Linear Koszul Duality and affine Hecke algebras

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LINEAR KOSZUL DUALITY AND AFFINE HECKE ALGEBRAS

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Abstract. In this paper we prove that the linear Koszul duality equivalence constructed in a previous paper provides a geometric realization of the Iwahori-Matsumoto involution of affine Hecke algebras.

Introduction

0.1. The Iwahori-Matsumoto involution of an affine Hecke algebra $\mathcal{H}_{\mathrm{aff}}$ is a certain involution of $\mathcal{H}_{\mathrm{aff}}$ which naturally appears in the study of representations of p-adic groups (see *e.g.* [BC], [BM]). This involution has a version for the corresponding *graded* affine Hecke algebra (*i.e.* the associated graded of $\mathcal{H}_{\mathrm{aff}}$, endowed with a certain filtration), which has been realized geometrically by Evens and the first author in [EM]. More precisely, the graded affine Hecke algebra is isomorphic to the equivariant homology of the Steinberg variety St, and the Iwahori-Matsumoto involution is essentially given by a Fourier transform on this homology.

In this paper we use the realization of $\mathcal{H}_{\mathrm{aff}}$ as the equivariant *K-theory* of the variety St. Using our constructions of [MR] for an appropriate choice of vector bundles, we obtain an equivalence between two triangulated categories whose K-theory is naturally isomorphic to $\mathcal{H}_{\mathrm{aff}}$. We show that the morphism induced in K-theory is essentially the Iwahori-Matsumoto involution of $\mathcal{H}_{\mathrm{aff}}$ (see Theorem 6.2.1 for a precise statement).

Our proof is based on the study of the behaviour of linear Koszul duality (in a general context) under natural operations, inspired by the results of the second author in [R2, Section 2], and quite similar to some compatibility properties of the Fourier transform on constructible sheaves (see [KS, 3.7]).

0.2. **Organization of the paper.** In Section 1 we introduce some basic results and useful technical tools. In Sections 2 and 3, we study the behaviour of linear Koszul duality under a morphism of vector bundles, and under base change. In sections 4 and 5 we deduce that a certain shift of linear Koszul duality is compatible with convolution, in a rather general context, and that it sends the unit (for the convolution product) to the unit. Finally, in section 6 we prove our main result, namely that linear Koszul duality gives a geometric realization of the Iwahori-Matsumoto involution of affine Hecke algebras.

0.3. **Notation.** If X is a variety and $\mathcal{F}$, $\mathcal{G}$ are sheaves of $\mathcal{O}_X$-modules, we denote by $\mathcal{F} \boxplus \mathcal{G}$ the $\mathcal{O}_{X^2}$-module $(p_1)^*\mathcal{F} \oplus (p_2)^*\mathcal{G}$ on X^2, where $p_1, p_2 : X \times X \to X$ are the first and second projections. We use the same notation as in [MR] for the categories of (quasi-coherent) sheaves of dg-modules over a (quasi-coherent) sheaf of dg-algebras. If X is a noetherian scheme and $Y \subseteq X$ is a closed subscheme, we denote by $\mathrm{Coh}_Y(X)$ the full subcategory of $\mathrm{Coh}(X)$ whose

objects are supported set-theoretically on Y. We use similar notation for G-equivariant sheaves (where G is an algebraic group acting on X).

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1. Preliminary results

1.1. **A simple lemma.** Let X be a noetherian scheme, and let $\mathcal{A}$ be a sheaf of dg-algebras on X, bounded and concentrated in non-positive degrees. Assume that $\mathcal{H}^0(\mathcal{A})$ is locally finitely generated as an $\mathcal{O}_X$-algebra, and that $\mathcal{H}(\mathcal{A})$ is locally finitely generated as an $\mathcal{H}^0(\mathcal{A})$-module. Let $\mathcal{D}^{\mathrm{c}}(\mathcal{A})$ be the subcategory of the derived category of quasi-coherent $\mathcal{A}$-dg-modules (the latter being defined as in [MR, 1.1]) whose objects have locally finitely generated cohomology (over $\mathcal{H}(\mathcal{A})$ or, equivalently, over $\mathcal{H}^0(\mathcal{A})$). Let $K(\mathcal{D}^{\mathrm{c}}(\mathcal{A}))$ be its Grothendieck group. Let also $K(\mathcal{H}^0(\mathcal{A}))$ be the Grothendieck group of the abelian category of quasi-coherent, locally finitely generated $\mathcal{H}^0(\mathcal{A})$-modules.

Lemma 1.1.1. *The natural morphism*

$$\begin{cases} K(\mathcal{D}^{\mathrm{c}}(\mathcal{A})) & \to & K(\mathcal{H}^0(\mathcal{A})) \\ [\mathcal{M}] & \mapsto & \sum_{i\in\mathbb{Z}}(-1)^i\cdot[\mathcal{H}^i(\mathcal{M})] \end{cases}$$

is an isomorphism of abelian groups.

Proof. Let us denote by ϕ the morphism of the lemma. Every object of $\mathcal{D}^{\mathrm{c}}(\mathcal{A})$ is isomorphic to the image in the derived category of a bounded $\mathcal{A}$-dg-module. (This follows from the fact that $\mathcal{A}$ is bounded and concentrated in non-positive degrees, using truncation functors, as defined in [MR, 2.1].) So let $\mathcal{M}$ be a bounded $\mathcal{A}$-dg-module, such that $\mathcal{M}^j = 0$ for $j \notin [[a,b]]$ for some integers $a < b$. Let $n = b - a$. Consider the following filtration of $\mathcal{M}$ as an $\mathcal{A}$-dg-module:

$$\{0\} = \mathcal{M}_0 \subset \mathcal{M}_1 \subset \mathcal{M}_2 \subset \cdots \subset \mathcal{M}_n = \mathcal{M},$$

where for $j \in [[0,n]]$ we put

$$\mathcal{M}_j := (\cdots\ 0 \to \mathcal{M}^a \to \cdots \to \mathcal{M}^{a+j-1} \xrightarrow{d^{a+j-1}} \mathrm{Ker}(d^{a+j}) \to 0\ \cdots).$$

Then, in $K(\mathcal{D}^{\mathrm{c}}(\mathcal{A}))$ we have

$$[\mathcal{M}] = \sum_{j=1}^{n} [\mathcal{M}_j/\mathcal{M}_{j-1}] = \sum_{i\in\mathbb{Z}} (-1)^i\cdot[\mathcal{H}^i(\mathcal{M})],$$

where $\mathcal{H}^i(\mathcal{M})$ is considered as an $\mathcal{A}$-dg-module concentrated in degree 0. It follows that the natural morphism $K(\mathcal{H}^0(\mathcal{A})) \to K(\mathcal{D}^{\mathrm{c}}(\mathcal{A}))$, which sends an $\mathcal{H}^0(\mathcal{A})$-module to itself, viewed as an $\mathcal{A}$-dg-module concentrated in degree 0, is an inverse to ϕ. □

1.2. **Functors for G-equivariant sheaves.** In the rest of this section we consider noetherian schemes X, Y over an algebraically closed field $\Bbbk$, endowed with an action of an algebraic group G (over $\Bbbk$), and a G-equivariant morphism $f_0 : X \to Y$. Let $\mathcal{A}$ (respectively $\mathcal{B}$) be a sheaf of $\mathcal{O}_X$-dg-algebras on X (respectively of $\mathcal{O}_Y$-dg-algebras on Y), which is quasi-coherent over $\mathcal{O}_X$ (respectively $\mathcal{O}_Y$), non-positively graded, graded-commutative, and G-equivariant. We assume that the multiplication morphism $\mathcal{A} \otimes_{\mathcal{O}_X} \mathcal{A} \to \mathcal{A}$ and the differential $d : \mathcal{A} \to \mathcal{A}$ are morphisms of complexes of G-equivariant quasi-coherent sheaves on X (and similarly for $\mathcal{B}$).

We denote by $\mathcal{C}_G(X, \mathcal{A})$ the category of G-equivariant quasi-coherent $\mathcal{A}$-dg-modules, and by $\mathcal{D}_G(X, \mathcal{A})$ the associated derived category, and similarly for $\mathcal{B}$-dg-modules (see [MR, 1.1] for details). We denote by $\mathcal{D}^{\mathrm{c}}_G(X, \mathcal{A})$ the full subcategory of $\mathcal{D}_G(X, \mathcal{A})$ whose objects have locally finitely generated cohomology, and similarly for $\mathcal{B}$. We also consider a morphism of G-equivariant dg-algebras $\phi : (f_0)^*\mathcal{B} \to \mathcal{A}$, and the associated morphism of dg-schemes $f : (X, \mathcal{A}) \to (Y, \mathcal{B})$. This morphism induces natural functors

$$f_* : \mathcal{C}_G(X, \mathcal{A}) \to \mathcal{C}_G(Y, \mathcal{B}), \qquad f^* : \mathcal{C}_G(Y, \mathcal{B}) \to \mathcal{C}_G(X, \mathcal{A}).$$

These functors should admit derived functors, which should satisfy nice properties (similar to those of [R2, Section 1]) in a general context. Here for simplicity we will only prove these facts under strong assumptions, which are always satisfied in the situations relevant to us.

We refer to [R2] and [MR] for the definitions of K-flat and K-injective dg-modules (see [Sp] for the original definition), and to [Ke] for generalities on derived functors.

1.3. **Inverse image.** Assume that Y is a quasi-projective variety. We will assume furthermore that for every G-equivariant quasi-coherent $\mathcal{O}_Y$-module $\mathcal{F}$ there exists a G-equivariant quasi-coherent $\mathcal{O}_Y$-module $\mathcal{P}$, which is flat as an $\mathcal{O}_Y$-module, and a surjection of G-equivariant quasi-coherent sheaves $\mathcal{P} \twoheadrightarrow \mathcal{F}$. (This assumption is satisfied in particular if Y is normal, see [CG, Proposition 5.1.26].) As in [R2, Theorem 1.3.5], one deduces:

Lemma 1.3.1. *Let $\mathcal{F}$ be an object of $\mathcal{C}_G(Y, \mathcal{B})$. Then there exists an object $\mathcal{P}$ in $\mathcal{C}_G(Y, \mathcal{B})$, which is K-flat as a $\mathcal{B}$-dg-module, and a quasi-isomorphism of G-equivariant $\mathcal{B}$-dg-modules $\mathcal{P} \xrightarrow{\mathrm{qis}} \mathcal{F}$.*

In particular, it follows from this lemma that f^* admits a derived functor

$$Lf^* : \mathcal{D}_G(Y, \mathcal{B}) \to \mathcal{D}_G(X, \mathcal{A}).$$

Moreover, the following diagram commutes by definition, where the lower arrow is the usual derived inverse image functor (*i.e.* when G is trivial):

$$\begin{array}{ccc} \mathcal{D}_G(Y, \mathcal{B}) & \xrightarrow{Lf^*} & \mathcal{D}_G(X, \mathcal{A}) \\ \downarrow{\scriptstyle \mathrm{For}} & & \downarrow{\scriptstyle \mathrm{For}} \\ \mathcal{D}(Y, \mathcal{B}) & \xrightarrow{Lf^*} & \mathcal{D}(X, \mathcal{A}). \end{array}$$

It is not clear to us under what general assumptions the functor Lf^* restricts to a functor from $\mathcal{D}^{\mathrm{c}}_G(Y, \mathcal{B})$ to $\mathcal{D}^{\mathrm{c}}_G(X, \mathcal{A})$. (This is already not the case for the natural morphism $(X, \mathcal{O}_X) \to (X, \Lambda_{\mathcal{O}_X}(\mathcal{V}))$ induced by the augmentation $\Lambda_{\mathcal{O}_X}(\mathcal{V}) \to \mathcal{O}_X$; here $\mathcal{V}$ is a non-zero locally free sheaf of finite rank over $\mathcal{O}_X$, and $\Lambda_{\mathcal{O}_X}(\mathcal{V})$ is its exterior algebra, endowed with the trivial differential

and with generators in degree -1.) We will prove this property by *ad hoc* arguments each time we need it.

1.4. **Direct image.** In this section, X and Y can be arbitrary noetherian G-schemes (over $\Bbbk$). Let $\mathcal{C}_G^+(X,\mathcal{A})$ be the subcategory of $\mathcal{C}_G(\mathcal{A})$ of bounded below dg-modules. Recall that the category of G-equivariant quasi-coherent sheaves on X has enough injective objects. (This follows from the fact that the averaging functor $\mathrm{Av} : \mathcal{F} \mapsto \alpha_*(p_X)^*\mathcal{F}$ is a right adjoint to the forgetful functor $\mathrm{QCoh}^G(X) \to \mathrm{QCoh}(X)$, hence sends injectives to injectives. Here $\alpha, p_X : G \times X \to X$ are the action map and the projection.) One easily deduces the following lemma, as in [R2, Lemma 1.3.7].

Lemma 1.4.1. *Let $\mathcal{M}$ be an object of $\mathcal{C}_G^+(X,\mathcal{A})$. Then there exists an object $\mathcal{I}$ of $\mathcal{C}_G^+(X,\mathcal{A})$, which is K-injective in the category $\mathcal{C}_G(X,\mathcal{A})$, and a quasi-isomorphism $\mathcal{M} \xrightarrow{\mathrm{qis}} \mathcal{I}$ of G-equivariant, quasi-coherent sheaves of $\mathcal{A}$-dg-modules.*

Assume now that $\mathcal{A}$ is bounded. Let $\mathcal{D}_G^{\mathrm{c}}(X,\mathcal{A})$ be the full subcategory of $\mathcal{D}_G(X,\mathcal{A})$ whose objects have locally finitely generated cohomology. In particular, an object of $\mathcal{D}_G^{\mathrm{c}}(X,\mathcal{A})$ has bounded cohomology, hence is isomorphic to a bounded $\mathcal{A}$-dg-module (use a truncation functor). As K-injective objects of $\mathcal{C}_G(X,\mathcal{A})$ are split on the right for the functor f_*, it follows in particular from Lemma 1.4.1 that the derived functor Rf_* is defined on the subcategory $\mathcal{D}_G^{\mathrm{c}}(X,\mathcal{A})$:

$$Rf_* : \mathcal{D}_G^{\mathrm{c}}(X,\mathcal{A}) \to \mathcal{D}_G(Y,\mathcal{B}).$$

In the rest of this subsection we prove that Rf_* takes values in the subcategory $\mathcal{D}_G^{\mathrm{c}}(Y,\mathcal{B})$, under some assumptions. More precisely we assume that $\mathcal{A}$ is K-flat over $\mathcal{A}^0$, that $\mathcal{H}(\mathcal{A})$ is locally finitely generated over $\mathcal{H}^0(\mathcal{A})$, and that $\mathcal{H}(\mathcal{B})$ is locally finitely generated over $\mathcal{H}^0(\mathcal{B})$. There exist G-schemes A, B, and G-equivariant affine morphisms $p_X : A \to X$, $p_Y : B \to Y$ such that $\mathcal{A}^0 = (p_X)_*\mathcal{O}_A$, $\mathcal{B}^0 = (p_X)_*\mathcal{O}_B$. We assume that A and B are noetherian schemes. The morphism f induces a morphism of schemes $\tilde{f}_0 : A \to B$. As $\mathcal{A}$ is concentrated in non-positive degrees, the morphism $\mathcal{A}^0 \to \mathcal{H}^0(\mathcal{A})$ is surjective. Hence $\mathcal{H}^0(\mathcal{A})$ is the structure sheaf of a (G-stable) closed subscheme $A' \subseteq A$. Our final (and most important) assumption is the following:

$$\text{the restriction } \tilde{f}_0 : A' \to B \text{ is proper.}$$

Lemma 1.4.2. *Under the assumptions above, $Rf_* : \mathcal{D}_G^{\mathrm{c}}(X,\mathcal{A}) \to \mathcal{D}_G(Y,\mathcal{B})$ takes values in the subcategory $\mathcal{D}_G^{\mathrm{c}}(Y,\mathcal{B})$.*

Proof. The natural direct image functor $(p_X)_* : \mathcal{D}\mathrm{QCoh}^G(A) \to \mathcal{D}_G(X,\mathcal{A}^0)$ is an equivalence of categories (because p_X is affine). Hence there is a forgetful functor $\mathrm{For} : \mathcal{D}_G^{\mathrm{c}}(X,\mathcal{A}) \to \mathcal{D}^b\mathrm{Coh}^G(A)$. (The fact that this functor takes values in $\mathcal{D}^b\mathrm{Coh}^G(A)$ follows from the fact that $\mathcal{H}(\mathcal{A})$ is locally finitely generated over $\mathcal{H}^0(\mathcal{A})$.) The same is true for $\mathcal{B}$. As $\mathcal{A}$ is K-flat over $\mathcal{A}^0$, a K-injective $\mathcal{A}$-dg-module is also K-injective over $\mathcal{A}^0$ (see [R2, Lemma 1.3.4]). It follows that the following diagram commutes:

$$\begin{array}{ccc} \mathcal{D}_G^{\mathrm{c}}(X,\mathcal{A}) & \xrightarrow{Rf_*} & \mathcal{D}_G(Y,\mathcal{B}) \\ \downarrow{\scriptstyle \mathrm{For}} & & \downarrow{\scriptstyle \mathrm{For}} \\ \mathcal{D}^b\mathrm{Coh}^G(A) & \xrightarrow{R(\tilde{f}_0)_*} & \mathcal{D}\mathrm{QCoh}^G(B). \end{array}$$

Moreover, as $\mathcal{H}(\mathcal{B})$ is locally finitely generated over $\mathcal{H}^0(\mathcal{B})$, an object $\mathcal{M}$ of $\mathcal{D}_G(Y,\mathcal{B})$ is in the subcategory $\mathcal{D}^{\mathrm{c}}_G(Y,\mathcal{B})$ if and only if $\mathrm{For}(\mathcal{M})$ is in the subcategory $\mathcal{D}^b\mathrm{Coh}^G(B) \subset \mathcal{D}\mathrm{QCoh}^G(B)$. Hence, as the left hand side functor For takes values in the subcategory $\mathcal{D}^b\mathrm{Coh}^G_{A'}(A)$, it is sufficient to prove that the restriction $R(\tilde{f}_0)_* : \mathcal{D}^b\mathrm{Coh}^G_{A'}(A) \to \mathcal{D}\mathrm{QCoh}^G(B)$ takes values in the subcategory $\mathcal{D}^b\mathrm{Coh}^G(B)$. The latter fact follows from the assumption that the restriction $\tilde{f}_0 : A' \to B$ is proper. □

2. Linear Koszul duality and morphisms of vector bundles

2.1. **Definitions.** In this section we consider a smooth quasi-projective variety X over $\Bbbk$ endowed with an action of a $\Bbbk$-algebraic group G. Let E and E' be two vector bundles over X, and let

$$\begin{array}{ccc} E & \xrightarrow{\quad\phi\quad} & E' \\ & \searrow \quad \swarrow & \\ & X & \end{array}$$

be a morphism of vector bundles. Let us stress that the morphism $X \to X$ induced by ϕ is assumed to be Id_X. We consider subbundles $F_1, F_2 \subseteq E$ and $F'_1, F'_2 \subseteq E'$, and assume that

$$\phi(F_1) \subseteq F'_1, \qquad \phi(F_2) \subseteq F'_2.$$

Let $\mathcal{E}$, $\mathcal{F}_1$, $\mathcal{F}_2$, $\mathcal{E}'$, $\mathcal{F}'_1$, $\mathcal{F}'_2$ be the respective sheaves of sections of E, F_1, F_2, E', F'_1, F'_2. We consider X as a $G \times \mathbb{G}_{\mathbf{m}}$-variety, with trivial $\mathbb{G}_{\mathbf{m}}$-action. We also consider E and E' as $G \times \mathbb{G}_{\mathbf{m}}$-equivariant vector bundles, where $t \in \Bbbk^\times$ acts by multiplication by t^{-2} in the fibers. Consider the $G \times \mathbb{G}_{\mathbf{m}}$-equivariant $\mathcal{O}_X$-dg-modules

$$\mathcal{X} := (0 \to \mathcal{F}_1^\perp \to \mathcal{F}_2^\vee \to 0), \qquad \mathcal{X}' := (0 \to (\mathcal{F}'_1)^\perp \to (\mathcal{F}'_2)^\vee \to 0),$$

where the (possibly) non-zero terms are in bidegrees $(-1,2)$ and $(0,2)$, and the differentials are the natural maps. Consider also

$$\mathcal{Y} := (0 \to \mathcal{F}_2 \to \mathcal{E}/\mathcal{F}_1 \to 0), \qquad \mathcal{Y}' := (0 \to \mathcal{F}'_2 \to \mathcal{E}'/\mathcal{F}'_1 \to 0),$$

where the (possibly) non-zero terms are in bidegrees $(-1,-2)$ and $(0,-2)$, and the differentials are the opposite of the natural maps. As in [MR] we will use the following $G \times \mathbb{G}_{\mathbf{m}}$-equivariant (sheaves of) dg-algebras:

$$\begin{aligned} \mathcal{T} &:= \mathrm{Sym}(\mathcal{X}), & \mathcal{T}' &:= \mathrm{Sym}(\mathcal{X}'), \\ \mathcal{R} &:= \mathrm{Sym}(\mathcal{Y}), & \mathcal{R}' &:= \mathrm{Sym}(\mathcal{Y}'), \\ \mathcal{S} &:= \mathrm{Sym}(\mathcal{Y}[-2]), & \mathcal{S}' &:= \mathrm{Sym}(\mathcal{Y}'[-2]), \end{aligned}$$

and the categories

$$\begin{aligned} \mathcal{D}^{\mathrm{c}}_{G\times\mathbb{G}_{\mathbf{m}}}(F_1 \overset{R}{\cap}_E F_2) &:= \mathcal{D}^{\mathrm{c}}_{G\times\mathbb{G}_{\mathbf{m}}}(X, \mathcal{T}), \\ \mathcal{D}^{\mathrm{c}}_{G\times\mathbb{G}_{\mathbf{m}}}(F'_1 \overset{R}{\cap}_{E'} F'_2) &:= \mathcal{D}^{\mathrm{c}}_{G\times\mathbb{G}_{\mathbf{m}}}(X, \mathcal{T}'), \\ \mathcal{D}^{\mathrm{c}}_{G\times\mathbb{G}_{\mathbf{m}}}(F_1^\perp \overset{R}{\cap}_{E^*} F_2^\perp) &:= \mathcal{D}^{\mathrm{c}}_{G\times\mathbb{G}_{\mathbf{m}}}(X, \mathcal{R}), \\ \mathcal{D}^{\mathrm{c}}_{G\times\mathbb{G}_{\mathbf{m}}}((F'_1)^\perp \overset{R}{\cap}_{(E')^*} (F'_2)^\perp) &:= \mathcal{D}^{\mathrm{c}}_{G\times\mathbb{G}_{\mathbf{m}}}(X, \mathcal{R}'). \end{aligned}$$

Then we have linear Koszul duality equivalences (see [MR, Theorem 4.3.1])

$$\kappa : \mathcal{D}^{\mathrm{c}}_{G\times\mathbb{G}_{\mathbf{m}}}(F_1 \overset{R}{\cap}_E F_2) \xrightarrow{\sim} \mathcal{D}^{\mathrm{c}}_{G\times\mathbb{G}_{\mathbf{m}}}(F_1^{\perp} \overset{R}{\cap}_{E^*} F_2^{\perp}),$$
$$\kappa' : \mathcal{D}^{\mathrm{c}}_{G\times\mathbb{G}_{\mathbf{m}}}(F_1' \overset{R}{\cap}_{E'} F_2') \to \mathcal{D}^{\mathrm{c}}_{G\times\mathbb{G}_{\mathbf{m}}}((F_1')^{\perp} \overset{R}{\cap}_{(E')^*} (F_2')^{\perp}).$$

We also define the following "regrading" equivalences as in [MR, 3.5]:

$$\xi : \mathcal{C}_{G\times\mathbb{G}_{\mathbf{m}}}(X, \mathcal{S}) \to \mathcal{C}_{G\times\mathbb{G}_{\mathbf{m}}}(X, \mathcal{R}),$$
$$\xi' : \mathcal{C}_{G\times\mathbb{G}_{\mathbf{m}}}(X, \mathcal{S}') \to \mathcal{C}_{G\times\mathbb{G}_{\mathbf{m}}}(X, \mathcal{R}').$$

We denote similarly the induced equivalences between the various derived categories.

The morphism ϕ induces a morphism of dg-schemes $\Phi : F_1 \overset{R}{\cap}_E F_2 \to F_1' \overset{R}{\cap}_{E'} F_2'$. Consider the (derived) direct image functor

$$R\Phi_* : \mathcal{D}^{\mathrm{c}}_{G\times\mathbb{G}_{\mathbf{m}}}(F_1 \overset{R}{\cap}_E F_2) \to \mathcal{D}_{G\times\mathbb{G}_{\mathbf{m}}}(X, \mathcal{T}').$$

(Note that this functor is just a functor of "restriction of scalars" for the morphism $\mathcal{T}' \to \mathcal{T}$.) If we assume that the induced morphism of schemes between non-derived intersections $F_1 \cap_E F_2 \to F_1' \cap_{E'} F_2'$ is proper, then by Lemma 1.4.2 the functor $R\Phi_*$ takes values in the subcategory $\mathcal{D}^{\mathrm{c}}_{G\times\mathbb{G}_{\mathbf{m}}}(F_1' \overset{R}{\cap}_{E'} F_2')$. We also consider the (derived) inverse image functor

$$L\Phi^* : \mathcal{D}^{\mathrm{c}}_{G\times\mathbb{G}_{\mathbf{m}}}(F_1' \overset{R}{\cap}_{E'} F_2') \to \mathcal{D}_{G\times\mathbb{G}_{\mathbf{m}}}(X, \mathcal{T}).$$

Similarly, ϕ induces a morphism of vector bundles

$$\phi^\vee : (E')^* \to E^*,$$

which satisfies $\phi^\vee((F_i')^\perp) \subset F_i^\perp$ for $i = 1, 2$. Hence the above constructions and results also apply to $\phi^\vee$. We use similar notation.

2.2. **Compatibility.**

Proposition 2.2.1. (i) *Assume that the induced morphism of schemes $F_1 \cap_E F_2 \to F_1' \cap_{E'} F_2'$ is proper. Then $L(\Phi^\vee)^*$ takes values in $\mathcal{D}^{\mathrm{c}}_{G\times\mathbb{G}_{\mathbf{m}}}((F_1')^{\perp} \overset{R}{\cap}_{(E')^*} (F_2')^{\perp})$. Moreover, there is a natural isomorphism of functors $L(\Phi^\vee)^* \circ \kappa \cong \kappa' \circ R\Phi_*$.*

(ii) *Assume that the induced morphism of schemes $(F_1')^{\perp} \cap_{(E')^*} (F_2')^{\perp} \to F_1^{\perp} \cap_{E'} F_2^{\perp}$ is proper. Then $L\Phi^*$ takes values in $\mathcal{D}^{\mathrm{c}}_{G\times\mathbb{G}_{\mathbf{m}}}(F_1 \overset{R}{\cap}_E F_2)$. Moreover, there is a natural isomorphism of functors $\kappa \circ L\Phi^* \cong R(\Phi^\vee)_* \circ \kappa'$.*

In particular, if both assumptions are satisfied, there is a commutative diagram:

$$\begin{array}{ccc}
\mathcal{D}^{\mathrm{c}}_{G\times\mathbb{G}_{\mathbf{m}}}(F_1 \overset{R}{\cap}_E F_2) & \xrightarrow[\sim]{\kappa} & \mathcal{D}^{\mathrm{c}}_{G\times\mathbb{G}_{\mathbf{m}}}(F_1^{\perp} \overset{R}{\cap}_{E^*} F_2^{\perp}) \\
{\scriptstyle L\Phi^*}\uparrow\downarrow{\scriptstyle R\Phi_*} & & {\scriptstyle R(\Phi^\vee)_*}\uparrow\downarrow{\scriptstyle L(\Phi^\vee)^*} \\
\mathcal{D}^{\mathrm{c}}_{G\times\mathbb{G}_{\mathbf{m}}}(F_1' \overset{R}{\cap}_{E'} F_2') & \xrightarrow[\sim]{\kappa'} & \mathcal{D}^{\mathrm{c}}_{G\times\mathbb{G}_{\mathbf{m}}}((F_1')^{\perp} \overset{R}{\cap}_{(E')^*} (F_2')^{\perp}).
\end{array}$$

Proof. We only prove (i). ((ii) can be proved similarly.) Recall that we have seen in Subsection 2.1 that the functor $R\Phi_*$ takes values in $\mathcal{D}^{\mathrm{c}}_{G\times\mathbb{G}_{\mathbf{m}}}(F_1' \overset{R}{\cap}_{E'} F_2')$. In this proof (as in Subsection 2.1) we consider $L(\Phi^\vee)^*$ as a functor from $\mathcal{D}^{\mathrm{c}}_{G\times\mathbb{G}_{\mathbf{m}}}(F_1^\perp \overset{R}{\cap}_{E^*} F_2^\perp)$ to $\mathcal{D}_{G\times\mathbb{G}_{\mathbf{m}}}(X,\mathcal{R}')$. As κ is an equivalence of categories it is enough, to prove both assertions of (i), to check that $L(\Phi^\vee)^*\circ\kappa$ is isomorphic to the composition of $\kappa'\circ R\Phi_*$ and the inclusion $\mathrm{Incl} : \mathcal{D}^{\mathrm{c}}_{G\times\mathbb{G}_{\mathbf{m}}}((F_1')^\perp \overset{R}{\cap}_{(E')^*} (F_2')^\perp) \hookrightarrow \mathcal{D}_{G\times\mathbb{G}_{\mathbf{m}}}(X,\mathcal{R}')$.

To prove this fact, it is sufficient to consider objects of the category $\mathcal{D}^{\mathrm{c}}_{G\times\mathbb{G}_{\mathbf{m}}}(F_1 \overset{R}{\cap}_E F_2)$ satisfying the assumptions of [MR, Lemma 3.7.2]. For such an object $\mathcal{M}$, $\mathrm{Incl}\circ\kappa'\circ R\Phi_*(\mathcal{M})$ is the image in the derived category of the $\mathcal{R}'$-dg-module

$$\xi'\big(\mathcal{S}'\otimes_{\mathcal{O}_X}\mathcal{H}om_{\mathcal{O}_X}(\mathcal{M},\mathcal{O}_X)\big),$$

with differential the sum of four terms: the differential induced by $d_{\mathcal{S}'}$, the one induced by $d_{\mathcal{M}}$, and the two "Koszul differentials". The first Koszul differential is induced by the composition of the natural morphisms

$$\mathcal{O}_X \to \mathcal{F}_2'\otimes_{\mathcal{O}_X}(\mathcal{F}_2')^\vee \to \mathcal{F}_2'\otimes_{\mathcal{O}_X}(\mathcal{F}_2)^\vee, \tag{2.2.2}$$

and the second one is defined similarly using $\mathcal{E}'/\mathcal{F}_1'$.

On the other hand, $L(\Phi^\vee)^*\circ\kappa(\mathcal{M})$ is the image in the derived category of the dg-module

$$\begin{aligned}\mathcal{R}'\otimes_{\mathcal{R}}\xi\big(\mathcal{S}\otimes_{\mathcal{O}_X}\mathcal{H}om_{\mathcal{O}_X}(\mathcal{M},\mathcal{O}_X)\big) &\cong \xi'\big(\mathcal{S}'\otimes_{\mathcal{S}}(\mathcal{S}\otimes_{\mathcal{O}_X}\mathcal{H}om_{\mathcal{O}_X}(\mathcal{M},\mathcal{O}_X))\big)\\ &\cong \xi'\big(\mathcal{S}'\otimes_{\mathcal{O}_X}\mathcal{H}om_{\mathcal{O}_X}(\mathcal{M},\mathcal{O}_X))\big).\end{aligned}$$

Via these identifications, the differential is also the sum of four terms: the differential induced by $d_{\mathcal{S}'}$, the one induced by $d_{\mathcal{M}}$, and the two "Koszul differentials". The first Koszul differential is induced by the composition of the natural morphisms

$$\mathcal{O}_X \to \mathcal{F}_2\otimes_{\mathcal{O}_X}\mathcal{F}_2^\vee \to \mathcal{F}_2'\otimes_{\mathcal{O}_X}\mathcal{F}_2^\vee, \tag{2.2.3}$$

and the second one is defined similarly using $\mathcal{E}/\mathcal{F}_1$.

These two dg-modules are clearly isomorphic (both (2.2.2) and (2.2.3) encode the morphism $\mathcal{F}_2\to\mathcal{F}_2'$ induced by ϕ), which finishes the proof. □

2.3. **Particular case: inclusion of a subbundle.** We will mainly use only a very special case of Proposition 2.2.1, which we state here for future reference. It is the case when $E=E'$, $\phi=\mathrm{Id}$, $F_1=F_1'$ (and F_2' is any subbundle containing F_2). In this case we denote by

$$f : F_1 \overset{R}{\cap}_E F_2 \to F_1 \overset{R}{\cap}_E F_2', \qquad g : F_1^\perp \overset{R}{\cap}_{E^*} (F_2')^\perp \to F_1^\perp \overset{R}{\cap}_{E^*} F_2^\perp$$

the morphisms of dg-schemes induced by $F_2\hookrightarrow F_2'$, $(F_2')^\perp\hookrightarrow F_2^\perp$. The assumption that the morphisms between non-derived intersections are proper is always satisfied here (because these morphisms are closed embeddings). Hence by Proposition 2.2.1 we have functors

$$\begin{aligned}Rf_* : \mathcal{D}^{\mathrm{c}}_{G\times\mathbb{G}_{\mathbf{m}}}(F_1 \overset{R}{\cap}_E F_2) &\to \mathcal{D}^{\mathrm{c}}_{G\times\mathbb{G}_{\mathbf{m}}}(F_1 \overset{R}{\cap}_E F_2'),\\ Lf^* : \mathcal{D}^{\mathrm{c}}_{G\times\mathbb{G}_{\mathbf{m}}}(F_1 \overset{R}{\cap}_E F_2') &\to \mathcal{D}^{\mathrm{c}}_{G\times\mathbb{G}_{\mathbf{m}}}(F_1 \overset{R}{\cap}_E F_2),\end{aligned}$$

and similarly for g. Moreover, the following proposition holds true.

Proposition 2.3.1. *Consider the following diagram:*

$$\begin{array}{ccc} \mathcal{D}^{\mathrm{c}}_{G\times\mathbb{G}_{\mathbf{m}}}(F_1 \overset{R}{\cap}_E F_2) & \xrightarrow[\sim]{\kappa} & \mathcal{D}^{\mathrm{c}}_{G\times\mathbb{G}_{\mathbf{m}}}(F_1^\perp \overset{R}{\cap}_{E^*} F_2^\perp) \\ Lf^* \uparrow\downarrow Rf_* & & Rg_* \uparrow\downarrow Lg^* \\ \mathcal{D}^{\mathrm{c}}_{G\times\mathbb{G}_{\mathbf{m}}}(F_1 \overset{R}{\cap}_E F_2') & \xrightarrow[\sim]{\kappa'} & \mathcal{D}^{\mathrm{c}}_{G\times\mathbb{G}_{\mathbf{m}}}(F_1^\perp \overset{R}{\cap}_{E^*} (F_2')^\perp). \end{array}$$

There are natural isomorphisms of functors

$$\begin{cases} \kappa' \circ Rf_* \cong Lg^* \circ \kappa \\ \kappa \circ Lf^* \cong Rg_* \circ \kappa' \end{cases}.$$

3. Linear Koszul duality and base change

3.1. **Definition of the functors.** Now let X and Y be smooth quasi-projective varieties over $\Bbbk$, endowed with actions of a $\Bbbk$-algebraic group G, and let $\pi : X \to Y$ be a G-equivariant morphism. Consider a G-equivariant vector bundle E on Y, and let $F_1, F_2 \subset E$ be G-equivariant subbundles. Consider also $E^X := E \times_Y X$, which is a G-equivariant vector bundle on X, and the subbundles $F_i^X := F_i \times_Y X \subset E^X$ $(i = 1, 2)$. If $\mathcal{E}$, $\mathcal{F}_1$, $\mathcal{F}_2$ are the respective sheaves of sections of E, F_1, F_2, then $\pi^*\mathcal{E}$, $\pi^*\mathcal{F}_1$, $\pi^*\mathcal{F}_2$ are the sheaves of sections of E^X, F_1^X, F_2^X, respectively. We consider X and Y as $G \times \mathbb{G}_{\mathbf{m}}$-varieties, with trivial $\mathbb{G}_{\mathbf{m}}$-action. We also consider E and E^X as $G \times \mathbb{G}_{\mathbf{m}}$-equivariant vector bundles, where $t \in \Bbbk^\times$ acts by multiplication by t^{-2} in the fibers.

As in [MR] we consider the $G \times \mathbb{G}_{\mathbf{m}}$-equivariant $\mathcal{O}_X$-dg-modules

$$\mathcal{X} := (0 \to \mathcal{F}_1^\perp \to \mathcal{F}_2^\vee \to 0)$$

where the (possibly) non-zero terms are in bidegrees $(-1, 2)$ and $(0, 2)$ and the differential is the natural map, and

$$\mathcal{Y} := (0 \to \mathcal{F}_2 \to \mathcal{E}/\mathcal{F}_1 \to 0)$$

where the (possibly) non-zero terms are in bidegrees $(-1, -2)$ and $(0, -2)$, and the differential is the opposite of the natural map. Consider the following sheaves of dg-algebras:

$$\begin{array}{ll} \mathcal{T}_Y := \mathrm{Sym}(\mathcal{X}), & \mathcal{T}_X := \mathrm{Sym}(\pi^*\mathcal{X}) \cong \pi^*\mathcal{T}_Y, \\ \mathcal{R}_Y := \mathrm{Sym}(\mathcal{Y}), & \mathcal{R}_X := \mathrm{Sym}(\pi^*\mathcal{Y}) \cong \pi^*\mathcal{R}_Y, \\ \mathcal{S}_Y := \mathrm{Sym}(\mathcal{Y}[-2]), & \mathcal{S}_X := \mathrm{Sym}(\pi^*\mathcal{Y}[-2]) \cong \pi^*\mathcal{S}_Y. \end{array}$$

As in [MR, 4.3] we define

$$\begin{aligned} \mathcal{D}^{\mathrm{c}}_{G\times\mathbb{G}_{\mathbf{m}}}(F_1 \overset{R}{\cap}_E F_2) &:= \mathcal{D}^{\mathrm{c}}_{G\times\mathbb{G}_{\mathbf{m}}}(Y, \mathcal{T}_Y), \\ \mathcal{D}^{\mathrm{c}}_{G\times\mathbb{G}_{\mathbf{m}}}(F_1^X \overset{R}{\cap}_{E^X} F_2^X) &:= \mathcal{D}^{\mathrm{c}}_{G\times\mathbb{G}_{\mathbf{m}}}(X, \mathcal{T}_X), \\ \mathcal{D}^{\mathrm{c}}_{G\times\mathbb{G}_{\mathbf{m}}}(F_1^\perp \overset{R}{\cap}_{E^*} F_2^\perp) &:= \mathcal{D}^{\mathrm{c}}_{G\times\mathbb{G}_{\mathbf{m}}}(Y, \mathcal{R}_Y), \\ \mathcal{D}^{\mathrm{c}}_{G\times\mathbb{G}_{\mathbf{m}}}((F_1^X)^\perp \overset{R}{\cap}_{(E^X)^*} (F_2^X)^\perp) &:= \mathcal{D}^{\mathrm{c}}_{G\times\mathbb{G}_{\mathbf{m}}}(X, \mathcal{R}_X). \end{aligned}$$

Then there are equivalences of categories

$$\kappa_X : \mathcal{D}^{\mathrm{c}}_{G\times\mathbb{G}_{\mathbf{m}}}(F_1^X \overset{R}{\cap}_{E^X} F_2^X) \ \xrightarrow{\sim}\ \mathcal{D}^{\mathrm{c}}_{G\times\mathbb{G}_{\mathbf{m}}}((F_1^X)^\perp \overset{R}{\cap}_{(E^X)^*} (F_2^X)^\perp),$$
$$\kappa_Y : \mathcal{D}^{\mathrm{c}}_{G\times\mathbb{G}_{\mathbf{m}}}(F_1 \overset{R}{\cap}_{E} F_2) \ \xrightarrow{\sim}\ \mathcal{D}^{\mathrm{c}}_{G\times\mathbb{G}_{\mathbf{m}}}(F_1^\perp \overset{R}{\cap}_{E^*} F_2^\perp)$$

(see [MR, Theorem 4.3.1]). We also denote by

$$\xi_X : \mathcal{C}_{G\times\mathbb{G}_{\mathbf{m}}}(X, \mathcal{S}_X) \ \xrightarrow{\sim}\ \mathcal{C}_{G\times\mathbb{G}_{\mathbf{m}}}(X, \mathcal{R}_X)$$
$$\xi_Y : \mathcal{C}_{G\times\mathbb{G}_{\mathbf{m}}}(Y, \mathcal{S}_Y) \ \xrightarrow{\sim}\ \mathcal{C}_{G\times\mathbb{G}_{\mathbf{m}}}(Y, \mathcal{R}_Y)$$

the "regrading" equivalences defined in [MR, 3.5]; we denote similarly the induced equivalences between the various derived categories.

The morphism of schemes π induces a morphism of dg-schemes

$$\hat{\pi} : F_1^X \overset{R}{\cap}_{E^X} F_2^X \to F_1 \overset{R}{\cap}_{E} F_2.$$

Via the equivalences above, it is represented by the natural morphism of dg-ringed spaces $(X, \mathcal{T}_X) \to (Y, \mathcal{T}_Y)$.

Lemma 3.1.1. (i) *The functor*

$$L\hat{\pi}^* : \mathcal{D}^{\mathrm{c}}_{G\times\mathbb{G}_{\mathbf{m}}}(F_1 \overset{R}{\cap}_{E} F_2) \ \to\ \mathcal{D}_{G\times\mathbb{G}_{\mathbf{m}}}(X, \mathcal{T}_X)$$

takes values in $\mathcal{D}^{\mathrm{c}}_{G\times\mathbb{G}_{\mathbf{m}}}(F_1^X \overset{R}{\cap}_{E^X} F_2^X)$.

(ii) *Assume π is proper. Then the functor*

$$R\hat{\pi}_* : \mathcal{D}^{\mathrm{c}}_{G\times\mathbb{G}_{\mathbf{m}}}(F_1^X \overset{R}{\cap}_{E^X} F_2^X) \ \to\ \mathcal{D}^{\mathrm{c}}_{G\times\mathbb{G}_{\mathbf{m}}}(Y, \mathcal{T}_Y),$$

takes values in $\mathcal{D}^{\mathrm{c}}_{G\times\mathbb{G}_{\mathbf{m}}}(F_1 \overset{R}{\cap}_{E} F_2)$.

Proof. (ii) follows from Lemma 1.4.2. Let us consider (i). As $\mathcal{T}_X \cong \pi^*\mathcal{T}_Y$ and $\mathcal{T}_Y$ is K-flat over $\mathcal{T}_Y^0 \cong \mathrm{S}(\mathcal{F}_2^\vee)$, the following diagram commutes:

$$\begin{array}{ccc}
\mathcal{D}^{\mathrm{c}}_{G\times\mathbb{G}_{\mathbf{m}}}(F_1 \overset{R}{\cap}_{E} F_2) & \xrightarrow{L\hat{\pi}^*} & \mathcal{D}_{G\times\mathbb{G}_{\mathbf{m}}}(X, \mathcal{T}_X) \\
\big\downarrow \mathrm{For} & & \big\downarrow \mathrm{For} \\
\mathcal{D}^{\mathrm{c}}_{G\times\mathbb{G}_{\mathbf{m}}}(Y, \mathrm{S}(\mathcal{F}_2^\vee)) & \xrightarrow{L\overline{\pi}^*} & \mathcal{D}_{G\times\mathbb{G}_{\mathbf{m}}}(X, \mathrm{S}(\pi^*\mathcal{F}_2^\vee)).
\end{array}$$

On the bottom line, $\overline{\pi}$ is the morphism of dg-schemes $(X, \mathrm{S}(\pi^*\mathcal{F}_2^\vee)) \to (Y, \mathrm{S}(\mathcal{F}_2^\vee))$ induced by π. But $\mathcal{D}^{\mathrm{c}}_{G\times\mathbb{G}_{\mathbf{m}}}(Y, \mathrm{S}(\mathcal{F}_2^\vee))$ is equivalent to $\mathcal{D}^b\mathrm{Coh}^{G\times\mathbb{G}_{\mathbf{m}}}(F_2)$, and $\mathcal{D}_{G\times\mathbb{G}_{\mathbf{m}}}(X, \mathrm{S}(\pi^*\mathcal{F}_2^\vee))$ to $\mathcal{D}\mathrm{QCoh}^{G\times\mathbb{G}_{\mathbf{m}}}(F_2^X)$. Moreover, via these indentifications, $L\overline{\pi}^*$ is the inverse image functor for the morphism $F_2^X \to F_2$ induced by π. Hence $L\overline{\pi}^*$ takes values in $\mathcal{D}^{\mathrm{c}}_{G\times\mathbb{G}_{\mathbf{m}}}(X, \mathrm{S}(\pi^*\mathcal{F}_2^\vee))$. Our result follows. □

Similarly, π induces a morphism of dg-schemes

$$\tilde{\pi} : (F_1^X)^\perp \overset{R}{\cap}_{(E^X)^*} (F_2^X)^\perp \to F_1^\perp \overset{R}{\cap}_{E^*} F_2^\perp,$$

hence a functor

$$L\tilde{\pi}^* : \mathcal{D}^{\mathrm{c}}_{G\times\mathbb{G}_{\mathbf{m}}}(F_1^\perp \overset{R}{\cap}_{E^*} F_2^\perp) \ \to\ \mathcal{D}^{\mathrm{c}}_{G\times\mathbb{G}_{\mathbf{m}}}((F_1^X)^\perp \overset{R}{\cap}_{(E^X)^*} (F_2^X)^\perp),$$

and, if π is proper, also a functor

$$R\tilde{\pi}_* : \mathcal{D}^{\mathrm{c}}_{G\times\mathbb{G}_{\mathbf{m}}}((F_1^X)^\perp \overset{R}{\cap}_{(E^X)^*} (F_2^X)^\perp) \to \mathcal{D}^{\mathrm{c}}_{G\times\mathbb{G}_{\mathbf{m}}}(F_1^\perp \overset{R}{\cap}_{E^*} F_2^\perp).$$

3.2. **Compatibility.** We denote by d_X and d_Y the dimensions of X and Y, respectively, and by ω_X and ω_Y the canonical line bundles on X and Y. If $\mathcal{M}$ is an object of $\mathcal{D}^{\mathrm{c}}_{G\times\mathbb{G}_{\mathbf{m}}}(F_1^X \overset{R}{\cap}_{E^X} F_2^X)$, then $\mathcal{M}\otimes_{\mathcal{O}_X}\omega_X^{-1}$ is also naturally an object of $\mathcal{D}^{\mathrm{c}}_{G\times\mathbb{G}_{\mathbf{m}}}(F_1^X \overset{R}{\cap}_{E^X} F_2^X)$ (with dg-module structure induced by that of $\mathcal{M}$). The same is true for other derived intersections.

Proposition 3.2.1. *Consider the diagram*

$$\begin{array}{ccc}
\mathcal{D}^{\mathrm{c}}_{G\times\mathbb{G}_{\mathbf{m}}}(F_1^X \overset{R}{\cap}_{E^X} F_2^X) & \xrightarrow[\sim]{\kappa_X} & \mathcal{D}^{\mathrm{c}}_{G\times\mathbb{G}_{\mathbf{m}}}((F_1^X)^\perp \overset{R}{\cap}_{(E^X)^*} (F_2^X)^\perp) \\
{\scriptstyle L\hat{\pi}^*}\uparrow\downarrow{\scriptstyle R\hat{\pi}_*} & & {\scriptstyle L\tilde{\pi}^*}\uparrow\downarrow{\scriptstyle R\tilde{\pi}_*} \\
\mathcal{D}^{\mathrm{c}}_{G\times\mathbb{G}_{\mathbf{m}}}(F_1 \overset{R}{\cap}_{E} F_2) & \xrightarrow[\sim]{\kappa_Y} & \mathcal{D}^{\mathrm{c}}_{G\times\mathbb{G}_{\mathbf{m}}}(F_1^\perp \overset{R}{\cap}_{E^*} F_2^\perp).
\end{array}$$

(i) *There is a natural isomorphism of functors*

$$L\tilde{\pi}^* \circ \kappa_Y \cong \kappa_X \circ L\hat{\pi}^*.$$

(ii) *Assume π is proper. Then there is a natural isomorphism of functors*

$$\kappa_Y \circ R\hat{\pi}_* \cong \big(R\tilde{\pi}_* \circ \kappa_X(- \otimes_{\mathcal{O}_X} \omega_X^{-1})\big) \otimes_{\mathcal{O}_Y} \omega_Y^{-1}[d_X - d_Y].$$

The first isomorphism is easy to prove (see below). The second one is not very difficult either, but to give a rigorous proof we need to take some care. We prove it in Subsection 3.3 below.

Proof of (i). It is sufficient to consider a sufficiently large family of well-behaved objects of $\mathcal{D}^{\mathrm{c}}_{G\times\mathbb{G}_{\mathbf{m}}}(F_1 \overset{R}{\cap}_{E} F_2)$. More precisely, any object of $\mathcal{D}^{\mathrm{c}}_{G\times\mathbb{G}_{\mathbf{m}}}(F_1 \overset{R}{\cap}_{E} F_2)$ is isomorphic to the image in the derived category of a $\mathcal{T}_Y$-dg-module which is K-flat over $\mathcal{T}_Y$ and which satisfies the conditions of [MR, Proposition 3.1.1] (see *loc. cit.* and its proof). For such an object $\mathcal{M}$, $L\tilde{\pi}^* \circ \kappa_Y(\mathcal{M})$ is the image in the derived category of the dg-module

$$\pi^* \circ \xi_Y\big(\mathcal{S}_Y \otimes_{\mathcal{O}_Y} \mathcal{H}om_{\mathcal{O}_Y}(\mathcal{M}, \mathcal{O}_Y)\big),$$

endowed with a certain differential. Now we have natural isomorphisms of dg-modules

$$\begin{aligned}
\pi^* \circ \xi_Y\big(\mathcal{S}_Y \otimes_{\mathcal{O}_Y} \mathcal{H}om_{\mathcal{O}_Y}(\mathcal{M}, \mathcal{O}_Y)\big) &\cong \xi_X \circ \pi^*\big(\mathcal{S}_Y \otimes_{\mathcal{O}_Y} \mathcal{H}om_{\mathcal{O}_Y}(\mathcal{M}, \mathcal{O}_Y)\big) \\
&\cong \xi_X\big(\mathcal{S}_X \otimes_{\mathcal{O}_X} \pi^*(\mathcal{H}om_{\mathcal{O}_Y}(\mathcal{M}, \mathcal{O}_Y))\big) \\
&\cong \xi_X\big(\mathcal{S}_X \otimes_{\mathcal{O}_X} \mathcal{H}om_{\mathcal{O}_X}(\pi^*\mathcal{M}, \mathcal{O}_X)\big).
\end{aligned}$$

(On each line, the differential is the natural one, defined as in [MR].) The image in the derived category of the latter dg-module is $\kappa_X \circ L\hat{\pi}^*(\mathcal{M})$; hence these isomorphisms prove statement (i) in Proposition 3.2.1. □

3.3. **Proof of** (ii) **in Proposition 3.2.1.** From now on we assume that π is proper. Recall the notation of [MR, 2.4, 3.2]. We also denote by $\mathcal{C}^+_{\mathrm{gr}}(Y,\,\mathcal{T}_Y)$ the category of $G\times\mathbb{G}_{\mathbf{m}}$-equivariant $\mathcal{T}_Y$-dg-modules, bounded below for the cohomological grading, and by $\mathcal{D}^+_{\mathrm{gr}}(Y,\,\mathcal{T}_Y)$ the corresponding derived category. We use similar notation for the other dg-algebras.

By definition, the category $\mathcal{D}^{\mathrm{c}}_{G\times\mathbb{G}_{\mathbf{m}}}(F_1^\perp \overset{R}{\cap}_{E^*} F_2^\perp)$ is a full subcategory of $\mathcal{D}_{G\times\mathbb{G}_{\mathbf{m}}}(Y,\,\mathcal{R}_Y)$. Hence we can consider κ_Y as a functor from $\mathcal{D}^{\mathrm{c}}_{G\times\mathbb{G}_{\mathbf{m}}}(F_1 \overset{R}{\cap}_E F_2)$ to $\mathcal{D}_{G\times\mathbb{G}_{\mathbf{m}}}(Y,\,\mathcal{R}_Y)$. Then κ_Y is built from the functor $\mathscr{B}_Y : \mathcal{C}^{\nwarrow}_{G\times\mathbb{G}_{\mathbf{m}}}(Y,\,\mathcal{T}_Y)\to\mathcal{C}^+_{G\times\mathbb{G}_{\mathbf{m}}}(Y,\,\mathcal{S}_Y)$ (see [MR, 2.4]), which is a composition of two functors. The first one, which we denote by $\mathscr{B}^1_Y$ here, takes $\mathcal{M}$ in $\mathcal{C}^{\nwarrow}_{G\times\mathbb{G}_{\mathbf{m}}}(Y,\,\mathcal{T}_Y)$, and sends it to $\mathcal{M}^\vee = \mathcal{H}om_{\mathcal{O}_Y}(\mathcal{M},\mathcal{O}_Y)$, which we consider as an object of $\mathcal{C}^+_{\mathrm{gr}}(Y,\,\mathcal{T}_Y)$. The second one, denoted $\mathscr{B}^2_Y$, takes an object $\mathcal{N}$ of $\mathcal{C}^+_{G\times\mathbb{G}_{\mathbf{m}}}(Y,\,\mathcal{T}_Y)$ and sends it to $\mathcal{S}_Y\otimes_{\mathcal{O}_Y}\mathcal{N}$ (with the differential defined in *loc. cit.*), an object of $\mathcal{C}^+_{G\times\mathbb{G}_{\mathbf{m}}}(Y,\,\mathcal{S}_Y)$.

By the arguments of the end of the proof of [MR, Corollary 3.2.1], the functor $\mathscr{B}^2_Y$ is exact, *i.e.* it sends quasi-isomorphisms to quasi-isomorphisms. Hence the derived functor $\overline{\mathscr{B}_Y}$ of $\mathscr{B}_Y$ is the composition of the derived functor

$$\overline{\mathscr{B}^1_Y} : \mathcal{D}^{\nwarrow}_{G\times\mathbb{G}_{\mathbf{m}}}(Y,\,\mathcal{T}_Y)\to\mathcal{D}^+_{\mathrm{gr}}(Y,\,\mathcal{T}_Y)$$

of $\mathscr{B}^1_Y$ (which exists by the arguments of the proof of [MR, Corollary 3.2.1]), and the functor induced by $\mathscr{B}^2_Y$ between the derived categories (which we denote similarly). (Beware that the notation "$\overline{\mathscr{B}}$" in [MR] does not exactly refer to the same functor as here.)

The same remarks apply to the functor κ_X, and we use the same notation (simply replacing Y by X).

Now, let $\mathcal{M}$ in $\mathcal{D}^{\mathrm{c}}_{G\times\mathbb{G}_{\mathbf{m}}}(F_1^X \overset{R}{\cap}_{E^X} F_2^X)$. Then $\kappa_Y\circ R\hat{\pi}_*\mathcal{M}$ is isomorphic to $\xi_Y\circ\mathscr{B}^2_Y\circ\overline{\mathscr{B}^1_Y}(R\hat{\pi}_*\mathcal{M})$. By duality (see [Ha]), there is an isomorphism in $\mathcal{D}^+_{\mathrm{gr}}(Y,\,\mathcal{T}_Y)$:

$$\overline{\mathscr{B}^1_Y}(R\hat{\pi}_*\mathcal{M}) \;\cong\; R\tilde{\pi}_*\big(\overline{\mathscr{B}^1_X}(\mathcal{M})\otimes_{\mathcal{O}_X}\pi^*\omega_Y^{-1}\otimes_{\mathcal{O}_X}\omega_X[d_X-d_Y]\big).$$

Statement (ii) in Proposition 3.2.1 easily follows, using the projection formula.

4. Convolution

4.1. **First definition.** Let X be a smooth projective variety over an algebraically closed field $\Bbbk$, endowed with an action of a $\Bbbk$-algebraic group G. Let V be a finite dimensional G-module, and $F\subset E := V\times X$ a G-equivariant subbundle of the trivial vector bundle with fiber V on X. Consider the diagonal $\Delta V\subset V\times V$. As in Sections 3 and 2 we consider X, E, F, ΔV as $G\times\mathbb{G}_{\mathbf{m}}$-varieties. We denote by $\mathcal{F},\mathcal{E}$ the sheaves of sections of F, E.

We want to define a convolution product on the category

$$\mathcal{D}^{\mathrm{c}}_{G\times\mathbb{G}_{\mathbf{m}}}\big((\Delta V\times X\times X)\overset{R}{\cap}_{E\times E}(F\times F)\big).$$

Let $\mathrm{S}(V^*)\otimes_{\Bbbk}\mathrm{S}(V^*)\otimes_{\Bbbk}\Lambda(V^*)$ be the Koszul resolution of the diagonal $\Delta V\subset V\times V$. Here V^* is identified naturally with the orthogonal of ΔV in $V\times V$, *i.e.* with the anti-diagonal copy of V^* in $V^*\times V^*$. Consider the inverse image of this dg-algebra under the morphism

$$F\times F\hookrightarrow E\times E\to V\times V.$$

This inverse image is a representative for the structure sheaf of the derived intersection $(\Delta V \times X \times X) \overset{R}{\cap}_{E\times E} (F \times F)$. Finally, taking the direct image of this dg-algebra to $X \times X$, we obtain the description

$$\mathcal{D}^{\mathrm{c}}_{G\times\mathbb{G}_{\mathbf{m}}}\big((\Delta V \times X \times X) \overset{R}{\cap}_{E\times E} (F \times F)\big) \;\cong\; \mathcal{D}^{\mathrm{c}}_{G\times\mathbb{G}_{\mathbf{m}}}(X \times X, \mathrm{S}_{\mathcal{O}_{X\times X}}(\mathcal{F}^\vee \boxplus \mathcal{F}^\vee) \otimes_{\Bbbk} \Lambda_{\Bbbk}(V^*))$$

(see [MR, Lemma 4.1.1]).

For $(i, j) = (1, 2)$, $(2, 3)$ or $(1, 3)$ we have the projection $p_{i,j} : X^3 \to X^2$ on the i-th and j-th factors. There are associated morphisms of dg-schemes

$$\widehat{p_{1,2}} : (\Delta V \times X^3) \overset{R}{\cap}_{E\times E\times X} (F \times F \times X) \to (\Delta V \times X^2) \overset{R}{\cap}_{E\times E} (F \times F),$$

$\widehat{p_{2,3}}$, $\widehat{p_{1,3}}$, and functors $L(\widehat{p_{1,2}})^*$, $L(\widehat{p_{2,3}})^*$, $R(\widehat{p_{1,3}})_*$ (see Section 3). For $i = 1, 2, 3$ we also denote by $p_i : X^3 \to X$ the projection on the i-th factor.

Next we consider a bifunctor

(4.1.1)

$$\mathcal{C}_{G\times\mathbb{G}_{\mathbf{m}}}\big(X^3, \mathrm{S}_{\mathcal{O}_{X^3}}(p_{1,2}^*(\mathcal{F}^\vee \boxplus \mathcal{F}^\vee)) \otimes_{\Bbbk} \Lambda(V^*)\big) \times \mathcal{C}_{G\times\mathbb{G}_{\mathbf{m}}}\big(X^3, \mathrm{S}_{\mathcal{O}_{X^3}}(p_{2,3}^*(\mathcal{F}^\vee \boxplus \mathcal{F}^\vee)) \otimes_{\Bbbk} \Lambda(V^*)\big) \to \mathcal{C}_{G\times\mathbb{G}_{\mathbf{m}}}(X^3, \mathrm{S}_{\mathcal{O}_{X^3}}(p_{1,3}^*(\mathcal{F}^\vee \boxplus \mathcal{F}^\vee)) \otimes_{\Bbbk} \Lambda(V^*)).$$

Here, in the first category the morphism $V^* \otimes_{\Bbbk} \mathcal{O}_{X^3} \to p_{1,2}^*(\mathcal{F}^\vee \boxplus \mathcal{F}^\vee)$ involved in the differential is the composition of the anti-diagonal embedding $V^* \to V^* \times V^*$ with the morphism induced by $F \times F \times X \hookrightarrow V \times V \times X^3 \to V \times V$, so that $\mathrm{S}_{\mathcal{O}_{X^3}}(p_{1,2}^*(\mathcal{F}^\vee \boxplus \mathcal{F}^\vee)) \otimes_{\Bbbk} \Lambda(V^*)$ is the structure sheaf of $(\Delta V \times X^3) \overset{R}{\cap}_{E\times E\times X} (F \times F \times X)$. Similarly, the second category corresponds to the dg-scheme $(\Delta V \times X^3) \overset{R}{\cap}_{X\times E\times E} (X \times F \times F)$, and the third one to the dg-scheme $(\Delta V \times X^3) \overset{R}{\cap}_{E\times X\times E} (F \times X \times F)$. The bifunctor (4.1.1) takes the dg-modules $\mathcal{M}_1$ and $\mathcal{M}_2$ to the dg-module $\mathcal{M}_1 \otimes_{\mathrm{S}_{\mathcal{O}_{X^3}}(p_2^*\mathcal{F}^\vee)} \mathcal{M}_2$, where the action of $\mathrm{S}_{\mathcal{O}_{X^3}}(p_{1,3}^*(\mathcal{F}^\vee \boxplus \mathcal{F}^\vee))$ is the natural one (*i.e.* we forget the action of the middle copy of $\mathrm{S}_{\mathcal{O}_X}(\mathcal{F}^\vee)$). To define the action of $\Lambda(V^*)$, we remark that $\mathcal{M}_1 \otimes_{\mathrm{S}_{\mathcal{O}_{X^3}}(p_2^*\mathcal{F}^\vee)} \mathcal{M}_2$ has a natural action of the dg-algebra $\Lambda(V^*) \otimes_{\Bbbk} \Lambda(V^*)$, which restricts to an action of $\Lambda(V^*)$ via the morphism of dg-algebras $\Lambda(V^*) \to \Lambda(V^*) \otimes_{\Bbbk} \Lambda(V^*)$ which sends an element $x \in V^*$ to $x \otimes 1 + 1 \otimes x$.

The bifunctor (4.1.1) has a derived bifunctor (which can be computed by means of K-flat resolutions), which induces a bifunctor $(- \overset{L}{\otimes}_{F^3} -)$:

$$\mathcal{D}^{\mathrm{c}}_{G\times\mathbb{G}_{\mathbf{m}}}\big(X^3, \mathrm{S}_{\mathcal{O}_{X^3}}(p_{1,2}^*(\mathcal{F}^\vee \boxplus \mathcal{F}^\vee)) \otimes_{\Bbbk} \Lambda(V^*)\big) \times \mathcal{D}^{\mathrm{c}}_{G\times\mathbb{G}_{\mathbf{m}}}\big(X^3, \mathrm{S}_{\mathcal{O}_{X^3}}(p_{2,3}^*(\mathcal{F}^\vee \boxplus \mathcal{F}^\vee)) \otimes_{\Bbbk} \Lambda(V^*)\big) \to \mathcal{D}^{\mathrm{c}}_{G\times\mathbb{G}_{\mathbf{m}}}\big(X^3, \mathrm{S}_{\mathcal{O}_{X^3}}(p_{1,3}^*(\mathcal{F}^\vee \boxplus \mathcal{F}^\vee)) \otimes_{\Bbbk} \Lambda(V^*)\big).$$

(This follows from the fact that the projection $\pi_{1,3} : F \times_V F \times_V F \to F \times_V F$ is proper).

Finally, we obtain a convolution product

$$(- * -) : \mathcal{D}^{\mathrm{c}}_{G\times\mathbb{G}_{\mathbf{m}}}\big((\Delta V \times X \times X) \overset{R}{\cap}_{E\times E} (F \times F)\big) \times \mathcal{D}^{\mathrm{c}}_{G\times\mathbb{G}_{\mathbf{m}}}\big((\Delta V \times X \times X) \overset{R}{\cap}_{E\times E} (F \times F)\big) \to \mathcal{D}^{\mathrm{c}}_{G\times\mathbb{G}_{\mathbf{m}}}\big((\Delta V \times X \times X) \overset{R}{\cap}_{E\times E} (F \times F)\big)$$

defined by the formula

$$\mathcal{M}_1 * \mathcal{M}_2 \;:=\; R(\widehat{p_{1,3}})_*\big(L(\widehat{p_{1,2}})^*\mathcal{M}_2 \overset{L}{\otimes}_{F^3} L(\widehat{p_{2,3}})^*\mathcal{M}_1\big).$$

This convolution is associative.

There is a natural $G \times \mathbb{G}_{\mathbf{m}}$-equivariant projection

$$p : (\Delta V \times X \times X) \overset{R}{\cap}_{E \times E} (F \times F) \to F \times F$$

corresponding to the morphism of $\mathcal{O}_{X^2}$-dg-algebras

$$\mathrm{S}_{\mathcal{O}_{X^2}}(\mathcal{F}^\vee \boxplus \mathcal{F}^\vee) \to \mathrm{S}_{\mathcal{O}_{X^2}}(\mathcal{F}^\vee \boxplus \mathcal{F}^\vee) \otimes_{\Bbbk} \Lambda_{\Bbbk}(V^*),$$

and an associated direct image functor Rp_*. The image of Rp_* lies in the full subcategory $\mathcal{D}^b_{\mathrm{prop}}\mathrm{Coh}(F \times F)$ of $\mathcal{D}^b\mathrm{Coh}(F \times F)$ whose objects are the complexes whose support is contained in a subvariety $Z \subset F \times F$ such that both projections $Z \to F$ are proper. This category $\mathcal{D}^b_{\mathrm{prop}}\mathrm{Coh}(F \times F)$ has a natural convolution product (see [R1]), and

$$Rp_* : \mathcal{D}^{\mathrm{c}}_{G \times \mathbb{G}_{\mathbf{m}}}\big((\Delta V \times X \times X) \overset{R}{\cap}_{E \times E} (F \times F)\big) \to \mathcal{D}^b_{\mathrm{prop}}\mathrm{Coh}(F \times F)$$

is compatible with the two convolution products.

4.2. **Alternative definition.** Before studying the compatibility of convolution with linear Koszul duality we give an alternative (and equivalent) definition of the convolution bifunctor. Consider the morphism

$$i : \begin{cases} X^3 & \hookrightarrow & X^4 \\ (x,y,z) & \mapsto & (x,y,y,z) \end{cases},$$

and the vector bundle E^4 over X^4. In Section 3 we have defined a "base change" functor

$$\begin{aligned} L\hat{i}^* : \mathcal{D}^{\mathrm{c}}_{G \times \mathbb{G}_{\mathbf{m}}}\big((\Delta V \times \Delta V \times X^4) \overset{R}{\cap}_{E^4} F^4\big) \\ \to\ \mathcal{D}^{\mathrm{c}}_{G \times \mathbb{G}_{\mathbf{m}}}\big((\Delta V \times \Delta V \times X^3) \overset{R}{\cap}_{E \times (E \times_X E) \times E} (F \times (F \times_X F) \times F)\big). \end{aligned}$$

Next, consider the inclusion of vector subbundles of $E \times (E \times_X E) \times E$

$$F \times F^{\mathrm{diag}} \times F \ \hookrightarrow\ F \times (F \times_X F) \times F,$$

where $F^{\mathrm{diag}} \subset F \times_X F$ is the diagonal copy of F. In Subsection 2.3 we have defined a functor

$$\begin{aligned} Lf^* : \mathcal{D}^{\mathrm{c}}_{G \times \mathbb{G}_{\mathbf{m}}}\big((\Delta V \times \Delta V \times X^3) \overset{R}{\cap}_{E \times (E \times_X E) \times E} (F \times (F \times_X F) \times F)\big) \\ \to\ \mathcal{D}^{\mathrm{c}}_{G \times \mathbb{G}_{\mathbf{m}}}\big((\Delta V \times \Delta V \times X^3) \overset{R}{\cap}_{E \times (E \times_X E) \times E} (F \times F^{\mathrm{diag}} \times F)\big). \end{aligned}$$

Finally, consider the morphism of vector bundles over X^3

$$\phi : E \times (E \times_X E) \times E \cong V^4 \times X^3 \ \to\ E \times X \times E \cong V^2 \times X^3$$

induced by

$$\begin{cases} V^4 & \to & V^2 \\ (a,b,c,d) & \mapsto & (a-b+c,d) \end{cases}.$$

We have $\phi(\Delta V \times \Delta V \times X^3) = \Delta V \times X^3$, and $\phi(F \times F^{\mathrm{diag}} \times F) = F \times X \times F$. In Section 2 we have defined a functor

$$\begin{aligned} R\Phi_* : \mathcal{D}^{\mathrm{c}}_{G \times \mathbb{G}_{\mathbf{m}}}\big((\Delta V \times \Delta V \times X^3) \overset{R}{\cap}_{E \times (E \times_X E) \times E} (F \times F^{\mathrm{diag}} \times F)\big) \\ \to\ \mathcal{D}^{\mathrm{c}}_{G \times \mathbb{G}_{\mathbf{m}}}\big((\Delta V \times X^3) \overset{R}{\cap}_{E \times X \times E} (F \times X \times F)\big). \end{aligned}$$

Consider two objects $\mathcal{M}_1$, $\mathcal{M}_2$ of $\mathcal{D}^{\mathrm{c}}_{G\times\mathbb{G}_{\mathbf{m}}}\big((\Delta V\times X^2)\overset{R}{\cap}_{E\times E}(F\times F)\big)$. The external tensor product $\mathcal{M}_2\boxtimes\mathcal{M}_1$ is naturally an object of the category $\mathcal{D}^{\mathrm{c}}_{G\times\mathbb{G}_{\mathbf{m}}}\big((\Delta V\times\Delta V\times X^4)\overset{R}{\cap}_{E^4}F^4\big)$. Then, with the definitions as above, we clearly have a (bifunctorial) isomorphism

$$\mathcal{M}_1*\mathcal{M}_2\ \cong\ R(\widehat{p_{1,3}})_*\circ R\Phi_*\circ Lf^*\circ L\hat{i}^*(\mathcal{M}_2\boxtimes\mathcal{M}_1) \tag{4.2.1}$$

in $\mathcal{D}^{\mathrm{c}}_{G\times\mathbb{G}_{\mathbf{m}}}\big((\Delta V\times X^2)\overset{R}{\cap}_{E\times E}(F\times F)\big)$.

4.3. **Compatibility with Koszul duality.** Consider the same situation as in Subsections 4.1 and 4.2, and assume in addition that the line bundle ω_X has a G-equivariant square root, *i.e.* there exists a G-equivariant line bundle $\omega_X^{1/2}$ on X such that $(\omega_X^{1/2})^{\otimes 2}\cong\omega_X$. We denote by d_X the dimension of X.

The orthogonal of $F\times F$ in $E\times E$ is $F^\perp\times F^\perp$. On the other hand, the orthogonal of $\Delta V\times X^2$ in $E\times E$ is the anti-diagonal $\overline{\Delta}V^*\times X^2\subset E^*\times E^*$. There is an automorphism of $E\times E$ sending $\overline{\Delta}V^*\times X^2$ to $\Delta V^*\times X^2$, and preserving $F^\perp\times F^\perp$ (namely multiplication by -1 on the second copy of V^*). Hence composing linear Koszul duality

$$\kappa:\mathcal{D}^{\mathrm{c}}_{G\times\mathbb{G}_{\mathbf{m}}}\big((\Delta V\times X\times X)\overset{R}{\cap}_{E\times E}(F\times F)\big)\ \xrightarrow{\sim}\ \mathcal{D}^{\mathrm{c}}_{G\times\mathbb{G}_{\mathbf{m}}}\big((\overline{\Delta}V^*\times X\times X)\overset{R}{\cap}_{E^*\times E^*}(F^\perp\times F^\perp)\big)$$

with the natural equivalence

$$\Xi:\mathcal{D}^{\mathrm{c}}_{G\times\mathbb{G}_{\mathbf{m}}}\big((\overline{\Delta}V^*\times X\times X)\overset{R}{\cap}_{E^*\times E^*}(F^\perp\times F^\perp)\big)\xrightarrow{\sim}\ \mathcal{D}^{\mathrm{c}}_{G\times\mathbb{G}_{\mathbf{m}}}\big((\Delta V^*\times X\times X)\overset{R}{\cap}_{E^*\times E^*}(F^\perp\times F^\perp)\big)$$

provides an equivalence

$$\widetilde{\kappa}:\mathcal{D}^{\mathrm{c}}_{G\times\mathbb{G}_{\mathbf{m}}}\big((\Delta V\times X\times X)\overset{R}{\cap}_{E\times E}(F\times F)\big)\xrightarrow{\sim}\mathcal{D}^{\mathrm{c}}_{G\times\mathbb{G}_{\mathbf{m}}}\big((\Delta V^*\times X\times X)\overset{R}{\cap}_{E^*\times E^*}(F^\perp\times F^\perp)\big).$$

The two categories related by $\widetilde{\kappa}$ are endowed with a convolution product (see Subsection 4.1). We denote by $\mathfrak{K}$ the following composition:

$$\begin{array}{ccc}\mathcal{D}^{\mathrm{c}}_{G\times\mathbb{G}_{\mathbf{m}}}\big((\Delta V\times X\times X)\overset{R}{\cap}_{E\times E}(F\times F)\big) & & \\ \Big\downarrow{\scriptstyle -\otimes\mathcal{O}_{X^2}(\omega_X^{-1/2}\boxtimes\omega_X^{-1/2})[-d_X]} & \searrow^{\mathfrak{K}} & \\ \mathcal{D}^{\mathrm{c}}_{G\times\mathbb{G}_{\mathbf{m}}}\big((\Delta V\times X\times X)\overset{R}{\cap}_{E\times E}(F\times F)\big) & \xrightarrow[\sim]{\widetilde{\kappa}} & \mathcal{D}^{\mathrm{c}}_{G\times\mathbb{G}_{\mathbf{m}}}\big((\Delta V^*\times X\times X)\overset{R}{\cap}_{E^*\times E^*}(F^\perp\times F^\perp)\big).\end{array}$$

The main result of this section is the following proposition. Its proof uses all the results proved so far.

Proposition 4.3.1. *The equivalence $\mathfrak{K}$ is compatible with convolution,* i.e. *for any objects $\mathcal{M}_1$, $\mathcal{M}_2$ of $\mathcal{D}^{\mathrm{c}}_{G\times\mathbb{G}_{\mathbf{m}}}\big((\Delta V\times X\times X)\overset{R}{\cap}_{E\times E}(F\times F)\big)$ there is a bifunctorial isomorphism*

$$\mathfrak{K}(\mathcal{M}_1*\mathcal{M}_2)\ \cong\ \mathfrak{K}(\mathcal{M}_1)*\mathfrak{K}(\mathcal{M}_2)$$

in $\mathcal{D}^{\mathrm{c}}_{G\times\mathbb{G}_{\mathbf{m}}}\big((\Delta V^*\times X\times X)\overset{R}{\cap}_{E^*\times E^*}(F^\perp\times F^\perp)\big)$.

Proof. On the left hand side we use isomorphism (4.2.1). First, consider the natural projection $p_{1,3} : X^3 \to X^2$. In Section 3 we have defined functors

$$R\widehat{p_{1,3}}_* : \mathcal{D}^{\mathrm{c}}_{G\times\mathbb{G}_{\mathbf{m}}}\big((\Delta V \times X^3) \overset{R}{\cap}_{E\times X\times E} (F \times X \times F)\big) \to \mathcal{D}^{\mathrm{c}}_{G\times\mathbb{G}_{\mathbf{m}}}\big((\Delta V \times X^2) \overset{R}{\cap}_{E\times E} (F \times F)\big),$$

$$R\widetilde{p_{1,3}}_* : \mathcal{D}^{\mathrm{c}}_{G\times\mathbb{G}_{\mathbf{m}}}\big((\overline{\Delta}V^* \times X^3) \overset{R}{\cap}_{E^*\times X\times E^*} (F^\perp \times X \times F^\perp)\big) \to \mathcal{D}^{\mathrm{c}}_{G\times\mathbb{G}_{\mathbf{m}}}\big((\overline{\Delta}V^* \times X^2) \overset{R}{\cap}_{E^*\times E^*} (F^\perp \times F^\perp)\big).$$

We denote by

$$\kappa_{1,3} : \mathcal{D}^{\mathrm{c}}_{G\times\mathbb{G}_{\mathbf{m}}}\big((\Delta V \times X^3) \overset{R}{\cap}_{E\times X\times E} (F \times X \times F)\big) \xrightarrow{\sim} \mathcal{D}^{\mathrm{c}}_{G\times\mathbb{G}_{\mathbf{m}}}\big((\overline{\Delta}V^* \times X^3) \overset{R}{\cap}_{E^*\times X\times E^*} (F^\perp \times X \times F^\perp)\big)$$

the linear Koszul duality functor. By Proposition 3.2.1 we have an isomorphism of functors

$$\kappa \circ R\widehat{p_{1,3}}_* \cong \big(R\widetilde{p_{1,3}}_* \circ \kappa_{1,3}(- \otimes_{\mathcal{O}_{X^3}} (\omega_X^{-1})^{\boxtimes 3})\big) \otimes_{\mathcal{O}_{X^2}} (\omega_X^{-1})^{\boxtimes 2}[d_X],$$

i.e. an isomorphism

$$\kappa \circ R\widehat{p_{1,3}}_* \cong R\widetilde{p_{1,3}}_*\big(\kappa_{1,3}(-) \otimes_{\mathcal{O}_{X^3}} (\mathcal{O}_X \boxtimes \omega_X \boxtimes \mathcal{O}_X)\big)[d_X]. \tag{4.3.2}$$

Next consider, as in Subsection 4.2, the inclusion $i : X^3 \hookrightarrow X^4$. In addition to the functor $L\hat{i}^*$, consider also

$$L\tilde{i}^* : \mathcal{D}^{\mathrm{c}}_{G\times\mathbb{G}_{\mathbf{m}}}\big((\overline{\Delta}V^* \times \overline{\Delta}V^* \times X^4) \overset{R}{\cap}_{(E^*)^4} (F^\perp)^4\big) \to \mathcal{D}^{\mathrm{c}}_{G\times\mathbb{G}_{\mathbf{m}}}\big((\overline{\Delta}V^* \times \overline{\Delta}V^* \times X^3) \overset{R}{\cap}_{E^*\times(E^*\times_X E^*)\times E^*} (F^\perp \times (F^\perp \times_X F^\perp) \times F^\perp)\big).$$

We denote by

$$\kappa_4 : \mathcal{D}^{\mathrm{c}}_{G\times\mathbb{G}_{\mathbf{m}}}\big((\Delta V \times \Delta V \times X^4) \overset{R}{\cap}_{E^4} F^4\big) \xrightarrow{\sim} \mathcal{D}^{\mathrm{c}}_{G\times\mathbb{G}_{\mathbf{m}}}\big((\overline{\Delta}V^* \times \overline{\Delta}V^* \times X^4) \overset{R}{\cap}_{(E^*)^4} (F^\perp)^4\big),$$

$$\kappa_3 : \mathcal{D}^{\mathrm{c}}_{G\times\mathbb{G}_{\mathbf{m}}}\big((\Delta V \times \Delta V \times X^3) \overset{R}{\cap}_{E\times(E\times_X E)\times E} (F \times (F \times_X F) \times F)\big) \xrightarrow{\sim} \mathcal{D}^{\mathrm{c}}_{G\times\mathbb{G}_{\mathbf{m}}}\big((\overline{\Delta}V^* \times \overline{\Delta}V^* \times X^3) \overset{R}{\cap}_{E^*\times(E^*\times_X E^*)\times E^*} (F^\perp \times (F^\perp \times_X F^\perp) \times F^\perp)\big)$$

the linear Koszul duality functors. By Proposition 3.2.1 we have an isomorphism of functors

$$L\tilde{i}^* \circ \kappa_4 \cong \kappa_3 \circ L\hat{i}^*. \tag{4.3.3}$$

As in Subsection 4.2 again, consider now the inclusion $F \times F^{\mathrm{diag}} \times F \hookrightarrow F \times (F \times_X F) \times F$, and the induced morphisms of dg-schemes

$$f : (\Delta V \times \Delta V \times X^3) \overset{R}{\cap}_{E\times(E\times_X E)\times E} (F \times F^{\mathrm{diag}} \times F) \to (\Delta V \times \Delta V \times X^3) \overset{R}{\cap}_{E\times(E\times_X E)\times E} (F \times (F \times_X F) \times F),$$

$$g : (\overline{\Delta}V^* \times \overline{\Delta}V^* \times X^3) \overset{R}{\cap}_{E^*\times(E^*\times_X E^*)\times E^*} (F^\perp \times (F^\perp \times_X F^\perp) \times F^\perp) \to (\overline{\Delta}V^* \times \overline{\Delta}V^* \times X^3) \overset{R}{\cap}_{E^*\times(E^*\times_X E^*)\times E^*} (F^\perp \times (F^{\mathrm{diag}})^\perp \times F^\perp).$$

In addition to the functor Lf^*, consider the functor

$$Rg_* : \mathcal{D}^{\mathrm{c}}_{G\times\mathbb{G}_{\mathbf{m}}}\big((\overline{\Delta}V^*\times\overline{\Delta}V^*\times X^3)\overset{R}{\cap}_{E^*\times(E^*\times_X E^*)\times E^*}(F^\perp\times(F^\perp\times_X F^\perp)\times F^\perp)\big)$$
$$\to\ \mathcal{D}^{\mathrm{c}}_{G\times\mathbb{G}_{\mathbf{m}}}\big((\overline{\Delta}V^*\times\overline{\Delta}V^*\times X^3)\overset{R}{\cap}_{E^*\times(E^*\times_X E^*)\times E^*}(F^\perp\times(F^{\mathrm{diag}})^\perp\times F^\perp)\big).$$

We denote by

$$\kappa'_3 : \mathcal{D}^{\mathrm{c}}_{G\times\mathbb{G}_{\mathbf{m}}}\big((\Delta V\times\Delta V\times X^3)\overset{R}{\cap}_{E\times(E\times_X E)\times E}(F\times F^{\mathrm{diag}}\times F)\big)$$
$$\xrightarrow{\sim}\ \mathcal{D}^{\mathrm{c}}_{G\times\mathbb{G}_{\mathbf{m}}}\big((\overline{\Delta}V^*\times\overline{\Delta}V^*\times X^3)\overset{R}{\cap}_{E^*\times(E^*\times_X E^*)\times E^*}(F^\perp\times(F^{\mathrm{diag}})^\perp\times F^\perp)\big)$$

the linear Koszul duality functor. Then by Proposition 2.3.1 we have an isomorphism of functors

$$\kappa'_3\circ Lf^*\ \cong\ Rg_*\circ\kappa_3. \tag{4.3.4}$$

Finally, consider the morphism of vector bundles

$$\phi : E\times(E\times_X E)\times E\to E\times X\times E$$

defined in Subsection 4.2. By Proposition 2.2.1, the dual morphism $\phi^\vee$ induces a functor

$$L(\Phi^\vee)^* : \mathcal{D}^{\mathrm{c}}_{G\times\mathbb{G}_{\mathbf{m}}}\big((\overline{\Delta}V^*\times\overline{\Delta}V^*\times X^3)\overset{R}{\cap}_{E^*\times(E^*\times_X E^*)\times E^*}(F^\perp\times(F^{\mathrm{diag}})^\perp\times F^\perp)\big)$$
$$\to\ \mathcal{D}^{\mathrm{c}}_{G\times\mathbb{G}_{\mathbf{m}}}\big((\overline{\Delta}V^*\times X^3)\overset{R}{\cap}_{E^*\times X\times E^*}(F^\perp\times X\times F^\perp)\big),$$

and we have an isomorphism of functors

$$\kappa_{1,3}\circ R\Phi_*\ \cong\ L(\Phi^\vee)^*\circ\kappa'_3. \tag{4.3.5}$$

Combining isomorphisms (4.2.1), (4.3.2), (4.3.3), (4.3.4) and (4.3.5) we obtain, for $\mathcal{M}_1$ and $\mathcal{M}_2$ in $\mathcal{D}^{\mathrm{c}}_{G\times\mathbb{G}_{\mathbf{m}}}\big((\Delta V\times X^2)\overset{R}{\cap}_{E\times E}(F\times F)\big)$,

$$\begin{aligned}\kappa(\mathcal{M}_1*\mathcal{M}_2)\ &\cong\ \kappa\circ R(\widehat{p_{1,3}})_*\circ R\Phi_*\circ Lf^*\circ L\hat{i}^*(\mathcal{M}_2\boxtimes\mathcal{M}_1)\\ &\cong\ R(\widetilde{p_{1,3}})_*\big(\big(L(\Phi^\vee)^*\circ Rg_*\circ L\tilde{i}^*\circ\kappa_4(\mathcal{M}_2\boxtimes\mathcal{M}_1)\big)\otimes_{\mathcal{O}_{X^3}}(\mathcal{O}_X\boxtimes\omega_X\boxtimes\mathcal{O}_X)\big)[d_X].\end{aligned}$$

It is clear by definition that $\kappa_4(\mathcal{M}_2\boxtimes\mathcal{M}_1)\cong\kappa(\mathcal{M}_2)\boxtimes\kappa(\mathcal{M}_1)$ in $\mathcal{D}^{\mathrm{c}}_{G\times\mathbb{G}_{\mathbf{m}}}\big((\overline{\Delta}V^*\times\overline{\Delta}V^*\times X^4)\overset{R}{\cap}_{(E^*)^4}(F^\perp)^4\big)$. Hence, to finish the proof we only have to check that for $\mathcal{N}_1$, $\mathcal{N}_2$ in $\mathcal{D}^{\mathrm{c}}_{G\times\mathbb{G}_{\mathbf{m}}}\big((\overline{\Delta}V^*\times X^2)\overset{R}{\cap}_{E^*\times E^*}(F^\perp\times F^\perp)\big)$ there is a bifunctorial isomorphism

$$\Xi'\circ L(\Phi^\vee)^*\circ Rg_*\circ L\tilde{i}^*(\mathcal{N}_1\boxtimes\mathcal{N}_2)\ \cong\ (\Xi\mathcal{N}_1)\overset{L}{\otimes}_{(F^\perp)^3}(\Xi\mathcal{N}_2), \tag{4.3.6}$$

where Ξ' is defined similarly as Ξ in the beginning of this subsection, and $(-\overset{L}{\otimes}_{(F^\perp)^3}-)$ is defined as in Subsection 4.1.

The functor $L(\Phi^\vee)^*$ is induced by the morphism of dg-algebras

$$\mathrm{Sym}\big(\mathcal{F}\boxplus\mathcal{F}^{\mathrm{diag}}\boxplus\mathcal{F}\to(V^4/\Delta V\times\Delta V)\otimes_{\Bbbk}\mathcal{O}_{X^3}\big)\ \longrightarrow\ \mathrm{Sym}\big(p_{1,3}^*(\mathcal{F}\boxplus\mathcal{F})\to(V^2/\Delta V)\otimes_{\Bbbk}\mathcal{O}_{X^3}\big)$$

induced by ϕ. There is a natural exact sequence of 2-term complexes of $\mathcal{O}_{X^3}$-modules

$$\begin{pmatrix} p_2^*\mathcal{F}\\ \downarrow\\ V\otimes_{\Bbbk}\mathcal{O}_{X^3}\end{pmatrix}\hookrightarrow\begin{pmatrix}\mathcal{F}\boxplus\mathcal{F}^{\mathrm{diag}}\boxplus\mathcal{F}\\ \downarrow\\ (V^4/\Delta V\times\Delta V)\otimes_{\Bbbk}\mathcal{O}_{X^3}\end{pmatrix}\twoheadrightarrow\begin{pmatrix}p_{1,3}^*(\mathcal{F}\boxplus\mathcal{F})\\ \downarrow\\ (V^2/\Delta V)\otimes_{\Bbbk}\mathcal{O}_{X^3}\end{pmatrix},$$

where the surjection is induced by ϕ, and the bottom arrow of the inclusion is induced by the morphism

$$\begin{cases} V & \to & V^4 \\ v & \mapsto & (0,v,v,0) \end{cases}.$$

On the other hand, the functor Rg_* is induced by the morphism of dg-algebras

$$\mathrm{Sym}\big(\mathcal{F}\boxplus\mathcal{F}^{\mathrm{diag}}\boxplus\mathcal{F}\to(V^4/\Delta V\times\Delta V)\otimes_{\Bbbk}\mathcal{O}_{X^3}\big)$$
$$\longrightarrow\ \mathrm{Sym}\big(\mathcal{F}\boxplus(\mathcal{F}\oplus\mathcal{F})\boxplus\mathcal{F}\to(V^4/\Delta V\times\Delta V)\otimes_{\Bbbk}\mathcal{O}_{X^3}\big),$$

which makes the second dg-algebra a K-flat dg-module over the first one. The isomorphism (4.3.6) follows from these two facts. □

5. Linear Koszul duality and the diagonal

5.1. **Image of the diagonal.** As in Subsection 4.3 we consider the duality

$$\widetilde{\kappa}:\mathcal{D}^{\mathrm{c}}_{G\times\mathbb{G}_{\mathbf{m}}}\big((\Delta V\times X\times X)\overset{R}{\cap}_{E\times E}(F\times F)\big)\ \xrightarrow{\sim}\ \mathcal{D}^{\mathrm{c}}_{G\times\mathbb{G}_{\mathbf{m}}}\big((\Delta V^*\times X\times X)\overset{R}{\cap}_{E^*\times E^*}(F^\perp\times F^\perp)\big)$$

and the functor $\mathfrak{K}$. Let us denote by $q:E^2\to X^2$ the projection. Consider the structure sheaf of the diagonal copy of F in E^2, denoted $\mathcal{O}_{\Delta F}$. Then $q_*\mathcal{O}_{\Delta F}$ is an object of the category

$$\mathcal{D}^{\mathrm{c}}_{G\times\mathbb{G}_{\mathbf{m}}}\big((\Delta V\times X\times X)\overset{R}{\cap}_{E\times E}(F\times F)\big),$$

where the structure of $\mathrm{S}_{\mathcal{O}_{X^2}}(\mathcal{F}^\vee\boxplus\mathcal{F}^\vee)\otimes_{\Bbbk}\Lambda(V^*)$-dg-module is given by the composition of $\mathrm{S}_{\mathcal{O}_{X^2}}(\mathcal{F}^\vee\boxplus\mathcal{F}^\vee)\otimes_{\Bbbk}\Lambda(V^*)\to q_*\mathcal{O}_{F\times_{V}F}$ (projection to the 0-cohomology) and $q_*\mathcal{O}_{F\times_{V}F}\to q_*\mathcal{O}_{\Delta F}$ (restriction). For simplicity, in the rest of this subsection we write $\mathcal{O}_{\Delta F}$ for $q_*\mathcal{O}_{\Delta F}$. Similarly we have an object $\mathcal{O}_{\Delta F^\perp}$ in $\mathcal{D}^{\mathrm{c}}_{G\times\mathbb{G}_{\mathbf{m}}}\big((\Delta V^*\times X\times X)\overset{R}{\cap}_{E^*\times E^*}(F^\perp\times F^\perp)\big)$.

The main result of this section is the following proposition. The idea of its proof is, using isomorphisms of functors proved in Propositions 2.3.1 and 3.2.1, to reduce to an explicit and very easy computation.

Proposition 5.1.1. *We have* $\mathfrak{K}(\mathcal{O}_{\Delta F})\cong\mathcal{O}_{\Delta F^\perp}$.

Proof. Consider the morphism $\Delta:X\hookrightarrow X\times X$ (inclusion of the diagonal). We denote by

$$\kappa_\Delta:\mathcal{D}^{\mathrm{c}}_{G\times\mathbb{G}_{\mathbf{m}}}\big((\Delta V\times X)\overset{R}{\cap}_{E\times_X E}(F\times_X F)\big)\ \xrightarrow{\sim}\ \mathcal{D}^{\mathrm{c}}_{G\times\mathbb{G}_{\mathbf{m}}}\big((\overline{\Delta}V^*\times X)\overset{R}{\cap}_{E^*\times_X E^*}(F^\perp\times_X F^\perp)\big)$$

the linear Koszul duality obtained by base change by Δ (see Section 3). By Proposition 3.2.1, there is an isomorphism of functors

$$\kappa\circ R\hat{\Delta}_*\cong\big(R\tilde{\Delta}_*\circ\kappa_\Delta(-\otimes_{\mathcal{O}_X}\omega_X^{-1})\big)\otimes_{\mathcal{O}_{X\times X}}\omega_{X\times X}^{-1}[-d_X],\tag{5.1.2}$$

where the functors $R\hat{\Delta}_*$ and $R\tilde{\Delta}_*$ are defined as in Subsection 3.1.

Consider the object $\mathrm{S}_{\mathcal{O}_X}(\mathcal{F}^\vee)$ of the category

$$\mathcal{D}^{\mathrm{c}}_{G\times\mathbb{G}_{\mathbf{m}}}\big((\Delta V\times X)\overset{R}{\cap}_{E\times_X E}(F\times_X F)\big)\ \cong\ \mathcal{D}^{\mathrm{c}}_{G\times\mathbb{G}_{\mathbf{m}}}\big(X,\mathrm{Sym}((\overline{\Delta}V^*)\otimes_{\Bbbk}\mathcal{O}_X\to\mathcal{F}^\vee\oplus\mathcal{F}^\vee)\big),$$

where the dg-module structure corresponds to the diagonal inclusion $\mathcal{F}\hookrightarrow\mathcal{F}\oplus\mathcal{F}$. Then by definition $\mathcal{O}_{\Delta F}\cong R\hat{\Delta}_*\,\mathrm{S}_{\mathcal{O}_X}(\mathcal{F}^\vee)$. Hence, using isomorphism (5.1.2),

$$K(\mathcal{O}_{\Delta F})\ \cong\ \Xi\circ\kappa(\mathcal{O}_{\Delta F}\otimes_{\mathcal{O}_{X^2}}(\omega_X^{-1/2}\boxtimes\omega_X^{-1/2})[-d_X])\ \cong\ \Xi\circ R\tilde{\Delta}_*\circ\kappa_\Delta(\mathrm{S}_{\mathcal{O}_X}(\mathcal{F}^\vee)),\tag{5.1.3}$$

where Ξ is defined as in Subsection 4.1.

Now consider the diagonal embedding $F \hookrightarrow F \times_X F$. This inclusion makes F a subbundle of $F \times_X F$. Taking the derived intersection with $\Delta V \times X$ inside $E \times_X E$, we are in the situation of Subsection 2.3. We consider the morphisms of dg-schemes[1]

$$f : (\Delta V \times X) \overset{R}{\cap}_{E \times_X E} F \ \to \ (\Delta V \times X) \overset{R}{\cap}_{E \times_X E} (F \times_X F),$$
$$g : (\overline{\Delta} V^* \times X) \overset{R}{\cap}_{E^* \times_X E^*} (F^\perp \times_X F^\perp) \ \to \ (\overline{\Delta} V^* \times X) \overset{R}{\cap}_{E^* \times_X E^*} F^{\perp, E \times_X E},$$

and the diagram:

$$\begin{array}{ccc} \mathcal{D}^{\mathrm{c}}_{G \times \mathbb{G}_{\mathbf{m}}}((\Delta V \times X) \overset{R}{\cap}_{E \times_X E} F) & \xrightarrow[\sim]{\kappa_F} & \mathcal{D}^{\mathrm{c}}_{G \times \mathbb{G}_{\mathbf{m}}}((\overline{\Delta} V^* \times X) \overset{R}{\cap}_{E^* \times_X E^*} F^{\perp, E \times_X E}) \\ {\scriptstyle Lf^*} \uparrow \downarrow {\scriptstyle Rf_*} & & {\scriptstyle Rg_*} \uparrow \downarrow {\scriptstyle Lg^*} \\ \mathcal{D}^{\mathrm{c}}_{G \times \mathbb{G}_{\mathbf{m}}}((\Delta V \times X) \overset{R}{\cap}_{E \times_X E} (F \times_X F)) & \xrightarrow[\sim]{\kappa_\Delta} & \mathcal{D}^{\mathrm{c}}_{G \times \mathbb{G}_{\mathbf{m}}}((\overline{\Delta} V^* \times X) \overset{R}{\cap}_{E^* \times_X E^*} (F^\perp \times_X F^\perp)). \end{array}$$

(Here, $F^{\perp, E \times_X E}$ is the orthogonal of F as a diagonal subbundle of $E \times_X E$.) The structure (dg-)sheaf of $(\Delta V \times X) \overset{R}{\cap}_{E \times_X E} F$ is $\Lambda(V^*) \otimes_{\Bbbk} \mathrm{S}_{\mathcal{O}_X}(\mathcal{F}^\vee)$, with trivial differential (because $F \subset \Delta V \times X$). In particular, $\mathrm{S}_{\mathcal{O}_X}(\mathcal{F}^\vee)$ is also an object of the top left category in the diagram, which we denote by $\mathcal{O}_F$. Then, by definition, $Rf_*\mathcal{O}_F$ is the object $\mathrm{S}_{\mathcal{O}_X}(\mathcal{F}^\vee)$ of (5.1.3). By Proposition 2.3.1, there is an isomorphism of functors

$$\kappa_\Delta \circ Rf_* \cong Lg^* \circ \kappa_F.$$

In particular we have $\kappa_\Delta(\mathrm{S}_{\mathcal{O}_X}(\mathcal{F}^\vee)) \cong Lg^* \circ \kappa_F(\mathcal{O}_F)$.

Now direct computation shows that $Lg^* \circ \kappa_F(\mathcal{O}_F)$ identifies with the structure sheaf of the anti-diagonal copy $\overline{\Delta} F^\perp \subset F^\perp \times_X F^\perp$. (The latter is naturally an object of the category $\mathcal{D}^{\mathrm{c}}_{G \times \mathbb{G}_{\mathbf{m}}}((\overline{\Delta} V^* \times X) \overset{R}{\cap}_{E^* \times_X E^*} (F^\perp \times_X F^\perp))$, for the same reasons as above.) One easily deduces, using isomorphism (5.1.3), that $\mathfrak{K}(\mathcal{O}_{\Delta F}) \cong \mathcal{O}_{\Delta F^\perp}$. □

5.2. **Image of line bundles on the diagonal.** From Proposition 5.1.1 one immediately deduces the following result.

Corollary 5.2.1. *Let $\mathcal{L}$ be a G-equivariant line bundle on X. Then $\mathcal{O}_{\Delta F} \otimes_{\mathcal{O}_X} \mathcal{L}$ is naturally an object of $\mathcal{D}^{\mathrm{c}}_{G \times \mathbb{G}_{\mathbf{m}}}\big((\Delta V \times X \times X) \overset{R}{\cap}_{E \times E} (F \times F)\big)$. We have $\mathfrak{K}(\mathcal{O}_{\Delta F} \otimes_{\mathcal{O}_X} \mathcal{L}) \cong \mathcal{O}_{\Delta F^\perp} \otimes_{\mathcal{O}_X} \mathcal{L}^\vee$.*

6. Linear Koszul duality and Iwahori-Matsumoto involution

6.1. **Reminder on affine Hecke algebras.** Let G be a connected, simply-connected, complex semi-simple algebraic group. Let $T \subset B \subset G$ be a torus and a Borel subgroup of G. Let also $\mathfrak{t} \subset \mathfrak{b} \subset \mathfrak{g}$ be their Lie algebras. Let U be the unipotent radical of B, and let $\mathfrak{n}$ be its Lie algebra. Let $\mathcal{B} := G/B$ be the flag variety of G. Consider the Springer variety $\widetilde{\mathcal{N}}$ and the Grothendieck resolution $\widetilde{\mathfrak{g}}$ defined as follows:

$$\widetilde{\mathcal{N}} := \{(X, gB) \in \mathfrak{g}^* \times \mathcal{B} \mid X_{|g \cdot \mathfrak{b}} = 0\}, \qquad \widetilde{\mathfrak{g}} := \{(X, gB) \in \mathfrak{g}^* \times \mathcal{B} \mid X_{|g \cdot \mathfrak{n}} = 0\}.$$

[1]Beware that f and g do *not* exactly denote the same morphisms as in Subsection 4.2.

(The variety $\widetilde{\mathcal{N}}$ is naturally isomorphic to the cotangent bundle of $\mathcal{B}$.) The varieties $\widetilde{\mathcal{N}}$ and $\widetilde{\mathfrak{g}}$ are subbundles of the trivial vector bundle $\mathfrak{g}^* \times \mathcal{B}$ over $\mathcal{B}$. In particular, there are natural maps $\widetilde{\mathcal{N}} \to \mathfrak{g}^*$ and $\widetilde{\mathfrak{g}} \to \mathfrak{g}^*$. Let us consider the varieties

$$Z := \widetilde{\mathcal{N}} \times_{\mathfrak{g}^*} \widetilde{\mathcal{N}}, \qquad \mathcal{Z} := \widetilde{\mathfrak{g}} \times_{\mathfrak{g}^*} \widetilde{\mathfrak{g}}.$$

There is a natural action of $G \times \mathbb{G}_{\mathbf{m}}$ on $\mathfrak{g}^* \times \mathcal{B}$, where (g,t) acts via:

$$(g,t) \cdot (X, hB) \ := \ (t^{-2}(g \cdot X), ghB).$$

The subbundles $\widetilde{\mathcal{N}}$ and $\widetilde{\mathfrak{g}}$ are $G \times \mathbb{G}_{\mathbf{m}}$-stable.

Let R be the root system of G, R^+ the positive roots (chosen as the weights of $\mathfrak{g}/\mathfrak{b}$), Φ the associated set of simple roots, $\mathbb{X}$ the weights of R (which naturally identify with the group of characters of T). Let also W be the Weyl group of R (or of (G,T)). For $\alpha \in \Phi$ we denote by $s_\alpha \in W$ the corresponding simple reflection. For $\alpha, \beta \in \Phi$, we let $n_{\alpha,\beta}$ be the order of $s_\alpha s_\beta$ in W. Then the (extended) affine Hecke algebra $\mathcal{H}_{\mathrm{aff}}$ associated to these data is the $\mathbb{Z}[v, v^{-1}]$-algebra generated by elements $\{T_\alpha,\ \alpha \in \Phi\} \cup \{\theta_x,\ x \in \mathbb{X}\}$, with defining relations

(i) $T_\alpha T_\beta \cdots = T_\beta T_\alpha \cdots$ ($n_{\alpha,\beta}$ elements on each side)
(ii) $\theta_0 = 1$
(iii) $\theta_x \theta_y = \theta_{x+y}$
(iv) $T_\alpha \theta_x = \theta_x T_\alpha$ if $s_\alpha(x) = x$
(v) $\theta_x = T_\alpha \theta_{x-\alpha} T_\alpha$ if $s_\alpha(x) = x - \alpha$
(vi) $(T_\alpha + v^{-1})(T_\alpha - v) = 0$

for $\alpha, \beta \in \Phi$ and $x, y \in \mathbb{X}$ (see *e.g.* [CG], [Lu]).

We will be interested in the *Iwahori-Matsumoto involution* IM of $\mathcal{H}_{\mathrm{aff}}$. This is the involution of $\mathbb{Z}[v, v^{-1}]$-algebra of $\mathcal{H}_{\mathrm{aff}}$ defined on the generators by

$$\begin{cases} \mathrm{IM}(T_\alpha) &= -T_\alpha^{-1}, \\ \mathrm{IM}(\theta_x) &= \theta_{-x}. \end{cases}$$

For $\alpha \in \Phi$ we also consider $t_\alpha := v \cdot T_\alpha$. Then we have $\mathrm{IM}(t_\alpha) = -q(t_\alpha)^{-1}$, with $q = v^2$.

Let $\alpha \in \Phi$. Let $P_\alpha \supset B$ be the minimal standard parabolic subgroup associated to α, let $\mathfrak{p}_\alpha$ be its Lie algebra, and let $\mathcal{P}_\alpha := G/P_\alpha$ be the associated partial flag variety. We define the following $G \times \mathbb{G}_{\mathbf{m}}$-subvariety of Z:

$$Y_\alpha := \{(X, g_1 B, g_2 B) \in \mathfrak{g}^* \times (\mathcal{B} \times_{\mathcal{P}_\alpha} \mathcal{B}) \mid X_{|g \cdot \mathfrak{p}_\alpha} = 0\}.$$

For any variety $X \to \mathcal{B}$ over $\mathcal{B}$ and for $x \in \mathbb{X}$, we denote by $\mathcal{O}_X(x)$ the inverse image to X of the line bundle on $\mathcal{B}$ associated to x. We use a similar notation for varieties over $\mathcal{B} \times \mathcal{B}$.

There is a natural isomorphism of $\mathbb{Z}[v, v^{-1}]$-algebras

$$\mathcal{H}_{\mathrm{aff}} \ \xrightarrow{\sim} K^{G \times \mathbb{G}_{\mathbf{m}}}(Z), \tag{6.1.1}$$

where $K^{G \times \mathbb{G}_{\mathbf{m}}}(Z)$ is endowed with the convolution product associated to the embedding $Z \subset \widetilde{\mathcal{N}} \times \widetilde{\mathcal{N}}$ (see [CG], [Lu]). It is defined by

$$\begin{cases} T_\alpha &\mapsto -v^{-1}[\mathcal{O}_{Y_\alpha}(-\rho, \rho - \alpha)] - v^{-1}[\Delta_* \mathcal{O}_{\widetilde{\mathcal{N}}}] \\ \theta_x &\mapsto [\Delta_* \mathcal{O}_{\widetilde{\mathcal{N}}}(x)] \end{cases}$$

for $\alpha \in \Phi$ and $x \in \mathbb{X}$. Here, Δ is the diagonal embedding, and for $\mathcal{F}$ in $\mathrm{Coh}^{G\times\mathbb{G}_{\mathbf{m}}}(Z)$ we denote by $[\mathcal{F}]$ its class in K-theory. Moreover, the action of v is induced by the functor $\langle 1\rangle : \mathrm{Coh}^{G\times\mathbb{G}_{\mathbf{m}}}(Z) \to \mathrm{Coh}^{G\times\mathbb{G}_{\mathbf{m}}}(Z)$ of tensoring with the one-dimensional $\mathbb{G}_{\mathbf{m}}$-module given by $\mathrm{Id}_{\mathbb{G}_{\mathbf{m}}}$.

For $\alpha \in \Phi$, let

$$\widetilde{\mathfrak{g}}_\alpha := \{(X, gP_\alpha) \in \mathfrak{g}^* \times \mathcal{P}_\alpha \mid X_{|g\cdot\mathfrak{p}_\alpha^{\mathrm{n}}} = 0\},$$

where $\mathfrak{p}_\alpha^{\mathrm{n}}$ is the nilpotent radical of $\mathfrak{p}_\alpha$. There is a natural morphism $\widetilde{\mathfrak{g}} \to \widetilde{\mathfrak{g}}_\alpha$.

Consider the embedding of smooth varieties $i : \widetilde{\mathcal{N}} \times \widetilde{\mathfrak{g}} \hookrightarrow \widetilde{\mathfrak{g}} \times \widetilde{\mathfrak{g}}$. Associated to this morphism, there a morphism of "restriction with supports" in K-theory

$$\eta : K^{G\times\mathbb{G}_{\mathbf{m}}}(\mathcal{Z}) \to K^{G\times\mathbb{G}_{\mathbf{m}}}(Z)$$

(see [CG, p. 246]). As above for $K^{G\times\mathbb{G}_{\mathbf{m}}}(Z)$, convolution endows $K^{G\times\mathbb{G}_{\mathbf{m}}}(\mathcal{Z})$ with the structure of a $\mathbb{Z}[v, v^{-1}]$-algebra. (Here we use the embedding $\mathcal{Z} \subset \widetilde{\mathfrak{g}} \times \widetilde{\mathfrak{g}}$ to define the product.) The following result is well-known. As we could not find a reference, we include a proof. Recall that if X is a scheme and $Y \subset X$ a closed subscheme, $\mathrm{Coh}_Y(X)$ denotes the category of $\mathcal{O}_X$-coherent sheaves supported set-theoretically on Y.

Lemma 6.1.2. *The morphism η is an isomorphism of $\mathbb{Z}[v, v^{-1}]$-algebras.*

Proof. Let us denote by $j : \widetilde{\mathcal{N}} \times \widetilde{\mathcal{N}} \hookrightarrow \widetilde{\mathcal{N}} \times \widetilde{\mathfrak{g}}$ and $k : \widetilde{\mathcal{N}} \hookrightarrow \widetilde{\mathfrak{g}}$ the embeddings. Let also Γ_k be the graph of k. Then η is the composition of the morphism in K-theory induced by the functor

$$Lj^* : \mathcal{D}^b\mathrm{Coh}_{\mathcal{Z}}(\widetilde{\mathfrak{g}} \times \widetilde{\mathfrak{g}}) \to \mathcal{D}^b\mathrm{Coh}_Z(\widetilde{\mathcal{N}} \times \widetilde{\mathfrak{g}})$$

and by the inverse of the morphism induced by

$$i_* : \mathcal{D}^b\mathrm{Coh}_Z(\widetilde{\mathcal{N}} \times \widetilde{\mathcal{N}}) \to \mathcal{D}^b\mathrm{Coh}_Z(\widetilde{\mathcal{N}} \times \widetilde{\mathfrak{g}}).$$

(It is well known that i_* induces an isomorphism in K-theory.) By [R1, Lemma 1.2.3], i_* is the product on the left (for convolution) by the kernel $\mathcal{O}_{\Gamma_k}$. By similar arguments, Lj^* is the product on the right by the kernel $\mathcal{O}_{\Gamma_k}$. It follows from these observations that η is a morphism of $\mathbb{Z}[v, v^{-1}]$-algebras.

Then we observe that Z and $\mathcal{Z}$ have compatible cellular fibrations (in the sense of [CG, 5.5]), defined using the partition of $\mathcal{B}\times\mathcal{B}$ in G-orbits. The stratas in Z are the transverse intersections of those of $\mathcal{Z}$ with $\widetilde{\mathcal{N}} \times \widetilde{\mathfrak{g}} \subset \widetilde{\mathfrak{g}} \times \widetilde{\mathfrak{g}}$. It follows, using the arguments of [CG, 6.2], that η is an isomorphism of $\mathbb{Z}[v, v^{-1}]$-modules, completing the proof. □

It follows in particular from this lemma that there is also an isomorphism

$$\mathcal{H}_{\mathrm{aff}} \xrightarrow{\sim} K^{G\times\mathbb{G}_{\mathbf{m}}}(\mathcal{Z}),$$

which satisfies

$$\begin{cases} T_\alpha & \mapsto & -v^{-1}[\mathcal{O}_{\widetilde{\mathfrak{g}}\times_{\widetilde{\mathfrak{g}}_\alpha}\widetilde{\mathfrak{g}}}] + v[\Delta_*\mathcal{O}_{\widetilde{\mathfrak{g}}}] \\ \theta_x & \mapsto & [\Delta_*\mathcal{O}_{\widetilde{\mathfrak{g}}}(x)] \end{cases}$$

(see *e.g.* [R1] for details).

Finally, we define $N := \#(R^+) = \dim(\mathcal{B})$.

6.2. **Main result.** From now on we consider a very special case of linear Koszul duality, namely the situation of Sections 4 and 5 with $X = \mathcal{B}$, $V = \mathfrak{g}^*$ and $F = \widetilde{\mathcal{N}}$. We identify $V^* = \mathfrak{g}$ with $\mathfrak{g}^*$ using the Killing form. Then $F^\perp$ identifies with $\widetilde{\mathfrak{g}}$. We obtain an equivalence

$$\widetilde{\kappa} : \mathcal{D}^{\mathrm{c}}_{G\times\mathbb{G}_{\mathbf{m}}}\big((\Delta\mathfrak{g}^*\times\mathcal{B}\times\mathcal{B})\overset{R}{\cap}_{(\mathfrak{g}^*\times\mathcal{B})^2}(\widetilde{\mathcal{N}}\times\widetilde{\mathcal{N}})\big) \xrightarrow{\sim} \mathcal{D}^{\mathrm{c}}_{G\times\mathbb{G}_{\mathbf{m}}}\big((\Delta\mathfrak{g}^*\times\mathcal{B}\times\mathcal{B})\overset{R}{\cap}_{(\mathfrak{g}^*\times\mathcal{B})^2}(\widetilde{\mathfrak{g}}\times\widetilde{\mathfrak{g}})\big),$$

and its shift $\mathfrak{K}$. Here the actions of $\mathbb{G}_{\mathbf{m}}$ on $\mathfrak{g}^*$ are not the same on the two sides. (They are "inverse", *i.e.* each one is the composition of the other one with $t \mapsto t^{-1}$.) We denote by κ_{IM} the composition of $\mathfrak{K}$ with the auto-equivalence of $\mathcal{D}^{\mathrm{c}}_{G\times\mathbb{G}_{\mathbf{m}}}\big((\Delta\mathfrak{g}^*\times\mathcal{B}\times\mathcal{B})\overset{R}{\cap}_{(\mathfrak{g}^*\times\mathcal{B})^2}(\widetilde{\mathfrak{g}}\times\widetilde{\mathfrak{g}})\big)$ which inverts the $\mathbb{G}_{\mathbf{m}}$-action. (In the realization as $\mathbb{G}_{\mathbf{m}}$-equivariant dg-modules on $\mathcal{B}\times\mathcal{B}$, this amounts to inverting the internal grading.)

By Lemma 1.1.1, the Grothendieck group of the triangulated category $\mathcal{D}^{\mathrm{c}}_{G\times\mathbb{G}_{\mathbf{m}}}\big((\Delta\mathfrak{g}^*\times\mathcal{B}\times\mathcal{B})\overset{R}{\cap}_{(\mathfrak{g}^*\times\mathcal{B})^2}(\widetilde{\mathcal{N}}\times\widetilde{\mathcal{N}})\big)$ is naturally isomorphic to $K^{G\times\mathbb{G}_{\mathbf{m}}}(Z)$, hence to the affine Hecke algebra $\mathcal{H}_{\mathrm{aff}}$ (see (6.1.1)). Similarly[2], the Grothendieck group of the category $\mathcal{D}^{\mathrm{c}}_{G\times\mathbb{G}_{\mathbf{m}}}\big((\Delta\mathfrak{g}^*\times\mathcal{B}\times\mathcal{B})\overset{R}{\cap}_{(\mathfrak{g}^*\times\mathcal{B})^2}(\widetilde{\mathfrak{g}}\times\widetilde{\mathfrak{g}})\big)$ is isomorphic to $K^{G\times\mathbb{G}_{\mathbf{m}}}(\mathcal{Z})$, hence also to $\mathcal{H}_{\mathrm{aff}}$ (see Lemma 6.1.2). Let us consider the automorphism $[\kappa_{\mathrm{IM}}] : \mathcal{H}_{\mathrm{aff}} \to \mathcal{H}_{\mathrm{aff}}$ induced by κ_{IM}.

In the presentation of $\mathcal{H}_{\mathrm{aff}}$ using the generators t_α and θ_x, the scalars appearing in the relations are polynomials in $q = v^2$. Hence we can define the involution ι of $\mathcal{H}_{\mathrm{aff}}$ (as an algebra) that fixes the t_α's and θ_x's, and sends v to $-v$. Our main result is the following.

Theorem 6.2.1. *The automorphism* $[\kappa_{\mathrm{IM}}]$ *of* $\mathcal{H}_{\mathrm{aff}}$ *is the composition of the Iwahori-Matsumoto involution* IM *and the involution* ι*:*

$$[\kappa_{\mathrm{IM}}] \;=\; \iota \circ \mathrm{IM}.$$

We will prove this theorem in Subsection 6.3. Before that, we need one more preliminary result.

Let α be a simple root. The coherent sheaf $\mathcal{O}_{Y_\alpha}(\rho-\alpha,-\rho)$ on Z has a natural structure of $G\times\mathbb{G}_{\mathbf{m}}$-equivariant dg-module over $\mathrm{Sym}\big(\overline{\Delta}\mathfrak{g}\otimes_{\Bbbk}\mathcal{O}_{\mathcal{B}\times\mathcal{B}}\to\mathcal{T}_{\mathcal{B}}\boxplus\mathcal{T}_{\mathcal{B}}\big)$, hence defines an object in the category $\mathcal{D}^{\mathrm{c}}_{G\times\mathbb{G}_{\mathbf{m}}}\big((\Delta\mathfrak{g}^*\times\mathcal{B}\times\mathcal{B})\overset{R}{\cap}_{(\mathfrak{g}^*\times\mathcal{B})^2}(\widetilde{\mathcal{N}}\times\widetilde{\mathcal{N}})\big)$. Similarly, $\mathcal{O}_{\widetilde{\mathfrak{g}}\times_{\widetilde{\mathfrak{g}}_\alpha}\widetilde{\mathfrak{g}}}$ is naturally an object of $\mathcal{D}^{\mathrm{c}}_{G\times\mathbb{G}_{\mathbf{m}}}\big((\Delta\mathfrak{g}^*\times\mathcal{B}\times\mathcal{B})\overset{R}{\cap}_{(\mathfrak{g}^*\times\mathcal{B})^2}(\widetilde{\mathfrak{g}}\times\widetilde{\mathfrak{g}})\big)$. The strategy of the proof of the next proposition is the same as that of Proposition 5.1.1.

Proposition 6.2.2. *We have* $\mathfrak{K}(\mathcal{O}_{Y_\alpha}(\rho-\alpha,-\rho)) \;\cong\; \mathcal{O}_{\widetilde{\mathfrak{g}}\times_{\widetilde{\mathfrak{g}}_\alpha}\widetilde{\mathfrak{g}}}[1]$.

Proof. Consider the inclusion $i : \mathcal{X}_\alpha := \mathcal{B}\times_{\mathcal{P}_\alpha}\mathcal{B} \hookrightarrow \mathcal{B}\times\mathcal{B}$. Applying the constructions of Section 3, we obtain the diagram

$$\begin{array}{ccc}
\mathcal{D}^{\mathrm{c}}_{G\times\mathbb{G}_{\mathbf{m}}}((\Delta\mathfrak{g}^*\times\mathcal{X}_\alpha)\overset{R}{\cap}_{(\mathfrak{g}^*)^2\times\mathcal{X}_\alpha}\widetilde{\mathcal{N}}\times_{\mathcal{P}_\alpha}\widetilde{\mathcal{N}}) & \xrightarrow[\sim]{\kappa_\alpha} & \mathcal{D}^{\mathrm{c}}_{G\times\mathbb{G}_{\mathbf{m}}}((\overline{\Delta}\mathfrak{g}^*\times\mathcal{X}_\alpha)\overset{R}{\cap}_{(\mathfrak{g}^*)^2\times\mathcal{X}_\alpha}\widetilde{\mathfrak{g}}\times_{\mathcal{P}_\alpha}\widetilde{\mathfrak{g}}) \\
{\scriptstyle L\hat{i}^*}\uparrow\downarrow{\scriptstyle R\hat{i}_*} & & {\scriptstyle L\tilde{i}^*}\uparrow\downarrow{\scriptstyle R\tilde{i}_*} \\
\mathcal{D}^{\mathrm{c}}_{G\times\mathbb{G}_{\mathbf{m}}}((\Delta\mathfrak{g}^*\times\mathcal{B}^2)\overset{R}{\cap}_{(\mathfrak{g}^*\times\mathcal{B})^2}\widetilde{\mathcal{N}}\times\widetilde{\mathcal{N}}) & \xrightarrow[\sim]{\kappa} & \mathcal{D}^{\mathrm{c}}_{G\times\mathbb{G}_{\mathbf{m}}}((\overline{\Delta}\mathfrak{g}^*\times\mathcal{B}^2)\overset{R}{\cap}_{(\mathfrak{g}^*\times\mathcal{B})^2}\widetilde{\mathfrak{g}}\times\widetilde{\mathfrak{g}}).
\end{array}$$

[2]In this case, a simple dimension counting shows that the derived intersection $(\Delta\mathfrak{g}^*\times\mathcal{B}\times\mathcal{B})\overset{R}{\cap}_{(\mathfrak{g}^*\times\mathcal{B})^2}(\widetilde{\mathfrak{g}}\times\widetilde{\mathfrak{g}})$ is quasi-isomorphic, as a dg-scheme, to $\widetilde{\mathfrak{g}}\times_{\mathfrak{g}^*}\widetilde{\mathfrak{g}}$. Hence we do not really need Lemma 1.1.1 here.

By Proposition 3.2.1 there is an isomorphism of functors

$$\kappa \circ R\hat{i}_* \ \cong\ \big(R\tilde{i}_* \circ \kappa_\alpha(- \otimes_{\mathcal{O}_{\mathcal{X}_\alpha}} \mathcal{O}_{\mathcal{X}_\alpha}(\alpha, 2\rho))\big) \otimes_{\mathcal{O}_{\mathcal{B}^2}} \mathcal{O}_{\mathcal{B}^2}(2\rho, 2\rho)[1-N].$$

In particular we obtain an isomorphism

$$\mathfrak{K}(\mathcal{O}_{Y_\alpha}(\rho-\alpha, -\rho)) \ \cong\ \Xi \circ R\tilde{i}_* \circ \kappa_\alpha(\mathcal{O}_{Y_\alpha})[1].$$

Here on the right hand side $\mathcal{O}_{Y_\alpha}$ is considered as an object of $\mathcal{D}^{\mathrm{c}}_{G\times\mathbb{G}_{\mathbf{m}}}((\Delta\mathfrak{g}^*\times\mathcal{X}_\alpha)\overset{R}{\cap}_{(\mathfrak{g}^*)^2\times\mathcal{X}_\alpha}\widetilde{\mathcal{N}}\times_{\mathcal{P}_\alpha}\widetilde{\mathcal{N}})$, with its natural structure of dg-module, and Ξ is defined as in Subsection 4.1.

Now Y_α is a subbundle of $\widetilde{\mathcal{N}}\times_{\mathcal{P}_\alpha}\widetilde{\mathcal{N}}$. Taking the derived intersection with $\Delta\mathfrak{g}^*\times\mathcal{X}_\alpha$, we can apply the results of Subsection 2.3. Denoting by

$$\begin{aligned} f : (\Delta\mathfrak{g}^*\times\mathcal{X}_\alpha)\overset{R}{\cap}_{(\mathfrak{g}^*)^2\times\mathcal{X}_\alpha} Y_\alpha \ &\to\ (\Delta\mathfrak{g}^*\times\mathcal{X}_\alpha)\overset{R}{\cap}_{(\mathfrak{g}^*)^2\times\mathcal{X}_\alpha}\widetilde{\mathcal{N}}\times_{\mathcal{P}_\alpha}\widetilde{\mathcal{N}}, \\ g : (\overline{\Delta}\mathfrak{g}^*\times\mathcal{X}_\alpha)\overset{R}{\cap}_{(\mathfrak{g}^*)^2\times\mathcal{X}_\alpha}\widetilde{\mathfrak{g}}\times_{\mathcal{P}_\alpha}\widetilde{\mathfrak{g}} \ &\to\ (\overline{\Delta}\mathfrak{g}^*\times\mathcal{X}_\alpha)\overset{R}{\cap}_{(\mathfrak{g}^*)^2\times\mathcal{X}_\alpha} Y_\alpha^\perp \end{aligned}$$

the morphisms of dg-schemes induced by inclusions, we obtain a diagram

$$\begin{array}{ccc} \mathcal{D}^{\mathrm{c}}_{G\times\mathbb{G}_{\mathbf{m}}}((\Delta\mathfrak{g}^*\times\mathcal{X}_\alpha)\overset{R}{\cap}_{(\mathfrak{g}^*)^2\times\mathcal{X}_\alpha} Y_\alpha) & \xrightarrow[\sim]{\kappa_Y} & \mathcal{D}^{\mathrm{c}}_{G\times\mathbb{G}_{\mathbf{m}}}((\overline{\Delta}\mathfrak{g}^*\times\mathcal{X}_\alpha)\overset{R}{\cap}_{(\mathfrak{g}^*)^2\times\mathcal{X}_\alpha} Y_\alpha^\perp) \\ {\scriptstyle Lf^*}\uparrow\downarrow{\scriptstyle Rf_*} & & {\scriptstyle Rg_*}\uparrow\downarrow{\scriptstyle Lg^*} \\ \mathcal{D}^{\mathrm{c}}_{G\times\mathbb{G}_{\mathbf{m}}}((\Delta\mathfrak{g}^*\times\mathcal{X}_\alpha)\overset{R}{\cap}_{(\mathfrak{g}^*)^2\times\mathcal{X}_\alpha}\widetilde{\mathcal{N}}\times_{\mathcal{P}_\alpha}\widetilde{\mathcal{N}}) & \xrightarrow[\sim]{\kappa_\alpha} & \mathcal{D}^{\mathrm{c}}_{G\times\mathbb{G}_{\mathbf{m}}}((\overline{\Delta}\mathfrak{g}^*\times\mathcal{X}_\alpha)\overset{R}{\cap}_{(\mathfrak{g}^*)^2\times\mathcal{X}_\alpha}\widetilde{\mathfrak{g}}\times_{\mathcal{P}_\alpha}\widetilde{\mathfrak{g}}). \end{array}$$

(Here, in the top right corner, $Y_\alpha^\perp$ is the orthogonal of Y_α as a subbundle of $(\mathfrak{g}^*)^2\times\mathcal{X}_\alpha$.) Let $\mathcal{Y}_\alpha$ denote the sheaf of sections of Y_α. The structure sheaf of $(\Delta\mathfrak{g}^*\times\mathcal{X}_\alpha)\overset{R}{\cap}_{(\mathfrak{g}^*)^2\times\mathcal{X}_\alpha} Y_\alpha$ is $\Lambda(\mathfrak{g})\otimes_{\Bbbk}\mathrm{S}_{\mathcal{O}_{\mathcal{X}_\alpha}}(\mathcal{Y}_\alpha^\vee)$, with trivial differential (because $Y_\alpha\subset\Delta\mathfrak{g}^*\times\mathcal{X}_\alpha$). In particular, $\mathrm{S}_{\mathcal{O}_{\mathcal{X}_\alpha}}(\mathcal{Y}_\alpha^\vee)$ is naturally an object of the top left category, and $Rf_*(\mathrm{S}_{\mathcal{O}_{\mathcal{X}_\alpha}}(\mathcal{Y}_\alpha^\vee))$ is the object $\mathcal{O}_{Y_\alpha}$ considered above. By Proposition 2.3.1 we have an isomorphism of functors

$$\kappa_\alpha \circ Rf_* \ \cong\ Lg^* \circ \kappa_Y.$$

In particular we obtain $\kappa_\alpha(\mathcal{O}_{Y_\alpha}) \cong Lg^* \circ \kappa_Y(\mathrm{S}_{\mathcal{O}_{\mathcal{X}_\alpha}}(\mathcal{Y}_\alpha^\vee))$.

Now the structure sheaf of $(\overline{\Delta}\mathfrak{g}^*\times\mathcal{X}_\alpha)\overset{R}{\cap}_{(\mathfrak{g}^*)^2\times\mathcal{X}_\alpha} Y_\alpha^\perp$ is $\Lambda_{\mathcal{O}_{\mathcal{X}_\alpha}}(\mathcal{Y}_\alpha)\otimes_{\Bbbk}\mathrm{S}_{\Bbbk}(\mathfrak{g})$, with trivial differential. And direct computation shows that $\kappa_Y(\mathrm{S}_{\mathcal{O}_{\mathcal{X}_\alpha}}(\mathcal{Y}_\alpha^\vee))$ is isomorphic to the dg-module $\mathcal{O}_{\mathcal{X}_\alpha}\otimes_{\Bbbk}\mathrm{S}(\mathfrak{g})$. Then $Lg^*(\mathcal{O}_{\mathcal{X}_\alpha}\otimes_{\Bbbk}\mathrm{S}(\mathfrak{g}))$ is the structure sheaf of the derived intersection of $\overline{\Delta}\mathfrak{g}^*\times\mathcal{X}_\alpha$ and $\widetilde{\mathfrak{g}}\times_{\mathcal{P}_\alpha}\widetilde{\mathfrak{g}}$ inside $Y_\alpha^\perp$. But $(\overline{\Delta}\mathfrak{g}^*\times\mathcal{X}_\alpha)\cap(\widetilde{\mathfrak{g}}\times_{\mathcal{P}_\alpha}\widetilde{\mathfrak{g}})\cong\widetilde{\mathfrak{g}}\times_{\widetilde{\mathfrak{g}}_\alpha}\widetilde{\mathfrak{g}}$, and

$$\dim(\overline{\Delta}\mathfrak{g}^*\times\mathcal{X}_\alpha)+\dim(\widetilde{\mathfrak{g}}\times_{\mathcal{P}_\alpha}\widetilde{\mathfrak{g}})-\dim(Y_\alpha^\perp) \ =\ \dim(\widetilde{\mathfrak{g}}\times_{\widetilde{\mathfrak{g}}_\alpha}\widetilde{\mathfrak{g}}) \quad (=\dim(\mathfrak{g})).$$

Hence the derived intersection is quasi-isomorphic to $(\overline{\Delta}\mathfrak{g}^*\times\mathcal{X}_\alpha)\cap(\widetilde{\mathfrak{g}}\times_{\mathcal{P}_\alpha}\widetilde{\mathfrak{g}})$. This completes the proof of the proposition. □

6.3. **Proof of Theorem 6.2.1.** By construction we have $\kappa(\mathcal{M}\langle m\rangle)\cong\kappa(\mathcal{M})[m]\langle -m\rangle$, hence

$$\kappa_{\mathrm{IM}}(\mathcal{M}\langle m\rangle) \ \cong\ \kappa_{\mathrm{IM}}(\mathcal{M})[m]\langle m\rangle.$$

In particular, for $a\in\mathcal{H}_{\mathrm{aff}}$ and $f(v)\in\mathbb{Z}[v,v^{-1}]$ we have $[\kappa_{\mathrm{IM}}](f(v)\cdot a)=f(-v)\cdot[\kappa_{\mathrm{IM}}](a)$.

By Proposition 4.3.1, the equivalence κ_{IM} is compatible with convolution, hence also the induced isomorphism $[\kappa_{\mathrm{IM}}]$. Also, by Proposition 5.1.1 it sends the unit to the unit. It follows that to

prove Theorem 6.2.1 we only have to check that $[\kappa_{\mathrm{IM}}]$ and $\iota \circ \mathrm{IM}$ coincide on the generators t_α and θ_x.

First, Corollary 5.2.1 implies that $[\kappa_{\mathrm{IM}}](\theta_x) = \theta_{-x}$.

Similarly, Proposition 6.2.2 implies that $[\kappa_{\mathrm{IM}}]([\mathcal{O}_{Y_\alpha}(\rho - \alpha, -\rho)]) = -[\mathcal{O}_{\widetilde{\mathfrak{g}} \times_{\widetilde{\mathfrak{g}}_\alpha} \widetilde{\mathfrak{g}}}]$. Hence

$$\begin{aligned} [\kappa_{\mathrm{IM}}](T_\alpha) &= -v^{-1}[\mathcal{O}_{\widetilde{\mathfrak{g}} \times_{\widetilde{\mathfrak{g}}_\alpha} \widetilde{\mathfrak{g}}}] + v^{-1} \\ &= T_\alpha - v + v^{-1}. \end{aligned}$$

Hence $[\kappa_{\mathrm{IM}}](t_\alpha) = -t_\alpha + v^2 - 1 = -q(t_\alpha)^{-1}$. This finishes the proof of the theorem.

References

[BC] D. Barbasch, D. Ciubotaru, *Whittaker unitary dual of affine graded Hecke algebras of type E*, preprint 0708.4036.

[BM] D. Barbasch, A. Moy, *A unitarity criterion for p-adic groups*, Inv. Math. **98** (1988), 19–37.

[CG] N. Chriss, V. Ginzburg, *Representation theory and complex geometry*, Birkhäuser, 1997.

[EM] S. Evens, I. Mirković, *Fourier transform and the Iwahori-Matsumoto involution*, Duke Math. J. **86** (1997), 435–464.

[Ha] R. Hartshorne, *Residues and duality*, Lecture Notes in Math. 20, Springer, 1966.

[KS] M. Kashiwara, P. Schapira, *Sheaves on manifolds*, Springer, 1990.

[Ke] B. Keller, *Derived categories and their uses*, in *Handbook of algebra, Vol. 1*, edited by M. Hazewinkel, Elsevier, 1996.

[Lu] G. Lusztig, *Bases in equivariant K-theory I*, Represent. Theory **2** (1998), 298–369.

[MR] I. Mirković, S. Riche, *Linear Koszul duality*, preprint 0804.0923.

[R1] S. Riche, *Geometric braid group action on derived categories of coherent sheaves*, with a joint appendix with R. Bezrukavnikov, Represent. Theory **12** (2008), 131–169.

[R2] S. Riche, *Koszul duality and modular representations of semi-simple Lie algebras*, preprint 0803.2076.

[Sp] N. Spaltenstein, *Resolutions of unbounded complexes*, Compos. Math. **65** (1988), 121–154.

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