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SEMICLASSICAL MEASURES FOR THE SCHRÖDINGER EQUATION ON THE TORUS

NALINI ANANTHARAMAN AND FABRICIO MACIÀ

ABSTRACT. Our main result is the following : let (u_n) be a sequence in $L^2(\mathbb{T}^d)$, such that $\|u_n\|_{L^2(\mathbb{T}^d)} = 1$ for all n . Consider the sequence of probability measures ν_n on \mathbb{T}^d , defined by $\nu_n(dx) = \left(\int_0^1 |e^{it\Delta/2}u_n(x)|^2 dt \right) dx$. Let ν be any weak-* limit of the sequence (ν_n) : then ν is absolutely continuous. This generalizes a former result of Bourgain and Jakobson, who considered the case when the functions u_n are eigenfunctions of the Laplacian. Our approach is different from theirs, it relies on the notion of (two-microlocal) semiclassical measures, and the properties of the geodesic flow on the torus.

1. INTRODUCTION

Consider the torus $\mathbb{T}^d := (\mathbb{R}/2\pi\mathbb{Z})^d$ equipped with the standard flat metric. We denote by Δ the associated Laplacian. We are interested in understanding the regularizing properties of the Schrödinger equation

$$i\frac{\partial u}{\partial t} = -\frac{1}{2}\Delta u, \quad u|_{t=0} \in L^2(\mathbb{T}^d).$$

More precisely, given a sequence of initial conditions $u_n \in L^2(\mathbb{T}^d)$, we shall investigate regularity properties that are uniform with respect to (u_n) . As a consequence of Theorem 2 below, we will prove :

Theorem 1. *Let (u_n) be a sequence in $L^2(\mathbb{T}^d)$, such that $\|u_n\|_{L^2(\mathbb{T}^d)} = 1$ for all n . Consider the sequence of probability measures ν_n on \mathbb{T}^d , defined by*

$$(1) \quad \nu_n(dx) = \left(\int_0^1 |e^{it\Delta/2}u_n(x)|^2 dt \right) dx.$$

Let ν be any weak- limit of the sequence (ν_n) : then ν is absolutely continuous.*

This is an expression of the dispersive properties of the unitary group $e^{it\Delta/2}$ on the flat torus \mathbb{T}^d . It should be noted that this result does not hold in a general compact Riemannian manifold (M, g) . For instance, in the class of manifolds with periodic geodesic flow (as is the standard sphere \mathbb{S}^d) the analogous of Theorem 1 is known to be false. In fact, any measure carried uniformly by a single geodesic may be realized as a weak-* limit

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of probability measures of the form (1), see [8]. This shows in particular that a dispersive Strichartz estimate

$$(2) \quad \|e^{it\Delta/2}u\|_{L^p([0,1]\times M)} \leq C \|u\|_{L^2(M)}$$

cannot hold on such a manifold for any $p > 2$ (see [10]). On the other hand, estimate (2) holds on \mathbb{T}^1 with $p = 4$, as shown by Zygmund [13].¹ Whether such an estimate holds on the flat torus \mathbb{T}^d , $d \geq 2$, for some $p > 2$ seems to be an open problem. In this direction, Theorem 1 shows that one cannot disprove the validity of the dispersive estimate (2) on \mathbb{T}^d by using the strategy explained above for the sphere. For the optimistic minds, this supports the idea that (2) might hold on \mathbb{T}^d for some $p = p(d) > 2$.

When the sequence (u_n) consists of eigenfunctions of Δ , we see that $\nu_n(dx) = |u_n(x)|^2 dx$; in this case, the theorem was proved by Bourgain and Jakobson [7]. Theorem 1 can be seen as a generalization of their result. However, our proof is quite different from theirs, and is completely self-contained. Instead of working directly on \mathbb{T}^d , we use the notion of *Wigner distributions*, defined on the cotangent bundle $T^*\mathbb{T}^d = \mathbb{T}^d \times \mathbb{R}^d$. This allows us to use the dynamics of the geodesic flow; since we are on a flat torus, the geodesic flow is completely explicit, and we use the decomposition of the momentum space into resonant vectors of various orders. We underline that, although we wrote everything working on the “square” torus, the arguments are valid on any flat torus.

Let us restate Theorem 1 using the notion of Wigner distributions. We will use semi-classical notations, and denote (u_h) our family of initial conditions, where $h > 0$ is a real parameter going to 0. We will always assume that the functions u_h are normalized in $L^2(\mathbb{T}^d)$. The *Wigner distribution* associated to u_h (at scale h) is a distribution on the cotangent bundle $T^*\mathbb{T}^d = \mathbb{T}^d \times \mathbb{R}^d$, defined by

$$\int_{T^*\mathbb{T}^d} a(x, \xi) w_{u_h}^h(dx, d\xi) = \langle u_h, \text{Op}_h(a)u_h \rangle_{L^2(\mathbb{T}^d)}, \quad \text{for all } a \in C_c^\infty(T^*\mathbb{T}^d),$$

where $\text{Op}_h(a)$ is the operator on $L^2(\mathbb{T}^d)$ associated to a by the Weyl quantization (Section 5). More explicitly, we have

$$\int_{T^*\mathbb{T}^d} a(x, \xi) w_{u_h}^h(dx, d\xi) = \frac{1}{(2\pi)^{d/2}} \sum_{k, j \in \mathbb{Z}^d} \hat{u}_h(k) \overline{\hat{u}_h(j)} \hat{a}_{j-k} \left(\frac{h}{2}(k+j) \right),$$

where $\hat{u}_h(k) := \int_{\mathbb{T}^d} u_h(x) \frac{e^{-ik \cdot x}}{(2\pi)^{d/2}} dx$ and $\hat{a}_k(\xi) := \int_{\mathbb{T}^d} a(x, \xi) \frac{e^{-ik \cdot x}}{(2\pi)^{d/2}} dx$ denote the respective Fourier coefficients of u_h and a , with respect to the variable $x \in \mathbb{T}^d$. We note that, if a is a function on $T^*\mathbb{T}^d = \mathbb{T}^d \times \mathbb{R}^d$ that depends only on the first coordinate, then

$$(3) \quad \int_{T^*\mathbb{T}^d} a(x) w_{u_h}^h(dx, d\xi) = \int_{\mathbb{T}^d} a(x) |u_h(x)|^2 dx.$$

The main object of our study will be the Wigner distributions $w_{e^{it\Delta/2}u_h}^h$, that, when no confusion arises, we will more simply denote by $w_h(t, \cdot)$. By standard estimates on the

¹Bourgain has shown in [1] that (2) is actually false when $d = 1$, $p = 6$, and $d = 2$, $p = 4$. In both cases, the estimate is known to hold on the whole Euclidean space \mathbb{R}^d .

norm of $\text{Op}_h(a)$ (the Calderón-Vaillancourt theorem, section 5), $t \mapsto w_h(t, \cdot)$ belongs to $L^\infty(\mathbb{R}; \mathcal{D}'(T^*\mathbb{T}^d))$, and is uniformly bounded in that space as $h \rightarrow 0^+$. Thus, one can extract subsequences that converge in the weak-* topology on $L^\infty(\mathbb{R}; \mathcal{D}'(T^*\mathbb{T}^d))$. In other words, after possibly extracting a subsequence, we have

$$\int_{\mathbb{R}} \varphi(t) a(x, \xi) w_h(t, dx, d\xi) dt \xrightarrow{h \rightarrow 0} \int_{\mathbb{R}} \varphi(t) a(x, \xi) \mu(t, dx, d\xi) dt$$

for all $\varphi \in L^1(\mathbb{R})$ and $a \in C_c^\infty(T^*\mathbb{T}^d)$. It also follows from standard properties of the Weyl quantization that the limit μ has the following properties :

- $\mu \in L^\infty(\mathbb{R}; \mathcal{M}_+(T^*\mathbb{T}^d))$, meaning that for almost all t , $\mu(t, \cdot)$ is a positive measure on $T^*\mathbb{T}^d$.
- The unitarity of $e^{it\Delta/2}$ implies that $\int_{T^*\mathbb{T}^d} \mu(t, dx, d\xi)$ does not depend on t ; from the normalization of u_h , we have $\int_{T^*\mathbb{T}^d} \mu(t, dx, d\xi) \leq 1$, the inequality coming from the fact that $T^*\mathbb{T}^d$ is not compact, and that there may be an escape of mass to infinity.
- Define the geodesic flow $\phi_s : T^*\mathbb{T}^d \rightarrow T^*\mathbb{T}^d$ by $\phi_s(x, \xi) := (x + s\xi, \xi)$. The Weyl quantization has the property that *Egorov's theorem* is satisfied in its exact form:

$$e^{-it\Delta/2} \text{Op}_h(a) e^{it\Delta/2} = \text{Op}_h(a \circ \phi_{t/h}).$$

This implies that $\mu(t, \cdot)$ is invariant under ϕ_s , for almost all t and all s .

We refer to [8] for details.

Theorem 2. *Let μ be a weak-* limit of the family w_h . Then, for almost all t , $\int_{\mathbb{R}^d} \mu(t, \cdot, d\xi)$ is an absolutely continuous measure on \mathbb{T}^d .*

Remark 3. *Theorem 1 is a consequence of Theorem 2 and (3), with one little subtlety. Because $T^*\mathbb{T}^d$ is not compact, if w_h converges weakly-* to μ and $\left(\int_0^1 |e^{it\Delta/2} u_h(x)|^2 dx\right) dt$ converges weakly-* to a probability measure ν on \mathbb{T}^d , it does not follow automatically that*

$$\nu = \int_0^1 \int_{\mathbb{R}^d} \mu(t, \cdot, d\xi) dt.$$

This is only true if we know a priori that $\int_{\mathbb{T}^d \times \mathbb{R}^d} \mu(t, dx, d\xi) = 1$ for almost all t , which means that there is no escape of mass to infinity. To check that Theorem 2 implies Theorem 1, we must explain why, for any normalized sequence $(u_n) \in L^2(\mathbb{T}^d)$, we can find a sequence of parameters $h_n \rightarrow 0$ such that the sequence $w_{u_n}^{h_n}$ does not escape to infinity. Let us choose h_n such that

$$(4) \quad \sum_{k \in \mathbb{Z}^d, \|k\| \leq h_n^{-1}} |\hat{u}_n(k)|^2 \xrightarrow{n \rightarrow +\infty} 1,$$

which is always possible. If we let $\tilde{u}_n(x) = \sum_{k \in \mathbb{Z}^d, \|k\| \leq h_n^{-1}} \hat{u}_n(k) \frac{e^{ik \cdot x}}{(2\pi)^{d/2}}$, equation (4) implies that $w_{\tilde{u}_n}^{h_n}$ has the same limit as $w_{u_n}^{h_n}$. On the other hand $w_{\tilde{u}_n}^{h_n}$ is supported in the compact set $\mathbb{T}^d \times B(0, 1) \subset \mathbb{T}^d \times \mathbb{R}^d$. Thus $w_{\tilde{u}_n}^{h_n}$ cannot escape to infinity.

Although $w_h(0, \cdot) = w_{u_h}^h$ completely determines $w_h(t, \cdot) = w_{e^{it\Delta/2}u_h}^h$ for all t , it is not true that the weak-* limits of $w_h(0, \cdot)$ determine $\mu(t, \cdot)$ for all t . In [8], one can find an example of two sequences (u_h) and (v_h) of initial conditions, such that $w_{u_h}^h$ and $w_{v_h}^h$ have the same limit in $\mathcal{D}'(T^*\mathbb{T}^d)$, but $w_{e^{it\Delta/2}u_h}^h$ and $w_{e^{it\Delta/2}v_h}^h$ have different limits in $L^\infty(\mathbb{R}; \mathcal{D}'(T^*\mathbb{T}^d))$. The proof of Theorem 2 will also enlighten the link between $\mu(0, \cdot)$ and $\mu(t, \cdot)$, expressed in Theorem 4 below. We call a submodule $\Lambda \subset \mathbb{Z}^d$ primitive if $\langle \Lambda \rangle \cap \mathbb{Z}^d = \Lambda$ (where $\langle \Lambda \rangle$ denotes the linear subspace of \mathbb{R}^d spanned by Λ). For such a submodule, $\langle \Lambda \rangle / 2\pi\mathbb{Z}^d$ is an embedded submanifold of \mathbb{T}^d , isomorphic to a torus of smaller dimension. If b is a function on \mathbb{T}^d such that $\hat{b}(k) = 0$ if $k \notin \Lambda$, we will say that b has only Fourier modes in Λ . This means that b is constant in the directions orthogonal to $\langle \Lambda \rangle$. Such a function defines naturally a function on the torus $\mathbb{T}_\Lambda = \langle \Lambda \rangle / 2\pi\mathbb{Z}^d$. We will denote m_b the multiplication operator by b , acting on $L^2(\mathbb{T}_\Lambda)$. We will also denote by Δ_Λ the Laplace operator on \mathbb{T}_Λ , for the metric that is inherited from \mathbb{T}^d .

Theorem 4. *For any sequence (u_h) , we can extract a subsequence such that the following hold :*

- *the subsequence $w_h(t, \cdot)$ converges weakly-* to a limit $\mu(t, \cdot)$;*
- *for each primitive submodule $\Lambda \subset \mathbb{Z}^d$, we can build a nonnegative trace class operator σ_Λ , acting on $L^2(\mathbb{T}_\Lambda)$; the operators σ_Λ are defined only in terms of the sequence of initial conditions (u_h) ;*
- *for almost all t , we have*

$$\int_{\mathbb{R}^d} \mu(t, \cdot, d\xi) = \sum_{\Lambda} \nu_\Lambda(t, \cdot),$$

where $\nu_\Lambda(t, \cdot)$ is the measure on \mathbb{T}^d , whose non-vanishing Fourier modes correspond to frequencies in Λ , defined by

$$\int_{\mathbb{T}^d} b(x) \nu_\Lambda(t, dx) = \text{Tr} (m_b e^{it\Delta_\Lambda/2} \sigma_\Lambda e^{-it\Delta_\Lambda/2}),$$

if $b \in C^\infty(\mathbb{T}^d)$.

We note that Theorem 4 implies Theorem 2. Besides, it tells us more about the dependence of $\mu(t, \cdot)$ with respect to t . If two sequences of initial conditions (u_h) and (v_h) give rise to the same family of operators σ_Λ , then they also give rise to the same limit $\mu(t, \cdot)$. Unfortunately, it is quite lengthy to describe how the operators σ_Λ are constructed from the functions (u_h) : the definition of σ_Λ will only be given at the end of Section 4.

Theorems 2 and 4 were proved for $d = 2$ in [9]. Our proof is an extension of the ideas therein.

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2. DECOMPOSITION OF AN INVARIANT MEASURE ON THE TORUS

Denote by \mathcal{L} the family of all submodules Λ of \mathbb{Z}^d which are *primitive*, in the sense that $\langle \Lambda \rangle \cap \mathbb{Z}^d = \Lambda$ (where $\langle \Lambda \rangle$ denotes the linear subspace of \mathbb{R}^d spanned by Λ). For each $\Lambda \in \mathcal{L}$, we define

$$\Lambda^\perp := \{ \xi \in \mathbb{R}^d : \xi \cdot k = 0, \quad k \in \Lambda \},$$

$$\mathbb{T}_\Lambda := \langle \Lambda \rangle / 2\pi\mathbb{Z}^d.$$

Note that \mathbb{T}_Λ is a submanifold of \mathbb{T}^d diffeomorphic to a torus of dimension $\text{rk } \Lambda$. Its cotangent bundle $T^*\mathbb{T}_\Lambda$ is $\mathbb{T}_\Lambda \times \langle \Lambda \rangle$. We shall use the notation $\mathbb{T}_{\Lambda^\perp}$ to refer to the torus $\Lambda^\perp / 2\pi\mathbb{Z}^d$. Denote by $\Omega_j \subset \mathbb{R}^d$, for $j = 0, \dots, d$, the set of resonant vectors of order exactly j , that is:

$$\Omega_j := \{ \xi \in \mathbb{R}^d : \text{rk } \Lambda_\xi = d - j \},$$

where $\Lambda_\xi := \{ k \in \mathbb{Z}^d : k \cdot \xi = 0 \}$. Note that the sets Ω_j form a partition of \mathbb{R}^d , and that $\Omega_0 = \{0\}$; more generally, $\xi \in \Omega_j$ if and only if the geodesic issued from any $x \in \mathbb{T}^d$ in the direction ξ is dense in a subtorus of \mathbb{T}^d of dimension j . The set $\Omega := \bigcup_{j=0}^{d-1} \Omega_j$ is usually called the set of *resonant* directions, whereas $\Omega_d = \mathbb{R}^d \setminus \Omega$ is referred to as the set of *non-resonant* vectors. Finally, write

$$R_\Lambda := \Lambda^\perp \cap \Omega_{d-\text{rk } \Lambda}.$$

Saying that $\xi \in R_\Lambda$ is equivalent to saying that (for any $x_0 \in \mathbb{T}^d$) the time-average $\frac{1}{T} \int_0^T \delta_{x_0+t\xi}(x) dt$ converges weakly to the Haar measure on the torus $x_0 + \mathbb{T}_{\Lambda^\perp}$, as $T \rightarrow \infty$. By construction, for $\xi \in R_\Lambda$ we have $\Lambda_\xi = \Lambda$; moreover, if $\text{rk } \Lambda = d - 1$ then $R_\Lambda = \Lambda^\perp \setminus \{0\}$. Finally,

$$(5) \quad \mathbb{R}^d = \bigsqcup_{\Lambda \in \mathcal{L}} R_\Lambda,$$

that is, the sets R_Λ form a partition of \mathbb{R}^d . As a consequence, the following result holds.

Lemma 5. *Let μ be a finite, positive Radon measure² on $T^*\mathbb{T}^d$. Then μ decomposes as a sum of positive measures:*

$$(6) \quad \mu = \sum_{\Lambda \in \mathcal{L}} \mu \llcorner_{\mathbb{T}^d \times R_\Lambda}.$$

Given any $\mu \in \mathcal{M}_+(T^*\mathbb{T}^d)$ we define the Fourier coefficients of μ as the complex measures on \mathbb{R}^d :

$$\widehat{\mu}(k, \cdot) := \int_{\mathbb{T}^d} \frac{e^{-ik \cdot x}}{(2\pi)^{d/2}} \mu(dx, \cdot), \quad k \in \mathbb{Z}^d.$$

One has, in the sense of distributions,

$$\mu(x, \xi) = \sum_{k \in \mathbb{Z}^d} \widehat{\mu}(k, \xi) \frac{e^{ik \cdot x}}{(2\pi)^{d/2}}.$$

²We denote by $\mathcal{M}_+(T^*\mathbb{T}^d)$ the set of all such measures.

Lemma 6. *Let $\mu \in \mathcal{M}_+(T^*\mathbb{T}^d)$ and $\Lambda \in \mathcal{L}$. The distribution:*

$$\langle \mu \rangle_\Lambda(x, \xi) := \sum_{k \in \Lambda} \widehat{\mu}(k, \xi) \frac{e^{ik \cdot x}}{(2\pi)^{d/2}}$$

is a finite, positive Radon measure on $T^\mathbb{T}^d$.*

Proof. Let $a \in C_c^\infty(T^*\mathbb{T}^d)$ and $\{v_1, \dots, v_n\}$ be a basis of Λ^\perp . Suppose

$$a(x, \xi) = \sum_{k \in \mathbb{Z}^d} \widehat{a}(k, \xi) \frac{e^{ik \cdot x}}{(2\pi)^{d/2}};$$

then it is not difficult to see that

$$\begin{aligned} \langle a \rangle_\Lambda(x, \xi) &:= \lim_{T_1, \dots, T_n \rightarrow \infty} \frac{1}{T_1 \dots T_n} \int_0^{T_1} \dots \int_0^{T_n} a\left(x + \sum_{j=1}^n t_j v_j, \xi\right) dt_1 \dots dt_n \\ &= \sum_{k \in \Lambda} \widehat{a}(k, \xi) \frac{e^{ik \cdot x}}{(2\pi)^{d/2}}, \end{aligned}$$

that $\langle a \rangle_\Lambda$ is non-negative as soon as a is, $\|\langle a \rangle_\Lambda\|_{L^\infty(T^*\mathbb{T}^d)} \leq \|a\|_{L^\infty(T^*\mathbb{T}^d)}$, and that $\langle a \rangle_\Lambda \in C_c^\infty(T^*\mathbb{T}^d)$ as well. Therefore,

$$\langle \langle \mu \rangle_\Lambda, a \rangle = \int_{T^*\mathbb{T}^d} \langle a \rangle_\Lambda(x, \xi) \mu(dx, d\xi)$$

defines a positive distribution, which is a positive Radon measure by Schwartz's theorem. \square

Recall that a measure $\mu \in \mathcal{M}_+(T^*\mathbb{T}^d)$ is invariant under the action of the geodesic flow³ on $T^*\mathbb{T}^d$ whenever:

$$(7) \quad (\phi_s)_* \mu = \mu, \quad \text{with } \phi_s(x, \xi) = (x + s\xi, \xi).$$

Let us also introduce, for $v \in \mathbb{R}^d$ and $s \in \mathbb{R}$ the translations $\tau_s^v : T^*\mathbb{T}^d \rightarrow T^*\mathbb{T}^d$ defined by:

$$\tau_s^v(x, \xi) = (x + sv, \xi).$$

Lemma 7. *Let μ be a positive invariant measure on $T^*\mathbb{T}^d$. Then every term in the decomposition (6) is a positive invariant measure, and*

$$(8) \quad \mu \llcorner_{\mathbb{T}^d \times R_\Lambda} = \langle \mu \rangle_\Lambda \llcorner_{\mathbb{T}^d \times R_\Lambda}.$$

Moreover, this last identity is equivalent to the following invariance property:

$$(\tau_s^v)_* \mu \llcorner_{\mathbb{T}^d \times R_\Lambda} = \mu \llcorner_{\mathbb{T}^d \times R_\Lambda}, \quad \text{for every } s \in \mathbb{R} \text{ and } v \in \Lambda^\perp.$$

³In what follows, we shall refer to such a measure simply as a *positive invariant measure*.

Proof. The invariance of the measures $\mu|_{\mathbb{T}^d \times R_\Lambda}$ is clearly a consequence of that of μ and of the form of the geodesic flow on $T^*\mathbb{T}^d$. To check (8) it suffices to show that $\widehat{\mu}(k, \cdot)|_{R_\Lambda} = 0$ as soon as $k \notin \Lambda$. Start noticing that (7) is equivalent to the fact that μ solves the equation:

$$\xi \cdot \nabla_x \mu(x, \xi) = 0.$$

This is in turn equivalent to:

$$i(k \cdot \xi) \widehat{\mu}(k, \xi) = 0, \quad \text{for every } k \in \mathbb{Z}^d,$$

from which we infer:

$$(9) \quad \text{supp } \widehat{\mu}(k, \cdot) \subset \{\xi \in \mathbb{R}^d : k \cdot \xi = 0\}.$$

Now remark that $R_\Lambda \cap \{\xi \in \mathbb{R}^d : k \cdot \xi = 0\} \neq \emptyset$ if and only if $k \in \Lambda$. This concludes the proof of the lemma. \square

3. SECOND MICROLOCALIZATION ON A RESONANT AFFINE SUBSPACE

Given $\Lambda \in \mathcal{L}$, we denote by \mathcal{S}_Λ^1 the class of smooth symbols $a(x, \xi, \eta)$ on $T^*\mathbb{T}^d \times \mathbb{R}^d$ that are:

- (i) compactly supported on $(x, \xi) \in T^*\mathbb{T}^d$,
- (ii) homogeneous of degree zero at infinity in $\eta \in \mathbb{R}^d$, *i.e.* such that there exist $R_0 > 0$ and $a_{\text{hom}} \in C_c^\infty(T^*\mathbb{T}^d \times \mathbb{S}^{d-1})$ with

$$a(x, \xi, \eta) = a_{\text{hom}}\left(x, \xi, \frac{\eta}{|\eta|}\right), \quad \text{for } |\eta| > R_0 \text{ and } (x, \xi) \in T^*\mathbb{T}^d;$$

- (iii) such that their non vanishing Fourier coefficients (in the x variable) correspond to frequencies $k \in \Lambda$:

$$a(x, \xi, \eta) = \sum_{k \in \Lambda} \widehat{a}(k, \xi, \eta) \frac{e^{ik \cdot x}}{(2\pi)^{d/2}}.$$

We will also express this fact by saying that a has only x -Fourier modes in Λ .

Let (u_h) be a bounded sequence in $L^2(\mathbb{T}^d)$ and suppose that its Wigner distributions $w_h(t) := w_{e^{it\Delta x/2}u_h}^h$ converge to a semiclassical measure $\mu \in L^\infty(\mathbb{R}; \mathcal{M}_+(T^*\mathbb{T}^d))$ in the weak-* topology of $L^\infty(\mathbb{R}; \mathcal{D}'(T^*\mathbb{T}^d))$.

Our purpose in this section is to analyse the structure of the restriction $\mu|_{\mathbb{T}^d \times R_\Lambda}$. To achieve this we shall introduce a two-microlocal distribution describing the concentration of the sequence $(e^{it\Delta/2}u_h)$ on the resonant subspaces:

$$\Lambda^\perp = \{\xi \in \mathbb{R}^d : P_\Lambda(\xi) = 0\},$$

where P_Λ denotes the orthogonal projection of \mathbb{R}^d onto $\langle \Lambda \rangle$. Similar objects have been introduced in the local, Euclidean, case by Fermanian-Kammerer [3, 4] under the name of two-microlocal semiclassical measures, by Gérard and Fermanian-Kammerer [5] when a specific concentration scale is preferred, and also by Miller [11] and Nier [12]. We shall follow the approach in [4], although it will be important to take into account the global nature of the objects we shall be dealing with.

Note that, by Lemma 7, it suffices to characterize the action of $\mu|_{\mathbb{T}^d \times R_\Lambda}$ on test functions having only x -Fourier modes in Λ . With this in mind, we introduce two auxiliary distributions which describe more precisely how $w_h(t)$ concentrates along $\mathbb{T}^d \times \Lambda^\perp$ and that act on symbols on the class \mathcal{S}_Λ^1 .

Let $\chi \in C_c^\infty(\mathbb{R})$ be a nonnegative cut-off function that is identically equal to one near the origin. For $a \in \mathcal{S}_\Lambda^1$, we define

$$\langle w_{h,R}^\Lambda(t), a \rangle := \int_{T^*\mathbb{T}^d} \left(1 - \chi\left(\frac{P_\Lambda(\xi)}{Rh}\right)\right) a\left(x, \xi, \frac{P_\Lambda(\xi)}{h}\right) w_h(t)(dx, d\xi),$$

and

$$(10) \quad \langle w_{\Lambda,h,R}(t), a \rangle := \int_{T^*\mathbb{T}^d} \chi\left(\frac{P_\Lambda(\xi)}{Rh}\right) a\left(x, \xi, \frac{P_\Lambda(\xi)}{h}\right) w_h(t)(dx, d\xi).$$

Remark 8. If $\Lambda = \{0\}$ then $w_{h,R}^\Lambda = 0$ and $w_{\Lambda,h,R}(t) = w_h(t) \otimes \delta_0$.

Remark 9. For every $R > 0$ and $a \in \mathcal{S}_\Lambda^1$ the following holds.

$$\int_{T^*\mathbb{T}^d} a\left(x, \xi, \frac{P_\Lambda(\xi)}{h}\right) w_h(t)(dx, d\xi) = \langle w_{h,R}^\Lambda(t), a \rangle + \langle w_{\Lambda,h,R}(t), a \rangle.$$

The Calderón-Vaillancourt theorem (see the appendix for a precise statement) ensures that both $w_{h,R}^\Lambda$ and $w_{\Lambda,h,R}$ are bounded in $L^\infty(\mathbb{R}; (\mathcal{S}_\Lambda^1)')$. After possibly extracting subsequences, we have for every $\varphi \in L^1(\mathbb{R})$ and $a \in \mathcal{S}_\Lambda^1$,

$$\int_{\mathbb{R}} \varphi(t) \langle \tilde{\mu}^\Lambda(t, \cdot), a \rangle dt := \lim_{R \rightarrow \infty} \lim_{h \rightarrow 0^+} \int_{\mathbb{R}} \varphi(t) \langle w_{h,R}^\Lambda(t), a \rangle dt,$$

and

$$\int_{\mathbb{R}} \varphi(t) \langle \tilde{\mu}_\Lambda(t, \cdot), a \rangle dt := \lim_{R \rightarrow \infty} \lim_{h \rightarrow 0^+} \int_{\mathbb{R}} \varphi(t) \langle w_{\Lambda,h,R}(t), a \rangle dt.$$

Both limiting distributions $\tilde{\mu}_\Lambda(t, \cdot)$ and $\tilde{\mu}^\Lambda(t, \cdot)$ are supported on $T^*\mathbb{T}^d \times \langle \Lambda \rangle$ since the same holds for each $w_{\Lambda,h,R}(t)$ and $w_{h,R}^\Lambda(t)$. Define, for $(x, \xi, \eta) \in T^*\mathbb{T}^d \times \mathbb{R}^d$ and $s \in \mathbb{R}$,

$$\phi_s^0(x, \xi, \eta) := (x + s\xi, \xi, \eta),$$

$$\phi_s^1(x, \xi, \eta) := (x + s\eta, \xi, \eta).$$

The distributions $w_{h,R}^\Lambda$ and $w_{\Lambda,h,R}$ satisfy a transport equation on the extra variable η .

Lemma 10. For every $a \in \mathcal{S}_\Lambda^1$ we have

$$\langle w_{h,R}^\Lambda(t), a \rangle = \langle w_{h,R}^\Lambda(0), a \circ \phi_{t/h}^0 \rangle = \langle w_{h,R}^\Lambda(0), a \circ \phi_t^1 \rangle,$$

$$\langle w_{\Lambda,h,R}(t), a \rangle = \langle w_{\Lambda,h,R}(0), a \circ \phi_{t/h}^0 \rangle = \langle w_{\Lambda,h,R}(0), a \circ \phi_t^1 \rangle.$$

Proof. Egorov's theorem for the Weyl quantization on the torus states that

$$\langle w_h(t), a \rangle = \langle w_h(0), a \circ \phi_{t/h} \rangle.$$

In view of the expressions defining $w_{h,R}^\Lambda$ and $w_{\Lambda,h,R}$ we conclude that the same holds for these distributions:

$$\langle w_{h,R}^\Lambda(t), a \rangle = \langle w_{h,R}^\Lambda(0), a \circ \phi_{t/h}^0 \rangle, \quad \langle w_{\Lambda,h,R}(t), a \rangle = \langle w_{\Lambda,h,R}(0), a \circ \phi_{t/h}^0 \rangle.$$

Since the Fourier coefficients of a only correspond to frequencies on Λ we have

$$a \left(\phi_{t/h}(x, \xi), \frac{P_\Lambda(\xi)}{h} \right) = a \left(x + t \frac{P_\Lambda(\xi)}{h}, \xi, \frac{P_\Lambda(\xi)}{h} \right) = a \left(\phi_t^1 \left(x, \xi, \frac{P_\Lambda(\xi)}{h} \right) \right),$$

and the result follows. \square

The structure of the accumulation points $\tilde{\mu}_\Lambda(t, \cdot)$ and $\tilde{\mu}^\Lambda(t, \cdot)$ is described in the next two results.

Theorem 11. (i) For a.e. $t \in \mathbb{R}$, $\tilde{\mu}_\Lambda(t, \cdot)$ and $\tilde{\mu}^\Lambda(t, \cdot)$ are distributions supported on $T^*\mathbb{T}^d \times \langle \Lambda \rangle$; in addition, $\tilde{\mu}^\Lambda(t, \cdot)$ is a positive measure, which is zero-homogeneous in the variable $\eta \in \langle \Lambda \rangle$. The projection of $\tilde{\mu}_\Lambda(t, \cdot)$ on $T^*\mathbb{T}^d$ is a positive measure.

(ii) Let

$$\mu^\Lambda(t, \cdot) := \int_{\langle \Lambda \rangle} \tilde{\mu}^\Lambda(t, \cdot, d\eta) \rfloor_{\mathbb{T}^d \times R_\Lambda}, \quad \mu_\Lambda(t, \cdot) := \int_{\langle \Lambda \rangle} \tilde{\mu}_\Lambda(t, \cdot, d\eta) \rfloor_{\mathbb{T}^d \times R_\Lambda}.$$

Then both $\mu^\Lambda(t, \cdot)$ and $\mu_\Lambda(t, \cdot)$ are positive measures on $T^*\mathbb{T}^d$, invariant by the geodesic flow, and satisfy:

$$(11) \quad \mu(t, \cdot) \rfloor_{\mathbb{T}^d \times R_\Lambda} = \mu^\Lambda(t, \cdot) + \mu_\Lambda(t, \cdot).$$

Note that identity (11) is a consequence of the decomposition property expressed in Remark 9.

The following is the key step of our proof, it says that both μ^Λ and μ_Λ have some extra regularity in the variable x , for two different reasons :

Theorem 12. (i) For a.e. $t \in \mathbb{R}$, the measure $\mu_\Lambda(t, \cdot)$ is concentrated on $\mathbb{T}^d \times \Lambda^\perp$ and its projection on \mathbb{T}^d is absolutely continuous with respect to the Lebesgue measure.

(ii) For a.e. $t \in \mathbb{R}$, the measure $\tilde{\mu}^\Lambda(t, \cdot)$ satisfies the invariance properties:

$$(12) \quad (\phi_s^0)_* \tilde{\mu}^\Lambda(t, \cdot) = \tilde{\mu}^\Lambda(t, \cdot), \quad (\phi_s^1)_* \tilde{\mu}^\Lambda(t, \cdot) = \tilde{\mu}^\Lambda(t, \cdot), \quad s \in \mathbb{R}.$$

Remark 13. If $\text{rk } \Lambda = 1$ then $\langle \Lambda \rangle = \langle \Lambda \rangle \cap \Omega_1$. Therefore, (12) and Lemma 7 imply that, for a.e. $t \in \mathbb{R}$, the measure $\tilde{\mu}^\Lambda(t, \cdot)$ satisfies for every $v \in \Lambda$:

$$(13) \quad (\tau_s^v)_* \tilde{\mu}^\Lambda(t, \cdot) \rfloor_{\mathbb{T}^d \times R_\Lambda \times \langle \Lambda \rangle} = \tilde{\mu}^\Lambda(t, \cdot) \rfloor_{\mathbb{T}^d \times R_\Lambda \times \langle \Lambda \rangle}, \quad s \in \mathbb{R}.$$

On the other hand, the invariance by the geodesic flow implies that (13) also holds for every $v \in \Lambda^\perp$. Therefore, we conclude that $\tilde{\mu}^\Lambda(t, \cdot) \rfloor_{\mathbb{T}^d \times R_\Lambda \times \langle \Lambda \rangle}$ is constant in $x \in \mathbb{T}^d$ in this case.

Remark 14. Theorems 11 (ii), and 12 (i), together with Lemma 5 imply that, for a.e. $t \in \mathbb{R}$, we have a decomposition:

$$\mu(t, \cdot) = \sum_{\Lambda \in \mathcal{L}} \mu^\Lambda(t, \cdot) + \sum_{\Lambda \in \mathcal{L}} \mu_\Lambda(t, \cdot),$$

where the second term in the above sum defines a positive measure whose projection on \mathbb{T}^d is absolutely continuous with respect to the Lebesgue measure.

The rest of this section is devoted to the proofs of Theorem 11 (i) and Theorem 12.

3.1. Computation and structure of $\tilde{\mu}_\Lambda$.

$$\pi_\Lambda : \mathbb{T}_{\Lambda^\perp} \times \mathbb{T}_\Lambda \rightarrow \mathbb{T}^d : (s, y) \mapsto s + y$$

is a smooth Riemannian covering; $\tilde{\pi}_\Lambda$ will denote its extension to the cotangent bundles $T^*\mathbb{T}_{\Lambda^\perp} \times T^*\mathbb{T}_\Lambda \rightarrow T^*\mathbb{T}^d$. Let us denote by $\chi_\Lambda : \Lambda^\perp \times \langle \Lambda \rangle \rightarrow \mathbb{R}^d$ the lift of π_Λ to the universal cover and by $\tilde{\chi}_\Lambda : T^*\Lambda^\perp \times T^*\langle \Lambda \rangle \rightarrow T^*\mathbb{R}^d$ the induced canonical transformation. There is a linear isomorphism $T_\Lambda : L^2_{\text{loc}}(\mathbb{R}^d) \rightarrow L^2_{\text{loc}}(\Lambda^\perp \times \langle \Lambda \rangle)$ given by $T_\Lambda u := u \circ \chi_\Lambda$. Note that T_Λ maps $L^2(\mathbb{T}^d)$ into $L^2(\mathbb{T}_{\Lambda^\perp} \times \mathbb{T}_\Lambda) = L^2(\mathbb{T}_{\Lambda^\perp}; L^2(\mathbb{T}_\Lambda))$; in other words, for any $u \in L^2(\mathbb{T}^d)$, the function $T_\Lambda u = u \circ \pi_\Lambda$ is a well-defined element of $L^2(\mathbb{T}_{\Lambda^\perp}; L^2(\mathbb{T}_\Lambda))$. If the non-vanishing Fourier modes of u correspond only to frequencies $k \in \Lambda$, then

$$(14) \quad T_\Lambda u(s, y) = u(y) \quad \text{for every } s \in \mathbb{T}_{\Lambda^\perp}.$$

Since $\tilde{\chi}_\Lambda$ is linear, the following holds for any $a \in C^\infty(T^*\mathbb{R}^d)$:

$$T_\Lambda \text{Op}_h(a) = \text{Op}_h(a \circ \tilde{\chi}_\Lambda) T_\Lambda.$$

Denote by $\text{Op}_h^{\Lambda^\perp}$ and Op_h^Λ the Weyl quantization operators defined on smooth test functions on $T^*\Lambda^\perp \times T^*\langle \Lambda \rangle$ which act on the variables $T^*\Lambda^\perp$ and $T^*\langle \Lambda \rangle$ respectively, leaving the other frozen. The composition $\text{Op}_h^{\Lambda^\perp} \text{Op}_h^\Lambda$ gives the whole Weyl quantization Op_h on $T^*\Lambda^\perp \times T^*\langle \Lambda \rangle$. Now, if $a \in C_c^\infty(\mathcal{S}_\Lambda^1)$ we have, in view of (14), that $a \circ \tilde{\pi}_\Lambda$ does not depend on $s \in \mathbb{T}_{\Lambda^\perp}$ and therefore:

$$(15) \quad T_\Lambda \text{Op}_h(a) = \text{Op}_h^\Lambda(a \circ \tilde{\pi}_\Lambda(hD_s, \cdot)) T_\Lambda.$$

Note that for every $\sigma \in \Lambda^\perp$, the operators $\text{Op}_h^\Lambda(a \circ \tilde{\pi}_\Lambda(\sigma, \cdot))$ map $L^2(\mathbb{T}_\Lambda)$ into itself. Denote by Δ_Λ the Laplacian on \mathbb{T}_Λ . We shall use the following version of Egorov's theorem adapted to our setting.

Lemma 15. *Let $a \in \mathcal{S}_\Lambda^1$; define*

$$b_h(t, x, \xi) := a \circ \phi_t^1 \left(x, \xi, \frac{P_\Lambda(\xi)}{h} \right),$$

and $a_\Lambda^h \in C_c^\infty(\Lambda^\perp \times T^*\mathbb{T}_\Lambda)$ by

$$a_\Lambda^h(\sigma, y, \eta) := a(\tilde{\pi}_\Lambda(\sigma, y, h\eta), \eta), \quad (y, \eta) \in T^*\mathbb{T}_\Lambda, \quad \sigma \in \Lambda^\perp.$$

Then we have

$$\text{Op}_h(b_h(t, \cdot)) = T_\Lambda^* e^{-it\Delta_\Lambda/2} \text{Op}_1^\Lambda(a_\Lambda^h(hD_s, \cdot)) e^{it\Delta_\Lambda/2} T_\Lambda.$$

Proof. Identity (15) applied to $b_h(t, \cdot)$ gives:

$$\begin{aligned} \text{Op}_h(b_h(t, \cdot)) &= T_\Lambda^* \text{Op}_h^\Lambda(b_h(t, \tilde{\pi}_\Lambda(hD_s, \cdot))) T_\Lambda \\ &= T_\Lambda^* \text{Op}_1^\Lambda(a(\tilde{\pi}_\Lambda(hD_s, y + t\eta, h\eta), \eta)) T_\Lambda. \end{aligned}$$

Egorov's theorem for the Weyl quantization Op_h^Λ gives, for every $\sigma \in \Lambda^\perp$,

$$\text{Op}_1^\Lambda (a(\tilde{\pi}_\Lambda(\sigma, y + t\eta, h\eta), \eta)) = e^{-it\Delta_\Lambda/2} \text{Op}_1^\Lambda (a_\Lambda^h(\sigma, \cdot)) e^{it\Delta_\Lambda/2},$$

and the result follows. \square

This, combined with Lemma 10 gives the following.

Lemma 16. *With the preceding notations, and setting $a_R(x, \xi, \eta) := \chi(\eta/R) a(x, \xi, \eta)$, we have:*

$$\langle w_{\Lambda, h, R}(t), a \rangle = \frac{1}{p_\Lambda} \langle T_\Lambda u_h, e^{-it\Delta_\Lambda/2} \text{Op}_1^\Lambda (a_{R, \Lambda}^h(hD_s, \cdot)) e^{it\Delta_\Lambda/2} T_\Lambda u_h \rangle_{L^2(\mathbb{T}_{\Lambda^\perp}; L^2(\mathbb{T}_\Lambda))},$$

where $p_\Lambda \in \mathbb{N}$ is the degree of π_Λ .

Remark 17. *Note that for every $R > 0$, $t \in \mathbb{R}$ and $\sigma \in \Lambda^\perp$, the operator*

$$e^{-it\Delta_\Lambda/2} \text{Op}_1^\Lambda (a_{R, \Lambda}^h(\sigma, \cdot)) e^{it\Delta_\Lambda/2}$$

is compact on $L^2(\mathbb{T}_\Lambda)$.

Given a Hilbert space H , denote respectively by $\mathcal{K}(H)$ and $\mathcal{L}^1(H)$ the spaces of compact and trace class operators on H . A measure on a polish space T , taking values in $\mathcal{L}^1(H)$, is a bounded linear functional ρ from $C_c(T)$ to $\mathcal{L}^1(H)$; ρ is said to be positive if, for every nonnegative $b \in C_c(T)$, $\rho(b)$ is a positive hermitian operator. The set of such measures is denoted by $\mathcal{M}_+(T; \mathcal{L}^1(H))$; they can be identified in a natural way to positive linear functionals on $C_c(T; \mathcal{K}(H))$. Background and further details on operator-valued measures may be found for instance in [6].

In view of Lemma 16 and Remark 17, it turns out that the limiting object relevant in the computation of $\tilde{\mu}_\Lambda$ is the one presented in the next result.

Lemma 18. *The functional n_h^Λ that associates to $K \in C_c(\Lambda^\perp; \mathcal{K}(L^2(\mathbb{T}_\Lambda)))$ the value*

$$n_h^\Lambda(K) := \frac{1}{p_\Lambda} \langle T_\Lambda u_h, K(hD_s) T_\Lambda u_h \rangle_{L^2(\mathbb{T}_{\Lambda^\perp}; L^2(\mathbb{T}_\Lambda))}$$

is a positive measure on Λ^\perp taking values on the set of trace-class operators $\mathcal{L}^1(L^2(\mathbb{T}_\Lambda))$. If (u_h) is bounded in $L^2(\mathbb{T}^d)$ then the measures (n_h^Λ) are uniformly bounded.

Proof. We have the following bound:

$$\begin{aligned} |n_h^\Lambda(K)| &\leq \frac{1}{p_\Lambda} \|T_\Lambda u_h\|_{L^2(\mathbb{T}_{\Lambda^\perp}; L^2(\mathbb{T}_\Lambda))}^2 \sup_{\sigma \in \Lambda^\perp} \|K(\sigma)\|_{\mathcal{L}(L^2(\mathbb{T}_\Lambda))} \\ &= \|u_h\|_{L^2(\mathbb{T}^d)}^2 \sup_{\sigma \in \Lambda^\perp} \|K(\sigma)\|_{\mathcal{L}(L^2(\mathbb{T}_\Lambda))}, \end{aligned}$$

that ensures that n_h^Λ are uniformly bounded functionals on $C_c(\Lambda^\perp; \mathcal{K}(L^2(\mathbb{T}_\Lambda)))$. Moreover, the measures n_h^Λ are positive in the sense that if $K(\sigma)$ is a positive operator for every $\sigma \in \Lambda^\perp$, then $n_h^\Lambda(K) \geq 0$. \square

Combining everything we have done so far we obtain:

Lemma 19. *Let $\tilde{\rho}_\Lambda \in \mathcal{M}_+(\Lambda^\perp; \mathcal{L}^1(L^2(\mathbb{T}_\Lambda)))$ be a weak-* limit of (n_h^Λ) . Let $\tilde{\mu}_\Lambda$ be defined by (10). Then, for every $a \in \mathcal{S}_\Lambda^1$ and a.e. $t \in \mathbb{R}$ we have:*

$$\int_{\mathbb{T}^d \times \Lambda^\perp \times \langle \Lambda \rangle} a(x, \xi, \eta) \tilde{\mu}_\Lambda(t, dx, d\xi, d\eta) = \text{Tr} \left(\int_{\Lambda^\perp} e^{-it\Delta_\Lambda/2} \text{Op}_1^\Lambda(a_\Lambda^0(\sigma, \cdot)) e^{it\Delta_\Lambda/2} \tilde{\rho}_\Lambda(d\sigma) \right).$$

Remark 20. *If $a \in \mathcal{S}_\Lambda^1$ does not depend on $\eta \in \mathbb{R}^d$ then the above identity can be rewritten as:*

$$\int_{\mathbb{T}^d \times \Lambda^\perp \times \langle \Lambda \rangle} a(x, \xi) \tilde{\mu}_\Lambda(t, dx, d\xi, d\eta) = \text{Tr} \left(e^{-it\Delta_\Lambda/2} \int_{\Lambda^\perp} m_a(\sigma) e^{it\Delta_\Lambda/2} \tilde{\rho}_\Lambda(d\sigma) \right),$$

where for $\sigma \in \Lambda^\perp$, $m_a(\sigma)$ denotes the operator of multiplication by $a(\cdot, \sigma)$ in $L^2(\mathbb{T}_\Lambda)$.

From this identity it follows that the projection of $\tilde{\mu}_\Lambda$ on \mathbb{T}^d acts on a function $b \in C(\mathbb{T}^d)$ as:

$$(16) \quad \int_{\mathbb{T}^d \times \Lambda^\perp \times \langle \Lambda \rangle} b(x) \tilde{\mu}_\Lambda(t, dx, d\xi, d\eta) = \text{Tr}(m_{b \circ \pi_\Lambda} \rho_\Lambda(t)),$$

where

$$i\partial_t \rho_\Lambda(t) = \frac{1}{2} [-\Delta_\Lambda, \rho_\Lambda(t)], \quad \rho_\Lambda(0) = \int_{\Lambda^\perp} \tilde{\rho}_\Lambda(d\sigma).$$

In particular, identity (16) shows that the projection of $\tilde{\mu}_\Lambda$ on \mathbb{T}^d can be extended to a linear functional on $L^\infty(\mathbb{T}^d)$; and in particular, it is absolutely continuous with respect to the Lebesgue measure (see Proposition 15 in [9] for a general result in this direction).

3.2. Computation and structure of $\tilde{\mu}^\Lambda$. The positivity of $\tilde{\mu}^\Lambda(t, \cdot)$ can be deduced following the lines of [5] §2.1, or those of the proof of Theorem 1 in [6]; the idea is recalled in Corollary 27 in the appendix. Given $a \in \mathcal{S}_\Lambda^1$ there exists $R_0 > 0$ and $a_{\text{hom}} \in C_c^\infty(T^*\mathbb{T}^d \times \mathbb{S}^{d-1})$ such that

$$a(x, \xi, \eta) = a_{\text{hom}} \left(x, \xi, \frac{\eta}{|\eta|} \right), \quad \text{for } |\eta| \geq R_0.$$

Clearly, for R large enough, the value $\langle w_{\Lambda, h, R}(t), a \rangle$ only depends on a_{hom} . Therefore, the limiting distribution $\tilde{\mu}^\Lambda(t, \cdot)$ is zero-homogeneous in the last variable $\eta \in \mathbb{R}^d$. Since each of the $w_{\Lambda, h, R}(t)$ is concentrated on $T^*\mathbb{T}^d \times \langle \Lambda \rangle$ so is $\tilde{\mu}^\Lambda(t, \cdot)$. Let us now check the invariance property (12). Recall that, by Lemma 10 we have

$$\langle w_{h, R}^\Lambda(t), a \rangle = \langle w_{h, R}^\Lambda(0), a^R \circ \phi_t^1 \rangle$$

with

$$a^R(x, \xi, \eta) := \left(1 - \chi \left(\frac{\eta}{R} \right) \right) a(x, \xi, \eta).$$

Writing $\eta = r\omega$ with $r > 0$ and $\omega \in \mathbb{S}^{d-1}$ we find, for R large enough:

$$a_R \circ \phi_t^1(x, \xi, \eta) = \left(1 - \chi \left(\frac{r}{R} \omega \right) \right) a_{\text{hom}}(x, +tr\omega, \xi, \omega).$$

Therefore, since $a^R \circ \phi_t^1$ vanishes near $r = 0$,

$$\frac{1}{r} \partial_t a^R \circ \phi_t^1 - \omega \cdot \nabla_x a^R \circ \phi_t^1 = 0,$$

in other words, for φ differentiable we have:

$$- \int_{\mathbb{R}} \varphi'(t) \left\langle \frac{1}{|\eta|} w_{h,R}^\Lambda(t), a \right\rangle dt + \int_{\mathbb{R}} \varphi(t) \langle \omega \cdot \nabla_x w_{h,R}^\Lambda(t), a_{\text{hom}} \rangle = 0$$

which gives, letting $h \rightarrow 0^+$ and $R \rightarrow \infty$,

$$\omega \cdot \nabla_x \tilde{\mu}^\Lambda = 0.$$

This is equivalent to (12).

4. SUCCESSIVE SECOND MICROLOCALIZATIONS CORRESPONDING TO A SEQUENCE OF LATTICES

Let us summarize what we have done in the previous section. The semiclassical measure $\mu(t, \cdot)$ has been decomposed as a sum

$$\mu(t, \cdot) = \sum_{\Lambda} \mu_{\Lambda}(t, \cdot) + \sum_{\Lambda} \mu^{\Lambda}(t, \cdot),$$

where Λ runs over the set of primitive submodules of \mathbb{Z}^d , and where

$$\mu_{\Lambda}(t, \cdot) = \int_{\mathbb{R}^d} \tilde{\mu}_{\Lambda}(t, \cdot, d\eta) \rfloor_{\mathbb{T}^d \times R_{\Lambda}}, \quad \mu^{\Lambda}(t, \cdot) = \int_{\mathbb{R}^d} \tilde{\mu}^{\Lambda}(t, \cdot, d\eta) \rfloor_{\mathbb{T}^d \times R_{\Lambda}}.$$

The distributions $\tilde{\mu}_{\Lambda}$ and $\tilde{\mu}^{\Lambda}$ have the following properties :

- $\tilde{\mu}_{\Lambda}(t, dx, d\xi, d\eta)$ is in $L^\infty(\mathbb{R}, \mathcal{D}'(T^*\mathbb{T}^d \times \mathbb{R}^d))$ and all its x -Fourier modes are in Λ ;
- $\int_{\mathbb{R}^d} \tilde{\mu}^{\Lambda}(t, \cdot, d\eta)$ is in $L^\infty(\mathbb{R}, \mathcal{M}_+(T^*\mathbb{T}^d))$;
- with respect to the variable η , $\tilde{\mu}_{\Lambda}(t, dx, d\xi, d\eta)$ is supported on $\{\eta \in \langle \Lambda \rangle\}$;
- $\int_{\mathbb{R}^d \times \mathbb{R}^d} \tilde{\mu}_{\Lambda}(t, \cdot, d\xi, d\eta)$ is an absolutely continuous measure on \mathbb{T}^d . In fact, with the notations of Section 3, we have

$$\int b(x) \tilde{\mu}_{\Lambda}(t, dx, d\xi, d\eta) = \text{Tr} \left(m_{b \circ \pi_{\Lambda}} \int_{\Lambda^\perp} e^{it\Delta_{\Lambda}/2} \tilde{\rho}_{\Lambda}(dr) e^{-it\Delta_{\Lambda}/2} \right)$$

if all the Fourier modes of b are in Λ , and where $m_{b \circ \pi_{\Lambda}}$ is the multiplication operator by $b \circ \pi_{\Lambda}$, acting on $L^2(\mathbb{T}_{\Lambda})$.

On the other hand,

- for $a \in \mathcal{S}_{\Lambda}^1$, $\langle \tilde{\mu}^{\Lambda}(t, dx, d\xi, d\eta), a(x, \xi, \eta) \rangle$ is obtained as the limit of

$$\langle w_{h,R}^{\Lambda}(t), a \rangle := \int_{T^*\mathbb{T}^d} \left(1 - \chi \left(\frac{P_{\Lambda}(\xi)}{Rh} \right) \right) a \left(x, \xi, \frac{P_{\Lambda}(\xi)}{h} \right) w_h(t)(dx, d\xi),$$

where the weak limit holds in $L^\infty(\mathbb{R}, \mathcal{S}_{\Lambda}^1)$, as $h \rightarrow 0$ then $R \rightarrow +\infty$ (along a subsequence);

- $\tilde{\mu}^\Lambda(t, dx, d\xi, d\eta)$ is in $L^\infty(\mathbb{R}, \mathcal{M}_+(T^*\mathbb{T}^d \times \mathbb{R}^d))$ and all its x -Fourier modes are in Λ ;
- with respect to the variable η , the measure $\tilde{\mu}^\Lambda(t, dx, d\xi, d\eta)$ is 0-homogeneous, and it is supported on $\{\eta \in \langle \Lambda \rangle\}$;
- $\tilde{\mu}^\Lambda$ is invariant by the two flows, $\phi_s^0 : (x, \xi, \eta) \mapsto (x + s\xi, \xi, \eta)$, and $\phi_s^1 : (x, \xi, \eta) \mapsto (x + s\eta, \xi, \eta)$.

This can be considered as the first step of an induction procedure, the k -th step of which will read as follows :

Step k of the induction : At step k , we have decomposed $\mu(t, \cdot)$ as a sum

$$\mu(t, \cdot) = \sum_{1 \leq l \leq k} \sum_{\Lambda_1 \supset \Lambda_2 \supset \dots \supset \Lambda_l} \mu_{\Lambda_l}^{\Lambda_1 \Lambda_2 \dots \Lambda_{l-1}}(t, \cdot) + \sum_{\Lambda_1 \supset \Lambda_2 \supset \dots \supset \Lambda_k} \mu^{\Lambda_1 \Lambda_2 \dots \Lambda_k}(t, \cdot),$$

where the sums run over the *strictly decreasing* sequences of primitive submodules of \mathbb{Z}^d (of lengths $l \leq k$ in the first term, of length k in the second term). We have

$$\begin{aligned} \mu_{\Lambda_l}^{\Lambda_1 \Lambda_2 \dots \Lambda_{l-1}}(t, \cdot) &= \int_{R_{\Lambda_2}(\Lambda_1) \times \dots \times R_{\Lambda_l}(\Lambda_{l-1}) \times \mathbb{R}^d} \tilde{\mu}_{\Lambda_l}^{\Lambda_1 \Lambda_2 \dots \Lambda_{l-1}}(t, \cdot, d\eta_1, \dots, d\eta_l) \Big|_{\mathbb{T}^d \times R_{\Lambda_1}}, \\ \mu^{\Lambda_1 \Lambda_2 \dots \Lambda_k}(t, \cdot) &= \int_{R_{\Lambda_2}(\Lambda_1) \times \dots \times R_{\Lambda_k}(\Lambda_{k-1}) \times \mathbb{R}^d} \tilde{\mu}^{\Lambda_1 \Lambda_2 \dots \Lambda_k}(t, \cdot, d\eta_1, \dots, d\eta_k) \Big|_{\mathbb{T}^d \times R_{\Lambda_1}}, \end{aligned}$$

where we denoted $R_\Lambda(\Lambda') := \Lambda'^\perp \cap \langle \Lambda' \rangle \cap \Omega_{\text{rk } \Lambda' - \text{rk } \Lambda}$, for $\Lambda \subset \Lambda'$. The distributions $\tilde{\mu}_{\Lambda_l}^{\Lambda_1 \Lambda_2 \dots \Lambda_{l-1}}$ and $\tilde{\mu}^{\Lambda_1 \Lambda_2 \dots \Lambda_k}$ have the following properties :

- $\tilde{\mu}_{\Lambda_l}^{\Lambda_1 \Lambda_2 \dots \Lambda_{l-1}}$ is in $L^\infty(\mathbb{R}, \mathcal{D}'(T^*\mathbb{T}^d \times (\mathbb{R}^d)^l))$ and all its x -Fourier modes are in Λ_l ;
- $\int_{\mathbb{R}^d} \tilde{\mu}_{\Lambda_l}^{\Lambda_1 \Lambda_2 \dots \Lambda_{l-1}}(t, \cdot, d\eta_l)$ is in $L^\infty(\mathbb{R}, \mathcal{M}_+(T^*\mathbb{T}^d \times (\mathbb{R}^d)^{l-1}))$;
- with respect to the variables η_1, \dots, η_l , $\tilde{\mu}_{\Lambda_l}(t, \cdot)$ is supported on $\{\eta_1 \in \langle \Lambda_1 \rangle, \dots, \eta_l \in \langle \Lambda_l \rangle\}$; besides, it is 0-homogeneous in each variable $\eta_1, \dots, \eta_{l-1}$;
- $\int_{(\mathbb{R}^d)^{l+1}} \tilde{\mu}_{\Lambda_l}^{\Lambda_1 \Lambda_2 \dots \Lambda_{l-1}}(t, \cdot, d\xi, d\eta_1, \dots, d\eta_l)$ is an absolutely continuous measure on \mathbb{T}^d .
In fact, for $b \in C(\mathbb{T}^d)$, it admits the expression

$$\begin{aligned} \int b(x) \tilde{\mu}_{\Lambda_l}^{\Lambda_1 \Lambda_2 \dots \Lambda_{l-1}}(t, dx, d\xi, d\eta_1, \dots, d\eta_l) \\ = \text{Tr} \left(m_{b \circ \pi_{\Lambda_l}} \cdot \int e^{it\Delta_{\Lambda_l}/2} \tilde{\rho}_{\Lambda_l}^{\Lambda_1 \Lambda_2 \dots \Lambda_{l-1}}(d\xi, d\eta_1, \dots, d\eta_{l-1}) e^{-it\Delta_{\Lambda_l}/2} \right), \end{aligned}$$

where $\tilde{\rho}_{\Lambda_l}^{\Lambda_1 \Lambda_2 \dots \Lambda_{l-1}}$ is a positive measure on $\Lambda_l^\perp \times (\langle \Lambda_1 \rangle \cap \Lambda_l^\perp) \times \dots \times (\langle \Lambda_{l-1} \rangle \cap \Lambda_l^\perp)$, taking values in $\mathcal{L}^1(L^2(\mathbb{T}_{\Lambda_l}))$. This expression holds if b has only Fourier modes in Λ_l , and $m_{b \circ \pi_{\Lambda_l}}$ denotes the multiplication operator by $b \circ \pi_{\Lambda_l}$, acting on $L^2(\mathbb{T}_{\Lambda_l})$.

In addition, let us denote by $\mathcal{S}_{\Lambda_k}^k$ the class of smooth functions $a(x, \xi, \eta_1, \dots, \eta_k)$ on $T^*\mathbb{T}^d \times (\mathbb{R}^d)^k$ that are (i) smooth and compactly supported in $(x, \xi) \in T^*\mathbb{T}^d$; (ii) homogeneous of degree 0 at infinity in each variable η_1, \dots, η_k ; (iii) such that their non-vanishing x -Fourier coefficients correspond to frequencies in Λ_k .

- For $a \in \mathcal{S}_{\Lambda_k}^k$, $\langle \tilde{\mu}^{\Lambda_1 \Lambda_2 \dots \Lambda_k}(t, dx, d\xi, d\eta_1, \dots, d\eta_k), a(x, \xi, \eta_1, \dots, \eta_k) \rangle$ is obtained as the limit of

$$\left\langle w_h(t, dx, d\xi), a \left(x, \xi, \frac{P_{\Lambda_1} \xi}{h}, \dots, \frac{P_{\Lambda_k} \xi}{h} \right) \left(1 - \chi \left(\frac{P_{\Lambda_1} \xi}{R_1 h} \right) \right) \dots \left(1 - \chi \left(\frac{P_{\Lambda_k} \xi}{R_k h} \right) \right) \right\rangle.$$

The weak limit holds in $L^\infty(\mathbb{R}, \mathcal{S}_{\Lambda_k}^k)$, as $h \rightarrow 0$ then $R_k \rightarrow +\infty, \dots, R_1 \rightarrow +\infty$ (along subsequences);

- $\tilde{\mu}^{\Lambda_1 \Lambda_2 \dots \Lambda_k}$ is in $L^\infty(\mathbb{R}, \mathcal{M}_+(T^* \mathbb{T}^d \times (\mathbb{R}^d)^k))$ and all its x -Fourier modes are in Λ_k ;
- with respect to the variables η_1, \dots, η_k , $\tilde{\mu}_\Lambda(t, \cdot)$ is supported on $\{\eta_1 \in \langle \Lambda_1 \rangle, \dots, \eta_k \in \langle \Lambda_k \rangle\}$; besides, it is 0-homogeneous in each variable η_1, \dots, η_k ;
- $\tilde{\mu}^{\Lambda_1 \Lambda_2 \dots \Lambda_k}$ is invariant by the $k+1$ flows, $\phi_s^0 : (x, \xi, \eta) \mapsto (x + s\xi, \xi, \eta_1, \dots, \eta_k)$, and $\phi_s^j : (x, \xi, \eta_1, \dots, \eta_k) \mapsto (x + s\eta_j, \xi, \eta_1, \dots, \eta_k)$ (where $j = 1, \dots, k$).

We now indicate how to go from step k to step $k+1$. The proofs are identical to those of Sections 2 and 3, and details will be omitted. Of course, the term $\sum_{1 \leq l \leq k} \sum_{\Lambda_1 \supset \Lambda_2 \supset \dots \supset \Lambda_l} \mu_{\Lambda_l}^{\Lambda_1 \Lambda_2 \dots \Lambda_{l-1}}$ remains untouched after step k . To decompose further the term $\sum_{\Lambda_1 \supset \Lambda_2 \supset \dots \supset \Lambda_k} \mu^{\Lambda_1 \Lambda_2 \dots \Lambda_k}$, we proceed as follows. Using the positivity of $\tilde{\mu}^{\Lambda_1 \Lambda_2 \dots \Lambda_k}$, we use the procedure described in Section 2 to write

$$\tilde{\mu}^{\Lambda_1 \Lambda_2 \dots \Lambda_k} = \sum_{\Lambda_{k+1} \subset \Lambda_k} \tilde{\mu}^{\Lambda_1 \Lambda_2 \dots \Lambda_k} \rfloor_{\eta_k \in R_{\Lambda_{k+1}}(\Lambda_k)},$$

where the sum runs over all primitive submodules Λ_{k+1} of Λ_k . Moreover, by the proof of Lemma 7, all the x -Fourier modes of $\tilde{\mu}^{\Lambda_1 \Lambda_2 \dots \Lambda_k} \rfloor_{\eta_k \in R_{\Lambda_{k+1}}(\Lambda_k)}$ are in Λ_{k+1} . To generalize the analysis of Section 3, we only need to consider test functions $a \in \mathcal{S}_{\Lambda_{k+1}}^{k+1}$. For such a function a , we let

$$\begin{aligned} & \left\langle w_{h, R_1, \dots, R_k}^{\Lambda_1 \Lambda_2 \dots \Lambda_{k+1}}(t), a \right\rangle \\ & := \int_{T^* \mathbb{T}^d} \left(1 - \chi \left(\frac{P_{\Lambda_1}(\xi)}{R_1 h} \right) \right) \dots \left(1 - \chi \left(\frac{P_{\Lambda_k}(\xi)}{R_k h} \right) \right) \left(1 - \chi \left(\frac{P_{\Lambda_{k+1}}(\xi)}{R_{k+1} h} \right) \right) \\ & \quad a \left(x, \xi, \frac{P_{\Lambda_1}(\xi)}{h}, \dots, \frac{P_{\Lambda_{k+1}}(\xi)}{h} \right) w_h(t)(dx, d\xi), \end{aligned}$$

and

$$\begin{aligned} & \left\langle w_{\Lambda_{k+1}, h, R_1, \dots, R_k}^{\Lambda_1 \Lambda_2 \dots \Lambda_k}(t), a \right\rangle \\ & := \int_{T^* \mathbb{T}^d} \left(1 - \chi \left(\frac{P_{\Lambda_1}(\xi)}{R_1 h} \right) \right) \dots \left(1 - \chi \left(\frac{P_{\Lambda_k}(\xi)}{R_k h} \right) \right) \chi \left(\frac{P_{\Lambda_{k+1}}(\xi)}{R_{k+1} h} \right) \\ & \quad a \left(x, \xi, \frac{P_{\Lambda_1}(\xi)}{h}, \dots, \frac{P_{\Lambda_{k+1}}(\xi)}{h} \right) w_h(t)(dx, d\xi). \end{aligned}$$

By the Calderón-Vaillancourt theorem, both $w_{\Lambda_{k+1}, h, R_1, \dots, R_k}^{\Lambda_1 \Lambda_2 \dots \Lambda_k}$ and $w_{h, R_1, \dots, R_k}^{\Lambda_1 \Lambda_2 \dots \Lambda_{k+1}}$ are bounded in $L^\infty(\mathbb{R}, (\mathcal{S}_{\Lambda_{k+1}}^{k+1})')$. After possibly extracting subsequences, we can take the following limits :

$$\lim_{R_1 \rightarrow +\infty} \dots \lim_{R_{k+1} \rightarrow +\infty} \lim_{h \rightarrow 0} \left\langle w_{h, R_1, \dots, R_k}^{\Lambda_1 \Lambda_2 \dots \Lambda_{k+1}}(t), a \right\rangle =: \left\langle \tilde{\mu}^{\Lambda_1 \Lambda_2 \dots \Lambda_{k+1}}, a \right\rangle,$$

and

$$\lim_{R_1 \rightarrow +\infty} \dots \lim_{R_{k+1} \rightarrow +\infty} \lim_{h \rightarrow 0} \left\langle w_{\Lambda_{k+1}, h, R_1, \dots, R_k}^{\Lambda_1 \Lambda_2 \dots \Lambda_k}(t), a \right\rangle =: \left\langle \tilde{\mu}_{\Lambda_{k+1}}^{\Lambda_1 \Lambda_2 \dots \Lambda_k}, a \right\rangle.$$

The following is a direct generalization of Theorem 11 :

Theorem 21. (i) For a.e. $t \in \mathbb{R}$, $\tilde{\mu}_{\Lambda_{k+1}}^{\Lambda_1 \Lambda_2 \dots \Lambda_k}(t, \cdot)$ and $\tilde{\mu}^{\Lambda_1 \Lambda_2 \dots \Lambda_{k+1}}(t, \cdot)$ are distributions supported on $T^*\mathbb{T}^d \times \langle \Lambda_1 \rangle \times \dots \times \langle \Lambda_{k+1} \rangle$; in addition, $\tilde{\mu}^{\Lambda_1 \Lambda_2 \dots \Lambda_{k+1}}(t, \cdot)$ is a positive measure, zero-homogeneous in the variables $\eta_1 \in \langle \Lambda_1 \rangle, \dots, \eta_{k+1} \in \langle \Lambda_{k+1} \rangle$. The projection of $\tilde{\mu}_{\Lambda_{k+1}}^{\Lambda_1 \Lambda_2 \dots \Lambda_k}(t, \cdot)$ on $T^*\mathbb{T}^d \times \langle \Lambda_1 \rangle \times \dots \times \langle \Lambda_k \rangle$ is positive, and zero-homogeneous in the variables $\eta_1 \in \langle \Lambda_1 \rangle, \dots, \eta_k \in \langle \Lambda_k \rangle$.

(ii) Let

$$\mu_{\Lambda_{k+1}}^{\Lambda_1 \Lambda_2 \dots \Lambda_k}(t, \cdot) = \int_{R_{\Lambda_2}(\Lambda_1) \times \dots \times R_{\Lambda_{k+1}}(\Lambda_k) \times \mathbb{R}^d} \tilde{\mu}_{\Lambda_{k+1}}^{\Lambda_1 \Lambda_2 \dots \Lambda_k}(t, \cdot, d\eta_1, \dots, d\eta_{k+1}) \rfloor_{\mathbb{T}^d \times R_{\Lambda_1}},$$

$$\mu^{\Lambda_1 \Lambda_2 \dots \Lambda_{k+1}}(t, \cdot) = \int_{R_{\Lambda_2}(\Lambda_1) \times \dots \times R_{\Lambda_{k+1}}(\Lambda_k) \times \mathbb{R}^d} \tilde{\mu}^{\Lambda_1 \Lambda_2 \dots \Lambda_{k+1}}(t, \cdot, d\eta_1, \dots, d\eta_{k+1}) \rfloor_{\mathbb{T}^d \times R_{\Lambda_1}}.$$

Then both $\mu_{\Lambda_{k+1}}^{\Lambda_1 \Lambda_2 \dots \Lambda_k}(t, \cdot)$ and $\mu^{\Lambda_1 \Lambda_2 \dots \Lambda_{k+1}}(t, \cdot)$ are positive measures on $T^*\mathbb{T}^d$, invariant by the geodesic flow, and satisfy:

$$(17) \quad \mu^{\Lambda_1 \Lambda_2 \dots \Lambda_k} \rfloor_{\eta_k \in R_{\Lambda_{k+1}}(\Lambda_k)}(t, \cdot) = \mu_{\Lambda_{k+1}}^{\Lambda_1 \Lambda_2 \dots \Lambda_k}(t, \cdot) + \mu^{\Lambda_1 \Lambda_2 \dots \Lambda_{k+1}}(t, \cdot).$$

As a generalization of Theorem 12, we get :

Theorem 22. (i) For a.e. $t \in \mathbb{R}$, the measure $\tilde{\mu}_{\Lambda_{k+1}}^{\Lambda_1 \Lambda_2 \dots \Lambda_k}(t, \cdot)$ is concentrated on $\mathbb{T}^d \times \Lambda_{k+1}^\perp \times (\Lambda_{k+1}^\perp \cap \langle \Lambda_1 \rangle) \times \dots \times (\Lambda_{k+1}^\perp \cap \langle \Lambda_k \rangle) \times \langle \Lambda_{k+1} \rangle$ and its projection on \mathbb{T}^d is absolutely continuous with respect to the Lebesgue measure.

(ii) For a.e. $t \in \mathbb{R}$, the measure $\tilde{\mu}^{\Lambda_1 \Lambda_2 \dots \Lambda_{k+1}}(t, \cdot)$ satisfies the invariance properties:

$$(\phi_s^j)_* \tilde{\mu}^{\Lambda_1 \Lambda_2 \dots \Lambda_{k+1}}(t, \cdot) = \tilde{\mu}^{\Lambda_1 \Lambda_2 \dots \Lambda_{k+1}}(t, \cdot),$$

for $j = 0, \dots, k+1$.

The absolute continuity in (i) is obtained by showing that

$$(18) \quad \int b(x) \tilde{\mu}_{\Lambda_{k+1}}^{\Lambda_1 \Lambda_2 \dots \Lambda_k}(t, dx, d\xi, d\eta_1, \dots, d\eta_{k+1}) \\ = \text{Tr} \left(m_{b \circ \pi_{\Lambda_{k+1}}} \cdot \int e^{it\Delta_{\Lambda_{k+1}}/2} \tilde{\rho}_{\Lambda_{k+1}}^{\Lambda_1 \Lambda_2 \dots \Lambda_k}(d\xi, d\eta_1, \dots, d\eta_k) e^{-it\Delta_{\Lambda_{k+1}}/2} \right),$$

where $\tilde{\rho}_{\Lambda_{k+1}}^{\Lambda_1 \Lambda_2 \dots \Lambda_k}$ is a positive measure on $\Lambda_{k+1}^\perp \times (\Lambda_{k+1}^\perp \cap \langle \Lambda_1 \rangle) \times \dots \times (\Lambda_{k+1}^\perp \cap \langle \Lambda_k \rangle)$, taking values in $\mathcal{L}^1(L^2(\mathbb{T}_{\Lambda_{k+1}}))$.

Remark 23. *By construction, if $\Lambda_{k+1} = \{0\}$, we have $\tilde{\mu}^{\Lambda_1\Lambda_2\dots\Lambda_{k+1}} = 0$, and the induction stops. Similarly to Remark 13, one can also see that if $\text{rk } \Lambda_{k+1} = 1$, the invariance properties of $\tilde{\mu}^{\Lambda_1\Lambda_2\dots\Lambda_{k+1}}$ imply that it is constant in x .*

End of proof of Theorem 4 : To end the proof of Theorem 4, we let

$$\nu_\Lambda(t, \cdot) = \sum_{0 \leq k \leq d-1} \sum_{\Lambda_1 \supset \Lambda_2 \supset \dots \supset \Lambda_k \supset \Lambda} \int_{\mathbb{R}^d} \mu_\Lambda^{\Lambda_1\Lambda_2\dots\Lambda_k}(t, \cdot, d\xi),$$

where $\Lambda_1, \dots, \Lambda_k$ run over the set of strictly decreasing sequences of submodules, such that $\Lambda_k \subset \Lambda$. And we let

$$\sigma_\Lambda = \sum_{0 \leq k \leq d-1} \sum_{\Lambda_1 \supset \Lambda_2 \supset \dots \supset \Lambda_k \supset \Lambda} \int \tilde{\rho}_\Lambda^{\Lambda_1\Lambda_2\dots\Lambda_k}(d\xi, d\eta_1, \dots, d\eta_k),$$

where the $\tilde{\rho}_\Lambda^{\Lambda_1\Lambda_2\dots\Lambda_k}$ are the operator-valued measures appearing in (18).

5. APPENDIX : PSEUDODIFFERENTIAL CALCULUS

In the paper, we use the Weyl quantization with parameter h , that associates to a function a on $T^*\mathbb{R}^d = \mathbb{R}^d \times \mathbb{R}^d$ an operator $\text{Op}_h(a)$, with kernel

$$K_a^h(x, y) = \frac{1}{(2\pi h)^d} \int_{\mathbb{R}^d} a\left(\frac{x+y}{2}, \xi\right) e^{\frac{i}{h}\xi \cdot (x-y)} d\xi.$$

If a is smooth and has uniformly bounded derivatives, then this defines a continuous operator $\mathcal{S}(\mathbb{R}^d) \rightarrow \mathcal{S}(\mathbb{R}^d)$, and also $\mathcal{S}'(\mathbb{R}^d) \rightarrow \mathcal{S}'(\mathbb{R}^d)$. If a is $(2\pi\mathbb{Z})^d$ -periodic with respect to the first variable (which is always the case in this paper), the operator preserves the space of $(2\pi\mathbb{Z})^d$ -periodic distributions on \mathbb{R}^d . We note the relation $\text{Op}_h(a(x, \xi)) = \text{Op}_1(a(x, h\xi))$.

We use two standard results of pseudodifferential calculus.

Theorem 24. *(The Calderón-Vaillancourt theorem)*

There exists an integer K_d , and a constant $C_d > 0$ (depending on the dimension d) such that, if a is a smooth function on $T^\mathbb{T}^d$, with uniformly bounded derivatives, then*

$$\|\text{Op}_1(a)\|_{L^2(\mathbb{T}^d) \rightarrow L^2(\mathbb{T}^d)} \leq \sum_{\alpha \in \mathbb{N}^{2d}, |\alpha| \leq K_d} \sup_{T^*\mathbb{T}^d} |\partial^\alpha a|.$$

A proof in the case of $L^2(\mathbb{R}^d)$ can be found in [2]. It can be adapted to the case of a compact manifold by working locally, in coordinate charts.

We also recall the following formula for the product of two pseudodifferential operators (see for instance [2], p. 79) : $\text{Op}_1(a) \circ \text{Op}_1(b) = \text{Op}_1(a\sharp b)$, where

$$a\sharp b(x, \xi) = \frac{1}{(2\pi)^{4d}} \int_{\mathbb{R}^{4d}} e^{\frac{i}{2}\sigma(u_1, u_2)} (\mathcal{F}a_z)(u_1) (\mathcal{F}b_z)(u_2) du_1 du_2,$$

where we let $z = (x, \xi) \in \mathbb{R}^{2d}$, a_z is the function $\omega \mapsto a(z + \omega)$, and \mathcal{F} is the Fourier transform. We can deduce from this formula and from the Calderón-Vaillancourt theorem the following estimate :

Proposition 25. *Let a and b be two smooth functions on $T^*\mathbb{T}^d$, with uniformly bounded derivatives.*

$$\|\mathrm{Op}_1(a) \circ \mathrm{Op}_1(b) - \mathrm{Op}_1(ab)\|_{L^2(\mathbb{T}^d) \rightarrow L^2(\mathbb{T}^d)} \leq C_d \sum_{\alpha \in \mathbb{N}^{2d}, |\alpha| \leq K_d} \sup_{T^*\mathbb{T}^d} |\partial^\alpha D(a, b)|,$$

where we denote $D(a, b)$ the function $D(a, b)(x, \xi) = (\partial_x \partial_\eta - \partial_y \partial_\xi)(a(x, \xi)b(y, \eta))|_{x=y, \eta=\xi}$.

We finally deduce the following corollary. We use the notations of Section 3.

Corollary 26. *Let $a \in C^\infty(\mathbb{T}^d \times \mathbb{R}^d)$ have uniformly bounded derivatives, and let $\chi \in C_c^\infty(\mathbb{R}^d)$ be a nonnegative cut-off function. Let $0 < h < 1$ and $R > 1$. Denote*

$$a_R(x, \xi) = a(x, \xi) \chi \left(\frac{P_\Lambda \xi}{hR} \right).$$

Assume that $a > 0$, and denote $b_R = \sqrt{a_R}$. Then

$$\|\mathrm{Op}_h(a_R) - \mathrm{Op}_h(b_R)^2\|_{L^2(\mathbb{T}^d) \rightarrow L^2(\mathbb{T}^d)} = \mathcal{O}(h) + \mathcal{O}(R^{-1})$$

in the limits $h \rightarrow 0$ and $R \rightarrow +\infty$.

Corollary 27. *Let $a \in C^\infty(\mathbb{T}^d \times \mathbb{R}^d \times \mathbb{R}^d)$, 0-homogeneous in the third variable outside a compact set, with uniformly bounded derivatives, and let $\chi \in C_c^\infty(\mathbb{R}^d)$ be a nonnegative cut-off function. Let $0 < h < 1$ and $R > 1$. Denote*

$$a^R(x, \xi) = a \left(x, \xi, \frac{P_\Lambda \xi}{h} \right) \left(1 - \chi \left(\frac{P_\Lambda \xi}{hR} \right) \right).$$

Assume that $a > 0$, and denote $b^R = \sqrt{a^R}$. Then

$$\|\mathrm{Op}_h(a^R) - \mathrm{Op}_h(b^R)^2\|_{L^2(\mathbb{T}^d) \rightarrow L^2(\mathbb{T}^d)} = \mathcal{O}(R^{-1})$$

in the limits $h \rightarrow 0$ and $R \rightarrow +\infty$.

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