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ABSTRACT. We construct two non isomorphic contractible affine threefolds X and Y with the property that their cylinders $X \times \mathbb{A}^1$ and $Y \times \mathbb{A}^1$ are isomorphic, showing that the generalized Cancellation Problem has a negative answer in general for contractible affine threefolds. We also establish that X and Y are actually biholomorphic as complex analytic varieties, providing the first example of a pair of biholomorphic but not isomorphic exotic $\mathbb{A}^3$'s.

INTRODUCTION

The Cancellation Problem asks if a complex algebraic variety X of dimension d such that $X \times \mathbb{A}^n$ is isomorphic to $\mathbb{A}^{n+d}$ is isomorphic to $\mathbb{A}^d$. This is a difficult problem in general and, apart form the trivial case $d = 1$, an affirmative answer is known only in dimension 2. One can ask more generally if two algebraic varieties X and Y such that $X \times \mathbb{A}^n$ is isomorphic to $Y \times \mathbb{A}^n$ for some $n \geq 1$ are isomorphic. This more general problem has an affirmative answer for a large class of varieties: intuitively, cancellation should hold provided that either X or Y does not contain too many rational curves. A precise characterization has been given by Iitaka and Fujita [10] in terms of logarithmic Kodaira dimension, namely, if either X or Y, say X, has non negative logarithmic Kodaira dimension $\bar{\kappa}(X) \geq 0$, then every isomorphism between $X \times \mathbb{A}^n$ and $Y \times \mathbb{A}^n$ descends to an isomorphism between X and Y. This assumption on the logarithmic Kodaira dimension turns out to be essential. Indeed, W. Danielewski [3] showed in 1989 that the rational affine surfaces $S_1 = \{xy = z^2 - 1\}$ and $S_2 = \{x^2 y = z^2 - 1\}$ in $\mathbb{A}^3$ are non isomorphic but have isomorphic cylinders $S_1 \times \mathbb{A}^1$ and $S_2 \times \mathbb{A}^1$. Since then, Danielewski's construction has been generalized and adapted to construct many new counter examples of the same type, in arbitrary dimension [6, 8, 22].

However, all counter-examples constructed so far using variants of Danielewski's idea are remote from affine spaces: for instance, the Danielewski surfaces have nontrivial Picard groups and their underlying euclidean topological spaces are not contractible. Therefore, one may expect that cancellation holds for affine varieties close to the affine space. This is actually the case for smooth contractible or factorial surfaces. For the first ones, this follows from an algebro-geometric characterization of $\mathbb{A}^2$ due to Miyanishi-Sugie [16, 17, 21] which says that a smooth acyclic surface S with $\bar{\kappa}(S) = -\infty$ is isomorphic to $\mathbb{A}^2$ (see also [2] for a purely algebraic self-contained proof). On the other hand, the fact that generalized cancellation holds for smooth factorial affine surfaces S seems to be folklore. Roughly, the argument goes as follows: first one may assume that S has logarithmic Kodaira dimension $\bar{\kappa} = -\infty$. By virtue of a characterization due to T. Sugie [20], it follows that S admits an $\mathbb{A}^1$-fibration $\pi : S \to C$ over a smooth curve C, that is, a surjective morphism with general fibers isomorphic to $\mathbb{A}^1$. The hypothesis that the Picard group of S is trivial implies that the same holds for C, and so, C is a factorial affine curve. Combined with the classification of germs of degenerate fibers of $\mathbb{A}^1$-fibrations given by K.-H. Fieseler [8], the factoriality of S implies that $\pi : S \to C$ has no degenerate fibers, whence is a locally trivial $\mathbb{A}^1$-bundle. Since C is affine and factorial, $\pi : S \to C$ is actually a trivial $\mathbb{A}^1$-bundle $S \simeq C \times \mathbb{A}^1 \to C$ and so, the result follows from the affirmative answer to the generalized Cancellation Problem for curves due to Abhyankar-Eakin-Heinzer [1].

The situation turns out to be very different in dimension 3. Indeed, recently, D. Finston and S. Maubach [9] constructed smooth factorial counter-examples to the generalized Cancellation Problem. The latter arise as total spaces of locally trivial $\mathbb{A}^1$-bundles over the complement of the isolated singularity of a Brieskorn surface $x^p + y^q + z^r = 0$ in $\mathbb{A}^3$, with $1/p + 1/q + 1/r < 1$. By construction, these counter-examples are not contractible, having the homology type of a 3-sphere, and so, the existence of contractible counter-examples remained open. A famous

candidate for being such a counter-example is the Russell cubic threefold V defined by the equation $x^2y + z^2 + t^3 + x = 0$ in $\mathbb{A}^4$. The latter is known to be contractible but not isomorphic to $\mathbb{A}^3$ (see e.g. [12] and [14]) and it is an open problem to decide whether $V \times \mathbb{A}^1$ is isomorphic to $\mathbb{A}^4$ or not. In this article, we show that a mild variation on the above candidate already leads to contractible counter-examples to the generalized Cancellation Problem in dimension 3. Namely we consider the smooth affine threefolds

$$X_a = \left\{ x^4y + z^2 + t^3 + x + x^2 + ax^3 = 0 \right\}$$

in $\mathbb{A}^4$, where a is a complex parameter. We establish the following result:

Theorem. *The threefolds X_a are contractible, non isomorphic to $\mathbb{A}^3$ and not isomorphic to each other. However, the cylinders $X_a \times \mathbb{A}^1$, $a \in \mathbb{C}$, are all isomorphic.*

Recall that by virtue of a characterization due to A. Dimca [4], the varieties X_a, $a \in \mathbb{C}$, are all diffeomorphic to $\mathbb{R}^6$ when equipped with the euclidean topology, whence give examples of non isomorphic exotic affine spaces. We show in contrast they are all biholomorphic when considered as complex analytic manifolds, thus answering an open problem raised by M. Zaidenberg [25].

The article is organized as follows. In the first section, we consider more general contractible affine threefolds $X_{n,p}$ in $\mathbb{A}^4$ defined by equations of the form $x^n y + z^2 + t^3 + xp(x) = 0$, where $n \geq 2$ and $p(x) \in \mathbb{C}[x]$. We provide, for each fixed integer $n \geq 2$, a complete classification of isomorphism classes of such varieties and their cylinders. As a corollary, we obtain that the varieties X_a, $a \in \mathbb{C}$, are pairwise non isomorphic and have isomorphic cylinders. The second section is devoted to a geometric interpretation of the existence of an isomorphism between the cylinders $X_a \times \mathbb{A}^1$, $a \in \mathbb{C}$, in terms of a Danielewski fiber product trick construction.

1. Main results

For any integer $n \geq 2$ and any polynomial $q \in \mathbb{C}[x, z, t]$, we consider the affine threefold $V_{n,q}$ in $\mathbb{A}^4 = \mathrm{Spec}(\mathbb{C}[x, y, z, t])$ defined by the equation

$$x^n y + z^2 + t^3 + xq(x, z, t) = 0.$$

Note that $V_{n,q}$ is smooth if and only if $q(0, 0, 0)$ is a nonzero constant. The morphism $\pi = \mathrm{pr}_x : V_{n,q} \to \mathbb{A}^1$ is a flat $\mathbb{A}^2$-fibration restricting to a trivial $\mathbb{A}^2$-bundle over $\mathbb{A}^1 \setminus \{0\}$ and with degenerate fiber $\pi^{-1}(0)$ isomorphic to the cylinder $\Gamma_{2,3} \times \mathbb{A}^1$ over the plane cuspidal curve $\Gamma_{2,3} = \{z^2 + t^3 = 0\}$. This implies in particular that $V_{n,q}$ is factorial. Moreover, via the natural localization homomorphism, one may identify the coordinate ring of $V_{n,q}$ with the sub-algebra $\mathbb{C}\left[x, z, t, x^{-n}\left(z^2 + t^3 + xq(x, y, t)\right)\right]$ of $\mathbb{C}\left[x^{\pm 1}, z, t\right]$. This says equivalently that $V_{n,q}$ is the *affine modification* $\sigma_{n,q} = \mathrm{pr}_{x,z,t} : V_{n,q} \to \mathbb{A}^3 = \mathrm{Spec}(\mathbb{C}[x, z, t])$ of $\mathbb{A}^3 = \mathrm{Spec}(\mathbb{C}[x, z, t])$ with center at the closed subscheme $Z_{n,q}$ with defining ideal $I_{n,q} = \left(x^n, z^2 + t^3 + xq(x, y, t)\right)$ and divisor $D = \{x^n = 0\}$ in the sense of [12], that is, $V_{n,q}$ is isomorphic to the complement of the proper transform of D in the blow-up of $\mathbb{A}^3$ with center at $Z_{n,q}$. It follows in particular from Theorem 3.1 in [12] that a smooth $V_{n,q}$ is contractible when considered as a complex manifold. As a consequence of the general methods developed in [11], we have the following useful criterion to decide which threefolds $V_{n,q}$ are isomorphic.

Lemma 1. *Every isomorphism $\Phi : V_{n_1,q_1} \xrightarrow{\sim} V_{n_2,q_2}$ is the lift via σ_{n_1,q_1} and σ_{n_2,q_2} of an automorphism of $\mathbb{A}^3$ which maps the locus of the modification σ_{n_1,q_1} isomorphically onto the one of the modification σ_{n_2,q_2}. Equivalently, there exists a commutative diagram*

$$
\begin{array}{ccc}
V_{n_1,q_1} & \xrightarrow{\ \Phi\ } & V_{n_2,q_2} \\
{\scriptstyle \sigma_{n_1,q_1}}\downarrow & & \downarrow{\scriptstyle \sigma_{n_2,q_2}} \\
\mathbb{A}^3 & \xrightarrow{\ \varphi\ } & \mathbb{A}^3
\end{array}
$$

where φ is an automorphism of $\mathbb{A}^3$ which preserves the hyperplane $\{x = 0\}$ and maps Z_{n_1,q_1} isomorphically onto Z_{n_2,q_2}.

Proof. The fact that every automorphism of $\mathbb{A}^3$ satisfying the above property lifts to an isomorphism between V_{n_1,q_1} and V_{n_2,q_2} is an immediate consequence of the universal property of affine modifications, Proposition 2.1 in [12]. For the converse we exploit two invariants of an affine variety V: the Makar-Limanov invariant (resp. the Derksen invariant) of V which is the sub-algebra $\mathrm{ML}\,(V)$ (resp. $\mathrm{Dk}\,(V)$) of $\Gamma\,(V,\mathcal{O}_V)$ generated by regular functions invariant under all (resp. at least one) non trivial algebraic $\mathbb{G}_a$-actions on V (see e.g. [24]). The same arguments as the ones used to treat the case of the Russell cubic threefold $V_{2,1}$ in [11] show more generally that $\mathrm{ML}\,(V_{n,q}) = \mathbb{C}\,[x]$ and $\mathrm{Dk}\,(V_{n,q}) = \mathbb{C}\,[x,z,t]$ for every $q \in \mathbb{C}\,[x,z,t]$. This implies that any isomorphism between the coordinate rings of V_{n_1,q_1} and V_{n_2,q_2} restricts to an automorphism φ^* of $\mathbb{C}\,[x,z,t]$ inducing a one $x \mapsto ax + b$ of $\mathbb{C}\,[x]$, where $a \in \mathbb{C}^*$ and $b \in \mathbb{C}$. Actually, $b = 0$ as the zero set of $ax + b$ in $V_{n_i,q_i}, i = 1, 2$, is singular if and only if $b = 0$. So φ^* stabilizes the ideal (x). In turn, the fact that $I_{n_i,q_i} = x^n \Gamma \left(V_{n_i,q_i}, \mathcal{O}_{V_{n_i,q_i}} \right) \cap \mathbb{C}\,[x,z,t]$ implies that $\varphi^* \left(I_{n_2,q_2} \right) = I_{n_1,q_1}$. Now the assertion follows since the modification morphism $\sigma_{n,q} : V_{n,q} \to \mathbb{A}^3$ defined above is precisely induced by the natural inclusion of $\mathrm{Dk}\,(V_{n,q})$ into $\Gamma \left(V_{n,q}, \mathcal{O}_{V_{n,q}} \right)$. $\qquad\square$

2. From now on, we only consider a very special case of smooth contractible threefolds $V_{n,q}$, namely, the ones $V_{n,p}$ defined by equations

$$x^n y + z^2 + t^3 + xp\,(x) = 0,$$

where $p \in \mathbb{C}\,[x]$ is a polynomial such that $p\,(0) \neq 0$. We have the following result.

Theorem 3. *For a fixed integer $n \geq 2$, the following hold:*
   (1) *The algebraic varieties V_{n,p_1} and V_{n,p_2} are isomorphic if and only if there exists $\lambda, \varepsilon \in \mathbb{C}^*$ such that $p_2\,(x) \equiv \varepsilon p_1\,(\lambda x)\ mod\ x^{n-1}$.*
   (2) *The cylinders $V_{n,p} \times \mathbb{A}^1$ are all isomorphic.*
   (3) *The varieties $V_{n,p}$ are all isomorphic as complex analytic manifolds.*

Proof. Letting $r = z^2 + t^3$, it follows from Lemma 1 above that $V_{n,p_2} \simeq V_{n,p_1}$ if and only if there exists an automorphism ϕ of $\mathbb{A}^3 = \mathrm{Spec}\,(\mathbb{C}\,[x,z,t])$ which preserves the hyperplane $\{x = 0\}$ and maps the closed subscheme with defining ideal $(x^n, r + xp_2\,(x))$ isomorphically onto the one with defining ideal $(x^n, r + xp_1\,(x))$. Since such an automorphism stabilizes the hyperplane $\{x = 0\}$, there exists $\lambda \in \mathbb{C}^*$ such that $\phi^*\,(x) = \lambda x$. Furthermore, ϕ maps the curve $\Gamma_{2,3} = \left\{ x = z^2 + t^3 = 0 \right\}$ isomorphically onto itself.

So there exists $\mu \in \mathbb{C}^*$ such that $\phi^* z \equiv \mu^3 z \ mod\ x$ and $\phi^* t \equiv \mu^2 t\ mod\ x$. Therefore, by composing ϕ with the automorphism $\theta : \mathbb{A}^3 \overset{\sim}{\to} \mathbb{A}^3$, $(x,z,t) \mapsto \left(\lambda^{-1} x, \mu^{-3} z, \mu^{-2} t \right)$, we get an automorphism ψ of $\mathbb{A}^3$ such that $\psi^* x = x$, $\psi^* z \equiv z\ mod\ x$, $\psi^* t \equiv t\ mod\ x$ and which maps the closed subscheme with defining ideal $(x^n, r + xp_2\,(x))$ isomorphically onto the one with defining ideal $\left(x^n, r + \mu^6 \lambda x p_1\,(\lambda x) \right)$. Letting $\varepsilon = \mu^6 \lambda$, this implies $p_2\,(x) \equiv \varepsilon p_1\,(\lambda x)\ mod\ x^{n-1}$ by virtue of Lemma 4 below.

Conversely, if $p_2\,(x) \equiv \varepsilon p_1\,(\lambda x)\ mod\ x^{n-1}$, we let $\mu \in \mathbb{C}^*$ be such that $\varepsilon = \mu^6 \lambda$. Then the automorphism

$$(x,y,z,t) \mapsto \left(\lambda x, \lambda^{-n} \mu^{-6} y + \lambda^{-n} x^{-n+1} \left(\mu^{-6} p_2(x) - \lambda p_1\,(\lambda x) \right), \mu^{-3} z, \mu^{-2} t \right)$$

of $\mathbb{A}^4$ maps V_{n,p_2} isomorphically onto V_{n,p_1}. This proves (1).

To prove (2) and (3), it is enough to show that for every $p \in \mathbb{C}\,[x]$ such that $p\,(0) \neq 0$, $V_{n,p}$ is biholomorphic to $V_{n,1}$ and that these two threefolds have algebraically isomorphic cylinders. The arguments are similar to arguments developed in [19].

First, up to the composition by an isomorphism induced by an automorphism of $\mathbb{A}^4$ of the form $(x,y,z,t) \mapsto (\lambda x, \lambda^{-n} y, z, t)$, we may assume that $p\,(0) = 1$. Remark also that the ideals $(x^n, z^2 + t^3 + x)$ and $(x^n, p(x)(z^2 + t^3 + x))$ are equal. Therefore, by virtue of Lemma 1, $V_{n,1}$ is isomorphic as an algebraic variety to the variety $W_{n,p}$ defined by the equation $x^n y + p(x)(z^2 + t^3 + x) = 0$.

Letting $f \in \mathbb{C}\,[x]$ be a polynomial such that $\exp\,(xf\,(x)) \equiv p\,(x)\ mod\ x^n$, one checks that the biholomorphism ψ of $\mathbb{A}^3$ defined by

$$\Psi\,(x,z,t) = \left(x, y - \frac{\exp\,(xf\,(x)) - p(x)}{x^n}(z^2 + t^3), \exp\left(\frac{1}{2} xf\,(x) \right) z, \exp\left(\frac{1}{3} xf\,(x) \right) t \right)$$

maps $W_{n,p}$ onto $V_{n,p}$. So (3) follows.

For (2), we choose polynomials $g_1 \in \mathbb{C}[x]$ such that $\exp\left(\frac{1}{2}xf(x)\right) \equiv g_1 \mod x^n$ and $g_2 \in \mathbb{C}[x]$ relatively prime to g_1 such that $\exp\left(\frac{1}{3}xf(x)\right) \equiv g_2 \mod x^n$. Since $g_1(0) = g_2(0) = 1$, the polynomials $x^n g_1$, $x^n g_2$ and $g_1 g_2$ generate the unit ideal in $\mathbb{C}[x]$. So there exist polynomials $h_1, h_2, h_3 \in \mathbb{C}[x]$ such that

$$\begin{pmatrix} g_1 & 0 & x^n \\ 0 & g_2 & x^n \\ h_1(x) & h_2(x) & h_3(x) \end{pmatrix} \in \mathrm{GL}_3\left(\mathbb{C}[x]\right).$$

This matrix defines a $\mathbb{C}[x]$-automorphism of $\mathbb{C}[x][z,t,w]$ which maps the ideal $(x^n, r + xp(x))$ of $\mathbb{C}[x][z,t,w]$ onto the one $(x^n, p(x)(r+x)) = (x^n, r + x)$. Since these ideals coincide with the centers of the affine modifications $\sigma_{n,p} \times \mathrm{id} : V_{n,p} \times \mathbb{A}^1 \to \mathbb{A}^4$ and $\sigma_{n,1} \times \mathrm{id} : V_{n,1} \times \mathbb{A}^1 \to \mathbb{A}^4$ respectively, we can conclude by Proposition 2.1 in [12] that the corresponding automorphism of $\mathbb{A}^4 = \mathrm{Spec}\left(\mathbb{C}[x,z,t,w]\right)$ lifts to an isomorphism between $V_{n,1} \times \mathbb{A}^1$ and $V_{n,p} \times \mathbb{A}^1$. This completes the proof. $\qquad\square$

Lemma 4. *Let $n \geq 2$ and $p_1, p_2 \in \mathbb{C}[x]$ be polynomials of degree $\leq n - 2$. If there exists a $\mathbb{C}[x]$-automorphism Φ of $\mathbb{C}[x][z,t]$ such that $\Phi \equiv \mathrm{id} \mod x$ and $\Phi\left(x^n, z^2 + t^3 + xp_1\right) = \left(x^n, z^2 + t^3 + xp_2\right)$ then $p_1 = p_2$.*

Proof. We let $r = z^2 + t^3$ and we let $p_i = \sum_{k=0}^{n-2} a_{ik} x^k$, $i = 1, 2$. We let $n_0 \geq 1$ be the largest integer such that $\Phi \equiv \mathrm{id} \mod x^{n_0}$. If $n_0 \geq n - 1$ then we are done. Otherwise, there exist $\alpha, \beta \in \mathbb{C}[x,z,t]$ such that $\Phi(r + xp_1) = (1 + x^{n_0}\alpha)(r + xp_2) + x^n\beta$. Since the determinant of the Jacobian of Φ is a nonzero constant, there exists $h \in \mathbb{C}[z,t]$ such that

$$\begin{cases} \Phi(z) & \equiv z + x^{n_0}\partial_t h \mod x^{n_0+1} \\ \Phi(t) & \equiv t - x^{n_0}\partial_z h \mod x^{n_0+1}. \end{cases}$$

It follows that $\Phi(r + xp_1) \equiv r + xp_1 + x^{n_0}\mathrm{Jac}(r,h) \mod x^{n_0+1}$. By comparing with the other expression for $\Phi(r + xp_1)$, we find that $p_1 \equiv p_2 \mod x^{n_0-1}$ and $a_{1,n_0-1} + \mathrm{Jac}(r,h) = \alpha(0,z,t)r + a_{2,n_0-1}$. Since $\mathrm{Jac}(r,h) \in (z,t)\mathbb{C}[z,t]$, we obtain $a_{1,n_0-1} = a_{2,n_0-1}$ and $\mathrm{Jac}(r,h) = \alpha(0,z,t)r$. Moreover, the condition $\mathrm{Jac}(r,h) \in r\mathbb{C}[z,t]$ implies that $h = \gamma(z,t)r + c$ for some $\gamma \in \mathbb{C}[z,t]$ and $c \in \mathbb{C}$. Now we consider the exponential $\mathbb{C}[x]/(x^n)$-automorphism $\exp(\delta)$ of $\mathbb{C}[x]/(x^n)[z,t]$ associated with the Jacobian derivation

$$\delta = x^{n_0}\mathrm{Jac}(\cdot, \gamma(z,t)(r + xp_1)).$$

Since the determinant of the Jacobian of $\exp(\delta)$ is equal to 1 (see [18]), it follows from [23] that there exists a $\mathbb{C}[x]$-automorphism Θ of $\mathbb{C}[x][z,t]$ such that $\Theta \equiv \exp(\delta) \mod x^n$. By construction, $\Theta \equiv \Phi \mod x^{n_0+1}$ and, since $r + xp_1 \in \mathrm{Ker}\delta$, Θ preserves the ideal $(x^n, r + xp_1)$. It follows that $\Psi = \Phi \circ \Theta^{-1}$ is a $\mathbb{C}[x]$-automorphism of $\mathbb{C}[x][z,t]$ such that $\Psi\left(x^n, z^2 + t^3 + xp_1\right) = \left(x^n, z^2 + t^3 + xp_2\right)$ and such that $\Psi \equiv \mathrm{id} \mod x^{n_0+1}$. Now the assertion follows by induction. $\quad\square$

Remark 5. In the proofs above, the crucial point is to characterize the existence of isomorphisms between the centers $Z_{n,p}$ of the affine modifications defining the threefolds $V_{n,p}$ that are induced by automorphisms of the ambient space $\mathbb{A}^3$. Note that for fixed integer $n \geq 2$, these closed subschemes $Z_{n,p}$ with defining ideals $I_{n,p} = \left(x^n, z^2 + t^3 + xp(x)\right)$ are all isomorphic as abstract schemes, and even as abstract infinitesimal deformations of the plane cubic $\Gamma_{2,3} = \{z^2 + t^3 = 0\}$ over $\mathrm{Spec}\left(\mathbb{C}[x]/(x^n)\right)$. Indeed, letting $g_1(x), g_2(x) \in \mathbb{C}[x]$ be polynomials such that $(g_1(x))^2 \equiv p(x) \mod x^n$ and $(g_2(x))^3 \equiv p(x) \mod x^n$, one checks for instance that the automorphism ξ of $\mathrm{Spec}\left(\mathbb{C}[x]/(x^n)[z,t]\right)$ defined by

$$\xi(x,z,t) = (x, g_1(x)z, g_2(x)t)$$

induces an isomorphism between $Z_{n,p}$ and $Z_{n,1}$. However, Theorem 3 says in particular that no isomorphism of this kind can be lifted to an automorphism of $\mathbb{A}^3 = \mathrm{Spec}\left(\mathbb{C}[x,z,t]\right)$. In other words, the $Z_{n,p}$'s can be considered as defining non-equivalent closed embeddings of $Z_{n,1}$ in $\mathbb{A}^3$. The above proof also gives counterexamples, in dimension four, to the so-called stable equivalence problem (see [15] and [19]). Indeed, it implies that for each $n \geq 2$ and each $p(x) \in \mathbb{C}[x]$ with $p(0) \neq 0$, the polynomials $x^n y + z^2 + t^3 + xp(x)$ and $x^n y + p(x)(z^2 + t^3 + x)$ are equivalent

by an automorphism of $\mathbb{C}[x, y, z, t, w]$ whereas they are not equivalent up to an automorphism of $\mathbb{C}[x, y, z, t]$. Their zero-sets are even non isomorphic smooth affine threefolds.

As a very particular case of the above discussion, we obtain the result announced in the introduction, namely:

Corollary 6. *The smooth contractible affine threefolds* $X_a = \left\{ x^4 y + z^2 + t^3 + x + x^2 + ax^3 = 0 \right\}$, $a \in \mathbb{C}$*, are pairwise non isomorphic. However, their cylinders* $X_a \times \mathbb{A}^1$*,* $a \in \mathbb{C}$*, are all isomorphic.*

2. A GEOMETRIC INTERPRETATION

Here we give a geometric interpretation of the existence of an isomorphism between the cylinders over the varieties X_a, $a \in \mathbb{C}$, in terms of a variant of the famous Danielewski fiber product trick [3]. Of course, the construction below can be adapted to cover the general case, but we find it more enlightening to only consider the particular case of the varities X_0 and X_1 in $\mathbb{A}^4 = \text{Spec} \left(\mathbb{C}[x, y, z, t] \right)$ defined respectively by the equations

$$x^4 y + z^2 + t^3 + x + x^2 = 0 \quad \text{and} \quad x^4 y + z^2 + t^3 + x + x^2 + x^3 = 0.$$

For our purpose, it is convenient to use the fact that X_0 and X_1 are isomorphic to the varieties X and Y in $\mathbb{A}^4$ defined respectively by the equations

$$x^4 z = y^2 + x + x^2 - t^3 \quad \text{and} \quad x^4 z = \left(1 + \alpha x^2 \right) y^2 + x + x^2 - t^3,$$

where $\alpha = -\frac{5}{3}$. Clearly, the first isomorphism is simply induced by the coordinate change $(x, y, z, t) \mapsto (x, z, -y, -t)$. For the second one, one checks first that for $\beta = -1/3$, the following matrix in $\text{GL}_2 \left(\mathbb{C}[x] \right)$

$$\begin{pmatrix} 1 - \beta x^2 + \frac{1}{2} \beta^2 x^4 & \frac{1}{2} \beta^2 x^4 \\ \frac{1}{2} \beta^2 x^4 & 1 + \beta x^2 + \frac{1}{2} \beta^2 x^4 \end{pmatrix}$$

defines a $\mathbb{C}[x]$-automorphism of $\mathbb{C}[x][z, t]$ which maps the ideal $\left(x^4, \left(1 + \alpha x^2 \right) z^2 + x + x^2 + t^3 \right)$ onto the one $\left(x^4, z^2 + x + x^2 + x^3 + t^3 \right)$. By virtue of Lemma 1, the corresponding automorphism of $\mathbb{A}^3$ lifts to an isomorphism between X_1 and the subvariety of $\mathbb{A}^4$ defined by the equation $x^4 y + \left(1 + \alpha x^2 \right) z^2 + x + x^2 + t^3 = 0$, and so, we eventually get the desired isomorphism by composing with the previous coordinate change.

7. Now, the principle is the following: we observe that both X and Y come equipped with $\mathbb{G}_a$-actions induced by the ones on $\mathbb{A}^4$ associated respectively with the locally nilpotent derivations $x^4 \partial_y + 2y \partial_z$ and $x^4 \partial_y + 2 \left(1 + \alpha x^2 \right) y \partial_z$ of $\mathbb{C}[x, y, z, t]$. The latter restrict to free actions on the open subsets $X^* = X \setminus \{x = t = 0\}$ and $Y^* = Y \setminus \{x = t = 0\}$ of X and Y respectively, and so, they admit quotients $X^* \to X^*/\mathbb{G}_a$ and $Y^* \to Y^*/\mathbb{G}_a$ in the form of étale locally trivial $\mathbb{G}_a$-bundles over suitable algebraic spaces. We first check that $X^*/\mathbb{G}_a$ and $Y^*/\mathbb{G}_a$ are isomorphic to a same algebraic space $\mathfrak{S}$. This implies that the fiber product $W = X^* \times_{\mathfrak{S}} Y^*$ has the structure of a locally trivial $\mathbb{G}_a$-bundle over both X^* and Y^* via the first and the second projection respectively. Since X^* and Y^* are both strictly quasi-affine, there is no guarantee a priori that these $\mathbb{G}_a$-bundles are trivial. But we check below that it is indeed the case. Therefore, since X and Y are affine, normal, and $X \setminus X^*$ and $Y \setminus Y^*$ have codimension 2 in X and Y respectively, the corresponding isomorphism $X^* \times \mathbb{A}^1 \simeq W \simeq Y^* \times \mathbb{A}^1$ extends to a one $X \times \mathbb{A}^1 \simeq Y \times \mathbb{A}^1$.

8. Let us check first that the quotient spaces $X^*/\mathbb{G}_a$ and $Y^*/\mathbb{G}_a$ are indeed isomorphic. The restriction of the projection $\text{pr}_{x,t} : \mathbb{A}^4 \to \mathbb{A}^2 = \text{Spec} \left(\mathbb{C}[x, t] \right)$ to X^* and Y^* induces $\mathbb{G}_a$-invariant morphisms $\alpha : X^* \to \mathbb{A}^2 \setminus \{(0, 0)\}$ and $\beta : Y^* \to \mathbb{A}^2 \setminus \{0, 0\}$. The latter restrict to trivial $\mathbb{G}_a$-bundles over $\mathbb{A}^2 \setminus \{x = 0\}$. In contrast, the fiber of each morphism over a closed point $(0, t) \in \mathbb{A}^2 \setminus \{(0, 0)\}$ consists of the disjoint union of two affine lines $\{x = 0, y = \pm \mu\}$ where μ is a square root of t^3 whereas the fiber over the non closed point $(x) \in \mathbb{C}[x, t]$ with residue field $\mathbb{C}(t)$ corresponding to the punctured line $\{x = 0\} \subset \mathbb{A}^2 \setminus \{(0, 0)\}$ is isomorphic to the affine line over the degree 2 Galois extension $\mathbb{C}(t)[y] / \left(y^2 - t^3 \right)$ of $\mathbb{C}(t)$. This indicates that the quotient spaces $X^*/\mathbb{G}_a$ and $Y^*/\mathbb{G}_a$ should be obtained from $\mathbb{A}^2 \setminus \{(0, 0)\}$ by replacing the punctured line $\{x = 0\}$ by a nontrivial double étale covering of itself. An algebraic space $\mathfrak{S}$ with this property can be constructed in two steps as follows : first we let $U_\lambda = \text{Spec} \left(\mathbb{C}\left[x, \lambda^{\pm 1} \right] \right)$, $U_{\lambda \lambda} = \text{Spec} \left(\mathbb{C}\left[x^{\pm 1}, \lambda^{\pm 1} \right] \right)$ and we define an algebraic space $\mathfrak{S}_\lambda$ as the quotient of U_λ by the following étale equivalence relation

$$(s,t): R_\lambda = U_\lambda \sqcup U_{\lambda\lambda} \longrightarrow U_\lambda \times U_\lambda, \quad \begin{cases} U_\lambda \ni (x,\lambda) & \mapsto ((x,\lambda),(x,\lambda)) \\ U_{\lambda\lambda} \ni (x,\lambda) & \mapsto ((x,\lambda),(x,-\lambda)). \end{cases}$$

By construction, the R_λ-invariant morphism $U_\lambda \to \mathrm{Spec}\left(\mathbb{C}\left[x,t^{\pm 1}\right]\right)$, $(x,\lambda) \mapsto \left(x,\lambda^2\right)$ descends to a morphism $\mathfrak{S}_\lambda \to \mathrm{Spec}\left(\mathbb{C}\left[x,t^{\pm 1}\right]\right)$ restricting to an isomorphism over $\mathrm{Spec}\left(\mathbb{C}\left[x^{\pm 1},t^{\pm 1}\right]\right)$. The fiber over the punctured line $\{x=0\}$ is isomorphic to $\mathrm{Spec}\left(\mathbb{C}\left(t\right)\left[\lambda\right]/\left(\lambda^2 - t^3\right)\right)$. Now we let $\mathfrak{S}$ be the algebraic space obtained by gluing $\mathfrak{S}_\lambda$ and $U_x = \mathrm{Spec}\left(\mathbb{C}\left[x^{\pm 1},t\right]\right)$ by the identity on $\mathrm{Spec}\left(\mathbb{C}\left[x^{\pm 1},t^{\pm 1}\right]\right)$. By construction, $\mathfrak{S}$ comes equipped with an étale cover $p: V \to \mathfrak{S}$ by the scheme $V = U_x \sqcup U_\lambda$. We let $U_{x,\lambda} = U_x \times_\mathfrak{S} U_\lambda \simeq \mathrm{Spec}\left(\mathbb{C}\left[x^{\pm 1},\lambda^{\pm 1}\right]\right)$.

Lemma 9. *The quotients spaces $X^*/\mathbb{G}_a$ and $Y^*/\mathbb{G}_a$ are both isomorphic to $\mathfrak{S}$.*

Proof. The argument is very similar to the one used in [5].

1) The case of X^*. This quasi-affine threefold is covered by two $\mathbb{G}_a$-invariant open subsets

$$V_x = X^* \setminus \{x=0\} = X \setminus \{x=0\} \qquad \text{and} \qquad V_t = X^* \setminus \{t=0\} = X \setminus \{t=0\}$$

Letting $U_x = \mathrm{Spec}\left(\mathbb{C}\left[x^{\pm 1},t\right]\right)$, one checks easily that the morphism

$$U_x \times \mathbb{G}_a \longrightarrow V_x, \quad (x,t,v) \mapsto \left(x, x^4 v, x^4 v^2 + x^{-4}\left(-t^3 + x + x^2\right), t\right)$$

is an isomorphism, equivariant for the $\mathbb{G}_a$-action on $U_x \times \mathbb{G}_a$ by translations on the second factor, which yields a trivialization of the induced $\mathbb{G}_a$-action on V_x. In contrast, the induced action on V_t is not trivial. However, letting $U_\lambda = \mathrm{Spec}\left(\mathbb{C}\left[x,\lambda^{\pm 1}\right]\right)$, we claim that there exists $\sigma, \xi \in \mathbb{C}\left[x,\lambda^{\pm 1}\right]$ such that the morphism

$$U_\lambda \times \mathbb{G}_a \longrightarrow V_t, \quad (x,\lambda,v) \mapsto \left(x, x^4 v + \sigma, \left(x^4 v + 2\sigma\right)v + \xi, \lambda^2\right)$$

is étale and equivariant for the $\mathbb{G}_a$-action on $U_\lambda \times \mathbb{G}_a$ by translations on the second factor, whence defines an étale trivialization on the induced action on V_t. This can be seen as follows : let

$$V_\lambda = V_t \times_{\mathbb{A}^1_*} \mathbb{A}^1_* \simeq \mathrm{Spec}\left(\mathbb{C}\left[x,y,z,\lambda^{\pm 1}\right]/\left(x^4 z - y^2 + \lambda^6 - x - x^2\right)\right)$$

be the pull-back of V_t by the Galois covering

$$\varphi: \mathbb{A}^1_* = \mathrm{Spec}\left(\mathbb{C}\left[\lambda^{\pm 1}\right]\right) \to \mathbb{A}^1_* = \mathrm{Spec}\left(\mathbb{C}\left[t^{\pm 1}\right]\right), \quad \lambda \mapsto t = \lambda^2.$$

Since $\lambda \in \mathbb{C}\left[x,\lambda^{\pm 1}\right]$ is invertible it follows that one can find $\sigma \in \mathbb{C}\left[x,\lambda^{\pm 1}\right]$ with $\deg_x \sigma \leq 3$ and $\sigma(0,\lambda) = \lambda^3$, and $\xi \in \mathbb{C}\left[x,\lambda^{\pm 1}\right]$ such that

$$y^2 - \lambda^6 + x + x^2 = (y-\sigma)(y+\sigma) + x^4 \xi.$$

Note that σ and ξ, considered as Laurent polynomials in the variable λ, are necessarily odd and even respectively. This identity implies in turn that V_λ is isomorphic to the subvariety of $\mathrm{Spec}\left(\mathbb{C}\left[x,y,z',\lambda^{\pm 1}\right]\right)$ defined by the equation $x^4 z' = (y-\sigma)(y+\sigma)$, where $z' = z - \xi$. The $\mathbb{G}_a$-action on V_t lift to the one on V_λ induced by the locally nilpotent derivation $x^4\partial_y + 2y\partial_{z'}$. The open subset $V_{\lambda+} = V_\lambda \setminus \{x = y+\sigma = 0\} \simeq \mathrm{Spec}\left(\mathbb{C}\left[x,\lambda^{\pm 1}\right][v]\right)$, where

$$v = x^{-4}(y-\sigma)\,|_{V_{\lambda+}} = (y-\sigma)^{-1} z'\,|_{V_{\lambda+}},$$

is equivariantly isomorphic to $U_\lambda \times \mathbb{G}_a$ where $\mathbb{G}_a$ acts on the second factor by translations, and the restriction of the étale morphism $\mathrm{pr}_1: V_t \times_{\mathbb{A}^1_*} \mathbb{A}^1_* \to V_t$ to $V_\lambda \setminus \{x = y+\sigma = 0\} \simeq U_\lambda \times \mathbb{G}_a$ yields the expected étale trivialization. It follows from this description that $X^*/\mathbb{G}_a$ is isomorphic to an algebraic space obtained as the quotient of disjoint union of $U_x = V_x/\mathbb{G}_a$ and $U_\lambda = V_{\lambda+}/\mathbb{G}_a$ by a certain étale equivalence relation. Clearly, the only nontrivial part is to check that $V_t/\mathbb{G}_a$ is isomorphic to the algebraic space $\mathfrak{S}_\lambda$ of 8 above. In view of I.5.8 in [13] it is enough to show that we have a cartesian square

$$\begin{array}{ccc} V_{\lambda+} \times_{V_t} V_{\lambda+} & \underset{\mathrm{pr}_2}{\overset{\mathrm{pr}_1}{\rightrightarrows}} & V_{\lambda+} = U_\lambda \times \mathbb{G}_a \\ \downarrow & & \downarrow {\scriptstyle \mathrm{pr}_1} \\ R_\lambda & \underset{t}{\overset{s}{\rightrightarrows}} & U_\lambda. \end{array}$$

Letting $g(x,\lambda,v) = x^4 v + \sigma(x,\lambda) \in \mathbb{C}\left[x,\lambda^{\pm 1},v\right]$ and $h = \left(x^4 v + 2\sigma(x,\lambda)\right)v + \xi(x,\lambda) \in \mathbb{C}\left[x,\lambda^{\pm 1},v\right]$, $V_{\lambda+} \times_{V_t} V_{\lambda+}$ is isomorphic to the spectrum of the ring

$$A = \mathbb{C}\left[x,\lambda^{\pm 1},\lambda_1^{\pm 1},v,v_1\right] / \left(g(x,\lambda,v) - g(x,\lambda_1,v_1), h(x,\lambda,v) - h(x,\lambda_1,v_1), \lambda^2 - \lambda_1^2\right)$$

Since λ is invertible and $\sigma(0,\lambda) = \lambda^3$, x and σ generate the unit ideal in $\mathbb{C}\left[x,\lambda^{\pm 1}\right]$. It follows that A decomposes as the direct product of the rings

$$
\begin{aligned}
A_0 &= \mathbb{C}\left[x,\lambda^{\pm 1},v,v_1\right] / \left(g(x,\lambda,v) - g(x,\lambda,v_1), h(x,\lambda,v) - h(x,\lambda,v_1)\right) \\
&\simeq \mathbb{C}\left[x,\lambda^{\pm 1},v,v_1\right] / \left(x^4(v - v_1), x^4(v^2 - v_1^2) + 2\sigma(x,\lambda)(v - v_1)\right) \\
&\simeq \mathbb{C}\left[x,\lambda^{\pm 1},v,v_1\right] / \left(x^4(v - v_1), 2\sigma(x,\lambda)(v - v_1)\right) \\
&\simeq \mathbb{C}\left[x,\lambda^{\pm 1}\right][v]
\end{aligned}
$$

and

$$
\begin{aligned}
A_1 &= \mathbb{C}\left[x,\lambda^{\pm 1},v,v_1\right] / \left(g(x,\lambda,v) - g(x,-\lambda,v_1), h(x,\lambda,v) - h(x,-\lambda,v_1)\right) \\
&\simeq \mathbb{C}\left[x,\lambda^{\pm 1},v,v_1\right] / \left(x^4(v - v_1) + 2\sigma(x,\lambda), x^4(v^2 - v_1^2) + 2\sigma(x,\lambda)(v + v_1)\right) \\
&\simeq \mathbb{C}\left[x,\lambda^{\pm 1},v,v_1\right] / \left(x^4(v - v_1) + 2\sigma(x,\lambda)\right) \\
&\simeq \mathbb{C}\left[x^{\pm 1},\lambda^{\pm 1},v,v_1\right] / \left(x^4(v - v_1) + 2\sigma(x,\lambda)\right) \\
&\simeq \mathbb{C}\left[x^{\pm 1},\lambda^{\pm 1}\right][v]
\end{aligned}
$$

Thus $V_{\lambda,+} \times_{V_t} V_{\lambda+}$ is isomorphic to $R_\lambda \times \mathbb{A}^1$ and the above diagram is clearly cartesian. This completes the proof for X^*.

2) The case of Y^*. Similarly as for the case of X^*, Y^* is covered by two $\mathbb{G}_a$-invariant open subsets

$$W_x = Y^* \setminus \{x = 0\} = Y \setminus \{x = 0\} \qquad \text{and} \qquad W_t = Y^* \setminus \{t = 0\} = Y \setminus \{t = 0\}$$

and the morphism

$$U_x \times \mathbb{G}_a \longrightarrow W_x, \ (x,t,v) \mapsto \left(x, x^4 v, x^4\left(1 + \alpha x^2\right)v^2 + x^{-4}\left(-t^3 + x + x^2\right), t\right)$$

defines a trivialization of the induced $\mathbb{G}_a$-action on W_x. To obtain an étale trivialization of the $\mathbb{G}_a$-action on W_t, one checks first that there exists $\zeta \in \mathbb{C}\left[x,\lambda^{\pm 1}\right]$ such that for

$$\tau = \left(1 - \frac{1}{2}\alpha x^2\right)\sigma(x,\lambda)$$

the identity

$$\left(1 + \alpha x^2\right)y^2 - \lambda^6 + x + x^2 = \left(1 + \alpha x^2\right)(y - \tau)(y + \tau) + x^4 \zeta(x,\lambda)$$

holds in $\mathbb{C}\left[x,\lambda^{\pm 1},y\right]$. Then one checks in a similar way as above that the morphism

$$U_\lambda \times \mathbb{G}_a \longrightarrow W_t, \ (x,\lambda,v) \mapsto \left(x, x^4 v + \tau, \left(x^4 v + 2\tau\right)v + \zeta, \lambda^2\right)$$

yields an étale trivialization, that $W_t / \mathbb{G}_a \simeq U_\lambda / R_\lambda = \mathfrak{S}_\lambda$ and that $Y^* / \mathbb{G}_a \simeq \mathfrak{S}$. $\qquad\square$

10. From now on, we identify $X^* / \mathbb{G}_a$ and $Y^* / \mathbb{G}_a$ with the algebraic space $\mathfrak{S}$ constructed in 8 above, and we let $W = X^* \times_{\mathfrak{S}} Y^*$. By construction, W is a scheme, equipped with a structure of Zariski locally trivial $\mathbb{G}_a$-bundle over X^* and Y^* via the first and the second projection respectively. The following completes the proof.

Lemma 11. *We have isomorphisms $X^* \times \mathbb{A}^1 \simeq W \simeq Y^* \times \mathbb{A}^1$.*

Proof. We will show more precisely that the $\mathbb{G}_a$-bundles $\mathrm{pr}_1 : W \to X^*$ and $\mathrm{pr}_2 : W \to Y^*$ are both trivial. It follows from the description of the étale trivialization given in the proof of Lemma 9 above that the isomorphy class of the $\mathbb{G}_a$-bundle $X^* \to \mathfrak{S}$ in $H^1_{\text{ét}}(\mathfrak{S}, \mathcal{O}_{\mathfrak{S}})$ is represented by the Čech 1-cocycle

$$\left(x^{-4}\sigma, 2x^{-4}\sigma\right) \in \Gamma\left(U_{x,\lambda}, \mathcal{O}_{U_{x,\lambda}}\right) \times \Gamma\left(U_{\lambda\lambda}, \mathcal{O}_{U_{\lambda\lambda}}\right) = \mathbb{C}\left[x^{\pm 1},\lambda^{\pm 1}\right]^2$$

with value in $\mathcal{O}_{\mathfrak{S}}$ for the étale cover $p : V \to \mathfrak{S}$ of $\mathfrak{S}$. Similarly, the isomorphy class of the $\mathbb{G}_a$-bundle $Y^* \to \mathfrak{S}$ is represented by the Čech 1-cocycle

$$\left(x^{-4}\tau, 2x^{-4}\tau\right) \in \Gamma\left(U_{x,\lambda}, \mathcal{O}_{U_{x,\lambda}}\right) \times \Gamma\left(U_{\lambda\lambda}, \mathcal{O}_{U_{\lambda\lambda}}\right) = \mathbb{C}\left[x^{\pm 1},\lambda^{\pm 1}\right]^2.$$

This implies in turn that the isomorphy class of the $\mathbb{G}_a$-bundle $\mathrm{pr}_1 : W \to X^*$ in $H^1_{\text{ét}}(X^*, \mathcal{O}_{X^*})$ is represented by the Čech 1-cocycle

$$\alpha = \left(x^{-4}\tau, 2x^{-4}\tau\right) \in \Gamma\left(U_{x,\lambda} \times \mathbb{G}_a, \mathcal{O}_{U_{x,\lambda} \times \mathbb{G}_a}\right) \times \Gamma\left(U_{\lambda\lambda} \times \mathbb{G}_a, \mathcal{O}_{U_{\lambda\lambda} \times \mathbb{G}_a}\right) = \left(\mathbb{C}\left[x^{\pm 1}, \lambda^{\pm 1}\right][v]\right)^2$$

with value in $\mathcal{O}_{X^*}$ for étale cover given by $U_x \times \mathbb{G}_a$ and $U_\lambda \times \mathbb{G}_a$. By definition, $\mathrm{pr}_1 : W \to X^*$ is a trivial $\mathbb{G}_a$-bundle if and only if α is coboundary. This is the case if and only if there exists

$$\beta_x \in \Gamma\left(U_x \times \mathbb{G}_a, \mathcal{O}_{U_x \times \mathbb{G}_a}\right) = \mathbb{C}\left[x^{\pm 1}, t\right][v] \quad \text{and} \quad \beta_\lambda \in \Gamma\left(U_\lambda \times \mathbb{G}_a, \mathcal{O}_{U_\lambda \times \mathbb{G}_a}\right) = \mathbb{C}\left[x, \lambda^{\pm 1}\right][v]$$

such that

$$\begin{cases} x^{-4}\tau & = \beta_\lambda\left(x, \lambda, v - x^{-4}\sigma\right) - \beta_x\left(x, \lambda^2, v\right) \\ 2x^{-4}\tau & = \beta_\lambda\left(x, \lambda, v\right) - \beta_\lambda\left(x, -\lambda, v + 2x^{-4}\sigma\right) \end{cases}$$

Since

$$\tau(x, \lambda) = \left(1 - \frac{1}{2}\alpha x^2\right)\sigma(x, \lambda),$$

one can choose for instance

$$\beta_x(x, t, v) = -\left(1 - \frac{1}{2}\alpha x^2\right)v \quad \text{and} \quad \beta_\lambda(x, \lambda, v) = -\left(1 - \frac{1}{2}\alpha x^2\right)v.$$

The fact that $\mathrm{pr}_2 : W \to Y^*$ is also a trivial $\mathbb{G}_a$-bundle follows from a similar argument using the identity

$$\sigma(x, \lambda) = \left(1 + \frac{1}{2}\alpha x^2\right)\tau(x, \lambda) + \frac{1}{4}\alpha^2 x^4 \sigma(x, \lambda).$$

$\square$

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