h_n}_{\tilde{u}_n}$ has the same limit as $w^{h_n}_{u_n}$. On the other hand $w^{h_n}_{\tilde{u}_n}$ is supported in the compact set $\mathbb{T}^d \times B(0,1) \subset \mathbb{T}^d \times \mathbb{R}^d$. Thus $w^{h_n}_{\tilde{u}_n}$ cannot escape to infinity. Let us point out that with this choice of scale (h_n), the sequence (u_n) becomes h_n-oscillating, in the terminology introduced in [13, 15].

5. Propagation law for $\tilde{\rho}_\Lambda$

We now study how $\tilde{\rho}_\Lambda(t,\cdot)$ (defined in Proposition 18 (21)) depends on t. This will allow us to complete the proof of Theorem 3 and will be crucial in the proof of the observability inequality, Theorem 4. We use the notation of §3.1. In particular, s will always be a variable in $\mathbb{T}_{\Lambda^\perp}$, and y a variable in $\mathbb{T}_\Lambda$.

In order to state our main result, let us introduce some notation. Let $\widehat{V}_k(t)$, $k \in \mathbb{Z}$, denote the Fourier coefficients of the potential $V(t,\cdot)$. We denote by $\langle V\rangle_\Lambda(t,\cdot)$ the average

of $V(t,\cdot)$ along $\Lambda^\perp$, in other words :

$$\langle V\rangle_\Lambda(t,\cdot) := \sum_{k\in\Lambda} \widehat{V}_k(t)\,\frac{e^{ik\cdot x}}{(2\pi)^{d/2}}.$$

We put $H^\Lambda_{\langle V\rangle_\Lambda}(t) := -\frac{1}{2}\Delta_\Lambda + \langle V\rangle_\Lambda(t,\cdot)$ where Δ_Λ is the Laplacian on $\langle\Lambda\rangle$, and denote by $U^\Lambda_{\langle V\rangle_\Lambda}(t)$ the unitary evolution in $L^2(\mathbb{T}_\Lambda)$, starting at $t=0$, generated by $H^\Lambda_{\langle V\rangle_\Lambda}(t)$.

Proposition 24. *Let $\tilde\rho_\Lambda \in L^\infty(\mathbb{R};\mathcal{M}_+(T^*\mathbb{T}_{\Lambda^\perp};\mathcal{L}^1(L^2(\mathbb{T}_\Lambda))))$ be a limit of $\left(n_h^\Lambda\right)$ as in Proposition 18.*

Let $(s,\sigma)\mapsto K(\sigma)$ be a function in $C_c^\infty(T^*\mathbb{T}_{\Lambda^\perp};\mathcal{K}(L^2(\mathbb{T}_\Lambda)))$ that does not depend on s. Then

$$\frac{d}{dt}\mathrm{Tr}\int_{\mathbb{T}_{\Lambda^\perp}\times R_\Lambda} K(\sigma)\,\tilde\rho_\Lambda(t,ds,d\sigma) = i\mathrm{Tr}\int_{\mathbb{T}_{\Lambda^\perp}\times R_\Lambda}\left[H^\Lambda_{\langle V\rangle_\Lambda}(t,\cdot),K(\sigma)\right]\tilde\rho_\Lambda(t,ds,d\sigma).$$

Corollary 25. *Let $\mu_\Lambda(t,\cdot)$ be the measure defined in Theorem 12.*

For any $a\in C_c^\infty\left(T^*\mathbb{T}^d\right)$ with Fourier coefficients in Λ the following holds:

$$\int_{T^*\mathbb{T}^d} a(x,\xi)\,\mu_\Lambda(t,dx,d\xi) = \mathrm{Tr}\left(\int_{\mathbb{T}_{\Lambda^\perp}\times R_\Lambda} U^\Lambda_{\langle V\rangle_\Lambda}(t)^*\, m_{a\circ\pi_\Lambda}(\sigma)\, U^\Lambda_{\langle V\rangle_\Lambda}(t)\,\tilde\rho_\Lambda(0,ds,d\sigma)\right).$$

Proposition 24 will be a consequence of a more general propagation law. For fixed $s\in\mathbb{T}_{\Lambda^\perp}$, denote by $U^\Lambda_V(t,s)$ $(t\in\mathbb{R})$ the propagator corresponding to the unitary evolution on $L^2(\mathbb{T}_\Lambda)$, starting at $t=0$, generated by

$$H^\Lambda_V(t,s) := -\frac{1}{2}\Delta_\Lambda + V(t,\pi_\Lambda(s,y)).$$

Our main goal in this section will be to establish the following result.

Lemma 26. *For all K as in Proposition 24,*

$$\frac{d}{dt}\mathrm{Tr}\int_{\mathbb{T}_{\Lambda^\perp}\times R_\Lambda}(\sigma)\,\tilde\rho_\Lambda(t,ds,d\sigma) = i\mathrm{Tr}\int_{\mathbb{T}_{\Lambda^\perp}\times R_\Lambda}\left[H^\Lambda_V(t,s),K(\sigma)\right]\tilde\rho_\Lambda(t,ds,d\sigma)$$

(where $\frac{d}{dt}$ is interpreted in distribution sense).

That Proposition 24 follows from Lemma 26 is a consequence of the invariance of $\tilde\rho_\Lambda(t,\cdot)$ with respect to the geodesic flow.

Proof that Lemma 26 implies Proposition 24. Assume that Lemma 26 holds. Since $\tilde\rho_\Lambda(t,\cdot)$ is invariant by $s\mapsto s+\tau\sigma$ $(\tau\in\mathbb{R})$, it follows from Lemma 8 that $\tilde\rho_\Lambda(t,\cdot)\rceil_{\mathbb{T}_{\Lambda^\perp}\times R_\Lambda}$ is invariant by all translations $s\mapsto s+v$ with $v\in\Lambda^\perp$. Therefore,

$$\tilde\rho_\Lambda(t,\cdot)\rceil_{\mathbb{T}_{\Lambda^\perp}\times R_\Lambda} = ds\otimes\int_{\mathbb{T}_{\Lambda^\perp}}\tilde\rho_\Lambda(t,ds,\cdot)\rceil_{R_\Lambda}.$$

As

$$\int_{\mathbb{T}_{\Lambda^\perp}} H^\Lambda_V(t,s)\,ds = -\frac{1}{2}\Delta_\Lambda + \int_{\mathbb{T}_{\Lambda^\perp}} V(t,\pi_\Lambda(s,y))\,ds = H^\Lambda_{\langle V\rangle_\Lambda}(t),$$

the result follows. □

Next we shall prove Lemma 26, first in the smooth case, then for continuous potentials and finally for potentials that satisfy assumption (R).

5.1. **The case of a C^∞ potential.** Here we shall assume that $V \in C^\infty(\mathbb{R}\times\mathbb{T}^d)$. The restriction of $n_h^\Lambda(t)$ to the class of test functions that do not depend on $s \in \mathbb{T}_{\Lambda^\perp}$ satisfies a certain propagation law, that we now describe. This generalizes statement (ii) in Theorem 2 of [19].

Lemma 27. *If $K \in C_c^\infty\left(\Lambda^\perp; \mathcal{K}\left(L^2\left(\mathbb{T}_\Lambda\right)\right)\right)$ is a function that does not depend on s then*

$$\frac{d}{dt}\left\langle n_h^\Lambda(t), K\right\rangle = i\left\langle T_\Lambda u_h, \left[H_V^\Lambda(t,\cdot), K(hD_s)\right] T_\Lambda u_h\right\rangle_{L^2\left(\mathbb{T}_{\Lambda^\perp}; L^2(\mathbb{T}_\Lambda)\right)}. \tag{31}$$

Proof. It is simple to check that (19) gives:

$$T_\Lambda \Delta T_\Lambda^* = \Delta_\Lambda + \Delta_{\Lambda^\perp}.$$

Moreover, it is clear that:

$$[\Delta_{\Lambda^\perp}, K(hD_s)] = 0.$$

Therefore, equation (20), in the case when K does not depend on s, gives (31). □

Taking limits in equation (31) and taking into account that we can restrict $\tilde{\rho}_\Lambda$ to $(s,\sigma) \in \mathbb{T}_{\Lambda^\perp}\times R_\Lambda$ (since it is a positive measure), concludes the proof of Lemma 26 in this case.

5.2. **The case of a continuous potential.** In this section, we assume that $V \in C(\mathbb{R}\times\mathbb{T}^d)$. In this case, Lemma 27 still holds, but we cannot obtain 26 by simply taking limits. Instead, we shall use an elementary approximation argument.

We introduce a sequence V_n of C^∞ potentials, such that

$$\|V - V_n\|_{L^\infty} \leq \frac{1}{n}.$$

We rewrite equation (31),

$$\frac{d}{dt}\left\langle n_h^\Lambda(t), K\right\rangle = i\left\langle T_\Lambda u_h(t), \left[H_{V_n}^\Lambda(t,\cdot), K(hD_s)\right] T_\Lambda u_h(t)\right\rangle + i\left\langle T_\Lambda u_h(t), [V - V_n, K(hD_s)] T_\Lambda u_h(t)\right\rangle.$$

We use the inequality

$$\left|\langle T_\Lambda u_h, [V - V_n, K(hD_s)] T_\Lambda u_h\rangle\right| \leq 2\|V - V_n\|_{L^\infty} \sup_{\sigma\in\Lambda^\perp}\|K(\sigma)\|$$

to estimate the error when replacing V by V_n.

In the limit $h \longrightarrow 0$,

$$\left\langle T_\Lambda u_h, \left[H_{V_n}^\Lambda(t,\cdot), K(hD_s)\right] T_\Lambda u_h\right\rangle \longrightarrow \mathrm{Tr}\int_{T^*\mathbb{T}_{\Lambda^\perp}}\left[H_{V_n}^\Lambda(t,\cdot), K(\sigma)\right]\tilde{\rho}_\Lambda(t, ds, d\sigma)$$

since V_n is smooth. We use again the inequality

$$\left|\mathrm{Tr}\int_{T^*\mathbb{T}_{\Lambda^\perp}}\left[V-V_n,K\left(\sigma\right)\right]\tilde{\rho}_\Lambda\left(t,ds,d\sigma\right)\right|\leq 2\|V-V_n\|_{L^\infty}\sup_{\sigma\in\Lambda^\perp}\|K(\sigma)\|$$

to estimate the error when replacing V_n by V.

Letting $h\longrightarrow 0$ and then $n\longrightarrow+\infty$, we find that

$$\frac{d}{dt}\mathrm{Tr}\int_{T^*\mathbb{T}_{\Lambda^\perp}}K\left(\sigma\right)\tilde{\rho}_\Lambda\left(t,ds,d\sigma\right)=i\mathrm{Tr}\int_{T^*\mathbb{T}_{\Lambda^\perp}}\left[H_V^\Lambda\left(t,s\right),K\left(\sigma\right)\right]\tilde{\rho}_\Lambda\left(t,ds,d\sigma\right)$$

where $\frac{d}{dt}$ is meant in the distribution sense.

Again, we can restrict $\tilde{\rho}_\Lambda$ to $(s,\sigma)\in\mathbb{T}_{\Lambda^\perp}\times R_\Lambda$ since it is a positive measure. This concludes the proof of Lemma 26 in the continuous case.

5.3. **Case of an L^∞ potential.** Let us turn to the case of a potential V that satisfies condition (R) of the introduction. We use again an approximation argument, but we have to use the fact that we already know that the limit measures are absolutely continuous.

It is enough to consider the restriction of $n_h^\Lambda(t)$ to $t\in[0,T]$, for any arbitrary T. For any $\epsilon>0$, we then consider the set K_ϵ and the function V_ϵ described in Assumption (R). Consider an open set $W_{2\epsilon}$ of Lebesgue measure $<2\epsilon$ such that $K_\epsilon\subset W_{2\epsilon}$. Let us introduce a continuous function χ_ϵ taking values in $[0,1]$, and which takes the value 1 on the complement of $W_{2\epsilon}$ and 0 on K_ϵ (this is where we use the fact that K_ϵ is closed).

Lemma 27 still holds. We use it to write

$$\begin{aligned}\frac{d}{dt}\left\langle n_h^\Lambda\left(t\right),K\right\rangle &= i\left\langle T_\Lambda u_h(t),\left[H^\Lambda_{\chi_\epsilon V_\epsilon}\left(t,\cdot\right),K\left(hD_s\right)\right]T_\Lambda u_h(t)\right\rangle\\ &\quad+i\left\langle T_\Lambda u_h(t),\left[\chi_\epsilon(t)\left(V(t)-V_\epsilon(t)\right),K\left(hD_s\right)\right]T_\Lambda u_h(t)\right\rangle\\ &\quad+i\left\langle T_\Lambda u_h(t),\left[V(1-\chi_\epsilon)(t),K\left(hD_s\right)\right]T_\Lambda u_h(t)\right\rangle.\end{aligned}\tag{32}$$

Arguing as in §5.2, we see that

$$\left\langle T_\Lambda u_h,\left[H^\Lambda_{\chi_\epsilon V_\epsilon}\left(t,\cdot\right),K\left(hD_s\right)\right]T_\Lambda u_h\right\rangle$$

converges to

$$\mathrm{Tr}\int_{T^*\mathbb{T}_{\Lambda^\perp}}\left[H^\Lambda_{\chi_\epsilon V_\epsilon}\left(t,\cdot\right),K\left(\sigma\right)\right]\tilde{\rho}_\Lambda\left(t,ds,d\sigma\right)\tag{33}$$

in the limit $h\longrightarrow 0$, since $\chi_\epsilon V_\epsilon$ is continuous. Note that we can replace V_ϵ by V in this limiting term (33), up to an error of $2\epsilon\sup_{\sigma\in\Lambda^\perp}\|K(\sigma)\|$. Analogously, we are going to show that in the limit $h\longrightarrow 0$ the remaining error terms give a contribution that vanishes as ϵ tends to zero. In other words, we are going to show that the following equation holds,

$$\frac{d}{dt}\mathrm{Tr}\int_{T^*\mathbb{T}_{\Lambda^\perp}}K\left(\sigma\right)\tilde{\rho}_\Lambda\left(t,ds,d\sigma\right)=i\mathrm{Tr}\int_{T^*\mathbb{T}_{\Lambda^\perp}}\left[H^\Lambda_{\chi_\epsilon V}\left(t,s\right),K\left(\sigma\right)\right]\tilde{\rho}_\Lambda\left(t,ds,d\sigma\right)+\sup_{\sigma\in\Lambda^\perp}\|K(\sigma)\|R_\epsilon,\tag{34}$$

where R_ϵ does not depend on K, and goes to 0 as $\epsilon\longrightarrow 0$. To do so, we estimate the error terms involved.

The term $|\langle T_\Lambda u_h(t), [\chi_\epsilon(V-V_\epsilon), K(hD_s)]\, T_\Lambda u_h(t)\rangle|$ is easily seen to be bounded from above by $2\epsilon \sup_{\sigma\in\Lambda^\perp}\|K(\sigma)\|$.

We now turn to the error term involving $V(1-\chi_\epsilon)$ in (32). We use the fact that this function is supported on a set of small measure, and that we know that the limit measures are absolutely continuous. We deal with the first term in the commutator, the second one may be treated analogously. Clearly

$$|\langle T_\Lambda u_h(t), V(1-\chi_\epsilon)K(hD_s)\, T_\Lambda u_h(t)\rangle| \leq \|V\|_{L^\infty} \sup_{\sigma\in\Lambda^\perp}\|K(\sigma)\|\|u_h(t)\|\|(1-\chi_\epsilon)u_h(t)\|.$$

Integrating along an L^1 function $\theta(t)$,

$$\begin{aligned}
&\left|\int_0^T \theta(t)\,\langle T_\Lambda u_h(t), V(1-\chi_\epsilon)K(hD_s)\, T_\Lambda u_h(t)\rangle\, dt\right| \\
&\qquad\leq \|V\|_{L^\infty}\sup_{\sigma\in\Lambda^\perp}\|K(\sigma)\|\int_0^T |\theta(t)|\|u_h(t)\|\|(1-\chi_\epsilon)u_h(t)\|dt \\
&\leq \|V\|_{L^\infty}\sup_{\sigma\in\Lambda^\perp}\|K(\sigma)\|\left(\int_0^T |\theta(t)|\|u_h(t)\|^2 dt\right)^{1/2}\left(\int_0^T |\theta(t)|\|(1-\chi_\epsilon)u_h(t)\|^2 dt\right)^{1/2} \\
&\qquad= \|V\|_{L^\infty}\sup_{\sigma\in\Lambda^\perp}\|K(\sigma)\|\left(\int_0^T |\theta(t)|dt\right)^{1/2}\left(\int_0^T |\theta(t)|\|(1-\chi_\epsilon)u_h(t)\|^2 dt\right)^{1/2}
\end{aligned}$$

By Corollary 2 we know that $\int_0^T |\theta(t)|\|(1-\chi_\epsilon)u_h(t)\|^2 dt$ converges as $h\longrightarrow 0$ (along a subsequence) to

$$\int_0^T\int_{\mathbb{T}^d} |\theta(t)||1-\chi_\epsilon(t,x)|^2\nu_t(dx)dt$$

where ν_t is an absolutely continuous probability measure on $\mathbb{T}^d$. The function $|1-\chi_\epsilon(t,x)|$ takes values in $[0,1]$ and is supported in $W_{2\epsilon}$, of measure $<2\epsilon$. Thus,

$$\int_0^T\int_{\mathbb{T}^d} |\theta(t)||1-\chi_\epsilon(t,x)|^2\nu_t(dx)dt \longrightarrow 0$$

as $\epsilon\longrightarrow 0$.

Equation (34) is now proved. Restricting $\tilde\rho_\Lambda$ to $(s,\sigma)\in\mathbb{T}_{\Lambda^\perp}\times R_\Lambda$, it follows that

(35)
$$\frac{d}{dt}\mathrm{Tr}\int_{\mathbb{T}_{\Lambda^\perp}\times R_\Lambda} K(\sigma)\,\tilde\rho_\Lambda(t,ds,d\sigma) = i\mathrm{Tr}\int_{\mathbb{T}_{\Lambda^\perp}\times R_\Lambda}\left[H^\Lambda_{\chi_\epsilon V}(t,s), K(\sigma)\right]\tilde\rho_\Lambda(t,ds,d\sigma) + \sup_{\sigma\in\Lambda^\perp}\|K(\sigma)\|R_\epsilon,$$

There remains to show how to conclude Lemma 26 from equation (35). To do so, we prove that

(36)
$$\mathrm{Tr}\int_{\mathbb{T}_{\Lambda^\perp}\times R_\Lambda}\left[H^\Lambda_{\chi_\epsilon V}(t,\cdot), K(\sigma)\right]\tilde\rho_\Lambda(t,ds,d\sigma)$$

is the same as

$$\mathrm{Tr}\int_{\mathbb{T}_{\Lambda^\perp}\times R_\Lambda}\left[H_V^\Lambda\left(t,\cdot\right),K\left(\sigma\right)\right]\tilde{\rho}_\Lambda\left(t,ds,d\sigma\right) \tag{37}$$

up to an error which goes to 0 with ϵ. The difference between both is

$$\mathrm{Tr}\int_{\mathbb{T}_{\Lambda^\perp}\times R_\Lambda}\left[V(1-\chi_\epsilon)(t),K\left(\sigma\right)\right]\tilde{\rho}_\Lambda\left(t,ds,d\sigma\right)=\mathrm{Tr}\int_{\mathbb{T}_{\Lambda^\perp}\times R_\Lambda}V(1-\chi_\epsilon)(t)K\left(\sigma\right)\tilde{\rho}_\Lambda\left(t,ds,d\sigma\right)$$
$$-\mathrm{Tr}\int_{\mathbb{T}_{\Lambda^\perp}\times R_\Lambda}K\left(\sigma\right)V(1-\chi_\epsilon)(t)\tilde{\rho}_\Lambda\left(t,ds,d\sigma\right).$$

Let us consider for instance

$$\mathrm{Tr}\int_{\mathbb{T}_{\Lambda^\perp}\times R_\Lambda}V(1-\chi_\epsilon)(t)K\left(\sigma\right)\tilde{\rho}_\Lambda\left(t,ds,d\sigma\right). \tag{38}$$

For any $\theta\in L^1(\mathbb{R})$, the measure

$$a\in C([0,T]\times\mathbb{T}^d)\mapsto\int_0^T\theta(t)\mathrm{Tr}\int_{\mathbb{T}_{\Lambda^\perp}\times R_\Lambda}m_aK\left(\sigma\right)\tilde{\rho}_\Lambda\left(t,ds,d\sigma\right)dt$$

is absolutely continuous, therefore

$$\int_0^T\theta(t)\mathrm{Tr}\int_{\mathbb{T}_{\Lambda^\perp}\times R_\Lambda}V(1-\chi_\epsilon)(t)K\left(\sigma\right)\tilde{\rho}_\Lambda\left(t,ds,d\sigma\right)dt$$

goes to 0 when $\epsilon\longrightarrow 0$.

This finishes the proof of Lemma 26.

Remark 28. *The same argument applies to show that the operator-valued measure*

$$\tilde{\rho}_{\Lambda_l}^{\Lambda_1\Lambda_2\ldots\Lambda_{l-1}}\left(t,ds,d\sigma,d\eta_1,\ldots,d\eta_l\right)$$

appearing in (28) *satisfies the propagation law analogous to Proposition 24*

$$\frac{d}{dt}\mathrm{Tr}\int_{T^*\mathbb{T}_{\Lambda_l^\perp}\times R_{\Lambda_2}(\Lambda_1)\times\ldots\times R_{\Lambda_l}(\Lambda_{l-1})}K\left(\sigma\right)\tilde{\rho}_{\Lambda_l}^{\Lambda_1\Lambda_2\ldots\Lambda_{l-1}}\left(t,ds,d\sigma,d\eta_1,\ldots,d\eta_{l-1}\right)$$
$$=i\mathrm{Tr}\int_{T^*\mathbb{T}_{\Lambda_l^\perp}\times R_{\Lambda_2}(\Lambda_1)\times\ldots\times R_{\Lambda_l}(\Lambda_{l-1})}\left[H_{\langle V\rangle_{\Lambda_l}}^{\Lambda_l}\left(t,\cdot\right),K\left(\sigma\right)\right]\tilde{\rho}_{\Lambda_l}^{\Lambda_1\Lambda_2\ldots\Lambda_{l-1}}\left(t,ds,d\sigma,d\eta_1,\ldots,d\eta_{l-1}\right).$$

5.4. **End of proof of Theorem 3.** To end the proof of Theorem 3, we let

$$\nu_\Lambda(t,\cdot)=\sum_{0\leq k\leq d-1}\ \sum_{\Lambda_1\supset\Lambda_2\supset\cdots\supset\Lambda_k\supset\Lambda}\int_{\mathbb{R}^d}\mu_\Lambda^{\Lambda_1\Lambda_2\ldots\Lambda_k}(t,\cdot,d\xi),$$

where $\Lambda_1,\ldots,\Lambda_k$ run over the set of strictly decreasing sequences of submodules, such that $\Lambda_k\subset\Lambda$. We also let

$$\sigma_\Lambda=\sum_{0\leq k\leq d-1}\ :\ \sum_{\Lambda_1\supset\Lambda_2\supset\cdots\supset\Lambda_k\supset\Lambda}\int_{\mathbb{T}^d\times R_{\Lambda_1}\times R_{\Lambda_2}(\Lambda_1)\times\ldots\times R_\Lambda(\Lambda_k)\times\langle\Lambda\rangle}\tilde{\rho}_\Lambda^{\Lambda_1\Lambda_2\ldots\Lambda_k}\left(0,ds,d\sigma,d\eta_1,\ldots,d\eta_k,d\eta\right),$$

where the $\tilde{\rho}_\Lambda^{\Lambda_1\Lambda_2\dots\Lambda_k}$ are the operator-valued measures appearing in (28).

6. Propagation of $\bar{\mu}$ and end of the proof of Theorem 1

We have already proved statement (i) of Theorem 1; we shall now concentrate on (ii). We shall need a preliminary result, which is of independent interest, that describes the propagation of $\bar{\mu}$, the projection of μ onto the variable $\xi \in \mathbb{R}^d$.

Proposition 29. *Suppose that $\mu_0 \in \mathcal{M}_+\left(T^*\mathbb{T}^d\right)$ is a semiclassical measure of (u_h). Then $\bar{\mu}$ is constant for a.e. t and,*

$$\bar{\mu} = \int_{\mathbb{T}^d} \mu_0\left(dy, \cdot\right). \tag{39}$$

Proof. We write for $a \in C_c^\infty(\mathbb{R}^d)$ and $T \in \mathbb{R}$:

$$\begin{aligned}&\langle U_V(T)u_h, a\left(hD_x\right) U_V(T)u_h\rangle - \langle u_h, a\left(hD_x\right) u_h\rangle \\ &= -i\int_0^T \langle U_V(t)u_h, \left[a\left(hD_x\right), -\frac{\Delta}{2} + V\right] U_V(t)u_h\rangle dt = -i\int_0^T \langle U_V(t)u_h, \left[a\left(hD_x\right), V\right] U_V(t)u_h\rangle dt.\end{aligned}$$

If $V \in C^\infty(\mathbb{R}\times\mathbb{T}^d)$, we have the estimate coming from pseudodifferential calculus,

$$\|[a(hD_x), V]\|_{L^2(\mathbb{T}^d)\longrightarrow L^2(\mathbb{T}^d)} = \mathcal{O}(h).$$

This implies that, for every $T \in \mathbb{R}$:

$$\lim_{h\to 0^+} \langle U_V(T)u_h, a\left(hD_x\right) U_V(T)u_h\rangle = \int_{T^*\mathbb{T}^d} a\left(\xi\right)\mu_0\left(dx, d\xi\right), \tag{40}$$

which in turn shows (39).

When $V \in C(\mathbb{R}\times\mathbb{T}^d)$, we establish (40) by showing that

$$\|[a(hD_x), V]\|_{L^2(\mathbb{T}^d)\longrightarrow L^2(\mathbb{T}^d)} \underset{h\longrightarrow 0}{\longrightarrow} 0.$$

This can be proved by an approximation argument as in §5.2 :

$$[a\left(hD_x\right), V] = [a\left(hD_x\right), V_n] + [a\left(hD_x\right), V - V_n],$$

with $[a\left(hD_x\right), V_n] \underset{h\longrightarrow 0}{\longrightarrow} 0$ if $V_n \in C^\infty(\mathbb{R}\times\mathbb{T}^d)$, and

$$\|[a(hD_x), V - V_n]\|_{L^2\longrightarrow L^2} \leq 2\|a(hD_x)\|_{L^2\longrightarrow L^2}\|V - V_n\|_{L^\infty}.$$

If V satisfies Assumption (R), we write with the same notation as in §5.3,

$$\begin{aligned}\int_0^T \langle U_V(t)u_h, [a\left(hD_x\right), V]U_V(t)u_h\rangle dt \\ &= \int_0^T \langle U_V(t)u_h, [a\left(hD_x\right), V_\epsilon\chi_\epsilon]U_V(t)u_h\rangle dt \\ &+ \int_0^T \langle U_V(t)u_h, [a\left(hD_x\right), (V-V_\epsilon)\chi_\epsilon]U_V(t)u_h\rangle dt \\ &+ \int_0^T \langle U_V(t)u_h, [a\left(hD_x\right), V(1-\chi_\epsilon)]U_V(t)u_h\rangle dt.\end{aligned}$$

For fixed ϵ, the term $\int_0^T \langle U_V(t)u_h, [a\left(hD_x\right), V_\epsilon\chi_\epsilon]U_V(t)u_h\rangle dt$ goes to 0 as $h \longrightarrow 0$. The term $|\int_0^T \langle U_V(t)u_h, [a\left(hD_x\right), (V-V_\epsilon)\chi_\epsilon]U_V(t)u_h\rangle dt|$ is less than $2\epsilon\|a\left(hD_x\right)\|$. Finally,

$$\begin{aligned}&\left|\int_0^T \langle U_V(t)u_h, [a\left(hD_x\right), V(1-\chi_\epsilon)]U_V(t)u_h\rangle dt\right| \\ &\qquad \leq 2\|V\|_\infty \int_0^T \|a(hD_x)U_V(t)u_h\|_{L^2(\mathbb{T}^d)}\|(1-\chi_\epsilon)U_V(t)u_h\|_{L^2(\mathbb{T}^d)}dt \\ &\qquad \leq 2\|V\|_\infty \left(\int_0^T \|a(hD_x)U_V(t)u_h\|^2_{L^2(\mathbb{T}^d)}dt\right)^{1/2}\left(\int_0^T \|(1-\chi_\epsilon)U_V(t)u_h\|^2_{L^2(\mathbb{T}^d)}dt\right)^{1/2}\end{aligned}$$

and this goes to 0 at the limits $h \longrightarrow 0$ and $\epsilon \longrightarrow 0$, by the same argument as in §5.3. Again, we conclude that (40) holds in this case. This concludes the proof of the proposition. □

Corollary 30. *Let Λ be a primitive submodule of $\mathbb{Z}^d$. If $\mu_0\left(\mathbb{T}^d \times \Lambda^\perp\right) = 0$ then $\sigma_\Lambda = 0$, where σ_Λ is the operator appearing in Theorem 3.*

6.1. **End of proof of Theorem 1.** Let us turn to the proof of the last assertion of Theorem 1. Let us consider the disintegration of the limit measure μ with respect to ξ. Here, to simplify the discussion, after normalizing μ we may assume that it is a probability measure (this is no loss of generality, since the result is trivially true when $\mu = 0$). We call $\bar{\mu}$ the probability measure on $\mathbb{R}^d$, image of $\mu(t,\cdot)$ under the projection map $(x,\xi) \mapsto \xi$. We know that it does not depend on t. We denote by $\mu_\xi(t,\cdot)$ the conditional law of x knowing ξ, when the pair (x,ξ) is distributed according to $\mu(t,\cdot)$. Starting from Theorem 1 (i), we now show that, for $\bar{\mu}$-almost every ξ, the probability measure $\mu_\xi(t,\cdot)$ is absolutely continuous.

We consider a filtration, that is to say, a sequence $\mathcal{F}_n \subset \mathcal{F}_{n+1}$ of Borel σ-fields of $\mathbb{R}^d$, such that $\cup_n \mathcal{F}_n$ generates the whole σ-field of Borel sets. We will choose $\mathcal{F}_n$ generated by a finite partition made of hypercubes (that is, a family of disjoint sets of the form $[a_1,b_1)\times\ldots\times[a_d,b_d)$, where $a_d < b_d$ can be finite or infinite). For every ξ, there is a unique such hypercube containing ξ, and we denote this hypercube by $\mathcal{F}_n(\xi)$. Finally, we choose $\mathcal{F}_n$ such that $\bar{\mu}$ does not put any weight on the boundary of each hypercube.

We know (by the martingale convergence theorem) that, for $\bar{\mu}$-almost every ξ, for every continuous compactly supported function f and every non-negative integrable θ,

$$\int_{\mathbb{T}^d} f(x,\xi)\mu_\xi(t,dx)\theta(t)dt = \lim_n \frac{\int_{\mathbb{T}^d\times\mathcal{F}_n(\xi)} f(x,\eta)\mu(t,dx,d\eta)\theta(t)dt}{\int \mu(t,\mathbb{T}^d\times\mathcal{F}_n(\xi))\theta(t)dt}. \tag{41}$$

Fix ξ such that (41) holds. Since $\mu(t,\cdot)$ is itself the limit of the Wigner distributions $w_h(t,\cdot)$ and since it does not put any weight on the boundary of $\mathcal{F}_n(\xi)$, we can choose
– a sequence of smooth compactly supported functions χ_n (obtained by convolution of the characteristic function of $\mathcal{F}_n(\xi)$ by a smooth kernel), and
– a sequence h_n, going to zero as fast as we wish,
such that

$$\int_{\mathbb{T}^d} f(x,\xi)\mu_\xi(t,dx)\theta(t)dt = \lim_n \frac{\int_{\mathbb{T}^d\times\mathbb{R}^d} \chi_n^2(\eta)f(x,\eta)w_{h_n}(t,dx,d\eta)\theta(t)dt}{\int_{\mathbb{T}^d\times\mathbb{R}^d} \chi_n^2(\eta)w_{h_n}(t,dx,d\eta)\theta(t)dt} \tag{42}$$

for all smooth compactly supported f and every θ.

The absolute continuity of μ_ξ now follows from Theorem 1 (i), applied to the sequence of functions

$$v_{h_n} = \frac{\mathrm{Op}_{h_n}(\chi_n)u_{h_n}}{||\,\mathrm{Op}_{h_n}(\chi_n)u_{h_n}||}.$$

7. Observability estimates

We now turn to the proof of Theorem 4. Using the uniqueness-compactness argument of Bardos, Lebeau and Rauch [4] and a Littlewood-Payley decomposition, one can reduce the proof of Theorem 4 to the following Proposition 31. This is clearly detailed in [7], from which we borrow the notation. This reduction requires the potential to be *time-independent* and this is why we make this assumption in Theorem 4.

Let $\chi \in C_c^\infty((-1/2,2))$ be a cut-off function equal to 1 close to 1 and define, for $h>0$:

$$\Pi_h u_0 := \chi\left(h^2\left(-\frac{1}{2}\Delta + V\right)\right)$$

Proposition 31. *Given any $T>0$ and any open set $\omega\subset\mathbb{T}^d$, there exist $C, h_0>0$ such that:*

$$\|\Pi_h u_0\|^2_{L^2(\mathbb{T}^d)} \le C\int_0^T \|U_V(t)\,\Pi_h u_0\|^2_{L^2(\omega)}\,dt, \tag{43}$$

for every $0<h<h_0$ and every $u_0\in L^2\left(\mathbb{T}^d\right)$.

Proof. We argue by contradiction; if (43) were false, then there would exist a sequence (h_n) tending to zero and $(u_{0,n})$ in $L^2\left(\mathbb{T}^d\right)$ such that $\Pi_{h_n}u_{0,n}=u_{0,n}$,

$$\|u_{0,n}\|_{L^2(\mathbb{T}^d)} = 1, \quad \lim_{n\to\infty}\int_0^T \|U_V(t)\,u_{0,n}\|^2_{L^2(\omega)}\,dt = 0.$$

After eventually extracting a subsequence, we can assume that $(u_{0,n})$ has a semiclassical measure μ_0 and that the Wigner distributions of $(U_V(t)\,u_{0,n})$ converge weak-$*$ to some $\mu \in L^\infty\left(\mathbb{R}; \mathcal{M}_+\left(T^*\mathbb{T}^d\right)\right)$. By construction, we have that:

$$\mu_0\left(T^*\mathbb{T}^d\right) = 1, \quad \mu_0\left(\mathbb{T}^d \times \{0\}\right) = 0;$$

and therefore, by Proposition 29, the same holds for $\mu(t,\cdot)$ for a.e. $t \in \mathbb{R}$. Moreover,

$$\int_0^T \mu\left(t, \omega \times \mathbb{R}^d\right) dt = 0. \tag{44}$$

Now, we shall use Theorem 3 to obtain a contradiction. We first establish the inequality for $d = 1$ and then use an induction on the dimension.

Case $d = 1$. Since $\mu(t, \mathbb{T} \times \{0\}) = 0$ and $\mu(t,\cdot)$ is invariant by the geodesic flow, it turns out that $\mu(t,\cdot)$ is constant. Since (44) holds, necessarily $\mu(t,\cdot) = 0$, which contradicts the fact that $\mu(t, T^*\mathbb{T}) = 1$. This establishes Proposition 31, and therefore, Theorem 4 for $d = 1$.

Case $d \geq 2$. We make the induction hypothesis that Proposition 31 holds for all tori $\mathbb{R}^n/2\pi\Gamma$ with $n \leq d-1$, and Γ a lattice in $\mathbb{R}^n$ such that $[\langle x, y\rangle \in \mathbb{Q}\ \forall y \in \mathbb{Q}\Gamma \Leftrightarrow x \in \mathbb{Q}\Gamma]$.

Now, as shown in Theorem 3, for $b \in L^\infty\left(\mathbb{T}^d\right)$ we have:

$$\int_{T^*\mathbb{T}^d} b(x)\,\mu(t, dx, d\xi) = \sum_\Lambda \int_{\mathbb{T}^d} b(x)\,\nu_\Lambda(t, dx) = \sum_\Lambda \operatorname{Tr}\left(m_{\langle b\rangle_\Lambda}\, U_{\langle V\rangle_\Lambda}(t)\,\sigma_\Lambda\, U_{\langle V\rangle_\Lambda}(t)^*\right),$$

where $m_{\langle b\rangle_\Lambda}$ denotes multiplication by $\langle b\rangle_\Lambda$ and σ_Λ is a trace-class positive operator on $L^2(\mathbb{T}_\Lambda)$, where recall, $\mathbb{T}_\Lambda = \langle\Lambda\rangle/2\pi\Lambda$.

For $\Lambda = 0$, the measure $\nu_\Lambda(t)$ is constant in x, and since $\nu_\Lambda(t,\omega) = 0$ we have $\nu_\Lambda(t) = 0$.

The fact that $\mu\left(t, \mathbb{T}^d \times \{0\}\right) = 0$ implies that $\sigma_\Lambda = 0$ for $\Lambda = \mathbb{Z}^d$. Therefore, it suffices to show that $\sigma_\Lambda = 0$ for every primitive non-zero submodule $\Lambda \subset \mathbb{Z}^d$ of rank $\leq d-1$.

The torus $\mathbb{T}_\Lambda$ has dimension $\leq d-1$ and falls into the range of our induction hypothesis. Since (44) holds, we conclude that:

$$\int_0^T \operatorname{Tr}\left(m_{\langle \mathbf{1}_\omega\rangle_\Lambda}\, U_{\langle V\rangle_\Lambda}(t)\,\sigma_\Lambda\, U_{\langle V\rangle_\Lambda}(t)^*\right) dt = 0,$$

and hence

$$\int_0^T \operatorname{Tr}\left(m_{\mathbf{1}_{\langle\omega\rangle_\Lambda}}\, U_{\langle V\rangle_\Lambda}(t)\,\sigma_\Lambda\, U_{\langle V\rangle_\Lambda}(t)^*\right) dt = 0,$$

where $\langle\omega\rangle_\Lambda$ is the open set where $\langle \mathbf{1}_\omega\rangle_\Lambda > 0$. By our induction hypothesis we have:[7]

$$\mathrm{Tr}\,(\sigma_\Lambda) \leq C(T,\langle\omega\rangle_\Lambda)\int_0^T \mathrm{Tr}\left(m_{\mathbf{1}_{\langle\omega\rangle_\Lambda}}\, U_{\langle V\rangle_\Lambda}(t)\,\sigma_\Lambda\, U_{\langle V\rangle_\Lambda}(t)^*\right)dt,$$

and thus $\sigma_\Lambda = 0$ (for all Λ) and $\mu(t, T^*\mathbb{T}^d) = 0$. This contradicts the fact that $\mu\left(t, T^*\mathbb{T}^d\right) = 1$. □

Coming back to the semiclassical measures of Theorem 1, it is now obvious that

$$\int_0^T \mu(t,\omega\times\mathbb{R}^d)dt \geq \frac{T}{C(T,\omega)}\mu_0(T^*\mathbb{T}^d).$$

Corollary 5 can then be derived by the same argument as in §6.1.

8. Appendix : pseudodifferential calculus

In the paper, we use the Weyl quantization with parameter h, that associates to a function a on $T^*\mathbb{R}^d = \mathbb{R}^d\times\mathbb{R}^d$ an operator $\mathrm{Op}_h(a)$, with kernel

$$K_a^h(x,y) = \frac{1}{(2\pi h)^d}\int_{\mathbb{R}^d} a\left(\frac{x+y}{2},\xi\right)e^{\frac{i}{h}\xi.(x-y)}d\xi.$$

If a is smooth and has uniformly bounded derivatives, then this defines a continuous operator $\mathcal{S}(\mathbb{R}^d)\longrightarrow\mathcal{S}(\mathbb{R}^d)$, and also $\mathcal{S}'(\mathbb{R}^d)\longrightarrow\mathcal{S}'(\mathbb{R}^d)$. If a is $(2\pi\mathbb{Z})^d$-periodic with respect to the first variable (which is always the case in this paper), the operator preserves the space of $(2\pi\mathbb{Z})^d$-periodic distributions on $\mathbb{R}^d$. We note the relation $\mathrm{Op}_h(a(x,\xi)) = \mathrm{Op}_1(a(x,h\xi))$.

We use two standard results of pseudodifferential calculus.

Theorem 32. *(The Calderón-Vaillancourt theorem)*
There exists an integer K_d, and a constant $C_d > 0$ (depending on the dimension d) such that, if a if a smooth function on $T^*\mathbb{T}^d$, with uniformly bounded derivatives, then

$$\|\mathrm{Op}_1(a)\|_{L^2(\mathbb{T}^d)\longrightarrow L^2(\mathbb{T}^d)} \leq C_d \sum_{\alpha\in\mathbb{N}^{2d},|\alpha|\leq K_d}\sup_{T^*\mathbb{T}^d}|\partial^\alpha a|.$$

A proof in the case of $L^2(\mathbb{R}^d)$ can be found in [9]. It can be adapted to the case of a compact manifold by working locally, in coordinate charts.

We also recall the following formula for the product of two pseudodifferential operators (see for instance [9], p. 79) : $\mathrm{Op}_1(a)\circ\mathrm{Op}_1(b) = \mathrm{Op}_1(a\sharp b)$, where

$$a\sharp b(x,\xi) = \frac{1}{(2\pi)^{4d}}\int_{\mathbb{R}^{4d}} e^{\frac{i}{2}\sigma(u_1,u_2)}(\mathcal{F}a_z)(u_1)(\mathcal{F}b_z)(u_2)du_1du_2,$$

[7] To deduce this from Theorem 4, it suffices to write σ_Λ as a linear combination of orthogonal projectors on an orthonormal basis of eigenfunctions of σ_Λ:

$$\sigma_\Lambda = \sum_{n\in\mathbb{N}}\lambda_n\,|\phi_n\rangle\,\langle\phi_n|\,;$$

since $\lambda_n \geq 0$ and $\sum_{n\in\mathbb{N}}\lambda_n < \infty$ the observability inequality for σ_Λ follows from the fact that it holds for every ϕ_n.

where we let $z = (x, \xi) \in \mathbb{R}^{2d}$, a_z is the function $\omega \mapsto a(z + \omega)$, and $\mathcal{F}$ is the Fourier transform. We can deduce from this formula and from the Calderón-Vaillancourt theorem the following estimate :

Proposition 33. *Let a and b be two smooth functions on $T^*\mathbb{T}^d$, with uniformly bounded derivatives.*

$$\|\mathrm{Op}_1(a) \circ \mathrm{Op}_1(b) - \mathrm{Op}_1(ab)\|_{L^2(\mathbb{T}^d) \longrightarrow L^2(\mathbb{T}^d)} \leq C_d \sum_{\alpha \in \mathbb{N}^{2d}, |\alpha| \leq K_d} \sup_{T^*\mathbb{T}^d} |\partial^\alpha D(a, b)|,$$

where we denote $D(a, b)$ the function $D(a, b)(x, \xi) = (\partial_x \partial_\eta - \partial_y \partial_\xi)\,(a(x, \xi)b(y, \eta))\rceil_{x=y, \eta=\xi}$.

We finally deduce the following corollary. We use the notations of Section 3.

Corollary 34. *Let $a \in C^\infty(\mathbb{T}^d \times \mathbb{R}^d)$ have uniformly bounded derivatives, and let $\chi \in C_c^\infty(\mathbb{R}^d)$ be a nonnegative cut-off function such that $\sqrt{\chi}$ is smooth. Let $0 < h < 1$ and $R > 1$. Denote*

$$a_R(x, \xi) = a(x, \xi)\chi\left(\frac{P_\Lambda \xi}{hR}\right).$$

Assume that $a > 0$, and denote $b_R = \sqrt{a_R}$, Then

$$\|\mathrm{Op}_h(a_R) - \mathrm{Op}_h(b_R)^2\|_{L^2(\mathbb{T}^d) \longrightarrow L^2(\mathbb{T}^d)} = \mathcal{O}(h) + \mathcal{O}(R^{-1})$$

in the limits $h \longrightarrow 0$ and $R \longrightarrow +\infty$.

Corollary 35. *Let $a \in C^\infty(\mathbb{T}^d \times \mathbb{R}^d \times \mathbb{R}^d)$, 0-homogeneous in the third variable outside a compact set, with uniformly bounded derivatives, and let $\chi \in C_c^\infty(\mathbb{R}^d)$ be a nonnegative cut-off function such that $\sqrt{\chi}$ is smooth. Let $0 < h < 1$ and $R > 1$. Denote*

$$a^R(x, \xi) = a\left(x, \xi, \frac{P_\Lambda \xi}{h}\right)\left(1 - \chi\left(\frac{P_\Lambda \xi}{hR}\right)\right).$$

Assume that $a > 0$, and denote $b^R = \sqrt{a^R}$. Then

$$\|\mathrm{Op}_h(a^R) - \mathrm{Op}_h(b^R)^2\|_{L^2(\mathbb{T}^d) \longrightarrow L^2(\mathbb{T}^d)} = \mathcal{O}(R^{-1})$$

in the limits $h \longrightarrow 0$ and $R \longrightarrow +\infty$.

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