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To cite this version:

Jean-Philippe Anker, Fatma Ayadi, Mohamed Sifi. Opdam's hypergeometric functions: product formula and convolution structure in dimension 1. Advances in Pure and Applied Mathematics, 2012, 3 (1), pp.11-44. 10.1515/apam.2011.008 . hal-00476400v3

HAL Id: hal-00476400

https://hal.science/hal-00476400v3

Submitted on 18 May 2011

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OPDAM'S HYPERGEOMETRIC FUNCTIONS : PRODUCT FORMULA AND CONVOLUTION STRUCTURE IN DIMENSION 1

J.-PH. ANKER, F. AYADI AND M. SIFI

Abstract. Let $\mathrm{G}_\lambda^{(\alpha,\beta)}$ be the eigenfunctions of the Dunkl–Cherednik operator $\mathrm{T}^{(\alpha,\beta)}$ on $\mathbb{R}$. In this paper we express the product $\mathrm{G}_\lambda^{(\alpha,\beta)}(x)\,\mathrm{G}_\lambda^{(\alpha,\beta)}(y)$ as an integral in terms of $\mathrm{G}_\lambda^{(\alpha,\beta)}(z)$ with an explicit kernel. In general this kernel is not positive. Furthermore, by taking the so–called rational limit, we recover the product formula of M. Rösler for the Dunkl kernel. We then define and study a convolution structure associated to $\mathrm{G}_\lambda^{(\alpha,\beta)}$.

1. Introduction

The Opdam hypergeometric functions $\mathrm{G}_\lambda^{(\alpha,\beta)}$ on $\mathbb{R}$ are normalized eigenfunctions

$$\left\{\begin{array}{l}\mathrm{T}^{(\alpha,\beta)}\mathrm{G}_\lambda^{(\alpha,\beta)}(x)=i\lambda\,\mathrm{G}_\lambda^{(\alpha,\beta)}(x)\\ \mathrm{G}_\lambda^{(\alpha,\beta)}(0)=1\end{array}\right. \tag{1.1}$$

of the differential–difference operator

$$\mathrm{T}^{(\alpha,\beta)}f(x)=f'(x)+\underbrace{\big\{(2\alpha{+}1)\coth x+(2\beta{+}1)\tanh x\big\}\,\tfrac{f(x)-f(-x)}{2}}_{\{(\alpha-\beta)\coth x+(2\beta+1)\coth 2x\}\,\{f(x)-f(-x)\}}-\rho\,f(-x)\,.$$

Here $\alpha\geq\beta\geq-\frac12$, $\alpha>-\frac12$, $\rho=\alpha+\beta+1$ and $\lambda\in\mathbb{C}$. Notice that, in Cherednik's notation, $\mathrm{T}^{(\alpha,\beta)}$ writes

$$\mathrm{T}(k_1,k_2)f(x)=f'(x)+\big\{\tfrac{2\,k_1}{1-e^{-2x}}+\tfrac{4\,k_2}{1-e^{-4x}}\big\}\,\big\{f(x)-f(-x)\big\}-(k_1{+}2k_2)\,f(x)\,,$$

with $\alpha=k_1+k_2-\frac12$ and $\beta=k_2-\frac12$. We use as main references the article [11] and the lecture notes [12] by Opdam.

2000 *Mathematics Subject Classification.* Primary 33C67, 43A62, 44A35 ; Secondary 33C45, 33C52, 43A15, 43A32.

Key words and phrases. Dunkl–Cherednik operator, Opdam–Cherednik transform, product formula, convolution product, Kunze–Stein phenomenon.

Acknowledgements. The authors were partially supported by the DGRST project 04/UR/15-02, the cooperation programs PHC Utique / CMCU 07G 1501 & 10G 1503 and by the *Fédération Denis Poisson* (FR 2964). They are grateful to the referee for his thorough checking and detailed comments.

The functions $\mathrm{G}_\lambda^{(\alpha,\beta)}$ are closely related to Jacobi or hypergeometric functions (see e.g. [11, p. 90], [12, Example 7.8], [6, Proposition 2.1]). Specifically,

$$\begin{aligned}\mathrm{G}_\lambda^{(\alpha,\beta)}(x) &= \varphi_\lambda^{(\alpha,\beta)}(x) - \frac{1}{\rho-i\lambda}\frac{\partial}{\partial x}\varphi_\lambda^{(\alpha,\beta)}(x)\\ &= \varphi_\lambda^{(\alpha,\beta)}(x) + \frac{\rho+i\lambda}{4(\alpha+1)}\sinh 2x\,\varphi_\lambda^{(\alpha+1,\beta+1)}(x)\,,\end{aligned} \tag{1.2}$$

where $\varphi_\lambda^{(\alpha,\beta)}(x) = {}_2\mathrm{F}_1\big(\frac{\rho+i\lambda}{2}, \frac{\rho-i\lambda}{2}\,;\alpha+1\,;-\sinh^2 x\big)$.

This paper deals with harmonic analysis for the functions $\mathrm{G}_\lambda^{(\alpha,\beta)}$. We derive mainly a product formula for $\mathrm{G}_\lambda^{(\alpha,\beta)}$, which is analogous to the corresponding result of Flensted-Jensen and Koornwinder [4] for Jacobi functions, and of Ben Salem and Ould Ahmed Salem [2] for Jacobi–Dunkl functions. The product formula is the key information needed in order to define an associated convolution structure on $\mathbb{R}$. More precisely, we deduce the product formula

$$\mathrm{G}_\lambda^{(\alpha,\beta)}(x)\,\mathrm{G}_\lambda^{(\alpha,\beta)}(y) = \int_{\mathbb{R}} \mathrm{G}_\lambda^{(\alpha,\beta)}(z)\,d\mu_{x,y}^{(\alpha,\beta)}(z) \quad \forall x,y\in\mathbb{R}, \quad \forall\lambda\in\mathbb{C}, \tag{1.3}$$

from the corresponding formula for $\varphi_\lambda^{(\alpha,\beta)}$ on $\mathbb{R}^+$. Here $\mu_{x,y}^{(\alpha,\beta)}$ is an explicit real valued measure with compact support on $\mathbb{R}$, which may not be positive and which is uniformly bounded in $x,y\in\mathbb{R}$. We conclude the first part of the paper by recovering as a limit case the product formula for the Dunkl kernel obtained in [13].

In the second part of the paper, we use the product formula (1.3) to define and study the translation operators

$$\tau_x^{(\alpha,\beta)} f(y) := \int_{\mathbb{R}} f(z)\,d\mu_{x,y}^{(\alpha,\beta)}(z).$$

We next define the convolution product of suitable functions f and g by

$$f *_{\alpha,\beta} g(x) := \int_{\mathbb{R}} \tau_x^{(\alpha,\beta)} f(-y)\,g(y)\,A_{\alpha,\beta}(|y|)\,dy,$$

where $A_{\alpha,\beta}(y) = (\sinh y)^{2\alpha+1}(\cosh y)^{2\beta+1}$. We show in particular that $f *_{\alpha,\beta} g = g *_{\alpha,\beta} f$ and that $\mathcal{F}(f *_{\alpha,\beta} g) = \mathcal{F}(f)\,\mathcal{F}(g)$, where $\mathcal{F}$ is the so–called Opdam–Cherednik transform. Eventually we prove an analog of the Kunze–Stein phenomenon for the $*_{\alpha,\beta}$-convolution product of L^p-spaces.

In the last part of the paper, we construct an orthogonal basis of the Hilbert space $L^2(\mathbb{R}, A_{\alpha,\beta}(|x|)\,dx)$, generalizing the corresponding result of Koornwinder [9] for $L^2(\mathbb{R}^+, A_{\alpha,\beta}(x)dx)$. As a limit case, we recover the Hermite functions constructed by Rosenblum [14] in $L^2(\mathbb{R}, |x|^{2\alpha+1}dx)$.

Our paper is organised as follows. In section 2, we recall some properties and formulas for Jacobi functions. In section 3, we give the proof of the product formula for $\mathrm{G}_\lambda^{(\alpha,\beta)}$. Section 4 is devoted to the translation operators and the associated convolution product. Section 5 contains a Kunze–Stein type phenomenon. In

Section 6, we construct an orthogonal basis of $L^2(\mathbb{R}, A_{\alpha,\beta}(|x|)\,dx)$ and compute its Opdam–Cherednik transform.

2. Preliminaries

In this section we recall some properties of the Jacobi functions. See [4] and [5] for more details, as well as the survey [9].

Let $\alpha \geq \beta \geq -\frac{1}{2}$ with $\alpha \neq -\frac{1}{2}$ and $\lambda \in \mathbb{C}$. The Jacobi function $\varphi_\lambda^{(\alpha,\beta)}$ is defined by

(2.1)
$$\begin{aligned}\varphi_\lambda^{(\alpha,\beta)}(x) &= {}_2\mathrm{F}_1\Big(\frac{\rho+i\lambda}{2}, \frac{\rho-i\lambda}{2}; \alpha+1; -\sinh^2 x\Big)\\ &= (\cosh x)^{-\rho-i\lambda}\, {}_2\mathrm{F}_1\Big(\frac{\rho+i\lambda}{2}, \frac{\alpha-\beta+1+i\lambda}{2}; \alpha+1; \tanh^2 x\Big) \quad \forall\, x\in\mathbb{R}\,,\end{aligned}$$

where $\rho = \alpha+\beta+1$ and ${}_2\mathrm{F}_1$ denotes the hypergeometric function.

Its asymptotic behavior is generically given by

$$\varphi_\lambda^{(\alpha,\beta)}(x) = \mathrm{c}_{\alpha,\beta}(\lambda)\,\Phi_\lambda^{(\alpha,\beta)}(x) + \mathrm{c}_{\alpha,\beta}(-\lambda)\,\Phi_{-\lambda}^{(\alpha,\beta)}(x) \qquad \forall\,\lambda\in\mathbb{C}\smallsetminus i\mathbb{Z}\,,\ \forall\, x\in\mathbb{R}^*, \tag{2.2}$$

where

$$\mathrm{c}_{\alpha,\beta}(\lambda) = \frac{\Gamma(2\alpha+1)}{\Gamma(\alpha+\frac{1}{2})}\frac{\Gamma(i\lambda)}{\Gamma(\alpha-\beta+i\lambda)}\frac{\Gamma(\frac{\alpha-\beta+i\lambda}{2})}{\Gamma(\frac{\rho+i\lambda}{2})} = \frac{\Gamma(\alpha+1)\,2^{\rho-i\lambda}\,\Gamma(i\lambda)}{\Gamma(\frac{\rho+i\lambda}{2})\,\Gamma(\frac{\alpha-\beta+1+i\lambda}{2})} \tag{2.3}$$

and

$$\Phi_\lambda^{(\alpha,\beta)}(x) = (2\cosh x)^{-\rho+i\lambda}\, {}_2\mathrm{F}_1\Big(\frac{\rho-i\lambda}{2}, \frac{\alpha-\beta+1-i\lambda}{2}; 1-i\lambda; \cosh^{-2} x\Big). \tag{2.4}$$

In the limit case $\lambda=0$, we obtain

$$\varphi_0^{(\alpha,\beta)}(x) = \frac{2^{\rho+1}\,\Gamma(\alpha+1)}{\Gamma(\frac{\rho}{2})\,\Gamma(\frac{\alpha-\beta+1}{2})}\,|x|\,e^{-\rho\,|x|} + \mathcal{O}\big(e^{-\rho\,|x|}\big) \qquad \text{as } |x|\to+\infty\,, \tag{2.5}$$

after multiplying (2.2) by λ and applying $\frac{\partial}{\partial\lambda}\big|_{\lambda=0}$.

The Jacobi functions satisfy the following product formula, for $\alpha > \beta > -\frac{1}{2}$ and $x, y \geq 0$:

$$\varphi_\lambda^{(\alpha,\beta)}(x)\,\varphi_\lambda^{(\alpha,\beta)}(y) = \int_0^1\int_0^\pi \varphi_\lambda^{(\alpha,\beta)}(\arg\cosh|\gamma(x,y,r,\psi)|)\,dm_{\alpha,\beta}(r,\psi), \tag{2.6}$$

where

$$\gamma(x,y,r,\psi) = \cosh x\cosh y + \sinh x\sinh y\, r e^{i\psi},$$

and

$$dm_{\alpha,\beta}(r,\psi) = 2\,M_{\alpha,\beta}\,(1-r^2)^{\alpha-\beta-1}(r\sin\psi)^{2\beta}\,r\,dr\,d\psi \tag{2.7}$$

with

$$M_{\alpha,\beta} = \frac{\Gamma(\alpha+1)}{\sqrt{\pi}\,\Gamma(\alpha-\beta)\,\Gamma(\beta+\frac{1}{2})}.$$

When $\alpha=\beta>-\frac{1}{2}$, the product formula becomes

$$\varphi_\lambda^{(\alpha,\alpha)}(x)\,\varphi_\lambda^{(\alpha,\alpha)}(y)=M_{\alpha,\alpha}\int_0^\pi \varphi_\lambda^{(\alpha,\alpha)}(\arg\cosh|\gamma(x,y,1,\psi)|)\,(\sin\psi)^{2\alpha}\,d\psi, \tag{2.8}$$

where $M_{\alpha,\alpha}=\frac{\Gamma(\alpha+1)}{\sqrt{\pi}\,\Gamma(\alpha+\frac{1}{2})}$. Notice that the limit cases $\alpha>\beta=-\frac{1}{2}$ and $\alpha=\beta>-\frac{1}{2}$ are connected by the quadratic transformation $\varphi_\lambda^{(\alpha,-\frac{1}{2})}(x)=\varphi_{2\lambda}^{(\alpha,\alpha)}\big(\frac{x}{2}\big)$.

For $\alpha>\beta>-\frac{1}{2}$ and fixed $x,y>0$, we perform the change of variables $]0,1[\times]0,\pi[\ni(r,\psi)\longmapsto(z,\chi)\in]0,+\infty[\times]0,\pi[$ defined by

$$\cosh z\; e^{i\chi}=\gamma(x,y,r,\psi)\quad\Longleftrightarrow\quad\begin{cases} r\cos\psi=\frac{\cosh z\cos\chi-\cosh x\cosh y}{\sinh x\,\sinh y}\,,\\ r\sin\psi=\frac{\cosh z\,\sin\chi}{\sinh x\,\sinh y}\,.\end{cases} \tag{2.9}$$

This implies in particular that

$$\cosh(x-y)\le\cosh(z)\le\cosh(x+y),$$

and therefore x, y, z satisfy the triangular inequality

$$|x-y|\le|z|\le x+y.$$

Moreover, an easy computation gives

$$1-r^2=(\sinh x\sinh y)^{-2}\,g(x,y,z,\chi),$$

where

$$g(x,y,z,\chi):=1-\cosh^2x-\cosh^2y-\cosh^2z+2\cosh x\cosh y\cosh z\cos\chi. \tag{2.10}$$

Furthermore, the measure $\sinh^2x\sinh^2y\,r\,drd\psi$ becomes $\cosh z\sinh z\,dz\,d\chi$ and therefore the measure (2.7) becomes

$$dm_{\alpha,\beta}(r,\psi)=2\,M_{\alpha,\beta}\,g(x,y,z,\chi)^{\alpha-\beta-1}\big(\sinh x\sinh y\sinh z\big)^{-2\alpha}(\sin\chi)^{2\beta}A_{\alpha,\beta}(z)\,dzd\chi,$$

where

$$A_{\alpha,\beta}(z):=(\sinh z)^{2\alpha+1}(\cosh z)^{2\beta+1}. \tag{2.11}$$

Hence, the product formula (2.6) reads

$$\varphi_\lambda^{(\alpha,\beta)}(x)\,\varphi_\lambda^{(\alpha,\beta)}(y)=\int_0^{+\infty}\varphi_\lambda^{(\alpha,\beta)}(z)\,W_{\alpha,\beta}(x,y,z)\,A_{\alpha,\beta}(z)\,dz,\qquad x,y>0, \tag{2.12}$$

where

$$W_{\alpha,\beta}(x,y,z):=2\,M_{\alpha,\beta}\,(\sinh x\sinh y\sinh z)^{-2\alpha}\int_0^\pi g(x,y,z,\chi)_+^{\alpha-\beta-1}(\sin\chi)^{2\beta}\,d\chi$$

if $x,y,z>0$ satisfy $|x-y|<z<x+y$ and $W_{\alpha,\beta}(x,y,z)=0$ otherwise. Here

$$g_+=\begin{cases} g & \text{if } g>0,\\ 0 & \text{if } g\le 0.\end{cases}$$

We point out that the function $W_{\alpha,\beta}(x,y,z)$ is nonnegative, symmetric in the variables x,y,z and that

$$\int_0^{+\infty} W_{\alpha,\beta}(x,y,z)\, A_{\alpha,\beta}(z)\, dz = 1.$$

Furthermore, in [4, Formula (4.19)] the authors express $W_{\alpha,\beta}$ as follows in terms of the hypergeometric function ${}_2\mathrm{F}_1$: For every $x,y,z>0$ satisfying the triangular inequality $|x-y|<z<x+y$,

$$\begin{aligned} W_{\alpha,\beta}(x,y,z) &= M_{\alpha,\alpha}\,(\cosh x\cosh y\cosh z)^{\alpha-\beta-1}\,(\sinh x\sinh y\sinh z)^{-2\alpha} \\ &\quad\times (1-B^2)^{\alpha-\frac12}\,{}_2\mathrm{F}_1\Big(\alpha+\beta,\alpha-\beta;\alpha+\frac12;\frac{1-B}{2}\Big), \end{aligned} \tag{2.13}$$

where

$$B := \frac{\cosh^2 x+\cosh^2 y+\cosh^2 z-1}{2\cosh x\cosh y\cosh z}.$$

Notice that

$$1\pm B = \frac{[\,\cosh(x+y)\pm\cosh z\,]\,[\,\cosh z\pm\cosh(x-y)\,]}{2\cosh x\cosh y\cosh z},$$

hence

$$\begin{aligned} 1-B^2 &= \frac{[\,\cosh 2(x+y)-\cosh 2z\,]\,[\,\cosh 2z-\cosh 2(x-y)\,]}{16\cosh^2 x\cosh^2 y\cosh^2 z} \\ &= \frac{\sinh(x+y+z)\sinh(-x+y+z)\sinh(x-y+z)\sinh(x+y-z)}{4\cosh^2 x\cosh^2 y\cosh^2 z}. \end{aligned} \tag{2.14}$$

In the case $\alpha=\beta>-\frac12$, we use instead the change of variables

$$\cosh z = |\gamma(x,y,1,\psi)| = |\cosh x\cosh y+\sinh x\sinh y\, e^{i\psi}\,|,$$

and we obtain the same product formula (2.12), where $W_{\alpha,\alpha}$ is given by

$$\begin{aligned} W_{\alpha,\alpha}(x,y,z) &= 2^{4\alpha+1}M_{\alpha,\alpha}\,\big[\sinh 2x\sinh 2y\sinh 2z\big]^{-2\alpha} \\ &\quad\times\big[\sinh(x+y+z)\sinh(-x+y+z)\sinh(x-y+z)\sinh(x+y-z)\big]^{\alpha-1/2}. \end{aligned} \tag{2.15}$$

In the case $\alpha>\beta=-\frac12$, we use the quadratic transformation

$$\varphi_\lambda^{(\alpha,-\frac12)}(2x) = \varphi_{2\lambda}^{(\alpha,\alpha)}(x),$$

and we obtain again the product formula (2.12), with

$$W_{\alpha,-\frac12}(x,y,z) = 2^{-2\alpha}\,W_{\alpha,\alpha}(\frac{x}{2},\frac{y}{2},\frac{z}{2}).$$

As noticed by Koornwinder [8] (see also [5]), the product formulas (2.6) and (2.12) are closely connected with the addition formula for the Jacobi functions,

that we recall now for later use:

$$\begin{aligned}\varphi_\lambda^{(\alpha,\beta)}(\arg\cosh|\gamma(x,y,r,\psi)|)\\ = \sum\nolimits_{0\le\ell\le k<\infty} \varphi_{\lambda,k,\ell}^{(\alpha,\beta)}(x)\,\varphi_{-\lambda,k,\ell}^{(\alpha,\beta)}(-y)\,\chi_{k,\ell}^{(\alpha,\beta)}(r,\psi)\,\Pi_{k,\ell}^{(\alpha,\beta)}\,,\end{aligned} \tag{2.16}$$

where

$$\varphi_{\lambda,k,\ell}^{(\alpha,\beta)}(x) = \frac{\mathrm{c}_{\alpha,\beta}(-\lambda)}{\mathrm{c}_{\alpha+k+\ell,\beta+k-\ell}(-\lambda)}\,(2\sinh x)^{k-\ell}\,(2\cosh x)^{k+\ell}\,\varphi_\lambda^{(\alpha+k+\ell,\beta+k-\ell)}(x)$$

are modified Jacobi functions, the functions

$$\chi_{k,\ell}^{(\alpha,\beta)}(r,\psi) = r^{k-\ell}\,\tfrac{\ell!}{(\alpha-\beta)_\ell}\,\mathrm{P}_\ell^{(\alpha-\beta-1,\beta+k-\ell)}(2r^2-1)\,\tfrac{(k-\ell)!}{(\beta+\frac12)_{k-\ell}}\,\mathrm{P}_{k-\ell}^{(\beta-\frac12,\beta-\frac12)}(\cos\psi)\,,$$

which are expressed in terms of Jacobi polynomials (see for instance [1])

$$\mathrm{P}_n^{(a,b)}(z) = \tfrac{(a+1)_n}{n!}\,{}_2\mathrm{F}_1\big(-n,a+b+n+1;a+1;\tfrac{1-z}{2}\big)\,, \tag{2.17}$$

are orthogonal with respect to the measure (2.7), and

$$\begin{aligned}\Pi_{k,\ell}^{(\alpha,\beta)} &= \Big(\int_0^1\int_0^\pi \chi_{k,\ell}^{(\alpha,\beta)}(r,\psi)^2\,dm_{\alpha,\beta}(r,\psi)\Big)^{-1}\\ &= \tfrac{(\alpha+k+\ell)\,(\beta+2k-2\ell)}{(\alpha+k)\,(2\beta+k-\ell)}\,\tfrac{(\alpha+1)_k\,(\alpha-\beta)_\ell\,(2\beta+1)_{k-\ell}}{(\beta+1)_k\,\ell!\,(k-\ell)!}\,.\end{aligned} \tag{2.18}$$

3. Product formula for $\mathrm{G}_\lambda^{(\alpha,\beta)}$

For $x,y,z\in\mathbb{R}$ and $\chi\in[0,\pi]$, let

$$\sigma_{x,y,z}^\chi = \begin{cases}\frac{\cosh x\,\cosh y-\cosh z\,\cos\chi}{\sinh x\,\sinh y} & \text{if } xy\neq 0\,,\\ 0 & \text{if } xy=0\,.\end{cases} \tag{3.1}$$

Furthermore, if $\alpha>\beta>-\frac12$, let us define $\mathcal{K}_{\alpha,\beta}$ by

$$\begin{aligned}\mathcal{K}_{\alpha,\beta}(x,y,z) = M_{\alpha,\beta}\,\big|\sinh x\,\sinh y\,\sinh z\,\big|^{-2\alpha}\int_0^\pi g(x,y,z,\chi)_+^{\alpha-\beta-1}\\ \times\Big[1-\sigma_{x,y,z}^\chi+\sigma_{x,z,y}^\chi+\sigma_{z,y,x}^\chi+\frac{\rho}{\beta+\frac12}\coth x\coth y\coth z\,(\sin\chi)^2\Big]\\ \times(\sin\chi)^{2\beta}\,d\chi\end{aligned}$$

if $x,y,z\in\mathbb{R}^*$ satisfy the triangular inequality $||x|-|y||<|z|<|x|+|y|$, and $\mathcal{K}_{\alpha,\beta}(x,y,z)=0$ otherwise. Here $g(x,y,z,\chi)$ is as in (2.10).

Remark 3.1. *The following symmetry properties are easy to check:*

$$\begin{cases}\mathcal{K}_{\alpha,\beta}(x,y,z)=\mathcal{K}_{\alpha,\beta}(y,x,z),\\ \mathcal{K}_{\alpha,\beta}(x,y,z)=\mathcal{K}_{\alpha,\beta}(-z,y,-x),\\ \mathcal{K}_{\alpha,\beta}(x,y,z)=\mathcal{K}_{\alpha,\beta}(x,-z,-y).\end{cases}$$

Recall the Opdam functions $\mathrm{G}_\lambda^{(\alpha,\beta)}$ defined in (1.2). This section is devoted to the proof of our main result, that we state first in the case $\alpha>\beta>-\frac12$.

Theorem 3.2. *Assume* $\alpha > \beta > -\frac{1}{2}$. *Then* $\mathrm{G}_\lambda^{(\alpha,\beta)}$ *satisfies the following product formula*

$$\mathrm{G}_\lambda^{(\alpha,\beta)}(x)\,\mathrm{G}_\lambda^{(\alpha,\beta)}(y) = \int_{-\infty}^{+\infty} \mathrm{G}_\lambda^{(\alpha,\beta)}(z)\,d\mu_{x,y}^{(\alpha,\beta)}(z),$$

for $x, y \in \mathbb{R}$ *and* $\lambda \in \mathbb{C}$. *Here*

$$d\mu_{x,y}^{(\alpha,\beta)}(z) = \begin{cases} \mathcal{K}_{\alpha,\beta}(x,y,z)\,A_{\alpha,\beta}(|z|)\,dz & \text{if } xy \neq 0 \\ d\delta_x(z) & \text{if } y = 0 \\ d\delta_y(z) & \text{if } x = 0 \end{cases} \tag{3.2}$$

and $A_{\alpha,\beta}$ *is as in* (2.11).

Let us split the Opdam function

$$\mathrm{G}_\lambda^{(\alpha,\beta)} = \mathrm{G}_{\lambda,\mathrm{e}}^{(\alpha,\beta)} + \mathrm{G}_{\lambda,\mathrm{o}}^{(\alpha,\beta)}$$

into its even part

$$\mathrm{G}_{\lambda,\mathrm{e}}^{(\alpha,\beta)}(x) = \varphi_\lambda^{(\alpha,\beta)}(x)$$

and odd part

$$\mathrm{G}_{\lambda,\mathrm{o}}^{(\alpha,\beta)}(x) = -\frac{1}{\rho - i\lambda}\frac{\partial}{\partial x}\varphi_\lambda^{(\alpha,\beta)}(x) = \frac{\rho + i\lambda}{4(\alpha+1)}\sinh 2x\,\varphi_\lambda^{(\alpha+1,\beta+1)}(x).$$

For $x, y \in \mathbb{R}^*$, the product formula (2.12) for the Jacobi functions yields

$$\begin{aligned}\mathrm{G}_{\lambda,\mathrm{e}}^{(\alpha,\beta)}(x)\,\mathrm{G}_{\lambda,\mathrm{e}}^{(\alpha,\beta)}(y) &= \int_{||x|-|y||}^{|x|+|y|} \mathrm{G}_{\lambda,\mathrm{e}}^{(\alpha,\beta)}(z)\,W_{\alpha,\beta}(|x|,|y|,z)\,A_{\alpha,\beta}(z)\,dz \\ &= \frac{1}{2}\int_{I_{x,y}} \mathrm{G}_\lambda^{(\alpha,\beta)}(z)\,W_{\alpha,\beta}(|x|,|y|,|z|)\,A_{\alpha,\beta}(|z|)\,dz,\end{aligned}$$

where

$$I_{x,y} := [-|x|-|y|, -||x|-|y||] \cup [||x|-|y||, |x|+|y|]\,. \tag{3.3}$$

Next let us turn to the mixed products. The following statement amounts to Lemma 2.3 in [2].

Lemma 3.3. *For* $\alpha > \beta > -\frac{1}{2}$, $\lambda \in \mathbb{C}$ *and* $x, y \in \mathbb{R}^*$, *we have*

$$\begin{aligned}\mathrm{G}_{\lambda,\mathrm{o}}^{(\alpha,\beta)}(x)\,\mathrm{G}_{\lambda,\mathrm{e}}^{(\alpha,\beta)}(y) = M_{\alpha,\beta}\int_{I_{x,y}} \mathrm{G}_\lambda^{(\alpha,\beta)}(z)\,\big|\sinh x\,\sinh y\,\sinh z\,\big|^{-2\alpha} \\ \times\Big\{\int_0^\pi g(x,y,z,\chi)_+^{\alpha-\beta-1}\,\sigma_{x,z,y}^\chi\,(\sin\chi)^{2\beta}\,d\chi\Big\}\,A_{\alpha,\beta}(|z|)\,dz\,,\end{aligned}$$

where $g(x,y,z,\chi)$ *is given by* (2.10), $\sigma_{x,z,y}^\chi$ *by* (3.1) *and* $I_{x,y}$ *by* (3.3).

We consider now purely odd products, which is the most difficult case.

Lemma 3.4. *For $\alpha>\beta>-\frac{1}{2}$, $\lambda\in\mathbb{C}$ and $x,y\in\mathbb{R}^*$, we have*

$$\mathrm{G}_{\lambda,\mathrm{o}}^{(\alpha,\beta)}(x)\,\mathrm{G}_{\lambda,\mathrm{o}}^{(\alpha,\beta)}(y) = M_{\alpha,\beta}\int_{I_{x,y}} \mathrm{G}_{\lambda}^{(\alpha,\beta)}(z)\,\big|\sinh x\,\sinh y\,\sinh z\,\big|^{-2\alpha}$$
$$\times\Big\{\int_0^\pi g(x,y,z,\chi)_+^{\alpha-\beta-1}\Big[-\sigma_{x,y,z}^{\chi}-\frac{\rho}{(\beta+\frac{1}{2})}\coth x\coth y\coth z\,(\sin\chi)^2\Big]$$
$$\times(\sin\chi)^{2\beta}d\chi\Big\}\,A_{\alpha,\beta}(|z|)\,dz.$$

Proof. For $x,y>0$, we have

$$\mathrm{G}_{\lambda,\mathrm{o}}^{(\alpha,\beta)}(x)\,\mathrm{G}_{\lambda,\mathrm{o}}^{(\alpha,\beta)}(y) = \frac{(\rho+i\lambda)^2}{16\,(\alpha+1)^2}\,\sinh 2x\,\sinh 2y\,\varphi_\lambda^{(\alpha+1,\beta+1)}(x)\,\varphi_\lambda^{(\alpha+1,\beta+1)}(y)$$
$$= \mathcal{I}_{\lambda,1}^{(\alpha,\beta)}(x,y)+\mathcal{I}_{\lambda,2}^{(\alpha,\beta)}(x,y), \tag{3.4}$$

where

$$\mathcal{I}_{\lambda,1}^{(\alpha,\beta)}(x,y) := -\frac{\rho^2+\lambda^2}{16\,(\alpha+1)^2}\,\sinh 2x\,\sinh 2y\,\varphi_\lambda^{(\alpha+1,\beta+1)}(x)\,\varphi_\lambda^{(\alpha+1,\beta+1)}(y),$$

and

$$\mathcal{I}_{\lambda,2}^{(\alpha,\beta)}(x,y) := \frac{\rho\,(\rho+i\lambda)}{8\,(\alpha+1)^2}\,\sinh 2x\,\sinh 2y\,\varphi_\lambda^{(\alpha+1,\beta+1)}(x)\,\varphi_\lambda^{(\alpha+1,\beta+1)}(y). \tag{3.5}$$

Consider first $\mathcal{I}_{\lambda,1}^{(\alpha,\beta)}$. We deduce from the addition formula (2.16) that

$$\int_0^1\int_0^\pi \varphi_\lambda^{(\alpha,\beta)}(\arg\cosh|\gamma(x,y,r,\psi)|)\,\chi_{1,0}^{(\alpha,\beta)}(r,\psi)\,dm_{\alpha,\beta}(r,\psi) = \varphi_{\lambda,1,0}^{(\alpha,\beta)}(x)\,\varphi_{-\lambda,1,0}^{(\alpha,\beta)}(-y),$$

where $\chi_{1,0}^{(\alpha,\beta)}(r,\psi)=r\cos\psi$ and $\varphi_{\pm\lambda,1,0}^{(\alpha,\beta)}(x)=\frac{\rho\mp i\lambda}{4(\alpha+1)}\sinh 2x\,\varphi_\lambda^{(\alpha+1,\beta+1)}(x)$. Hence

$$\mathcal{I}_{\lambda,1}^{(\alpha,\beta)}(x,y) = \int_0^1\int_0^\pi \varphi_\lambda^{(\alpha,\beta)}(\arg\cosh|\gamma(x,y,r,\psi)|)\,r\cos\psi\,dm_{\alpha,\beta}(r,\psi). \tag{3.6}$$

By performing the change of variables (2.9) and arguing as in Section 2, (3.6) becomes

$$\mathcal{I}_{\lambda,1}^{(\alpha,\beta)}(x,y) = -\,2\,M_{\alpha,\beta}\int_{|x-y|}^{x+y}\mathrm{G}_{\lambda,\mathrm{e}}^{(\alpha,\beta)}(z)\,(\sinh x\sinh y\sinh z)^{-2\alpha}$$
$$\times\Big\{\int_0^\pi \sigma_{x,y,z}^{\chi}\,g(x,y,z,\chi)_+^{\alpha-\beta-1}\,(\sin\chi)^{2\beta}\,d\chi\Big\}\,A_{\alpha,\beta}(z)\,dz.$$

By using the symmetries

$$\begin{cases} g(x,y,z,\chi) = g(|x|,|y|,|z|,\chi)\,,\\ \mathcal{I}_{\lambda,1}^{(\alpha,\beta)}(x,y) = \mathrm{sign}(xy)\,\mathcal{I}_{\lambda,1}^{(\alpha,\beta)}(|x|,|y|)\,,\\ \sigma_{x,y,z}^{\chi} = \mathrm{sign}(xy)\,\sigma_{|x|,|y|,|z|}^{\chi}\,,\end{cases}$$

we conclude, for all $x, y \in \mathbb{R}^*$, that

$$\mathcal{I}_{\lambda,1}^{(\alpha,\beta)}(x,y) = -\, M_{\alpha,\beta} \int_{I_{x,y}} \mathrm{G}_\lambda^{(\alpha,\beta)}(z) \,\big|\sinh x \,\sinh y \,\sinh z\,\big|^{-2\alpha}$$
$$\times \Big\{ \int_0^\pi \sigma_{x,y,z}^\chi \, g(x,y,z,\chi)_+^{\alpha-\beta-1} (\sin\chi)^{2\beta}\, d\chi \Big\} \, A_{\alpha,\beta}(|z|)\, dz.$$

Consider next $\mathcal{I}_{\lambda,2}^{(\alpha,\beta)}$. By using this time the product formula (2.12) for $\varphi_\lambda^{(\alpha+1,\beta+1)}$, we obtain, for $x, y > 0$,

$$\begin{aligned}
\mathcal{I}_{\lambda,2}^{(\alpha,\beta)}(x,y) &= \frac{\rho\,(\rho+i\lambda)}{4\,(\alpha+1)^2}\, M_{\alpha+1,\beta+1} \sinh 2x \sinh 2y \\
&\times \int_{|x-y|}^{x+y} \varphi_\lambda^{(\alpha+1,\beta+1)}(z) (\sinh x \sinh y \sinh z)^{-2\alpha-2} \\
&\times \Big\{ \int_0^\pi g(x,y,z,\chi)_+^{\alpha-\beta-1} (\sin\chi)^{2\beta+2} d\chi \Big\} \, A_{\alpha+1,\beta+1}(z)\, dz \\
&= 2\, M_{\alpha,\beta} \int_{|x-y|}^{x+y} \mathrm{G}_{\lambda,\mathrm{o}}^{(\alpha,\beta)}(z)\, (\sinh x \sinh y \sinh z)^{-2\alpha} \frac{\rho}{\beta+\frac12} \\
&\times \coth x \coth y \coth z \Big\{ \int_0^\pi g(x,y,z,\chi)_+^{\alpha-\beta-1} (\sin\chi)^{2\beta+2} d\chi \Big\} \, A_{\alpha,\beta}(z)\, dz\,.
\end{aligned}$$

By arguing again by evenness and oddness, we deduce, for all $x, y \in \mathbb{R}^*$,

$$\begin{aligned}
\mathcal{I}_{\lambda,2}^{(\alpha,\beta)}(x,y) &= M_{\alpha,\beta} \int_{I_{x,y}} \mathrm{G}_\lambda^{(\alpha,\beta)}(z) \,\big|\sinh x \,\sinh y \,\sinh z\,\big|^{-2\alpha} \\
&\times \frac{\rho}{\beta+\frac12}\, (\coth x \coth y \coth z) \Big\{ \int_0^\pi g(x,y,z,\chi)_+^{\alpha-\beta-1} (\sin\chi)^{2\beta+2} d\chi \Big\} \, A_{\alpha,\beta}(|z|)\, dz\,.
\end{aligned}$$

This concludes the proof of Lemma 3.4 and hence the proof of Theorem 3.2. □

Next we turn our attention to the case $\alpha = \beta > -\frac12$. For $x, y, z \in \mathbb{R}$, let

$$\sigma_{x,y,z} = \begin{cases} \frac{\cosh 2x \,\cosh 2y - \cosh 2z}{\sinh 2x \,\sinh 2y} & \text{if } \; xy \neq 0\,, \\ 0 & \text{if } \; xy = 0\,. \end{cases} \tag{3.7}$$

Moreover, we define the kernel $\mathcal{K}_{\alpha,\alpha}$ by

$$\begin{aligned}
\mathcal{K}_{\alpha,\alpha}(x,y,z) &= 2^{4\alpha+2} M_{\alpha,\alpha}\, e^{\,x+y-z} \\
&\times \frac{\big[\sinh(x+y+z)\sinh(-x+y+z)\sinh(x-y+z)\sinh(x+y-z)\big]^{\alpha-1/2}}{\big|\sinh 2x \,\sinh 2y \,\sinh 2z\,\big|^{2\alpha}} \\
&\times \frac{\sinh(x+y+z)\sinh(-x+y+z)\sinh(x-y+z)}{\sinh 2x \,\sinh 2y \,\sinh 2z}
\end{aligned} \tag{3.8}$$

if $||x| - |y|| < |z| < |x| + |y|$, and $\mathcal{K}_{\alpha,\alpha}(x,y,z) = 0$ otherwise. The symmetry properties of $\mathcal{K}_{\alpha,\beta}$ (see Remark 3.1) remain true for $\mathcal{K}_{\alpha,\alpha}$.

Theorem 3.5. *In the case $\alpha=\beta>-\frac{1}{2}$, the product formula reads*

$$\mathrm{G}_\lambda^{(\alpha,\alpha)}(x)\,\mathrm{G}_\lambda^{(\alpha,\alpha)}(y)=\int_{-\infty}^{+\infty}\mathrm{G}_\lambda^{(\alpha,\alpha)}(z)\,d\mu_{x,y}^{(\alpha,\alpha)}(z), \tag{3.9}$$

for $x,y\in\mathbb{R}$ and $\lambda\in\mathbb{C}$. Here

$$d\mu_{x,y}^{(\alpha,\alpha)}(z)=\begin{cases}\mathcal{K}_{\alpha,\alpha}(x,y,z)\,A_{\alpha,\alpha}(|z|)\,dz & \text{if } xy\neq 0\,,\\ d\delta_x(z) & \text{if } y=0\,,\\ d\delta_y(z) & \text{if } x=0\,.\end{cases} \tag{3.10}$$

Proof. The even product formula

$$\mathrm{G}_{\lambda,\mathrm{e}}^{(\alpha,\alpha)}(x)\,\mathrm{G}_{\lambda,\mathrm{e}}^{(\alpha,\alpha)}(y)=\frac{1}{2}\int_{I_{x,y}}\mathrm{G}_\lambda^{(\alpha,\alpha)}(z)\,W_{\alpha,\alpha}(|x|,|y|,|z|)\,A_{\alpha,\alpha}(|z|)\,dz$$

and the mixed product formulae

$$\mathrm{G}_{\lambda,\mathrm{e}}^{(\alpha,\alpha)}(x)\,\mathrm{G}_{\lambda,\mathrm{o}}^{(\alpha,\alpha)}(y)=\frac{1}{2}\int_{I_{x,y}}\mathrm{G}_\lambda^{(\alpha,\alpha)}(z)\,\sigma_{z,y,x}\,W_{\alpha,\alpha}(|x|,|y|,|z|)\,A_{\alpha,\alpha}(|z|)\,dz\,,$$

$$\mathrm{G}_{\lambda,\mathrm{o}}^{(\alpha,\alpha)}(x)\,\mathrm{G}_{\lambda,\mathrm{e}}^{(\alpha,\alpha)}(y)=\frac{1}{2}\int_{I_{x,y}}\mathrm{G}_\lambda^{(\alpha,\alpha)}(z)\,\sigma_{x,z,y}\,W_{\alpha,\alpha}(|x|,|y|,|z|)\,A_{\alpha,\alpha}(|z|)\,dz$$

are obtained as in the case $\alpha>\beta$. Here $W_{\alpha,\alpha}$ is given by (2.15), $I_{x,y}$ by (3.3) and $\sigma_{x,z,y}$ by (3.7). As far as they are concerned, odd products are splitted up as in (3.4):

$$\mathrm{G}_{\lambda,\mathrm{o}}^{(\alpha,\alpha)}(x)\,\mathrm{G}_{\lambda,\mathrm{o}}^{(\alpha,\alpha)}(y)=\mathcal{I}_{\lambda,1}^{(\alpha,\alpha)}(x,y)+\mathcal{I}_{\lambda,2}^{(\alpha,\alpha)}(x,y).$$

The first expression $\mathcal{I}_{\lambda,1}^{(\alpha,\alpha)}$ is handled as $\mathcal{I}_{\lambda,1}^{(\alpha,\beta)}$ in the case $\alpha>\beta$. We perform this time the change of variables $]0,\pi[\,\ni\psi\longmapsto z\in\,]0,+\infty[$ defined by $\cosh z=|\gamma(x,y,1,\psi)|$ and we obtain this way

$$\begin{aligned}\mathcal{I}_{\lambda,1}^{(\alpha,\alpha)}(x,y)&=-\,M_{\alpha,\alpha}\int_0^\pi\mathrm{G}_{\lambda,\mathrm{e}}^{(\alpha,\alpha)}(\arg\cosh|\gamma(x,y,1,\psi)|)\,\cos\psi\,(\sin\psi)^{2\alpha}\,d\psi\\&=-\int_{|x-y|}^{x+y}\mathrm{G}_{\lambda,\mathrm{e}}^{(\alpha,\alpha)}(z)\,\sigma_{x,y,z}\,W_{\alpha,\alpha}(x,y,z)\,A_{\alpha,\alpha}(z)\,dz\,,\end{aligned}$$

hence

$$\mathcal{I}_{\lambda,1}^{(\alpha,\alpha)}(x,y)=-\,\frac{1}{2}\int_{I_{x,y}}\mathrm{G}_\lambda^{(\alpha,\alpha)}(z)\,\sigma_{x,y,z}\,W_{\alpha,\alpha}(|x|,|y|,|z|)\,A_{\alpha,\alpha}(|z|)\,dz\,,$$

first for $x,y>0$ and next for $x,y\in\mathbb{R}^*$. According to the product formula for $\varphi_\lambda^{(\alpha+1,\alpha+1)}$, the second expression $\mathcal{I}_{\lambda,2}^{(\alpha,\alpha)}$ becomes

$$\begin{aligned}\mathcal{I}_{\lambda,2}^{(\alpha,\alpha)}(x,y)=\frac{2\alpha+1}{2(\alpha+1)}&\int_{|x-y|}^{x+y}\mathrm{G}_{\lambda,\mathrm{o}}^{(\alpha,\alpha)}(z)\\&\times\frac{\sinh 2x\,\sinh 2y}{\sinh 2z}\,W_{\alpha+1,\alpha+1}(x,y,z)\,A_{\alpha+1,\alpha+1}(z)\,dz\end{aligned}$$

for all $x, y > 0$. By using

$$W_{\alpha+1,\alpha+1}(x,y,z) = 16\,\frac{\alpha+1}{\alpha+1/2} \times \frac{\sinh(x+y+z)\sinh(-x+y+z)\sinh(x-y+z)\sinh(x+y-z)}{\sinh^2 2x \sinh^2 2y \sinh^2 2z}\, W_{\alpha,\alpha}(x,y,z)$$

and

$$A_{\alpha+1,\alpha+1}(z) = \frac{\sinh^2 2z}{4}\, A_{\alpha,\alpha}(z)\,,$$

we obtain

$$\begin{aligned}\mathcal{I}^{(\alpha,\alpha)}_{\lambda,2}(x,y) = 2\int_{I_{x,y}} & \mathrm{G}^{(\alpha,\alpha)}_{\lambda}(z) \\ & \times \frac{\sinh(x+y+z)\sinh(-x+y+z)\sinh(x-y+z)\sinh(x+y-z)}{\sinh 2x \sinh 2y \sinh 2z} \\ & \times W_{\alpha,\alpha}(|x|,|y|,|z|)\, A_{\alpha,\alpha}(|z|)\, dz\,,\end{aligned}$$

first for $x, y > 0$ and next for $x, y \in \mathbb{R}^*$. We conclude the proof of Theorem 3.5 by summing all partial product formulas and by using the remarkable identity

$$\begin{aligned}\varrho(x,y,z) &= 1 - \sigma_{x,y,z} + \sigma_{z,y,x} + \sigma_{x,z,y} \\ &= 4\,\frac{\sinh(x+y+z)\sinh(-x+y+z)\sinh(x-y+z)\cosh(x+y-z)}{\sinh 2x \sinh 2y \sinh 2z}\,.\end{aligned}$$

□

Consider next the rational limit of the product formula (3.9). It is well known that the hypergeometric function ${}_2\mathrm{F}_1\big(a,b;c;z\big)$ tends to the confluent hypergeometric limit function ${}_0\mathrm{F}_1\big(c;Z\big)$ as $a, b \to \infty$ and $z \to 0$ in such a way that $abz \to Z$. Consequently, as $\varepsilon \to 0$,

$$\varphi^{(\alpha,\alpha)}_{\lambda/\varepsilon}(\varepsilon x) = {}_2\mathrm{F}_1\Big(\alpha+\frac{1}{2}+i\frac{\lambda}{2\varepsilon}, \alpha+\frac{1}{2}-i\frac{\lambda}{2\varepsilon}; \alpha+1; -(\sinh \varepsilon x)^2\Big)$$

tends to the normalized Bessel function

$$\mathrm{j}_\alpha(\lambda x) = {}_0\mathrm{F}_1\Big(\alpha+1; -\big(\frac{\lambda x}{2}\big)^2\Big) = \Gamma(\alpha+1)\sum_{m=0}^{+\infty} \frac{(-1)^m}{m!\,\Gamma(\alpha+1+m)}\,\big(\frac{\lambda x}{2}\big)^{2m},$$

hence

$$\mathrm{G}^{(\alpha,\alpha)}_{\lambda/\varepsilon}(\varepsilon x) = \varphi^{(\alpha,\alpha)}_{\lambda/\varepsilon}(\varepsilon x) + \frac{2\alpha+1+i\lambda/\varepsilon}{4\,(\alpha+1)}\sinh(2\varepsilon x)\,\varphi^{(\alpha,\alpha)}_{\lambda/\varepsilon}(\varepsilon x)$$

tends to

$$\mathrm{E}_\alpha(i\lambda, x) = \mathrm{j}_\alpha(\lambda x) + \frac{i\lambda x}{2\,(\alpha+1)}\,\mathrm{j}_{\alpha+1}(\lambda x).$$

The latter expression is the so–called Dunkl kernel in dimension 1, whose product formula was obtained in [13] :

$$\mathrm{E}_\alpha(i\lambda,x)\,\mathrm{E}_\alpha(i\lambda,y)=\int_{\mathbb{R}}\mathrm{E}_\alpha(i\lambda,z)\,k_\alpha(x,y,z)\,|z|^{2\alpha+1}\,dz\,, \tag{3.11}$$

where

$$k_\alpha(x,y,z)=2^{-2\alpha}\,\frac{\Gamma(\alpha{+}1)}{\sqrt{\pi}\,\Gamma(\alpha{+}1/2)}\,\big[\,1-\varsigma_{x,y,z}+\varsigma_{z,y,x}+\varsigma_{x,z,y}\,\big]$$
$$\times\,\frac{\big[\,(x{+}y{+}z)\,(-x{+}y{+}z)\,(x{-}y{+}z)\,(x{+}y{-}z)\,\big]^{\alpha-\frac12}}{|\,x\,y\,z\,|^{2\alpha}}$$

with

$$\varsigma_{x,y,z}=\begin{cases}\frac{x^2+y^2-z^2}{2\,x\,y} & \text{if } x\,y\neq 0\,,\\ 0 & \text{if } x\,y=0\,,\end{cases}$$

hence

$$k_\alpha(x,y,z)=2^{-2\alpha-1}\,\frac{\Gamma(\alpha{+}1)}{\sqrt{\pi}\,\Gamma(\alpha{+}1/2)}\,\frac{[(x{+}y{+}z)(-x{+}y{+}z)(x{-}y{+}z)(x{+}y{-}z)]^{\alpha-\frac12}}{|xyz|^{2\alpha}}$$
$$\times\,\frac{(x{+}y{+}z)(-x{+}y{+}z)(x{-}y{+}z)}{xyz}\,. \tag{3.12}$$

Here is an immediate consequence of (3.8) and (3.12).

Lemma 3.6. *For every* $\alpha>-\frac12$ *and* $x,y,z\in\mathbb{R}^*$, *we have*

$$\lim_{\varepsilon\to0}\,\varepsilon^{\,2\alpha+2}\,\mathcal{K}_{\alpha,\alpha}(\varepsilon x,\varepsilon y,\varepsilon z)=k_\alpha(x,y,z).$$

We deduce the following result, which was announced in the abstract and in the introduction.

Corollary 3.7. *The product formula* (3.11) *is the rational limit of the product formula* (3.9). *More precisely,* (3.11) *is obtained by replacing* λ *by* λ/ε *and* (x,y) *by* $(\varepsilon x,\varepsilon y)$ *in* (3.9), *and by letting* $\varepsilon\to0$.

Theorem 3.8. *Let* $x,y\in\mathbb{R}$.

(i) *For* $\alpha\geq\beta\geq-\frac12$ *with* $\alpha>-\frac12$, *we have* $\operatorname{supp}\mu_{x,y}^{(\alpha,\beta)}\subset I_{x,y}$.
(ii) *For* $\alpha\geq\beta\geq-\frac12$ *with* $\alpha>-\frac12$, *we have* $\mu_{x,y}^{(\alpha,\beta)}(\mathbb{R})=1$.
(iii) *For* $\alpha>\beta>-\frac12$, *we have* $\|\mu_{x,y}^{(\alpha,\beta)}\|\leq 4+\dfrac{\Gamma(\alpha+1)}{\Gamma(\alpha+\frac12)}\dfrac{\Gamma(\beta+\frac12)}{\Gamma(\beta+1)}$.
(iv) *For* $\alpha=\beta>-\frac12$, *we have* $\|\mu_{x,y}^{(\alpha,\alpha)}\|\leq\frac52$.

Proof. (i) is obvious.
(ii) This claim follows from Theorems 3.2 and 3.5 and the fact that $\mathrm{G}_{i\rho}^{(\alpha,\beta)}\equiv1$.

(iii) From the proof of Theorem 3.2, we may rewrite the product formula for $\mathrm{G}_\lambda^{(\alpha,\beta)}$ as follows:

$$\mathrm{G}_\lambda^{(\alpha,\beta)}(x)\,\mathrm{G}_\lambda^{(\alpha,\beta)}(y) = \int_{I_{x,y}} \mathrm{G}_\lambda^{(\alpha,\beta)}(z)\,\widetilde{\mathcal{K}}_{\alpha,\beta}(x,y,z)\,A_{\alpha,\beta}(|z|)\,dz \,+\, \mathcal{I}_{\lambda,2}^{(\alpha,\beta)}(x,y),$$

where $\mathcal{I}_{\lambda,2}^{(\alpha,\beta)}$ is given by (3.5) and

$$\begin{aligned}\widetilde{\mathcal{K}}_{\alpha,\beta}(x,y,z) &:= M_{\alpha,\beta}\,\big|\sinh x\,\sinh y\,\sinh z\,\big|^{-2\alpha}\\ &\quad\times\int_0^\pi (1-\sigma_{x,y,z}^\chi+\sigma_{x,z,y}^\chi+\sigma_{z,y,x}^\chi)\,g(x,y,z,\chi)_+^{\alpha-\beta-1}\,(\sin\chi)^{2\beta}\,d\chi.\end{aligned}$$

By [2, Proposition 2.7], we have

$$\int_{I_{x,y}} |\,\widetilde{\mathcal{K}}_{\alpha,\beta}(x,y,z)|\,A_{\alpha,\beta}(|z|)\,dz \le 4.$$

On the other hand, using the product formula (2.6) for the Jacobi functions, we may rewrite $\mathcal{I}_{\lambda,2}^{(\alpha,\beta)}$ as follows :

$$\begin{aligned}\mathcal{I}_{\lambda,2}^{(\alpha,\beta)}(x,y) &= \frac{\rho(\rho+i\lambda)}{8(\alpha+1)^2}\,\sinh 2x\,\sinh 2y\,\varphi_\lambda^{(\alpha+1,\beta+1)}(x)\,\varphi_\lambda^{(\alpha+1,\beta+1)}(y)\\ &= \frac{\rho(\rho+i\lambda)}{8(\alpha+1)^2}\sinh 2x\sinh 2y\int_0^1\int_0^\pi \varphi_\lambda^{(\alpha+1,\beta+1)}(\arg\cosh|\gamma(x,y,r,\psi)|)\,dm_{\alpha+1,\beta+1}(r,\psi)\\ &= \frac{\rho}{4(\alpha+1)}\sinh 2x\sinh 2y\int_0^1\int_0^\pi \frac{\mathrm{G}_{\lambda,\mathrm{o}}^{(\alpha,\beta)}(\arg\cosh|\gamma(x,y,r,\psi)|)}{|\gamma(x,y,r,\psi)|\sqrt{|\gamma(x,y,r,\psi)|^2-1}}\,dm_{\alpha+1,\beta+1}(r,\psi),\end{aligned}$$

where $\gamma(x,y,r,\psi)=\cosh x\cosh y+\sinh x\sinh y\,re^{i\psi}$. In order to conclude, it remains for us to prove the following inequality

$$\frac{\rho}{4(\alpha+1)}\sinh 2x\sinh 2y\int_0^1\int_0^\pi \frac{dm_{\alpha+1,\beta+1}(r,\psi)}{|\gamma(x,y,r,\psi)|\sqrt{|\gamma(x,y,r,\psi)|^2-1}} \le \frac{\Gamma(\alpha+1)}{\Gamma(\alpha+\frac12)}\frac{\Gamma(\beta+\frac12)}{\Gamma(\beta+1)}.$$

By expressing $|\gamma(x,y,r,\psi)|$ and $dm_{\alpha+1,\beta+1}$, the left hand side becomes

$$\begin{aligned}&\frac{\rho}{4(\alpha+1)}\sinh 2x\,\sinh 2y\int_0^1\int_0^\pi \frac{dm_{\alpha+1,\beta+1}(r,\psi)}{|\gamma(x,y,r,\psi)|\sqrt{|\gamma(x,y,r,\psi)|^2-1}}\\ &= \frac{\rho}{4(\alpha+1)}\frac{2\,\Gamma(\alpha+2)}{\sqrt{\pi}\,\Gamma(\alpha-\beta)\,\Gamma(\beta+\frac32)}\,\sinh 2x\,\sinh 2y\int_0^1\int_0^\pi (1-r^2)^{\alpha-\beta-1}(r\sin\psi)^{2\beta+2}\\ &\times\frac{1}{\sqrt{(\cosh x\cosh y+r\cos\psi\sinh x\sinh y)^2+(r\sin\psi\sinh x\sinh y)^2}}\\ &\times\frac{1}{\sqrt{(\cosh x\cosh y+r\cos\psi\sinh x\sinh y)^2+(r\sin\psi\sinh x\sinh y)^2-1}}\,r\,dr\,d\psi\\ &= \frac{\rho\,\Gamma(\alpha+1)}{\sqrt{\pi}\,\Gamma(\alpha-\beta)\,\Gamma(\beta+\frac32)}\int_0^1\int_0^\pi (1-r^2)^{\alpha-\beta-1}(r\sin\psi)^{2\beta+2}\frac{dr\,d\psi}{\sqrt{U+\cos\psi}\sqrt{V+\cos\psi}},\end{aligned}$$

where

$$U = \frac{\cosh^2 x \cosh^2 y + r^2 \sinh^2 x \sinh^2 y}{2r \cosh x \cosh y \sinh x \sinh y},$$

and

$$V = \frac{\cosh^2 x \cosh^2 y + r^2 \sinh^2 x \sinh^2 y - 1}{2r \cosh x \cosh y \sinh x \sinh y}.$$

Since

$$U - 1 > V - 1 = \frac{(\cosh x \cosh y - r \sinh x \sinh y)^2 - 1}{2r \cosh x \cosh y \sinh x \sinh y} \geq 0,$$

we can estimate

$$\begin{aligned}
&\frac{\rho}{\alpha+1} \sinh x \cosh x \sinh y \cosh y \int_0^1 \int_0^\pi \frac{dm_{\alpha+1,\beta+1}(r,\psi)}{|\gamma(x,y,r,\psi)|\sqrt{|\gamma(x,y,r,\psi)|^2 - 1}} \\
&\leq \frac{\rho\,\Gamma(\alpha+1)}{\sqrt{\pi}\,\Gamma(\alpha-\beta)\,\Gamma(\beta+\frac{3}{2})} \int_0^1 \int_0^\pi (1-r^2)^{\alpha-\beta-1} (r \sin\psi)^{2\beta+2} (1+\cos\psi)^{-1} dr d\psi \\
&= \frac{\rho}{2} \frac{\Gamma(\alpha+1)}{\Gamma(\alpha+\frac{3}{2})} \frac{\Gamma(\beta+\frac{1}{2})}{\Gamma(\beta+1)} \leq \frac{\Gamma(\alpha+1)}{\Gamma(\alpha+\frac{1}{2})} \frac{\Gamma(\beta+\frac{1}{2})}{\Gamma(\beta+1)},
\end{aligned}$$

using classical formulas for the Beta and Gamma functions.
(iv) is proved in a similar way, using the product formula (2.8) for $\varphi_\lambda^{(\alpha,\alpha)}$ instead of (2.6). □

Remark 3.9. *The measure $\mu_{x,y}^{(\alpha,\beta)}$ is not positive, for any $\alpha \geq \beta > -\frac{1}{2}$ and $x, y > 0$. More precisely, let us show that $\mathcal{K}_{\alpha,\beta}(x,y,z) < 0$ if $-x-y < z < -|x-y|$, while $\mathcal{K}_{\alpha,\beta}(x,y,z) > 0$ if $|x-y| < z < x+y$. In the limit case $\alpha = \beta$, our claim follows immediately from the expression* (3.8)*. Thus we may restrict to the case $\alpha > \beta$. Assume first that $-x-y < z < -|x-y|$ and let us split up*

$$\mathcal{K}_{\alpha,\beta}(x,y,z) = M_{\alpha,\beta}\,(\sinh x\, \sinh y\, \sinh(-z))^{-2\alpha} \Big[\mathcal{K}_{\alpha,\beta}^{(1)}(x,y,z) + \mathcal{K}_{\alpha,\beta}^{(2)}(x,y,z)\Big],$$

where

$$\mathcal{K}_{\alpha,\beta}^{(1)}(x,y,z) = \int_0^\pi g(x,y,-z,\chi)_+^{\alpha-\beta-1} \big(1 - \sigma_{x,y,z}^\chi + \sigma_{x,z,y}^\chi + \sigma_{z,y,x}^\chi\big) (\sin\chi)^{2\beta}\, d\chi$$

and

$$\mathcal{K}_{\alpha,\beta}^{(2)}(x,y,z) = \frac{\rho}{\beta+1/2} \coth x \coth y \coth z \int_0^\pi g(x,y,-z,\chi)_+^{\alpha-\beta-1} (\sin\chi)^{2\beta+2}\, d\chi\,.$$

On one hand, $\coth x\, \coth y\, \coth z < -1$ *and*

$$\int_0^\pi g(x,y,-z,\chi)_+^{\alpha-\beta-1}\, (\sin\chi)^{2\beta+2}\, d\chi > 0\,, \tag{3.13}$$

as the change of variables (2.9) *holds for χ in an interval starting at* 0*, where*

$$g(x,y,-z,\chi) = \sinh^2 x\, \sinh^2 y\, (1 - r^2) > 0\,.$$

Hence $\mathcal{K}^{(2)}_{\alpha,\beta}(x,y,z)<0$. *On the other hand, as*

$$\begin{aligned}\varrho^{\chi}(x,y,z) &= 1-\sigma^{\chi}_{x,y,z}+\sigma^{\chi}_{x,z,y}+\sigma^{\chi}_{z,y,x}\\ &= \tfrac{1}{\sinh x\sinh y\sinh z}\big[\sinh x\sinh y\sinh z+\sinh x\cosh y\cosh z\\ &\qquad +\cosh x\sinh y\cosh z-\cosh x\cosh y\sinh z\\ &\qquad +\tfrac{\cos\chi}{2}(-\sinh 2x-\sinh 2y+\sinh 2z)\big]\end{aligned}$$

is a decreasing function of χ, *we have*

$$\begin{aligned}\varrho^{\chi}(x,y,z) &\le \varrho^{0}(x,y,z)\\ &= \tfrac{4}{\sinh x\sinh y\sinh z}\,\sinh\tfrac{x+y+z}{2}\,\sinh\tfrac{-x+y+z}{2}\,\sinh\tfrac{x-y+z}{2}\,\cosh\tfrac{x+y-z}{2}\\ &< 0\,.\end{aligned}$$

Hence $\mathcal{K}^{(2)}_{\alpha,\beta}(x,y,z)<0$. *When* $|x-y|<z<x+y$, *the positivity of* $\mathcal{K}_{\alpha,\beta}(x,y,z)$ *is proved along the same lines. If* $\sinh 2z\le\sinh 2x+\sinh 2y$, *we have now*

$$\varrho^{\chi}(x,y,z)\ge\varrho^{0}(x,y,z)>0$$

while, if $\sinh 2z\ge\sinh 2x+\sinh 2y$,

$$\begin{aligned}\varrho^{\chi}(x,y,z) &\ge \varrho^{\pi}(x,y,z)\\ &= \tfrac{4}{\sinh x\sinh y\sinh z}\,\cosh\tfrac{x+y+z}{2}\,\cosh\tfrac{-x+y+z}{2}\,\cosh\tfrac{x-y+z}{2}\,\sinh\tfrac{x+y-z}{2}\\ &> 0\,.\end{aligned}$$

4. Generalized translations and convolution product

Let us denote by $\mathcal{C}_c(\mathbb{R})$ the space of continuous functions on $\mathbb{R}$ with compact support.

Let $\alpha\ge\beta\ge-\frac12$ with $\alpha>-\frac12$. The Opdam–Cherednik transform is the Fourier transform in the trigonometric Dunkl setting. It is defined for $f\in\mathcal{C}_c(\mathbb{R})$ by

$$\mathcal{F}(f)(\lambda)=\int_{\mathbb{R}} f(x)\,\mathrm{G}^{(\alpha,\beta)}_{\lambda}(-x)\,A_{\alpha,\beta}(|x|)\,dx\qquad\forall\,\lambda\in\mathbb{C}\tag{4.1}$$

and the inverse transform writes

$$\mathcal{J}g(x)=\int_{\mathbb{R}} g(\lambda)\,\mathrm{G}^{(\alpha,\beta)}_{\lambda}(x)\left(1-\frac{\rho}{i\lambda}\right)\frac{d\lambda}{8\pi\,|\mathrm{c}_{\alpha,\beta}(\lambda)|^2}\,.$$

Here $A_{\alpha,\beta}$ and $\mathrm{c}_{\alpha,\beta}$ are given by (2.11) and (2.3). See [11] for more details.

The Fourier transform $\mathcal{F}$ can be expressed in terms of the Jacobi transform

$$\mathcal{F}_{\alpha,\beta}(f)(\lambda)\;=\int_0^{+\infty} f(x)\,\varphi^{(\alpha,\beta)}_{\lambda}(x)\,A_{\alpha,\beta}(x)\,dx\,.\tag{4.2}$$

More precisely:

Lemma 4.1. *For $\lambda \in \mathbb{C}$ and $f \in C_c(\mathbb{R})$, we have*

$$\mathcal{F}(f)(\lambda) = 2\,\mathcal{F}_{\alpha,\beta}(f_{\mathrm{e}})(\lambda) + 2\,(\rho{+}i\lambda)\,\mathcal{F}_{\alpha,\beta}(Jf_{\mathrm{o}})(\lambda)\,,$$

where f_{e} (resp. f_{o}) denotes the even (resp. odd) part of f, and

$$Jf_{\mathrm{o}}(x) := \int_{-\infty}^{x} f_{\mathrm{o}}(t)\,dt\,.$$

Proof. Write $f = f_{\mathrm{e}} + f_{\mathrm{o}}$. Firstly, if $\lambda = -i\rho$, then

$$\mathcal{F}(f)(\lambda) = \int_{\mathbb{R}} f(x)\,A_{\alpha,\beta}(|x|)\,dx = 2\,\mathcal{F}_{\alpha,\beta}(f_{\mathrm{e}})(i\rho)\,.$$

Secondly, if $\lambda{\neq}{-i\rho}$, we have

$$\mathcal{F}(f)(\lambda) = 2\,\mathcal{F}_{\alpha,\beta}(f_{\mathrm{e}})(\lambda) + \frac{2}{\rho{-}i\lambda}\int_0^{+\infty} f_{\mathrm{o}}(x)\,\frac{\partial}{\partial x}\varphi_\lambda^{(\alpha,\beta)}(x)\,A_{\alpha,\beta}(x)\,dx\,.$$

Recall the Jacobi operator

$$\Delta_{\alpha,\beta} = \frac{1}{A_{\alpha,\beta}(x)}\frac{\partial}{\partial x}\Big[A_{\alpha,\beta}(x)\frac{\partial}{\partial x}\Big] = \frac{\partial^2}{\partial x^2} + \Big[(2\alpha{+}1)\coth x + (2\beta{+}1)\tanh x\Big]\frac{\partial}{\partial x}\,.$$

By integration by parts, we obtain

$$\begin{aligned}
&\int_0^{+\infty} f_{\mathrm{o}}(x)\,\frac{\partial}{\partial x}\varphi_\lambda^{(\alpha,\beta)}(x)\,A_{\alpha,\beta}(x)\,dx\\
&= -\int_0^{+\infty} \varphi_\lambda^{(\alpha,\beta)}(x)\,\frac{1}{A_{\alpha,\beta}(x)}\,\frac{\partial}{\partial x}\Big[A_{\alpha,\beta}(x)\,\frac{\partial}{\partial x}\,Jf_{\mathrm{o}}(x)\Big]A_{\alpha,\beta}(x)\,dx\\
&= -\,\mathcal{F}_{\alpha,\beta}(\Delta_{\alpha,\beta}\,Jf_{\mathrm{o}})(\lambda) = (\rho^2{+}\lambda^2)\,\mathcal{F}_{\alpha,\beta}(Jf_{\mathrm{o}})(\lambda)\,.
\end{aligned}$$

□

The following Plancherel formula was proved by Opdam [11, Theorem 9.13(3)]:

$$\begin{aligned}
\int_{\mathbb{R}} |f(x)|^2 A_{\alpha,\beta}(|x|)\,dx &= \int_0^{+\infty} \big(\,|\mathcal{F}(f)(\lambda)|^2 + |\mathcal{F}(\check{f})(\lambda)|^2\,\big)\,\frac{d\lambda}{16\pi\,|\mathrm{c}_{\alpha,\beta}(\lambda)|^2}\\
&= \int_{\mathbb{R}} \mathcal{F}(f)(\lambda)\,\overline{\mathcal{F}(\check{f})(-\lambda)}\,(1-\frac{\rho}{i\lambda})\,\frac{d\lambda}{8\pi\,|\mathrm{c}_{\alpha,\beta}(\lambda)|^2}\,,
\end{aligned}$$

where $\check{f}(x) := f(-x)$. The following result is obtained by specializing [15, Theorem 4.1].

Theorem 4.2. *The Opdam–Cherednik transform $\mathcal{F}$ and its inverse $\mathcal{J}$ are topological isomorphisms between the Schwartz space $\mathcal{S}_{\alpha,\beta}(\mathbb{R}) = (\cosh x)^{-\rho}\,\mathcal{S}(\mathbb{R})$ and the Schwartz space $\mathcal{S}(\mathbb{R})$. Recall that $\rho{=}\alpha{+}\beta{+}1$.*

Let us denote by $\mathcal{C}_b(\mathbb{R})$ the space of bounded continuous functions on $\mathbb{R}$.

Definition 4.3. *Let $x\in\mathbb{R}$ and let $f\in\mathcal{C}_b(\mathbb{R})$. For $\alpha\geq\beta\geq-\frac{1}{2}$ with $\alpha\neq-\frac{1}{2}$, we define the generalized translation operator $\tau_x^{(\alpha,\beta)}$ by*

$$\tau_x^{(\alpha,\beta)}f(y)\,=\int_{\mathbb{R}}f(z)\,d\mu_{x,y}^{(\alpha,\beta)}(z)\,,$$

where $d\mu_{x,y}^{(\alpha,\beta)}$ is given by (3.2) for $\alpha>\beta$, and by (3.10) for $\alpha=\beta$.

The following properties are clear. However for completeness we will sketch their proof.

Proposition 4.4. *Let $\alpha\geq\beta\geq-\frac{1}{2}$ with $\alpha\neq-\frac{1}{2}$, $x,\,y\in\mathbb{R}$ and $f\in\mathcal{C}_b(\mathbb{R})$. Then*

(i) $\tau_x^{(\alpha,\beta)}f(y)=\tau_y^{(\alpha,\beta)}f(x)$.
(ii) $\tau_0^{(\alpha,\beta)}f=f$.
(iii) $\tau_x^{(\alpha,\beta)}\tau_y^{(\alpha,\beta)}=\tau_y^{(\alpha,\beta)}\tau_x^{(\alpha,\beta)}$.
(iv) $\tau_x^{(\alpha,\beta)}\mathrm{G}_\lambda^{(\alpha,\beta)}(y)=\mathrm{G}_\lambda^{(\alpha,\beta)}(x)\,\mathrm{G}_\lambda^{(\alpha,\beta)}(y)$.
If we suppose also that f belongs to $\mathcal{C}_c(\mathbb{R})$, then
(v) $\mathcal{F}(\tau_x^{(\alpha,\beta)}f)(\lambda)=\mathrm{G}_\lambda^{(\alpha,\beta)}(x)\,\mathcal{F}(f)(\lambda)$.
(vi) $\mathrm{T}^{(\alpha,\beta)}\,\tau_x^{(\alpha,\beta)}=\tau_x^{(\alpha,\beta)}\,\mathrm{T}^{(\alpha,\beta)}$.

Proof. (i) follows from the property $\mathcal{K}_{\alpha,\beta}(x,y,z)=\mathcal{K}_{\alpha,\beta}(y,x,z)$.
(ii) follows from the fact that $\mathcal{K}_{\alpha,\beta}(0,y,z)=\delta_y(z)$.
(iii) follows from the fact that the function

$$H(x_1,y_1,x_2,y_2):=\int_{\mathbb{R}}\mathcal{K}_{\alpha,\beta}(x_1,y_1,z)\,\mathcal{K}_{\alpha,\beta}(x_2,y_2,z)\,A_{\alpha,\beta}(|z|)\,dz$$

is symmetric in the four variables.
(iv) follows from the product formula for $\mathrm{G}_\lambda^{(\alpha,\beta)}$.
(v) For $f\in\mathcal{C}_c(\mathbb{R})$, we have

$$\begin{aligned}\mathcal{F}(\tau_x^{(\alpha,\beta)}f)(\lambda)&=\int_{\mathbb{R}}\tau_x^{(\alpha,\beta)}f(y)\,\mathrm{G}_\lambda^{(\alpha,\beta)}(-y)\,A_{\alpha,\beta}(|y|)\,dy\\&=\int_{\mathbb{R}}\Big[\int_{\mathbb{R}}f(z)\,\mathcal{K}_{\alpha,\beta}(x,y,z)\,A_{\alpha,\beta}(|z|)\,dz\Big]\mathrm{G}_\lambda^{(\alpha,\beta)}(-y)\,A_{\alpha,\beta}(|y|)\,dy\\&=\int_{\mathbb{R}}f(z)\Big[\int_{\mathbb{R}}\mathrm{G}_\lambda^{(\alpha,\beta)}(-y)\,\mathcal{K}_{\alpha,\beta}(x,y,z)\,A_{\alpha,\beta}(|y|)\,dy\Big]A_{\alpha,\beta}(|z|)\,dz\,.\end{aligned}$$

Since $\mathcal{K}_{\alpha,\beta}(x,y,z)=\mathcal{K}_{\alpha,\beta}(x,-z,-y)$, it follows from the product formula that

$$\begin{aligned}\mathcal{F}(\tau_x^{(\alpha,\beta)}f)(\lambda)&=\mathrm{G}_\lambda^{(\alpha,\beta)}(x)\int_{\mathbb{R}}f(z)\,\mathrm{G}_\lambda^{(\alpha,\beta)}(-z)\,A_{\alpha,\beta}(|z|)\,dz\\&=\mathrm{G}_\lambda^{(\alpha,\beta)}(x)\,\mathcal{F}(f)(\lambda).\end{aligned}$$

(vi) This property follows from the injectivity of $\mathcal{F}$ and the fact that $\tau_x^{(\alpha,\beta)}(\mathrm{T}^{(\alpha,\beta)}f)$ and $\mathrm{T}^{(\alpha,\beta)}(\tau_x^{(\alpha,\beta)}f)$ have the same Fourier transform, namely

$$\lambda\longmapsto i\,\lambda\,\mathrm{G}_\lambda^{(\alpha,\beta)}(x)\,\mathcal{F}(f)(\lambda)\,.$$

□

Remark 4.5. *Generalized translations in the Dunkl setting were first introduced by Trimèche, using transmutation operators. This approach is resumed in* [10], *which deals with a generalization of Dunkl analysis in dimension* 1.

Lemma 4.6. *Let* $1 \le p \le \infty$, $f \in L^p(\mathbb{R}, A_{\alpha,\beta}(|z|)dz)$ *and* $x \in \mathbb{R}$. *Then*

$$\|\tau_x^{(\alpha,\beta)} f\|_p \le C_{\alpha,\beta}\, \|f\|_p\,, \tag{4.3}$$

where

$$C_{\alpha,\beta} = \begin{cases} 4 + \frac{\Gamma(\alpha+1)}{\Gamma(\alpha+\frac{1}{2})}\,\frac{\Gamma(\beta+\frac{1}{2})}{\Gamma(\beta+1)} & \text{if } \alpha > \beta > -\frac{1}{2}\,, \\ \frac{5}{2} & \text{if } \alpha = \beta > -\frac{1}{2}\,. \end{cases} \tag{4.4}$$

Proof. The inequality (4.3) follows from Theorem 3.8. More precisely, the cases $p = 1$ and $p = \infty$ are elementary, while the intermediate case $1 < p < \infty$ is obtained by interpolation or by using Hölder's inequality, as follows:

$$\begin{aligned} \|\tau_x^{(\alpha,\beta)} f\|_p^p &\le \Big(\int_{\mathbb{R}} |\mathcal{K}_{\alpha,\beta}(x,y,z)|\, A_{\alpha,\beta}(|z|)\, dz\Big)^{p-1} \\ &\quad \times \int_{\mathbb{R}}\int_{\mathbb{R}} |\mathcal{K}_{\alpha,\beta}(x,y,z)|\, |f(z)|^p\, A_{\alpha,\beta}(|z|)\, A_{\alpha,\beta}(|y|)\, dz\, dy \\ &\le C_{\alpha,\beta}^p\, \|f\|_p^p\,. \end{aligned}$$

□

Definition 4.7. *The convolution product of suitable functions* f *and* g *is defined by*

$$(f *_{\alpha,\beta} g)(x) = \int_{\mathbb{R}} \tau_x^{(\alpha,\beta)} f(-y)\, g(y)\, A_{\alpha,\beta}(|y|)\, dy\,.$$

Remark 4.8. *It is clear that this convolution product is both commutative and associative:*

(i) $f *_{\alpha,\beta} g = g *_{\alpha,\beta} f$.
(ii) $(f *_{\alpha,\beta} g) *_{\alpha,\beta} h = f *_{\alpha,\beta} (g *_{\alpha,\beta} h)$.

For every $a > 0$, let us denote by $\mathcal{D}_a(\mathbb{R})$ the space of smooth functions on $\mathbb{R}$ which are supported in $[-a, a]$.

Proposition 4.9. *Let* $f \in \mathcal{D}_a(\mathbb{R})$ *and* $g \in \mathcal{D}_b(\mathbb{R})$. *Then* $f *_{\alpha,\beta} g \in \mathcal{D}_{a+b}(\mathbb{R})$ *and*

$$\mathcal{F}(f *_{\alpha,\beta} g)(\lambda) = \mathcal{F}(f)(\lambda)\, \mathcal{F}(g)(\lambda)\,.$$

Proof. By definition we have

$$\mathcal{F}(f *_{\alpha,\beta} g)(\lambda) = \int_{\mathbb{R}}\int_{\mathbb{R}} \tau_x^{(\alpha,\beta)} f(-y)\, g(y)\, \mathrm{G}_\lambda^{(\alpha,\beta)}(-x)\, A_{\alpha,\beta}(|x|)\, A_{\alpha,\beta}(|y|)\, dx\, dy\,.$$

Using the product formula for $\mathrm{G}_\lambda^{(\alpha,\beta)}$ and Remark 3.1, we deduce that

$$\begin{aligned}\mathcal{F}(f *_{\alpha,\beta} g)(\lambda) &= \int_{\mathbb{R}} f(z) \int_{\mathbb{R}} g(y) \int_{\mathbb{R}} \mathrm{G}_\lambda^{(\alpha,\beta)}(-x)\, \mathcal{K}_{\alpha,\beta}(-z,-y,-x) \\ &\quad \times\ A_{\alpha,\beta}(|x|)\, A_{\alpha,\beta}(|y|)\, A_{\alpha,\beta}(|z|)\, dx\, dy\, dz \\ &= \int_{\mathbb{R}} f(z)\, \mathrm{G}_\lambda^{(\alpha,\beta)}(-z)\, A_{\alpha,\beta}(|z|)\, dz \int_{\mathbb{R}} g(y)\, \mathrm{G}_\lambda^{(\alpha,\beta)}(-y)\, A_{\alpha,\beta}(|y|)\, dy \\ &= \mathcal{F}(f)(\lambda)\, \mathcal{F}(g)(\lambda)\,.\end{aligned}$$

□

By standard arguments, the following statement follows from Lemma 4.5.

Proposition 4.10. *Assume that* $1 \le p, q, r \le \infty$ *satisfy* $\frac{1}{p} + \frac{1}{q} - 1 = \frac{1}{r}$. *Then, for every* $f \in L^p(\mathbb{R}, A_{\alpha,\beta}(|x|)dx)$ *and* $g \in L^q(\mathbb{R}, A_{\alpha,\beta}(|x|)dx)$, *we have* $f *_{\alpha,\beta} g \in L^r(\mathbb{R}, A_{\alpha,\beta}(|x|)dx)$, *and*

$$\| f *_{\alpha,\beta} g \|_r \le C_{\alpha,\beta} \|f\|_p \|g\|_q\,,$$

where $C_{\alpha,\beta}$ *is as in* (4.4).

5. The Kunze–Stein phenomenon

This remarkable phenomenon was first observed by Kunze and Stein [7] for the group $G = SL(2,\mathbb{R})$ equipped with its Haar measure. They proved that

$$L^p(G) * L^2(G) \subset L^2(G) \quad \forall\, 1 \le p < 2\,.$$

By such an inclusion, we mean the existence of a constant $C_p > 0$ such that the following inequality holds:

$$\| f * g \|_2 \le C_p \|f\|_p \|g\|_2 \quad \forall\, f \in L^p(G),\ \forall\, g \in L^2(G).$$

This result was generalized by Cowling [3] to all connected noncompact semisimple Lie groups with finite center. We prove the following analog in our setting (we understand that Trimèche has recently extended this result to higher dimensions).

Theorem 5.1. *Let* $1 \le p < 2 < q \le \infty$. *Then*

$$L^p(\mathbb{R}, A_{\alpha,\beta}(|x|)dx) *_{\alpha,\beta} L^2(\mathbb{R}, A_{\alpha,\beta}(|x|)dx) \subset L^2(\mathbb{R}, A_{\alpha,\beta}(|x|)dx) \tag{5.1}$$

and

$$L^2(\mathbb{R}, A_{\alpha,\beta}(|x|)dx) *_{\alpha,\beta} L^2(\mathbb{R}, A_{\alpha,\beta}(|x|)dx) \subset L^q(\mathbb{R}, A_{\alpha,\beta}(|x|)dx)\,. \tag{5.2}$$

Proof. (i) Let $f,g\in\mathcal{C}_c(\mathbb{R})$. Then, by the Plancherel formula, we have

$$\begin{aligned}
&\int_{\mathbb{R}} |(f*_{\alpha,\beta} g)|^2(x)\, A_{\alpha,\beta}(|x|)\, dx\\
&= \int_{\mathbb{R}^+} |\mathcal{F}(f*_{\alpha,\beta} g)(\lambda)|^2 \frac{d\lambda}{16\pi\,|\mathrm{c}(\lambda)|^2} + \int_{\mathbb{R}^+} |\mathcal{F}(f*_{\alpha,\beta} g)\check{}(\lambda)|^2 \frac{d\lambda}{16\pi\,|\mathrm{c}(\lambda)|^2}\\
&\le \sup_{\lambda\in\mathbb{R},\, w\in\{\pm1\}} |\mathcal{F}(w\cdot g)(\lambda)|^2 \Big[\int_{\mathbb{R}^+} |\mathcal{F}(f)(\lambda)|^2 \frac{d\lambda}{16\pi\,|\mathrm{c}(\lambda)|^2} + \int_{\mathbb{R}^+} |\mathcal{F}(\check{f})(\lambda)|^2 \frac{d\lambda}{16\pi\,|\mathrm{c}(\lambda)|^2}\Big]\\
&= \sup_{\lambda\in\mathbb{R},\, w\in\{\pm1\}} |\mathcal{F}(w\cdot g)(\lambda)|^2\, \|f\|_2^2\,.
\end{aligned}$$

Here we have used the fact that $\mathcal{F}(f*_{\alpha,\beta} g)\check{}=\mathcal{F}(\check{f})\,\mathcal{F}(\check{g})$. Next, if $1\le p<2$ and $2<q\le\infty$ are dual indices, we estimate

$$\begin{aligned}
|\mathcal{F}(w\cdot g)(\lambda)| &\le \int_{\mathbb{R}} |g(wx)|\, |\mathrm{G}_\lambda^{(\alpha,\beta)}(-x)|\, A_{\alpha,\beta}(|x|)dx\\
&\le \|g\|_p\, \|\mathrm{G}_\lambda^{(\alpha,\beta)}\|_q
\end{aligned}$$

using Hölder's inequality. We conclude by using Lemma 5.2 below, which implies that $\|\mathrm{G}_\lambda^{(\alpha,\beta)}\|_q$ is bounded uniformly in $\lambda\in\mathbb{R}$.
(ii) Let $f,g,k\in\mathcal{C}_c(\mathbb{R})$. Using the Cauchy-Schwartz inequality and (5.1), we get

$$\begin{aligned}
\Big|\int_{\mathbb{R}} (f*_{\alpha,\beta} g)(x)\, k(x)\, A_{\alpha,\beta}(|x|)\, dx\Big| &\le C\,\|g\|_2\, \|\, f*\check{k}\,\|_2\\
&\le C_p\, \|f\|_2\, \|g\|_2\, \|k\|_p\,.
\end{aligned}$$

Hence $\|\, f*_{\alpha,\beta} g\,\|_q \le C_q\, \|f\|_2\, \|g\|_2$. □

Lemma 5.2. (i) *The function* $\mathrm{G}_0^{(\alpha,\beta)}$ *is stricly positive and is bounded above by*

$$\begin{cases} C\,(1+x)\,e^{-\rho x} & \text{if } x\ge 0\,,\\ C\,e^{\rho x} & \text{if } x\le 0\,.\end{cases}$$

(ii) *For every* $\lambda\in\mathbb{R}$ *and* $x\in\mathbb{R}$, *we have*

$$\big|\mathrm{G}_\lambda^{(\alpha,\beta)}(x)\big| \le \mathrm{G}_0^{(\alpha,\beta)}(x)\,.$$

Proof. These estimates are proved in full generality in [15] (see Lemma 3.1, Proposition 3.1.a and Theorem 3.2). For the reader's convenience, we include a proof in dimension 1.
(i) Firstly, by specializing (2.1) and (1.2) for $\lambda=0$, we obtain

$$\varphi_0^{(\alpha,\beta)}(x) = {}_2\mathrm{F}_1\big(\tfrac{\rho}{2}, \tfrac{\alpha-\beta+1}{2}; \alpha+1; \tanh^2 x\big)\,(\cosh x)^{-\rho} \tag{5.3}$$

and

$$\mathrm{G}_0^{(\alpha,\beta)}(x) = \varphi_0^{(\alpha,\beta)}(x) + \frac{\frac{\rho}{2}}{\alpha+1}\sinh x\cosh x\,\varphi_0^{(\alpha+1,\beta+1)}(x)\,. \tag{5.4}$$

It is clear that (5.3) is stricly positive, hence (5.4) when $x \geq 0$. By looking more carefully at their expansions, we observe that the expression

$$\Psi_1(x) = {}_2\mathrm{F}_1\big(\tfrac{\rho}{2}, \tfrac{\alpha-\beta+1}{2}; \alpha+1; \tanh^2 x\big) = \sum\nolimits_{n=0}^{+\infty} \frac{(\frac{\rho}{2})_n\,(\frac{\alpha-\beta+1}{2})_n}{(\alpha+1)_n\,n!}\,(\tanh x)^{2n}$$

is stricly larger than the expression

$$\begin{aligned}\Psi_2(x) &= \frac{\frac{\rho}{2}}{\alpha+1}\,{}_2\mathrm{F}_1\big(\tfrac{\rho}{2}+1, \tfrac{\alpha-\beta+1}{2}; \alpha+2; \tanh^2 x\big)\\ &= \sum\nolimits_{n=0}^{+\infty} \frac{(\frac{\rho}{2})_{n+1}(\frac{\alpha-\beta+1}{2})_n}{(\alpha+1)_{n+1}\,n!}\,(\tanh x)^{2n}\,.\end{aligned}$$

Hence

$$\mathrm{G}_0^{(\alpha,\beta)}(x) = (\cosh x)^{-\rho}\,\big\{\Psi_1(x) + \tanh x\,\Psi_2(x)\big\} > (\cosh x)^{-\rho}\,\big\{\Psi_1(x) - \Psi_2(x)\big\}$$

is strictly positive on $\mathbb{R}$. Secondly, by combining (5.4) with (2.5), we obtain

$$\mathrm{G}_0^{(\alpha,\beta)}(x) = \frac{2^{\rho+2}\,\Gamma(\alpha+1)}{\Gamma(\frac{\rho}{2})\,\Gamma(\frac{\alpha-\beta+1}{2})}\,x\,e^{-\rho\,x} + \mathcal{O}\big(e^{-\rho\,x}\big) \qquad \text{as } \ x \to +\infty$$

and

$$\mathrm{G}_0^{(\alpha,\beta)}(x) = \mathcal{O}\big(e^{\rho\,x}\big) \qquad \text{as } \ x \to -\infty\,,$$

which yields the announced upper bounds.
(ii) Consider the quotient

$$\mathrm{Q}_\lambda^{(\alpha,\beta)}(x) = \tfrac{\mathrm{G}_\lambda^{(\alpha,\beta)}(x)}{\mathrm{G}_0^{(\alpha,\beta)}(x)}\,.$$

By using the equation (1.1) for $\mathrm{G}_\lambda^{(\alpha,\beta)}$ and $\mathrm{G}_0^{(\alpha,\beta)}$, we obtain

$$\begin{aligned}&\tfrac{\partial}{\partial x}\,\mathrm{Q}_\lambda^{(\alpha,\beta)}(x) = \tfrac{\frac{\partial}{\partial x}\,\mathrm{G}_\lambda^{(\alpha,\beta)}(x)}{\mathrm{G}_0^{(\alpha,\beta)}(x)} - \mathrm{Q}_\lambda^{(\alpha,\beta)}(x)\,\tfrac{\frac{\partial}{\partial x}\,\mathrm{G}_0^{(\alpha,\beta)}(x)}{\mathrm{G}_0^{(\alpha,\beta)}(x)}\\ &= \big\{(\alpha-\beta)\coth x + (2\beta+1)\coth 2x + \rho\big\}\,\tfrac{\mathrm{G}_0^{(\alpha,\beta)}(-x)}{\mathrm{G}_0^{(\alpha,\beta)}(x)}\,\big\{\mathrm{Q}_\lambda^{(\alpha,\beta)}(-x) - \mathrm{Q}_\lambda^{(\alpha,\beta)}(x)\big\}\\ &+ i\lambda\,\mathrm{Q}_\lambda^{(\alpha,\beta)}(x)\,.\end{aligned}$$

Hence

$$\begin{aligned}\tfrac{\partial}{\partial x}\,\big|\mathrm{Q}_\lambda^{(\alpha,\beta)}(x)\big|^2 &= 2\,\mathrm{Re}\,\Big[\tfrac{\partial}{\partial x}\,\mathrm{Q}_\lambda^{(\alpha,\beta)}(x)\,\overline{\mathrm{Q}_\lambda^{(\alpha,\beta)}(x)}\,\Big]\\ &= -\,2\,\big\{(\alpha-\beta)\coth x + (2\beta+1)\coth 2x + \rho\big\}\,\tfrac{\mathrm{G}_0^{(\alpha,\beta)}(-x)}{\mathrm{G}_0^{(\alpha,\beta)}(x)}\\ &\times\Big\{\,\big|\mathrm{Q}_\lambda^{(\alpha,\beta)}(x)\big|^2 - \mathrm{Re}\,\Big[\mathrm{Q}_\lambda^{(\alpha,\beta)}(-x)\,\overline{\mathrm{Q}_\lambda^{(\alpha,\beta)}(x)}\,\Big]\Big\}\end{aligned}$$

and

$$\begin{aligned}\tfrac{\partial}{\partial x}\,\big|\mathrm{Q}_\lambda^{(\alpha,\beta)}(-x)\big|^2 &= -\,2\,\big\{(\alpha-\beta)\coth x + (2\beta+1)\coth 2x - \rho\big\}\,\tfrac{\mathrm{G}_0^{(\alpha,\beta)}(x)}{\mathrm{G}_0^{(\alpha,\beta)}(-x)}\\ &\times\Big\{\,\big|\mathrm{Q}_\lambda^{(\alpha,\beta)}(-x)\big|^2 - \mathrm{Re}\,\Big[\mathrm{Q}_\lambda^{(\alpha,\beta)}(x)\,\overline{\mathrm{Q}_\lambda^{(\alpha,\beta)}(-x)}\,\Big]\Big\}\,.\end{aligned}$$

Thus, for every $x>0$, we have

$$\begin{aligned}\tfrac{\partial}{\partial x}\left|\mathrm{Q}_\lambda^{(\alpha,\beta)}(x)\right|^2 &\leq -2\left\{(\alpha-\beta)\coth x+(2\beta+1)\coth 2x+\rho\right\}\tfrac{\mathrm{G}_0^{(\alpha,\beta)}(-x)}{\mathrm{G}_0^{(\alpha,\beta)}(x)}\\ &\quad\times\left|\mathrm{Q}_\lambda^{(\alpha,\beta)}(x)\right|\left\{\left|\mathrm{Q}_\lambda^{(\alpha,\beta)}(x)\right|-\left|\mathrm{Q}_\lambda^{(\alpha,\beta)}(-x)\right|\right\}\\ &\leq 0\end{aligned}\tag{5.5}$$

if $\left|\mathrm{Q}_\lambda^{(\alpha,\beta)}(x)\right|\geq\left|\mathrm{Q}_\lambda^{(\alpha,\beta)}(-x)\right|$ and

$$\begin{aligned}\tfrac{\partial}{\partial x}\left|\mathrm{Q}_\lambda^{(\alpha,\beta)}(-x)\right|^2 &\leq -2\left\{(\alpha-\beta)\coth x+(2\beta+1)\coth 2x-\rho\right\}\tfrac{\mathrm{G}_0^{(\alpha,\beta)}(x)}{\mathrm{G}_0^{(\alpha,\beta)}(-x)}\\ &\quad\times\left|\mathrm{Q}_\lambda^{(\alpha,\beta)}(-x)\right|\left\{\left|\mathrm{Q}_\lambda^{(\alpha,\beta)}(-x)\right|-\left|\mathrm{Q}_\lambda^{(\alpha,\beta)}(x)\right|\right\}\\ &\leq 0\end{aligned}\tag{5.6}$$

if $|\mathrm{Q}_\lambda^{(\alpha,\beta)}(-x)|\geq|\mathrm{Q}_\lambda^{(\alpha,\beta)}(x)|$. As real analytic functions of x, $|\mathrm{Q}_\lambda^{(\alpha,\beta)}(x)|^2$ and $|\mathrm{Q}_\lambda^{(\alpha,\beta)}(-x)|^2$ coincide either everywhere or on a discrete subset of $\mathbb{R}$ with no accumulation point. In the first case, $|\mathrm{Q}_\lambda^{(\alpha,\beta)}(x)|^2=|\mathrm{Q}_\lambda^{(\alpha,\beta)}(-x)|^2$ is a decreasing function of x on $[0,+\infty)$, according to (5.5) or (5.6). In the second case, consider the continuous and piecewise differentiable function

$$M(x)=\max\left\{\left|\mathrm{Q}_\lambda^{(\alpha,\beta)}(x)\right|^2,\left|\mathrm{Q}_\lambda^{(\alpha,\beta)}(-x)\right|^2\right\}$$

on $[0,+\infty)$. Firstly, if $\left|\mathrm{Q}_\lambda^{(\alpha,\beta)}(x)\right|>\left|\mathrm{Q}_\lambda^{(\alpha,\beta)}(-x)\right|$, then

$$\tfrac{\partial}{\partial x}M(x)=\tfrac{\partial}{\partial x}\left|\mathrm{Q}_\lambda^{(\alpha,\beta)}(x)\right|^2<0,$$

according to (5.5). Secondly, if $\left|\mathrm{Q}_\lambda^{(\alpha,\beta)}(x)\right|<\left|\mathrm{Q}_\lambda^{(\alpha,\beta)}(-x)\right|$, then

$$\tfrac{\partial}{\partial x}M(x)=\tfrac{\partial}{\partial x}\left|\mathrm{Q}_\lambda^{(\alpha,\beta)}(-x)\right|^2<0,$$

according to (5.6). Thirdly, if $\left|\mathrm{Q}_\lambda^{(\alpha,\beta)}(x)\right|=\left|\mathrm{Q}_\lambda^{(\alpha,\beta)}(-x)\right|$ for some $x>0$, then M has left and right derivatives at x, which are nonpositive, according to (5.5) and (5.6). Thus M is a decreasing function on $[0,+\infty)$. In all cases, we conclude in particular that, for every $x\in\mathbb{R}$,

$$\left|\mathrm{Q}_\lambda^{(\alpha,\beta)}(x)\right|\leq\left|\mathrm{Q}_\lambda^{(\alpha,\beta)}(0)\right|=1\quad\text{i.e.}\quad\left|\mathrm{G}_\lambda^{(\alpha,\beta)}(x)\right|\leq\mathrm{G}_0^{(\alpha,\beta)}(x).$$

□

The following results are deduced by interpolation and duality from Theorem 5.1 and Proposition 4.10.

Corollary 5.3. (i) *Let $1\leq p<q\leq 2$. Then*

$$L^p(\mathbb{R},A_{\alpha,\beta}(|x|)\,dx)*_{\alpha,\beta}L^q(\mathbb{R},A_{\alpha,\beta}(|x|)\,dx)\subset L^q(\mathbb{R},A_{\alpha,\beta}(|x|)\,dx).$$

(ii) *Let $1<p<2$ and $p<q\leq\frac{p}{2-p}$. Then*

$$L^p(\mathbb{R},A_{\alpha,\beta}(|x|)\,dx)*_{\alpha,\beta}L^p(\mathbb{R},A_{\alpha,\beta}(|x|)\,dx)\subset L^q(\mathbb{R},A_{\alpha,\beta}(|x|)\,dx).$$

(iii) *Let* $2<p,q<\infty$ *such that* $\frac{q}{2}\leq p<q$. *Then*

$$L^p(\mathbb{R},A_{\alpha,\beta}(|x|)\,dx)*_{\alpha,\beta}L^{q'}(\mathbb{R},A_{\alpha,\beta}(|x|)\,dx)\subset L^q(\mathbb{R},A_{\alpha,\beta}(|x|)\,dx).$$

6. A special orthogonal system

In this section we construct an orthogonal basis of $L^2(\mathbb{R},A_{\alpha,\beta}(|x|)\,dx)$ and we compute its Opdam–Cherednik transform. As limits, we recover the Hermite functions constructed by Rosenblum [14].

Proposition 6.1. *Let* $\alpha\geq\beta\geq-\frac{1}{2}$ *with* $\alpha>-\frac{1}{2}$. *For any fixed* $\delta>0$, *consider the sequence of functions*

$$\begin{cases}\mathrm{H}^{\delta}_{2n}(x)=(\cosh x)^{-\alpha-\beta-\delta-2}\,\mathrm{P}^{(\alpha,\delta)}_n(1-2\tanh^2x),\\ \mathrm{H}^{\delta}_{2n+1}(x)=(\cosh x)^{-\alpha-\beta-\delta-2}\,\mathrm{P}^{(\alpha+1,\delta-1)}_n(1-2\tanh^2x)\,\tanh x,\end{cases}\tag{6.1}$$

whose definition involves the Jacobi polynomials (2.17). *Then* $\{\mathrm{H}^{\delta}_n\}_{n\in\mathbb{N}}$ *is an orthogonal basis of* $L^2(\mathbb{R},A_{\alpha,\beta}(|x|)\,dx)$.

Proof. To begin with, let us prove the orthogonality of $\{\mathrm{H}^{\delta}_n\}_{n\in\mathbb{N}}$ in $L^2(\mathbb{R},A_{\alpha,\beta}(|x|)\,dx)$. Firstly, by oddness

$$\int_{\mathbb{R}}\mathrm{H}^{\delta}_{2m}(x)\,\mathrm{H}^{\delta}_{2n+1}(x)\,A_{\alpha,\beta}(|x|)\,dx=0.$$

Secondly, we obtain

$$\begin{aligned}&\int_{\mathbb{R}}\mathrm{H}^{\delta}_{2m}(x)\,\mathrm{H}^{\delta}_{2n}(x)\,A_{\alpha,\beta}(|x|)\,dx\\ &=2\int_0^1\mathrm{P}^{(\alpha,\delta)}_m(1-2y^2)\,\mathrm{P}^{(\alpha,\delta)}_n(1-2y^2)\,y^{2\alpha+1}\,(1-y^2)^{\delta}\,dy\\ &=2^{-\alpha-\delta-1}\int_{-1}^1\mathrm{P}^{(\alpha,\delta)}_m(z)\,\mathrm{P}^{(\alpha,\delta)}_n(z)\,(1-z)^{\alpha}\,(1+z)^{\delta}\,dz\\ &=\frac{\Gamma(\alpha+n+1)\,\Gamma(\delta+n+1)}{(\alpha+\delta+2n+1)\,n!\,\Gamma(\alpha+\delta+n+1)}\,\delta_{m,n}\,,\end{aligned}$$

by performing the changes of variables $y=\tanh x$, $z=1-y^2$ and by using the orthogonality of Jacobi polynomials (see for instance [1]) :

$$\begin{aligned}&\int_{-1}^{+1}\mathrm{P}^{(\alpha,\delta)}_m(z)\,\mathrm{P}^{(\alpha,\delta)}_n(z)\,(1-z)^{\alpha}\,(1+z)^{\delta}\,dz\\ &=2^{\alpha+\delta+1}\,\frac{\Gamma(\alpha+n+1)\,\Gamma(\delta+n+1)}{(\alpha+\delta+2n+1)\,n!\,\Gamma(\alpha+\delta+n+1)}\,\delta_{m,n}.\end{aligned}$$

Thirdly, by the same arguments

$$\begin{aligned}
&\int_{\mathbb{R}} \mathrm{H}^{\delta}_{2m+1}(x)\,\mathrm{H}^{\delta}_{2n+1}(x)\,A_{\alpha,\beta}(|x|)\,dx\\
&= 2\int_0^1 \mathrm{P}^{(\alpha+1,\delta-1)}_m(1-2y^2)\,\mathrm{P}^{(\alpha+1,\delta-1)}_n(1-2y^2)\,y^{2\alpha+3}\,(1-y^2)^{\delta-1}\,dy\\
&= 2^{-\alpha-\delta-1}\int_{-1}^1 \mathrm{P}^{(\alpha+1,\delta-1)}_m(z)\,\mathrm{P}^{(\alpha+1,\delta-1)}_n(z)\,(1-z)^{\alpha+1}\,(1+z)^{\delta-1}\,dz\\
&= \frac{\Gamma(\alpha+n+2)\,\Gamma(\delta+n)}{(\alpha+\delta+2n+1)\,n!\,\Gamma(\alpha+\delta+n+1)}\,\delta_{m,n}\,.
\end{aligned}$$

Let us turn to the completeness of $\{\mathrm{H}^{\delta}_n\}_{n\in\mathbb{N}}$ in $L^2(\mathbb{R}, A_{\alpha,\beta}(|x|)dx)$. Recall (see for instance [1]) that the Jacobi polynomials $\{\mathrm{P}^{(\tilde\alpha,\tilde\delta)}_n\}_{n\in\mathbb{N}}$ span a dense subspace of $L^2(\,]-1,1[\,,(1-z)^{\tilde\alpha}(1+z)^{\tilde\delta}dz)$. By the above changes of variables, we deduce that $\{\mathrm{H}^{\delta}_{2n}\}_{n\in\mathbb{N}}$ and $\{\mathrm{H}^{\delta}_{2n+1}\}_{n\in\mathbb{N}}$ span dense subspaces of $L^2(\mathbb{R}, A_{\alpha,\beta}(|x|)dx)_{\mathrm{e}}$ and $L^2(\mathbb{R}, A_{\alpha,\beta}(|x|)dx)_{\mathrm{o}}$ respectively. □

Remark 6.2. *In* (6.1)*, let us replace* δ *by* ε^{-2}*,* x *by* εx *and let* $\varepsilon\searrow 0$*. As*

$$(\cosh\varepsilon x)^{-\alpha-\beta-\varepsilon^{-2}-2} \longrightarrow e^{-\frac{x^2}{2}}$$

and

$$\mathrm{P}^{(a,b+\varepsilon^{-2})}_n(1-2\tanh^2\varepsilon x) = \tfrac{(a+1)_n}{n!}\,{}_2\mathrm{F}_1(-n, a+b+\varepsilon^{-2}+n+1; a+1; \tanh^2\varepsilon x)$$

tends to the Laguerre polynomial

$$\mathrm{L}^a_n(x^2) = \tfrac{(a+1)_n}{n!}\,{}_1\mathrm{F}_1(-n\,; a+1; x^2)\,,$$

we recover in the limit the even and odd Hermite functions constructed by Rosenblum [14, Definition 3.4] *in the rational Dunkl setting*:

$$\begin{cases} \mathrm{H}^{\varepsilon^{-2}}_{2n}(\varepsilon x) \longrightarrow e^{-\frac{x^2}{2}}\,\mathrm{L}^{\alpha}_n(x^2)\,,\\ \varepsilon^{-1}\,\mathrm{H}^{\varepsilon^{-2}}_{2n+1}(\varepsilon x) \longrightarrow e^{-\frac{x^2}{2}}\,\mathrm{L}^{\alpha+1}_n(x^2)\,x\,. \end{cases}$$

Theorem 6.3. *The Opdam–Cherednik transform of* $\{H^{\delta}_n\}_{n\in\mathbb{N}}$ *is given by*

$$\begin{aligned}
\mathcal{F}(\mathrm{H}^{\delta}_{2n})(\lambda) &= \frac{(-1)^n}{n!}\,\frac{\Gamma(\alpha+1)\,\Gamma(\frac{\delta+1+i\lambda}{2})\,\Gamma(\frac{\delta+1-i\lambda}{2})}{\Gamma(\frac{\alpha+\beta+\delta}{2}+n+1)\,\Gamma(\frac{\alpha-\beta+\delta}{2}+n+1)}\\
&\quad\times \mathrm{P}_n\Big(-\frac{\lambda^2}{4}; \frac{\delta+1}{2}, \frac{\delta+1}{2}, \frac{\alpha+\beta+1}{2}, \frac{\alpha-\beta+1}{2}\Big)\,,
\end{aligned} \tag{6.2}$$

and

$$\begin{aligned}
\mathcal{F}(\mathrm{H}^{\delta}_{2n+1})(\lambda) &= \frac{(-1)^n}{n!}\,\frac{(\rho+i\lambda)\,\Gamma(\alpha+1)\,\Gamma(\frac{\delta+1+i\lambda}{2})\,\Gamma(\frac{\delta+1-i\lambda}{2})}{2\,\Gamma(\frac{\alpha+\beta+\delta}{2}+n+2)\,\Gamma(\frac{\alpha-\beta+\delta}{2}+n+1)}\\
&\quad\times \mathrm{P}_n\Big(-\frac{\lambda^2}{4}; \frac{\delta+1}{2}, \frac{\delta-1}{2}, \frac{\alpha+\beta+3}{2}, \frac{\alpha-\beta+1}{2}\Big)
\end{aligned} \tag{6.3}$$

where

$$\begin{aligned}\mathrm{P}_n(t^2;a,b,c,d) &= (a{+}b)_n\,(a{+}c)_n\,(a{+}d)_n \\ &\qquad\times\ {}_4\mathrm{F}_3\Big(\begin{matrix}-n,\,a{+}b{+}c{+}d{+}n{-}1,\,a{+}t,\,a{-}t\\ a{+}b,\,a{+}c,\,a{+}d\end{matrix};1\Big)\end{aligned}$$

denotes the Wilson polynomials.

Proof. By evenness, the Opdam–Cherednik transform $\mathcal{F}(\mathrm{H}^\delta_{2n})$ coincides with the Jacobi transform $\mathcal{F}_{\alpha,\beta}(\mathrm{H}^\delta_{2n})$. Thus (6.2) amounts to Formula (9.4) in [9]. Let us recall its proof, which was sketched in [9, Section 9] and which will be used for (6.3). On one hand, we expand

$$\begin{aligned}\mathrm{H}^\delta_{2n}(x) &= (-1)^n\,(\cosh x)^{-\alpha-\beta-\delta-2}\,\mathrm{P}^{(\delta,\alpha)}_n(2\tanh^2 x-1)\\ &= (-1)^n\,\frac{(\delta{+}1)_n}{n!}\,(\cosh x)^{-\alpha-\beta-\delta-2}\,{}_2\mathrm{F}_1(-n,\alpha{+}\delta{+}n{+}1;\delta{+}1;\cosh^{-2}x)\\ &= \frac{(-1)^n}{n!}\,(\delta{+}1)_n\sum_{m=0}^{n}\frac{(-n)_m\,(\alpha{+}\delta{+}n{+}1)_m}{(\delta{+}1)_m\,m!}\,(\cosh x)^{-\alpha-\beta-\delta-2m-2}\end{aligned}\tag{6.4}$$

in negative powers of $\cosh x$, using the symmetry

$$\mathrm{P}^{(\alpha,\delta)}_n(x) = (-1)^n\,\mathrm{P}^{(\delta,\alpha)}_n(-x)$$

and the definition (2.17) of Jacobi polynomials. On the other hand, recall the following Jacobi transform [9, Formula (9.1)] :

$$\int_0^{+\infty}(\cosh x)^{-\alpha-\beta-\mu-1}\,\varphi^{(\alpha,\beta)}_\lambda(x)\,A_{\alpha,\beta}(x)\,dx = \frac{\Gamma(\alpha{+}1)\,\Gamma(\frac{\mu+i\lambda}{2})\,\Gamma(\frac{\mu-i\lambda}{2})}{2\,\Gamma(\frac{\alpha+\beta+\mu+1}{2})\,\Gamma(\frac{\alpha-\beta+\mu+1}{2})}\,.\tag{6.5}$$

We conclude by combining (6.4) and (6.5) :

$$
\begin{aligned}
\mathcal{F}(\mathrm{H}^\delta_{2n})(\lambda) &= \int_{\mathbb{R}} \mathrm{H}^\delta_{2n}(x)\, \varphi^{(\alpha,\beta)}_\lambda(x)\, A_{\alpha,\beta}(|x|)\, dx \\
&= \frac{(-1)^n}{n!}\, (\delta{+}1)_n \sum_{m=0}^{n} \frac{(-n)_m\, (\alpha{+}\delta{+}n{+}1)_m}{(\delta{+}1)_m\, m!} \\
&\times \underbrace{2\int_0^{+\infty} (\cosh x)^{-\alpha-\beta-\delta-2m-2}\, \varphi^{(\alpha,\beta)}_\lambda(x)\, A_{\alpha,\beta}(x)\, dx}_{\dfrac{\Gamma(\alpha+1)\,\Gamma(\frac{\delta+1+i\lambda}{2}+m)\,\Gamma(\frac{\delta+1-i\lambda}{2}+m)}{\Gamma(\frac{\alpha+\beta+\delta}{2}+m+1)\,\Gamma(\frac{\alpha-\beta+\delta}{2}+m+1)}} \\
&= \frac{(-1)^n}{n!}\, \frac{\Gamma(\alpha+1)\,\Gamma(\frac{\delta+1+i\lambda}{2})\,\Gamma(\frac{\delta+1-i\lambda}{2})}{\Gamma(\frac{\alpha+\beta+\delta}{2}+1)\,\Gamma(\frac{\alpha-\beta+\delta}{2}+1)}\, (\delta{+}1)_n \\
&\times \underbrace{\sum_{m=0}^{n} \frac{(-n)_m\, (\alpha{+}\delta{+}n{+}1)_m\, (\frac{\delta+1+i\lambda}{2})_m\, (\frac{\delta+1-i\lambda}{2})_m}{(\delta{+}1)_m\, (\frac{\alpha+\beta+\delta}{2}+1)_m\, (\frac{\alpha-\beta+\delta}{2}+1)_m\, m!}}_{{}_4\mathrm{F}_3\left(\begin{smallmatrix} -n,\, \alpha+\delta+n+1,\, \frac{\delta+1+i\lambda}{2},\, \frac{\delta+1-i\lambda}{2} \\ \delta+1,\, \frac{\alpha+\beta+\delta}{2}+1,\, \frac{\alpha-\beta+\delta}{2}+1 \end{smallmatrix}\,;1\right)} \\
&= \frac{(-1)^n}{n!}\, \frac{\Gamma(\alpha+1)\,\Gamma(\frac{\delta+1+i\lambda}{2})\,\Gamma(\frac{\delta+1-i\lambda}{2})}{\Gamma(\frac{\alpha+\beta+\delta}{2}+n+1)\,\Gamma(\frac{\alpha-\beta+\delta}{2}+n+1)} \\
&\times \mathrm{P}_n\Big(-\frac{\lambda^2}{4}; \frac{\delta+1}{2}, \frac{\delta+1}{2}, \frac{\alpha+\beta+1}{2}, \frac{\alpha-\beta+1}{2}\Big)
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
\mathrm{H}^\delta_{2n+1}(x) &= (-1)^n\, (\cosh x)^{-\alpha-\beta-\delta-2}\, \mathrm{P}^{(\delta-1,\alpha+1)}_n(2\tanh^2 x - 1)\ \tanh x \\
&= \frac{(-1)^n}{n!}\, (\delta)_n\, (\sinh x)\, (\cosh x)^{-\alpha-\beta-\delta-3}\, {}_2\mathrm{F}_1(-n, \alpha{+}\delta{+}n{+}1; \delta; \cosh^{-2} x) \\
&= \frac{(-1)^n}{n!}\, (\delta)_n\, (\sinh x) \sum_{m=0}^{n} \frac{(-n)_m\, (\alpha{+}\delta{+}n{+}1)_m}{(\delta)_m\, m!}\, (\cosh x)^{-\alpha-\beta-\delta-2m-3}
\end{aligned}
$$

and

$$
\begin{aligned}
\mathcal{F}(\mathrm{H}^\delta_{2n+1})(\lambda) &= \int_{\mathbb{R}} \mathrm{H}^\delta_{2n+1}(x)\, \mathrm{G}^{(\alpha,\beta)}_{\lambda,\mathrm{o}}(x)\, A_{\alpha,\beta}(|x|)\, dx \\
&= \frac{\rho+i\lambda}{4(\alpha+1)} \int_{\mathbb{R}} \mathrm{H}^\delta_{2n+1}(x)\, \varphi^{(\alpha+1,\beta+1)}_\lambda(x)\, (\sinh 2x)\, A_{\alpha,\beta}(|x|)\, dx \\
&= \frac{\rho+i\lambda}{\alpha+1} \int_0^{+\infty} (\sinh x \cosh x)^{-1}\, \mathrm{H}^\delta_{2n+1}(x)\, \varphi^{(\alpha+1,\beta+1)}_\lambda(x)\, A_{\alpha+1,\beta+1}(x)\, dx
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{(-1)^n}{n!}\,\frac{\rho+i\lambda}{\alpha+1}\,(\delta)_n \sum_{m=0}^{n} \frac{(-n)_m\,(\alpha+\delta+n+1)_m}{(\delta)_m\,m!} \\
&\times \underbrace{\int_0^{+\infty} (\cosh x)^{-\alpha-\beta-\delta-2m-4}\,\varphi_\lambda^{(\alpha+1,\beta+1)}(x)\,A_{\alpha+1,\beta+1}(x)\,dx}_{\dfrac{\Gamma(\alpha+2)\,\Gamma(\frac{\delta+1+i\lambda}{2}+m)\,\Gamma(\frac{\delta+1-i\lambda}{2}+m)}{2\,\Gamma(\frac{\alpha+\beta+\delta}{2}+m+2)\,\Gamma(\frac{\alpha-\beta+\delta}{2}+m+1)}} \\
&= \frac{(-1)^n}{n!}\,\frac{(\rho+i\lambda)\,\Gamma(\alpha+1)\,\Gamma(\frac{\delta+1+i\lambda}{2})\,\Gamma(\frac{\delta+1-i\lambda}{2})}{2\,\Gamma(\frac{\alpha+\beta+\delta}{2}+2)\,\Gamma(\frac{\alpha-\beta+\delta}{2}+1)}\,(\delta)_n \\
&\times \underbrace{\sum_{m=0}^{n} \frac{(-n)_m\,(\alpha+\delta+n+1)_m\,(\frac{\delta+1+i\lambda}{2})_m\,(\frac{\delta+1-i\lambda}{2})_m}{(\delta)_m\,(\frac{\alpha+\beta+\delta}{2}+2)_m\,(\frac{\alpha-\beta+\delta}{2}+1)_m\,m!}}_{{}_4\mathrm{F}_3\left(\begin{matrix}-n,\,\alpha+\delta+n+1,\,\frac{\delta+1+i\lambda}{2},\,\frac{\delta+1-i\lambda}{2}\\ \delta,\,\frac{\alpha+\beta+\delta}{2}+2,\,\frac{\alpha-\beta+\delta}{2}+1\end{matrix};1\right)} \\
&= \frac{(-1)^n}{n!}\,\frac{(\rho+i\lambda)\,\Gamma(\alpha+1)\,\Gamma(\frac{\delta+1+i\lambda}{2})\,\Gamma(\frac{\delta+1-i\lambda}{2})}{2\,\Gamma(\frac{\alpha+\beta+\delta}{2}+n+2)\,\Gamma(\frac{\alpha-\beta+\delta}{2}+n+1)} \\
&\times \mathrm{P}_n\Big(-\frac{\lambda^2}{4};\frac{\delta+1}{2},\frac{\delta-1}{2},\frac{\alpha+\beta+3}{2},\frac{\alpha-\beta+1}{2}\Big)
\end{aligned}
$$

□

By comparing the Opdam–Cherednik transform of H_n^δ with the particular case $\mathrm{H}_0^\delta(x) = (\cosh x)^{-\alpha-\beta-\delta-2}$, we obtain the following Rodrigues type formula.

Corollary 6.4. *Consider the polynomials*

$$
\begin{cases}
\tilde{\mathrm{P}}_{2n}^\delta(t) = \dfrac{(-1)^n}{n!\,(\frac{\alpha+\beta+\delta}{2}+1)_n\,(\frac{\alpha-\beta+\delta}{2}+1)_n}\,\mathrm{P}_n\big(\frac{t^2}{4};\frac{\delta+1}{2},\frac{\delta+1}{2},\frac{\alpha+\beta+1}{2},\frac{\alpha-\beta+1}{2}\big), \\[2ex]
\tilde{\mathrm{P}}_{2n+1}^\delta(t) = \dfrac{(-1)^n\,(\rho+t)}{2\,n!\,(\frac{\alpha+\beta+\delta}{2}+1)_{n+1}\,(\frac{\alpha-\beta+\delta}{2}+1)_n}\,\mathrm{P}_n\big(\frac{t^2}{4};\frac{\delta+1}{2},\frac{\delta-1}{2},\frac{\alpha+\beta+3}{2},\frac{\alpha-\beta+1}{2}\big).
\end{cases}
$$

Then

$$
\mathrm{H}_n^\delta(x) = \tilde{\mathrm{P}}_n^\delta(\mathrm{T}_x^{(\alpha,\beta)})(\cosh x)^{-\alpha-\beta-\delta-2} \qquad \forall\, n\in\mathbb{N}. \tag{6.6}
$$

In other words, by replacing in the expansion of the polynomial $\tilde{\mathrm{P}}_n^\delta(t)$ the variable t by the Dunkl–Cherednik operator $T^{(\alpha,\beta)}$, one obtains a differential–difference operator, whose action on the function $(\cosh x)^{-\alpha-\beta-\delta-2}$ yields the function $\mathrm{H}_n^\delta(x)$.

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