On the zeros of the Riemann Zeta function

Lazhar Fekih-Ahmed

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LAZHAR FEKIH-AHMED

Abstract. This paper is divided into two independent parts. The first part presents a new integral and series representation of the Riemaan zeta function. An equivalent formulation of the Riemann hypothesis is given and few results on this formulation are briefly outlined. The second part exposes a totally different approach. Using the new series representation of the zeta function of the first part, exact information on its zeros is provided based on Tauberian-like results.

PART I

1. Introduction

It is well-known that the Riemaan zeta function defined by the Dirichlet series:

$$\zeta(s) = \frac{1}{1^s} + \frac{1}{2^s} + \cdots + \frac{1}{k^s} + \cdots = \sum_{n=1}^{\infty} n^{-s} \tag{1.1}$$

converges for $\Re(s) > 1$, and can be analytically continued to the whole complex plane with one singularity, a simple pole with residue 1 at $s = 1$. It is also well known that $\zeta(s)$ satisfies the functional equation:

$$\chi(s)\zeta(s) = \zeta(1-s)\chi(1-s) \quad \text{with} \quad \chi(s) = \pi^{-\frac{s}{2}}\Gamma(\frac{s}{2}), \tag{1.2}$$

and that the zeros of $\zeta(s)$ come into two types. The trivial zeros which occur at all negative even integers $s = -2, -4, \cdots$, and the nontrivial zeros which occur at certain values of $s \in \mathbb{C}$, $0 < \Re(s) < 1$.

The Riemann hypothesis states that the nontrivial zeros of $\zeta(s)$ all have real part $\Re(s) = \frac{1}{2}$. From the functional equation (1.2), the Riemann hypothesis is equivalent to $\zeta(s)$ not having any zeros in the half plane $\Re(s) > \frac{1}{2}$.

2. An Analytic Continuation of $\zeta(s)$

Let

$$S_n(s) = 1 - \binom{n-1}{1}2^{-s} + \binom{n-1}{2}3^{-s} - \cdots + (-1)^{n-1}(n)^{-s}, n \geq 2 \tag{2.1}$$

with $S_1(s) = 1$.

Using the well-known identity, valid for $\Re(s) > 0$:

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$$n^{-s} = \frac{1}{\Gamma(s)} \int_0^\infty e^{-nt} t^{s-1}\, dt, \tag{2.2}$$

we can rewrite $S_n(s)$ in (2.1) as:

$$\begin{aligned} S_n(s) &= \sum_{k=0}^{n-1} (-1)^k \binom{n-1}{k} (k+1)^{-s} \\ &= \frac{1}{\Gamma(s)} \int_0^\infty \sum_{k=0}^{n-1} (-1)^k \binom{n-1}{k} e^{-(k+1)t} t^{s-1}\, dt \\ &= \frac{1}{\Gamma(s)} \int_0^\infty (1-e^{-t})^{n-1} e^{-t} t^{s-1}\, dt, \end{aligned} \tag{2.3}$$

since $\sum_{k=0}^{n-1} (-1)^k \binom{n-1}{k} e^{-(k+1)t} = e^{-t}(1-e^{-t})^{n-1}$.

We have

$$\sum_{n=1}^\infty \frac{S_n(s)}{n+1} = \frac{1}{\Gamma(s)} \sum_{n=1}^\infty \int_0^\infty \frac{(1-e^{-t})^{n-1}}{n+1} e^{-t} t^{s-1}\, dt \tag{2.4}$$

$$= \frac{1}{\Gamma(s)} \int_0^\infty \sum_{n=1}^\infty \frac{(1-e^{-t})^{n-1}}{n+1} e^{-t} t^{s-1}\, dt \tag{2.5}$$

$$= \frac{1}{\Gamma(s)} \int_0^\infty \left(\frac{t}{(1-e^{-t})^2} - \frac{1}{1-e^{-t}} \right) e^{-t} t^{s-1}\, dt. \tag{2.6}$$

Before we proceed further, some remarks are in order:

Remark 2.1. The interchange of the summation and integration in equation (2.5) is valid because the series $\sum_{n=1}^\infty \int_0^\infty \frac{(1-e^{-t})^{n-1}}{n+1} e^{-t} t^{s-1}\, dt$ converges absolutely and uniformly for $0 < t < \infty$. To see this, we show uniform convergence for the dominating series $\sum_{n=1}^\infty \int_0^\infty \frac{(1-e^{-t})^{n-1}}{n+1} e^{-t} t^{\sigma-1}\, dt$, $\sigma = \Re(s)$. Indeed, let $K = \max((1-e^{-t})^{n-1} e^{-t/2})$, $0 < t < \infty$. A straightforward calculation of the derivative shows that $K = (1 - \frac{1}{2n-1})^{n-1} \frac{1}{\sqrt{2n-1}}$ and is attained when $e^{-t} = \frac{1}{2n-1}$. Now, for $n \geq 2$, we have

$$\begin{aligned} \frac{1}{n+1} \int_0^\infty (1-e^{-t})^{n-1} e^{-t} t^{\sigma-1}\, dt &= \frac{1}{n+1} \int_0^\infty (1-e^{-t})^{n-1} e^{-t/2} (e^{-t/2} t^{\sigma-1})\, dt \\ &\leq \frac{K}{n+1} \int_0^\infty e^{-t/2} t^{\sigma-1}\, dt \\ &= \frac{1}{n+1} (1 - \frac{1}{2n-1})^{n-1} \frac{2^\sigma \Gamma(\sigma)}{\sqrt{2n-1}} \\ &\leq \frac{2^\sigma \Gamma(\sigma)}{(n+1)\sqrt{2n-1}}. \end{aligned} \tag{2.7}$$

The last inequality implies that the dominating series converges by the comparison test.

Remark 2.2. To get equation (2.6) we used the identity:

$$\sum_{n=1}^{\infty} \frac{(1-e^{-t})^{n-1}}{n+1} = \frac{t}{(1-e^{-t})^2} - \frac{1}{1-e^{-t}}, \tag{2.8}$$

which can be obtained by putting $X = 1-e^{-t}$ into $-\dfrac{\log(1-X)}{X^2} - \dfrac{1}{X} = \displaystyle\sum_{n=1}^{\infty} \frac{X^{n-1}}{n+1}$.

Now, since

$$\frac{d}{dt}\frac{-te^{-t}}{1-e^{-t}} = \frac{te^{-t}}{(1-e^{-t})^2} - \frac{e^{-t}}{1-e^{-t}}, \tag{2.9}$$

an integration by parts in (2.6) yields

$$\sum_{n=1}^{\infty} \frac{S_n(s)}{n+1} = \frac{s-1}{\Gamma(s)} \int_0^{\infty} \frac{e^{-t}t^{s-1}}{1-e^{-t}}\, dt = (s-1)\zeta(s), \tag{2.10}$$

which is valid when $\Re(s) > 1$. And since the integral (2.6) is valid for $\Re(s) > 0$, then we have proved

Theorem 2.3. *Let* $\phi(t) = \frac{t}{(1-e^{-t})^2} - \frac{1}{1-e^{-t}}$, *then for* $\Re(s) > 0$,

$$(s-1)\zeta(s) = \frac{1}{\Gamma(s)} \int_0^{\infty} \phi(t) e^{-t} t^{s-1}\, dt. \tag{2.11}$$

Remark 2.4. Although we will not need it in the rest of the paper, we can also obtain an analytic continuation of $(s-1)\zeta(s)$ when $\Re(s) \leq 0$. We simply rewrite (2.11) as a contour integral

$$\frac{\Gamma(1-s)}{2\pi i} \int_{\mathcal{C}} \phi(t) e^{-t} t^{s-1}\, dt, \tag{2.12}$$

where $\mathcal{C}$ is the Hankel contour consisting of the three parts $C = C_- \cup C_\epsilon \cup C_+$: a path which extends from $(-\infty, -\epsilon)$, around the origin counter clockwise on a circle of center the origin and of radius ϵ and back to $(-\epsilon, -\infty)$, where ϵ is an arbitrarily small positive number. The integral (2.12) now defines $(s-1)\zeta(s)$ for all $s \in \mathbb{C}$.

Remark 2.5. In particular, when $s = k$ is a positive integer, we have yet another formula for $\zeta(k)$:

$$(k-1)\zeta(k) = \frac{1}{(k-1)!} \int_0^{\infty} \phi(t) e^{-t} t^{k-1}\, dt. \tag{2.13}$$

Remark 2.6. The above integral formula for $(s-1)\zeta(s)$, although obtained by elementary means, does not seem to be found in the literature. As for the series formula, it has been obtained by a different method in [10]. A series formula that is different but similar in form and always mentioned in the literature is that of Hasse [7].

3. A Series Expansion of $(s-1)\zeta(s)\Gamma(s)$

The analytic function $(s-1)\zeta(s)\Gamma(s)$ can be represented by a Taylor series around any point $s_0 = 1 + iy$ on the vertical line $\sigma = 1$. In particular, for $s_0 = 1$ we obtain the well-known power series [1]:

$$(s-1)\zeta(s)\Gamma(s) = a_0 + a_1(s-1) + a_2(s-1)^2 + a_3(s-1)^3 + \cdots \tag{3.1}$$

where the coefficients $a_0 = 1$ and a_n are defined by

$$a_n = \lim_{s\to 1}\frac{d^n}{ds^n}\Big\{\int_0^\infty \phi(t)e^{-t}t^{s-1}\,dt\Big\} \tag{3.2}$$

$$= \int_0^\infty \phi(t)e^{-t}\lim_{s\to 1}\frac{d^n}{ds^n}\big\{t^{s-1}\big\}\,dt \tag{3.3}$$

$$= \frac{1}{n!}\int_0^\infty \phi(t)e^{-t}(\log t)^n\,dt, \tag{3.4}$$

with $\phi(t)$ being the function defined in Theorem 2.3.

The coefficients a_n are very important in the evaluation of $\zeta^{(k)}(0)$ as given by Apostol in [1]. Up to now the a_n are regarded as unknowns and as difficult to approximate as $\zeta^{(k)}(0)$ itself as pointed out by Lehmer [9]. The formula above solves the exact evaluation problem of the $\zeta^{(k)}(0)$ and many other variant formulae. The following proposition provides more information on the sequence $\{a_n\}$.

Proposition 3.1. *For n large enough, the coefficients a_n are given by*

$$a_n = (-1)^n\left(\frac{1}{2} - \frac{1}{6}\frac{1}{2^{n+1}}\right) + \mathcal{O}(\frac{1}{4^n}).$$

Proof. The expression of a_n can be split into the sum

$$a_n = \frac{1}{n!}\int_0^1 \phi(t)e^{-t}(\log t)^n\,dt + \frac{1}{n!}\int_1^\infty \phi(t)e^{-t}(\log t)^n\,dt$$

$$= \frac{(-1)^n}{2} + \frac{(-1)^n}{n!}\int_0^1 \big[\phi(t)e^{-t} - \frac{1}{2}\big]\log\left(\frac{1}{t}\right)^n dt + \frac{1}{n!}\int_1^\infty \phi(t)e^{-t}(\log t)^n\,dt. \tag{3.5}$$

To obtain an estimate the first integral in (3.5), we use equation (2.9). A differentiation with respect to t of the following expansion which defines the Bernoulli numbers[1]

$$\frac{te^{-t}}{1-e^{-t}} = \sum_{n=0}^\infty \frac{B_n}{n!}t^n \tag{3.6}$$

gives

$$\phi(t)e^{-t} - \frac{1}{2} = \sum_{n=2}^\infty \frac{-B_n}{(n-1)!}t^{n-1} = -\frac{1}{6}t + \frac{1}{180}t^3 - \frac{1}{5040}t^5 + \cdots \tag{3.7}$$

Now, since

$$\frac{1}{n!}\int_0^1 t^m \log\left(\frac{1}{t}\right)^n dt = \frac{1}{(m+1)^{n+1}} \tag{3.8}$$

[1] $B_0 = 1$ $B_1 = 1/2$, $B_2 = 1/6$, $B_3 = 0$, $B_4 = -1/30$, $B_5 = 0$, $B_6 = 1/42$, $B_7 = 0$, $B_8 = -1/30$ etc.

for all n, m positive, the first integral in (3.5) without the factor $(-1)^n$ has an expansion

$$\frac{1}{n!}\int_0^1 \big[\phi(t)e^{-t}-\frac{1}{2}\big]\log\left(\frac{1}{t}\right)^n dt = \sum_{n=2}^{\infty}\frac{-B_n}{(n-1)!(n+1)^{n+1}} \tag{3.9}$$
$$= -\frac{1}{6}\frac{1}{2^{n+1}}+\frac{1}{180}\frac{1}{4^{n+1}}-\frac{1}{5040}\frac{1}{6^{n+1}}+\cdots$$

To estimate the second integral in (3.5), we use the bound

$$(\log t)^n < e^{\epsilon t} \tag{3.10}$$

which is valid for all $t \geq n^{1+\epsilon}$ and for n large enough and where ϵ is any positive small number. We split the integral into two parts

$$\frac{1}{n!}\int_1^{\infty}\phi(t)e^{-t}(\log t)^n\,dt \leq \frac{1}{n!}\int_1^{n^{1+\epsilon}}\phi(t)e^{-t}(\log t)^n\,dt + \tag{3.11}$$
$$\frac{1}{n!}\int_{n^{1+\epsilon}}^{\infty}\phi(t)e^{-t}(\log t)^n\,dt$$
$$\leq \frac{C_0}{n!}(1+\epsilon)^n\log(n)^n+\frac{1}{2n!}\frac{e^{-(1-\epsilon)n^{1+\epsilon}}}{1-\epsilon}, \tag{3.12}$$

where $C_0=\int_1^{\infty}\phi(t)e^{-t}\,dt = 0.58$.

Clearly, the term $\frac{1}{2n!}\frac{e^{-(1-\epsilon)n^{1+\epsilon}}}{1-\epsilon}$ is extremely small for n large enough. In particular, for $n \geq 2$, it is less than $\frac{C}{a^n}$ where C is a positive constant and a is any positive constant greater than 2.

Using Stirling formula $n! \sim \sqrt{2\pi n}\big(\frac{n}{e}\big)^n$, we can verify that the term $\frac{C_0}{n!}(1+\epsilon)^n\log(n)^n$ can also be made less than $\frac{C}{a^n}$ for n large enough. Taking $a=4$ for example, we obtain

$$\frac{1}{n!}\int_1^{\infty}\phi(t)e^{-t}(\log t)^n\,dt \leq \frac{C}{4^n}. \tag{3.13}$$

By combining the above estimates, we obtain

$$a_n = (-1)^n\left(\frac{1}{2}-\frac{1}{6}\frac{1}{2^{n+1}}\right)+\mathcal{O}(\frac{1}{4^n}). \tag{3.14}$$

□

For any s_0 of the form $s_0 = 1+iy$ the corresponding Taylor series is

$$(s-1)\zeta(s)\Gamma(s) = b_0+b_1(s-s_0)+b_2(s-s_0)^2+b_3(s-s_0)^3+\cdots \tag{3.15}$$

where the b_n is expressed as

$$b_n = \frac{1}{n!}\int_0^{\infty}\phi(t)e^{-t}(\log t)^n t^{iy}\,dt. \tag{3.16}$$

An asymptotic estimate of the coefficients b_n is given by the following proposition whose proof is the same as the proof of Proposition 3.1.

Proposition 3.2. *The radius of convergence of the series (3.15) is* $\sqrt{1+y^2}$ *and for* n *large enough, the coefficients* b_n *are given by*

$$b_n = \frac{1}{2}\frac{(-1)^n}{(1+iy)^{n+1}} - \frac{1}{6}\frac{(-1)^n}{(2+iy)^{n+1}} + \mathcal{O}(\frac{1}{(\sqrt{16+y^2})^n}).$$

4. The zeros of $\zeta(s)$

The Taylor series expansion $(s-1)\zeta(s)\Gamma(s)$ provides us with a tool to study the zeros of $\zeta(s)$ is a neighborhood of $s_0 = 1+iy$. We have

$$(s-1)\zeta(s)\Gamma(s) = b_0 + b_1(s-s_0) + b_2(s-s_0)^2 + b_3(s-s_0)^3 + \cdots \tag{4.1}$$

with

$$b_0 = iy\zeta(1+iy)\Gamma(1+iy), \tag{4.2}$$

$$b_n = \frac{1}{n!}\int_0^\infty \phi(t)e^{-t}(\log t)^n e^{iy}\,dt. \tag{4.3}$$

It is a well-know fact [6] that $\zeta(1+iy) \neq 0$ for all y. Therefore $b_0 \neq 0$ for all y and the inverse of $(s-1)\zeta(s)\Gamma(s)$ is well-defined and can be expanded into a power series of the form

$$\frac{1}{(s-1)\zeta(s)\Gamma(s)} = c_0 + c_1(s-s_0) + c_2(s-s_0)^2 + c_3(s-s_0)^3 + \cdots, \tag{4.4}$$

where the coefficients c_n are given by

$$c_n = (-1)^n\frac{\Delta_n}{b_0^{n+1}}, \tag{4.5}$$

with

$$\Delta_n = \begin{vmatrix} b_1 & b_0 & \cdots & \cdots & 0 \\ b_2 & b_1 & b_0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ b_{n-1} & b_{n-2} & b_{n-3} & \cdots & b_0 \\ b_n & b_{n-1} & b_{n-2} & \cdots & b_1 \end{vmatrix}. \tag{4.6}$$

Let $D(s_0, \frac{1}{2})$ be the open disk of center s_0 and radius $\frac{1}{2}$. The zeros of $\zeta(s)$ are the same as those of $(s-1)\zeta(s)\Gamma(s)$ in the right half plane; therefore, $\zeta(s) \neq 0$ in $D(s_0, \frac{1}{2})$ for any y is equivalent to the fact that the radius of convergence of the series (4.4) is at least $\frac{1}{2}$ for any y.

Now, the union of the strips $S_1 = \{s \in \mathbb{C} : \frac{1}{2} < \sigma < 1\}$ and $S_2 = \{s \in \mathbb{C} : 0 < \sigma < \frac{1}{2}\}$ form the critical strip minus the critical line $\sigma = \frac{1}{2}$. Moreover, the strip $S = \{s \in \mathbb{C} : \frac{1}{2} < \sigma < \frac{3}{2}\}$ can be written as

$$S = \bigcup_y D(1+iy, \frac{1}{2}) \tag{4.7}$$

We conclude from the above, that if $\zeta(s)$ doest not have a zero inside the strip S and a fortiori does not have a zero in the strip S_1, then by the functional equation $\zeta(s)$ cannot have a zero inside S_2 neither. We thus have proved

Theorem 4.1. *The Riemann hypothesis is equivalent to either*

(1) *the series (4.1) does not have any zero in the disk $D(s_0, \frac{1}{2})$ for any y.*
(2) *the radius of convergence of the series (4.4) is at least $\frac{1}{2}$ for any y.*

Remark 4.2. We have been able to prove that the series (4.1) does not have any zero in the disk $D(1, \frac{1}{2})$ (i.e $y = 0$). The proof is trivial and uses the criterion of Petrovitch [8] for power series. We also have been able to prove the well-known result that $D(1, 1)$ is a zero-free region. We have been unable to generalize the proof for any y because as y gets large the values of $|b_0|$ become very small compared to that of $|b_1|$. The typical power series and polynomial non-zero regions criteria are inapplicable. More knowledge on the ratios $|b_n|/|b_0|$ is needed.

Remark 4.3. Another criterion of the Riemann hypothesis can be formulated using the conformal mapping $s = \frac{1}{1-z}$ which maps the plane $\Re(s) > \frac{1}{2}$ onto the unit disk $|z| < 1$. We wan write

$$\begin{aligned} (\frac{z}{1-z})\zeta(\frac{1}{1-z})\Gamma(\frac{1}{1-z}) \triangleq f(z) &= \int_0^\infty \phi(t) e^{-t} t^{\frac{z}{1-z}}\, dt \\ &= \sum_{n=0}^\infty \tilde{a}_n z^n, \end{aligned} \tag{4.8}$$

where the coefficients $\tilde{a}_n$ are given by

$$\tilde{a}_n = \int_0^\infty \phi(t) e^{-t} L_n(-\ln t)\, dt, \tag{4.9}$$

$L_n(x) = L_n^0(x)$ being the Laguerre polynomial of order 0.

The Riemann hypothesis is equivalent to the function $f(z)$ having no zeros in the unit disk.

Although the above formulations of the Riemann hypothesis seem to be promising since exact information on the coefficients is known, we will not pursue this approach. The new approach that we will adopt is presented next.

PART II

In this part we will pursue a completely different approach from the one presented in PART I. Using the new series representation of the zeta function of the first part, exact information on its zeros is provided based on Tauberian-like results.

5. The Series representation of $(s-1)\zeta(s)$

In PART I, we showed that $(s-1)\zeta(s)$ process both an integral and a series representation valid for $\Re(s) > 0$. In the remaining of the paper we will only consider the series representation. We recall the series representation valid when $\Re(s) > 1$:

$$\sum_{n=1}^{\infty} \frac{S_n(s)}{n+1} = \frac{s-1}{\Gamma(s)} \int_0^{\infty} \frac{e^{-t}t^{s-1}}{1-e^{-t}}\, dt = (s-1)\zeta(s), \tag{5.1}$$

where $S_n(s)$ is given by

$$S_n(s) = \sum_{k=0}^{n-1} (-1)^k \binom{n-1}{k} (k+1)^{-s}. \tag{5.2}$$

First, we will provide another proof on the validity of the series representation for $\Re(s) > 0$. To prove the analytic continuation when $\Re(s) > 0$, we need to evaluate the sum when $\Re(s) > 0$. The next lemma, which will also be needed in the rest of the paper, provides such an estimation. It provides an estimate of the exact asymptotic order of growth of $\frac{S_n(s)}{n+1}$ when n is large.

Lemma 5.1. $\dfrac{S_n(s)}{n+1} \sim \dfrac{1}{n(n+1)(\log n)^{1-s}\Gamma(s)}$ *for n large enough and for all $s = \sigma + it$, $\Re(s) > 0$, $s \notin \{1, 2, \cdots\}$.*

Proof. By putting $k = m-1$ in (5.2), we have by definition

$$\begin{aligned} S_n(s) &= \sum_{m=1}^{n} \binom{n-1}{m-1} (-1)^{m-1} m^{-s} = \sum_{m=1}^{n} \frac{m}{n} \binom{n}{m} (-1)^{m-1} m^{-s} \\ &= \frac{-1}{n} \sum_{m=1}^{n} \binom{n}{m} (-1)^m m^{1-s} = \frac{-1}{n} \Delta_n(s-1), \end{aligned} \tag{5.3}$$

where $\Delta_n(\lambda) \triangleq \displaystyle\sum_{m=1}^{n} \binom{n}{m} (-1)^m m^{-\lambda}$.

The asymptotic expansion of sums of the form $\Delta_n(\lambda)$, with $\lambda \in \mathbb{C}$ being nonintegral has been given in Theorem 3 of Flajolet et al. [4]. With a slight modification of notation, the authors in [4] have shown that $\Delta_n(\lambda)$ has an asymptotic expansion in descending powers of $\log n$ of the form

$$-\Delta_n(\lambda) \sim (\log n)^{\lambda} \sum_{j=0}^{\infty} (-1)^j \frac{\Gamma^{(j)}(1)}{j!\Gamma(1+\lambda-j)} \frac{1}{(\log n)^j} \tag{5.4}$$

We apply the theorem to $\Delta_n(\lambda)$ with $\lambda = s-1$ to get

$$\Delta_n(s-1) \sim \frac{-(\log n)^{s-1}}{\Gamma(s)}, \tag{5.5}$$

which leads to the result

$$S_n(s) \sim \frac{1}{n(\log n)^{1-s}\Gamma(s)}. \tag{5.6}$$

The Lemma follows from dividing equation (5.6) by $n+1$. □

Now to obtain an analytic continuation when $\Re(s) > 0$, we simply observe that the logarithmic test of series in combination with the asymptotic value of $S_n(s)$ provided by Lemma 5.1 imply that the absolute value of the series on the left hand side of (5.1) is dominated by a uniformly convergent series for all finite s whose real part is greater than 0.

Remark 5.2. By Weierstrass theorem, we can see that the function $(s-1)\zeta(s)$ can be extended outside of the domain $\Re(s) > 1$ and that it does not have any singularity when $\Re(s) > 0$. Moreover, by repeating the same process for $\Re(s) > -k$, $k \in \mathbb{N}$, it is clear that the series defines an analytic continuation of $\zeta(s)$ valid for all $s \in \mathbb{C}$.

6. Preparation Lemmas

Throughout this section, we suppose that $0 < \Re(s) < 1$. For a fixed $s = \sigma + it$, we associate with $\zeta(s)$, the following power series:

$$(s-1)\zeta(s,x) \triangleq \frac{S_1(s)}{2}x + \frac{S_2(s)}{3}x^2 + \cdots + \frac{S_{n-1}(s)}{n}x^{n-1} + \frac{S_n(s)}{n+1}x^n + \cdots \tag{6.1}$$

$x \in \mathbb{R}$.

Let's also further define the "comparison" power series by

$$\Phi(x) \triangleq (1-x)\left(\frac{\log(1-x)}{-x}\right)^s = \phi_0 + \phi_1 x + \phi_2 x^2 + \cdots + \phi_n x^n + \cdots \tag{6.2}$$

It is easy to verify that for $\sigma > 0$

$$\lim_{x\to 1}(1-x)\left(\frac{\log(1-x)}{-x}\right)^s = 0. \tag{6.3}$$

Furthermore, direct calculation of $\Phi'(x)$ yields the expression

$$\Phi'(x) = \left(\frac{\log(1-x)}{-x}\right)^s \big(-1 + s - \frac{s}{\log(1-x)} - \frac{s}{x}\big). \tag{6.4}$$

Clearly, $\Phi'(x)$ is well-defined for all $x \in [0,1)$ and satisfies

$$\lim_{x\to 1}|\Phi'(x)| = \infty. \tag{6.5}$$

In other words the power series $\Phi'(x)$ is a continuous well defined function of x, converges for all values of $x \in [0,1)$ and diverges when $x \to 1$. Moreover, because $\Phi(x)$ is analytic at $x = 0$, $\Phi'(x)$ must possess the following power series expansion around $x = 0$:

$$\Phi'(x) = \phi_1 + 2\phi_2 x + \cdots + n\phi_n x^{n-1} + \cdots \tag{6.6}$$

Finally, we associate to the series (6.6) the following positive coefficients power series:

$$\tilde{\Phi}(x) \triangleq |\phi_1| + 2|\phi_2|x + \cdots + n|\phi_n|x^{n-1} + \cdots \tag{6.7}$$

The proofs in the remaining of this section will be based on two theorems. The first theorem, which is due to Nörlund and generalized by Flajolet et al., estimates the asymptotic behavior of the coefficients of certain powers series:

Theorem 6.1 ([5]). *Let α be a positive integer and β be a real or complex number, $\beta \notin \{0, 1, 2, \cdots\}$. Define the function $f(z)$ by*

$$f(z) = (1-z)^{\alpha}\Big(\frac{1}{z}\big(\log\frac{1}{1-z}\big)^{\beta}. \tag{6.8}$$

Then, the Taylor coefficients f_n of $f(z)$ satisfy

$$f_n \sim n^{-\alpha-1}(\log n)^{\beta}\Big(\frac{e_1}{1!}\frac{-(\beta)}{(\log n)} + \frac{e_2}{2!}\frac{\beta(\beta-1)}{(\log n)^2} + \cdots\Big), \tag{6.9}$$

with

$$e_k = \frac{d^k}{ds^k}\left(\frac{1}{\Gamma(s)}\right)\Big|_{s=\alpha}. \tag{6.10}$$

The second theorem, due to Appell, is the counterpart of l'Hospital's rule for divergent positive coefficients power series:

Theorem 6.2 ([2] p. 66). *Let $f(x), g(x)$ be two real power series of the form*

$$f(x) = \sum_{n=1}^{\infty} a_n x^n, g(x) = \sum_{n=1}^{\infty} b_n x^n, a_n, b_n > 0 \text{ for all } n > N, 0 < x < 1. \tag{6.11}$$

We further suppose that

- *the series $\sum_{n=1}^{\infty} a_n$, $\sum_{n=1}^{\infty} b_n$ are both divergent so that $x = 1$ is a singular point of both $f(x)$ and $g(x)$.*
- $\lim\limits_{n\to\infty} \dfrac{a_n}{b_n} = l$,

then,

$$\lim_{x\to 1} \frac{f(x)}{g(x)} = l. \tag{6.12}$$

Our first result establishes an important property on the behavior of the derivative of the function $\Phi(x)$ when x is close to 1:

Lemma 6.3. *There exists an $x_0 \in (0, 1)$ and a constant C independent of x such that for all $x \in (x_0, 1)$ we have $\dfrac{|\Phi'(x)|}{\tilde{\Phi}(x)} > C > 0$.*

Proof. From (6.4),

$$|\Phi'(x)| = \left(\frac{\log(1-x)}{-x}\right)^{\sigma}\Big| - 1 + s - \frac{s}{\log(1-x)} - \frac{s}{x}\Big|. \tag{6.13}$$

Let's suppose that the power series expansion of $\left(\dfrac{\log(1-x)}{-x}\right)^{\sigma} \triangleq \Psi(x)$ is given by

$$\Psi(x) = \psi_0 + \psi_1 x + \psi_2 x^2 + \cdots + \psi_n x^n + \cdots, \tag{6.14}$$

then applying Theorem 6.1 with $\alpha = 0$ and $\beta = \sigma$, implies that for large values of n, the coefficients ψ_n satisfy the following asymptotic value:

$$\psi_n \sim \frac{\sigma}{n(\log n)^{1-\sigma}}. \tag{6.15}$$

Similarly, for $\Phi'(x) = \phi_1 + 2\phi_2 x + \cdots + n\phi_n x^{n-1} + \cdots$, Theorem 6.1 with $\alpha = 1$ and $\beta = s$, implies that for large values of n, the coefficients ϕ_n satisfy the following asymptotic estimates:

$$\phi_n \sim \frac{-s\gamma}{n^2(\log n)^{1-s}}, \tag{6.16}$$

where γ is Euler's constant.

The asymptotic value of $n|\phi_n|$ imply by Abel's Theorem and the logarithmic test of series that, like $\Psi(x)$, the series $\tilde{\Phi}(x)$ goes to infinity as x approaches 1.

We thus have

$$\frac{|\Phi'(x)|}{\tilde{\Phi}(x)} = \frac{\Psi(x)x}{\sum_{n=1}^{\infty} n|\phi_n|x^n}\Big| - 1 + s - \frac{s}{\log(1-x)} - \frac{s}{x}\Big|. \tag{6.17}$$

Now as x approaches 1, $\Big| - 1 + s - \frac{s}{\log(1-x)} - \frac{s}{x}\Big|$ approaches 1 so that given any small $\epsilon > 0$ we can find x_1 such that for $x \in (x_1, 1)$, $\Big| - 1 + s - \frac{s}{\log(1-x)} - \frac{s}{x}\Big| > 1 - \epsilon$.

Moreover, since $\Psi(x)x = \sum_{n=1}^{\infty} \psi_{n-1}x^n$ and $\sum_{n=1}^{\infty} n|\phi_n|x^n$ both go to infinity as x approches 1, and since the asymptotic estimates (6.15)-(6.16) of ψ_{n-1} and ϕ_n verify

$$\lim_{n\to\infty} \frac{\psi_{n-1}}{n|\phi_n|} = \frac{\sigma}{\gamma|s|}, \tag{6.18}$$

then Theorem 6.2 gives

$$\lim_{x\to 1} \frac{\sum_{n=1}^{\infty} \psi_{n-1}x^n}{\sum_{n=1}^{\infty} n|\phi_n|x^n} = \frac{\sigma}{\gamma|s|}. \tag{6.19}$$

In other words, given any small $\epsilon > 0$ we can find x_2 such that for $x \in (x_2, 1)$, $\frac{\Psi(x)x}{\sum_{n=1}^{\infty} n|\phi_n|x^n} > \frac{\sigma}{\gamma|s|} - \epsilon$.

To complete the proof take for example $C = \frac{\sigma}{4\gamma|s|}$ and $x_0 = max\{x_1, x_2\}$.

□

The second lemma that we need establishes a relationship between the derivative of $\zeta(s, x)$ and that of $\Phi(x)$:

Lemma 6.4. *Let Φ be defined as above and let $\zeta'(s, x)$ be $\frac{d}{dx}\zeta(s, x)$, then*

$$\lim_{x\to 1} \frac{(s-1)\zeta'(s, x)}{\Phi'(x)} = \frac{-1}{\gamma s\Gamma(s)}, \tag{6.20}$$

where the limit is taken from below.

Proof. We have

$$(s-1)\zeta'(s, x) = \frac{S_1(s)}{2} + \frac{2S_2(s)}{3}x + \cdots + \frac{(n-1)S_{n-1}(s)}{n}x^{n-2} + \frac{nS_n(s)}{n+1}x^{n-1} + \cdots \tag{6.21}$$

$x \in \mathbb{R}$.

Lemma 5.1 gives $\dfrac{S_n(s)}{n+1} \sim \dfrac{1}{n(n+1)(\log n)^{1-s}\Gamma(s)}$ for n large enough and for all $s = \sigma + it$, $0 < \Re(s) < 1$. Combining with the estimate (6.16) yields

$$\lim_{n\to\infty} \frac{n\frac{S_n(s)}{n+1}}{n\phi_n} = \frac{-1}{\gamma s\Gamma(s)}. \tag{6.22}$$

The limit (6.22) is equivalent to saying that there exists a complex sequence ϵ_n with $\lim\limits_{n\to\infty} \epsilon_n = 0$ such that

$$n\frac{S_n(s)}{n+1} = \frac{-1}{\gamma s\Gamma(s)} n\phi_n + \epsilon_n n\phi_n \tag{6.23}$$

or equivalently

$$n\frac{S_n(s)}{n+1}x^{n-1} = \frac{-1}{\gamma s\Gamma(s)} n\phi_n x^{n-1} + \epsilon_n n\phi_n x^{n-1}, \tag{6.24}$$

and finally adding the equalities for $n = 1, 2, \cdots$, yields

$$(s-1)\zeta'(s,x) = \frac{-1}{\gamma s\Gamma(s)}\Phi'(x) + \sum_{n=1}^{\infty} \epsilon_n n\phi_n x^{n-1}. \tag{6.25}$$

Now by dividing all sides of (6.25) by $\Phi'(x)$, $0 < x < 1$, and taking the limit as $x \to 1$, we get

$$\lim_{x\to 1} \frac{(s-1)\zeta'(s,x)}{\Phi'(x)} = \frac{-1}{\gamma s\Gamma(s)} + \lim_{x\to 1} \frac{\sum_{n=1}^{\infty} \epsilon_n n\phi_n x^{n-1}}{\Phi'(x)}. \tag{6.26}$$

To prove the lemma it suffices to show that

$$\lim_{x\to 1} \left| \frac{\sum_{n=1}^{\infty} \epsilon_n n\phi_n x^{n-1}}{\Phi'(x)} \right| = 0. \tag{6.27}$$

Indeed, using our first preparation Lemma 6.3, simple calculations yield

$$\begin{aligned} \lim_{x\to 1} \frac{|\sum_{n=1}^{\infty} \epsilon_n n\phi_n x^{n-1}|}{|\Phi'(x)|} &\leq \lim_{x\to 1} \frac{\sum_{n=1}^{\infty} n|\epsilon_n||\phi_n| x^{n-1}}{|\Phi'(x)|} \\ &\leq \frac{1}{C} \lim_{x\to 1} \frac{\sum_{n=1}^{\infty} n|\epsilon_n||\phi_n| x^{n-1}}{\tilde{\Phi}(x)}, \end{aligned} \tag{6.28}$$

where $\tilde{\Phi}(x)$ is defined in (6.7).

If the series in the numerator is convergent, the result is obvious. If not, the two series in the right hand side of the last inequality are both divergent positive coefficients power series. An application of Theorem 6.2 shows that the limit in (6.28) is equal to the limit of

$$\lim_{n\to\infty} \frac{n|\epsilon_n||\phi_n|}{n|\phi_n|} = 0, \tag{6.29}$$

and the lemma is proved.

□

Now let s be a nontrivial zero of $\zeta(s)$. By Abel's theorem $\lim_{x\to 1}(s-1)\zeta(s,x) = (s-1)\zeta(s,1) = 0$. In addition, $\lim_{x\to 1}\Phi(x) = 0$. The next and last preparation lemma shows that because of the particular function $\Phi(x)$, l'Hospital's rule which usually does not apply to vector valued or complex valued function, does apply for this particular case:

$$\lim_{x\to 1}\frac{(s-1)\zeta(s,x)}{\Phi(x)} = \lim_{x\to 1}\frac{(s-1)\zeta'(s,x)}{\Phi'(x)} = \frac{-1}{\gamma s\Gamma(s)}. \tag{6.30}$$

Lemma 6.5. *Let Φ be defined as above and let s be a nontrivial zero of $\zeta(s)$, then*

$$\lim_{x\to 1}\frac{(s-1)\zeta(s,x)}{\Phi(x)} = \frac{-1}{\gamma s\Gamma(s)} \tag{6.31}$$

where the limit is taken from below.

Proof. From Lemma 6.4, we have

$$\lim_{x\to 1}\frac{(s-1)\zeta'(s,x)}{\Phi'(x)} = \frac{-1}{\gamma s\Gamma(s)}. \tag{6.32}$$

Define $\delta(x)$ by

$$\delta(x) \triangleq \frac{(s-1)\zeta'(s,x)}{\Phi'(x)} + \frac{1}{\gamma s\Gamma(s)} \tag{6.33}$$

so that (6.32) can be written as

$$\lim_{x\to 1}\delta(x) = 0. \tag{6.34}$$

Multiplying equation (6.33) by $\Phi'(x)$, and integrating[2] from x to 1, we obtain

$$(s-1)\big(\lim_{\epsilon\to 0}\zeta(s,1-\epsilon)-\zeta(s,x)\big)+\frac{\lim_{\epsilon\to 0}\Phi(1-\epsilon)-\Phi(x)}{\gamma s\Gamma(s)} = \int_x^1 \delta(y)\Phi'(y)\,dy. \tag{6.35}$$

Now recalling that $\lim_{\epsilon\to 0}\Phi(1-\epsilon) = 0$, and that s is a zero of $\zeta(s)$ so that $\lim_{\epsilon\to 0}\zeta(s,1-\epsilon) = 0$, and dividing both sides of (6.35) by $\Phi(x)$, we finally get

$$\frac{(s-1)\zeta(s,x)}{\Phi(x)} + \frac{1}{\gamma s\Gamma(s)} = -\frac{\int_x^1 \delta(y)\Phi'(y)\,dy}{\Phi(x)}, \tag{6.36}$$

by which we obtain:

$$\begin{aligned}\lim_{x\to 1}\left|\frac{(s-1)\zeta(s,x)}{\Phi(x)} + \frac{1}{\gamma s\Gamma(s)}\right| &\le \lim_{x\to 1}\frac{\left|\int_x^1 \delta(y)\Phi'(y)dy\right|}{|\Phi(x)|} \\ &\le \lim_{x\to 1}\frac{\int_x^1 |\delta(y)||\Phi'(y)|\,dy}{|\Phi(x)|} \end{aligned} \tag{6.37}$$

By observing that the ratio

$$\frac{|\Phi'(y)|}{|\Phi(y)|'} = \frac{\left|-1+s-\frac{s}{\log(1-y)}-\frac{s}{y}\right|}{\left|-1+\sigma-\frac{\sigma}{\log(1-y)}-\frac{\sigma}{y}\right|}, \tag{6.38}$$

[2]The integral is an improper integral, i.e. it is defined as $\lim_{\epsilon\to 0}\int_x^{1-\epsilon} f(y)\,dy$

where $\big|\Phi(y)\big|' = \dfrac{d}{dy}\big|\Phi(y)\big|$ is always bounded by a suitable constant, say K, for $y \in (x_0, 1)$, x_0 close to 1, the limit in (6.37) is less than or equal to

$$K \lim_{x\to 1} \frac{\int_x^1 \big|\delta(y)\big|\big|\Phi(y)\big|' \, dy}{|\Phi(x)|}. \tag{6.39}$$

The last limit in (6.39) consists of a limit of the ratio of two real functions that satisfy the hypothesis of l'Hospital's rule. That is

$$K \lim_{x\to 1} \frac{\int_x^1 \big|\delta(y)\big|\big|\Phi(y)\big|' \, dy}{|\Phi(x)|} = K \lim_{x\to 1} \frac{\big|\delta(x)\big|\big|\Phi(x)\big|'}{|\Phi(x)|'} = K \lim_{x\to 1} \big|\delta(x)\big| = 0. \tag{6.40}$$

Consequently,

$$\lim_{x\to 1} \left| \frac{(s-1)\zeta(s,x)}{\Phi(x)} + \frac{1}{\gamma s \Gamma(s)} \right| = 0, \tag{6.41}$$

and the lemma is proved. □

Equation (6.31) in Lemma 6.5 is quite a remarquable identity. It is in fact the key formula for proving the Riemann hypothesis. It says that if s is a zero of $\zeta(s)$ and even though for this particular value of s, $\lim_{x\to 1}(s-1)\zeta(s,x) = 0$ and $\lim_{x\to 1} \Phi(x) = 0$, the limit $\lim\limits_{x\to 1} \dfrac{(s-1)\zeta(s,x)}{\Phi(x)}$ is well defined and is equal to $\dfrac{-1}{\gamma s \Gamma(s)}$. Using this identity, we are now ready to prove the Riemann hypothesis.

7. Proof of Riemann Hypothesis

We proceed by contradiction. Suppose that $s = \sigma + it$ is a zero of $\zeta(s)$ with $0 < \sigma < \dfrac{1}{2}$. From the functional equation (1.2), $1 - s$ must also be a zero of $\zeta(s)$.

We have from the previous analysis:

$$\lim_{x\to 1} \frac{(s-1)\zeta(s,x)}{\Phi(x)} = \frac{-1}{\gamma s \Gamma(s)}, \quad \text{where} \qquad \Phi(x) \triangleq (1-x)\left(\frac{\log(1-x)}{-x}\right)^s. \tag{7.1}$$

Similarly, for $1 - s$, we define the comparison function $\hat{\Phi}(x)$ which is analogous to $\Phi(x)$ but with $1 - s$ in place of s to get

$$\lim_{x\to 1} \frac{-s\zeta(1-s,x)}{\hat{\Phi}(x)} = \frac{-1}{\gamma(1-s)\Gamma(1-s)}, \quad \text{where} \qquad \hat{\Phi}(x) \triangleq (1-x)\big(\frac{\log(1-x)}{-x}\big)^{1-s}. \tag{7.2}$$

Taking absolute values and dividing equation (7.1) by (7.2) [3], we must then have

[3] We could have evaluated the quotient directly without taking absolute values, but then the limit of the quotient exists under the condition that $\lim_{x\to 1} \frac{\hat{\Phi}(x)}{\Phi(x)}$ exists; and this is not always

$$\lim_{x\to 1}\left|\frac{(1-s)\zeta(s,x)}{s\zeta(1-s,x)}\frac{\hat{\Phi}(x)}{\Phi(x)}\right| = \left|\frac{(1-s)\zeta(s)}{s\zeta(1-s)}\right|\lim_{x\to 1}\left|\frac{\hat{\Phi}(x)}{\Phi(x)}\right| = \left|\frac{\Gamma(1-s)}{\Gamma(s)}\frac{1-s}{s}\right|; \tag{7.3}$$

consequently,

$$\lim_{x\to 1}\left|\frac{\hat{\Phi}(x)}{\Phi(x)}\right| = \left|\frac{\zeta(1-s)}{\zeta(s)}\frac{\Gamma(1-s)}{\Gamma(s)}\right|. \tag{7.4}$$

The right hand side of (7.4) has a finite value for any zero of $\zeta(s)$ since the functional equation (1.2) implies that:

$$\frac{\zeta(1-s)}{\zeta(s)} = \frac{\pi^{-\frac{s}{2}}\Gamma(\frac{s}{2})}{\pi^{-\frac{1-s}{2}}\Gamma(\frac{1-s}{2})} \tag{7.5}$$

so that $\frac{\zeta(1-s)}{\zeta(s)}$ has a finite value for any zero of $\zeta(s)$. But exact calculation of the left hand side of (7.4) gives:

$$\lim_{x\to 1}\left|\frac{\hat{\Phi}(x)}{\Phi(x)}\right| = \lim_{x\to 1}\left(\frac{\log(1-x)}{-x}\right)^{1-2\sigma} = \infty. \tag{7.6}$$

This contradicts equation (7.4) unless $\sigma = \frac{1}{2}$ in which case the limit in (7.4) is equal to 1. So there cannot be a zero such that $0 < \sigma < \frac{1}{2}$ and therefore all the zeros must lie on the line $\sigma = \frac{1}{2}$. The proof is complete.

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École Nationale d'Ingénieurs de Tunis, BP 37, Le Belvédère 1002 , Tunis, Tunisia

Current address: École Nationale d'Ingénieurs de Tunis, BP 37, Le Belvédère 1002 , Tunis, Tunisia

E-mail address: lazhar.fekihahmed@enit.rnu.tn

true for complex-valued functions (e.g. for $s = 0.5 + it, t \neq 0$, the limit does not exist). By taking the absolute values, the quotients are real-valued and we circumvent such cases.