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A compact cell-centered Galerkin method with subgrid stabilization

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Abstract

In this work we propose a compact cell-centered Galerkin method with subgrid stabilization for anisotropic heterogeneous diffusion problems on general meshes. Both essential theoretical results and numerical validation are provided.

Résumé

Une méthode de Galerkin centrée aux mailles avec stencil compact et stabilisation de sous-grille.

On propose une méthode de Galerkin centrée aux mailles avec stencil compact et stabilisation de sous-grille pour des problèmes de diffusion anisotrope et hétérogène. On présente à la fois les résultats théoriques essentiels et une validation numérique.

1. Introduction

Soit $\Omega \subset \mathbb{R}^d$, $d \geq 1$, un ouvert polygonal borné. On considère le problème modèle suivant :

$$-\nabla \cdot (\kappa \nabla u) = f \text{ dans } \Omega, \quad u = 0 \text{ sur } \partial\Omega, \quad (1)$$

où $f \in L^2(\Omega)$ et κ est un champ de diffusion tensoriel uniformément défini positif et constant par morceaux sur Ω . En posant $V \stackrel{\text{def}}{=} H_0^1(\Omega)$, la formulation faible de (1) s'écrit

$$\text{Trouver } u \in V \text{ t.q. } \int_{\Omega} \kappa \nabla u \cdot \nabla v = \int_{\Omega} f v \text{ pour tout } v \in V. \quad (2)$$

La discrétisation de (1) par des méthodes de bas ordre sur des maillages généraux qui soient robustes par rapport à l'hétérogénéité et à l'anisotropie du tenseur de diffusion κ , et qui aient en même temps des bonnes propriétés de stabilité est un sujet actif de recherche. Les méthodes de volumes finis classiques reposent sur des bilans par maille faisant intervenir des *flux numériques*. L'idée d'obtenir des flux numériques à

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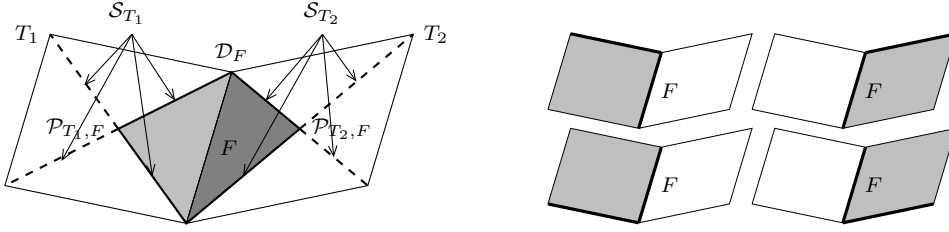


Figure 1. *Left panel.* Pyramids $\mathcal{P}_{T_1, F}$ (light gray) and $\mathcal{P}_{T_2, F}$ (dark gray), and diamond \mathcal{D}_F for $F \in \mathcal{F}_h^i$ in $d = 2$. *Right panel.* Face groups G in \mathcal{G}_F for $F \in \mathcal{F}_h^i$ (bold line) in $d = 2$. The unique $T_F \in \mathcal{T}_h$ such that $G \subset \partial T_F$ is in gray.

l'aide une reconstruction locale peut être attribuée à Aavatsmark *et al.* (voir [1] pour une bibliographie) et à Edwards *et al.* (voir [6] pour une bibliographie). Le problème principal de ces méthodes est qu'il n'existe souvent pas de conditions simples à vérifier qui en garantissent la stabilité. Plus récemment, des approches variationnelles ont été proposées par Eymard, Gallouët et Herbin [7]. Les méthodes de ce type sont inconditionnellement coercives au prix d'un stencil élargi. En outre, dans tous les cas cités, les méthodes ne sont définies qu'au niveau discret, ce qui ne permet pas d'appliquer les techniques d'analyse typiques des méthodes de Galerkin/éléments finis. L'analyse de convergence repose alors sur un argument de compacité, qui ne fournit pas une estimation de l'ordre de convergence [7, 2]. Un possible remède à cette difficulté a été suggéré dans [4], où on propose une famille de méthodes variationnelles centrées aux mailles dites *cell-centered Galerkin* (ccG), et basées sur des espaces polynomiaux par morceaux incomplets (mais qui ont cependant des propriétés d'approximation suffisantes). Ces méthodes sont inconditionnellement coercives. De plus, étant formulées comme des méthodes d'éléments finis non conformes, il est possible d'étendre la forme bilinéaire à un espace contenant la solution exacte, et d'effectuer une analyse de convergence dans l'esprit du Lemme de Céa. Le désavantage principal de la famille de méthodes ccG de [4] consiste à avoir un stencil élargi, ce qui réduit la sparsité du système linéaire. Dans cette note, on propose une variation qui réduit significativement le stencil tout en gardant les bonnes propriétés de la méthode originale (coercivité inconditionnelle, robustesse par rapport à l'hétérogénéité et à l'anisotropie du tenseur de diffusion). L'idée principale est de réduire le support des fonctions de base de l'espace fonctionnel sur lequel repose la méthode en introduisant une reconstruction opportune sur une sous-grille. La consistence et la symétrie sont garanties par des termes de bord contrôlés par une stabilisation aux sous-faces. Le plan de la note est le suivant : dans §2 on présente la méthode puis on cite les résultats théoriques principaux ; dans §3 on propose une validation numérique.

2. Discrete setting

Notation. The discretization of problem (2) is based on general polygonal meshes. More specifically, in the context of the h -convergence analysis, we denote by $\mathcal{H} \subset \mathbb{R}_+^*$ a countable set having 0 as its unique accumulation point, and we adopt the concept of *admissible mesh sequence* $(\mathcal{T}_h)_{h \in \mathcal{H}}$ introduced in [2, Definition 2.1] (which may be consulted for further details on the regularity assumptions). We recall here only the relevant notations. For a given $h \in \mathcal{H}$, we let $\mathcal{T}_h = \{T\}$ denote a partition of Ω into polygonal elements whose faces are portions of hyperplanes and such that κ is constant inside each element. An *interface* is a portion of hyperplane $F \subset \partial T_1 \cap \partial T_2$ with $T_1, T_2 \in \mathcal{T}_h$, whereas a *boundary face* is a portion of hyperplane $F \subset \partial T \cap \partial \Omega$ with $T \in \mathcal{T}_h$. Interfaces are collected in the set \mathcal{F}_h^i , boundary faces in \mathcal{F}_h^b . We additionally set $\mathcal{F}_h \stackrel{\text{def}}{=} \mathcal{F}_h^i \cup \mathcal{F}_h^b$ and, for all $T \in \mathcal{T}_h$, $\mathcal{F}_T \stackrel{\text{def}}{=} \{F \in \mathcal{F}_h \mid F \subset \partial T\}$. The set of elements whose boundary F belongs to is denoted by \mathcal{T}_F , i.e., $\mathcal{T}_F \stackrel{\text{def}}{=} \{T \in \mathcal{T}_h \mid F \subset \partial T\}$. For each $T \in \mathcal{T}_h$ we assume that

there exists $x_T \in T$ (the *cell-center*) such that T is star-shaped with respect to x_T , and x_T is uniformly away from the boundary of T . For all $F \in \mathcal{F}_T$, we let $\mathcal{P}_{T,F}$ denote the pyramid of apex x_T and base F , that is to say (see the left panel of Figure 2),

$$\forall T \in \mathcal{T}_h, \forall F \in \mathcal{F}_T, \quad \mathcal{P}_{T,F} \stackrel{\text{def}}{=} \{x \in T \mid \exists y \in F, \exists \theta \in [0, 1] \text{ s.t. } x = \theta x_T + (1 - \theta)y\}.$$

The subgrid obtained from the decomposition of the elements of \mathcal{T}_h into pyramids is denoted by \mathcal{S}_h , i.e. $\mathcal{S}_h = \{\mathcal{P}_{T,F}\}_{T \in \mathcal{T}_h, F \in \mathcal{F}_T}$. The set of lateral pyramid faces (*subfaces*) S included in T is denoted by \mathcal{S}_T (see the left panel of Figure 2). For all $S \in \mathcal{S}_T$, $T \in \mathcal{T}_h$, we fix a unit normal \mathbf{n}_S . Moreover, for all $\varphi \in L^1(S)$, we let $\varphi^\pm = \lim_{\epsilon \rightarrow 0} \varphi(x \pm \epsilon \mathbf{n}_S)$ for a.e. $x \in S$, and define the following jump and average trace operators:

$$[[\varphi]] \stackrel{\text{def}}{=} \varphi^- - \varphi^+, \quad \{\varphi\} \stackrel{\text{def}}{=} \frac{\varphi^- + \varphi^+}{2}.$$

For all $F \in \mathcal{F}_h$ we define \mathcal{D}_F as the diamond obtained by the union of all pyramids of base F , i.e., $\mathcal{D}_F \stackrel{\text{def}}{=} \bigcup_{T \in \mathcal{T}_F} \mathcal{P}_{T,F}$ (see the left panel of Figure 2). The diameters of the generic face $F \in \mathcal{F}_h$ and of a the generic subface $S \in \mathcal{S}_T$, $T \in \mathcal{T}_h$, are denoted by h_F and h_S respectively, and, for all $T \in \mathcal{T}_h$, we introduce the following notation for the diffusion in the normal direction:

$$\forall F \in \mathcal{F}_T, \lambda_F^T \stackrel{\text{def}}{=} \kappa|_{T \mathbf{n}_F \cdot \mathbf{n}_F}, \quad \forall S \in \mathcal{S}_T, \lambda_S \stackrel{\text{def}}{=} \kappa|_{T \mathbf{n}_S \cdot \mathbf{n}_S}.$$

Since, for boundary faces, there exists a unique $T \in \mathcal{T}_h$ such that $F \subset \partial T$, we shall alleviate the notation and simply write λ_F instead of λ_F^T whenever possible. Finally, the space of the cell-centered unknowns is denoted by $\mathbb{R}^{\mathcal{T}_h}$, its elements are indexed by the mesh elements, i.e., we write $\mathbf{v} = (v_T)_{T \in \mathcal{T}_h} \in \mathbb{R}^{\mathcal{T}_h}$.

A local reconstruction. The main ingredient for the design of the incomplete polynomial space upon which the method is built is the piecewise affine local reconstruction introduced in [1] and analyzed in [2, §3.1]. To illustrate the construction, for all $F \in \mathcal{F}_h$ we introduce the family \mathcal{G}_F of groups of d faces of a cell in \mathcal{T}_F including F . More precisely, let

$$\forall F \in \mathcal{F}_h, \forall T \in \mathcal{T}_F, \quad \mathcal{G}_{F,T} \stackrel{\text{def}}{=} \{F' \in \mathcal{F}_T \mid F' \cap F \neq \emptyset\},$$

and define (see the right panel of Figure 2)

$$\forall F \in \mathcal{F}_h, \quad \mathcal{G}_F \stackrel{\text{def}}{=} \{G \subset \mathcal{F}_h \mid \exists T \in \mathcal{T}_F \text{ s.t. } G \subset \mathcal{F}_{T,F}, F \in G \text{ and } \text{card}(G) = d\}.$$

For every $F \in \mathcal{F}_h$ we select the group $G_F \in \mathcal{G}_F$ with the best approximation properties (cf. [2, eq. (B9)]), and use the unknowns associated to the cells sharing the faces in G_F to reconstruct a continuous piecewise affine function on \mathcal{D}_F with continuous flux across F . To this end, we need some additional notation. The unique element whose boundary all faces of G_F belong to is denoted by T_F , i.e., $G_F \subset \mathcal{F}_{T_F}$. Moreover, for every interface $F' \in G_F \cap \mathcal{F}_h^i$, $F' \neq F$, we let $T_{F'}$ denote the unique element of \mathcal{T}_h such that $F' \subset \partial T_F \cap \partial T_{F'}$. We then introduce the matrix $\mathbf{A}_F \in \mathbb{R}^{d,d}$ and the linear operator $\mathbf{b}_F \in \mathcal{L}(\mathbb{R}^{\mathcal{T}_h}, \mathbb{R}^d)$ such that

$$\mathbf{A}_F \stackrel{\text{def}}{=} \begin{bmatrix} \left(\lambda_{F'}^{T_{F'}} (x_{T_{F'}} - x_{T_F}) + (\kappa_{T_F} - \kappa_{T_{F'}}) \mathbf{n}_{T_{F'}} \right)_{F' \in G \cap \mathcal{F}_h^i}^t \\ \left(\lambda_{F'}^{T_F} (x_{F'} - x_{T_F}) \right)_{F' \in G \cap \mathcal{F}_h^b}^t \end{bmatrix}, \quad \mathbf{b}_F(\mathbf{v}) \stackrel{\text{def}}{=} \begin{bmatrix} \left(\lambda_{F'}^{T_{F'}} (v_{T_G} - v_{T_{F'}}) \right)_{F' \in G \cap \mathcal{F}_h^i} \\ \left(-\lambda_{T_F}^{F'} v_{T_F} \right)_{F' \in G \cap \mathcal{F}_h^b} \end{bmatrix}.$$

The gradient reconstruction on $\mathcal{P}_{T_F,F}$ is given by $\nabla_{\mathcal{P}_{T_F,F}}(\mathbf{v}) = \mathbf{A}_F^{-1} \mathbf{b}_F(\mathbf{v})$. Whenever F is an internal face, i.e., there exists $\bar{T}_F \in \mathcal{T}_h$, $\bar{T}_F \neq T_F$, such that $F \subset \partial T_F \cap \partial \bar{T}_F$, we define the gradient reconstruction on $\mathcal{P}_{\bar{T}_F,F}$ as $\nabla_{\mathcal{P}_{\bar{T}_F,F}}(\mathbf{v}) = \nabla_{\mathcal{P}_{T_F,F}}(\mathbf{v}) - \text{dist}(x_{\bar{T}_F}, F)^{-1} \left(v_{\bar{T}_F} - v_{T_F} - \nabla_{\mathcal{P}_{T_F,F}}(\mathbf{v}) \cdot (x_{\bar{T}_F} - x_{T_F}) \right) \mathbf{n}_{\bar{T}_F}$. The following relevant property can be proved proceeding as in [2, §3.1].

Proposition 1 (Local reconstruction) *Fix $\mathbf{v} \in \mathbb{R}^{\mathcal{T}_h}$. For a given $F \in \mathcal{F}_h$, let $\xi_{\mathbf{v}}$ be the unique piecewise affine function such that, for all $T \in \mathcal{T}_F$, $\xi_{\mathbf{v}}|_T(x) = v_T + \nabla_{\mathcal{P}_{T,F}}(\mathbf{v}) \cdot (x - x_T)$. Then, $\xi_{\mathbf{v}}$ is continuous in \mathcal{D}_F and, if $F \subset \partial T_1 \cap \partial T_2$, it satisfies $\kappa_{T_1} \nabla_{\mathcal{P}_{T_1,F}} \cdot \mathbf{n}_{T_1} + \kappa_{T_2} \nabla_{\mathcal{P}_{T_2,F}} \cdot \mathbf{n}_{T_2} = 0$ for a.e. $x \in F$.*

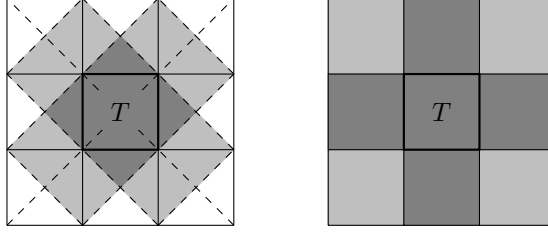


Figure 2. Comparison between minimal (dark gray) and maximal (light gray) support of the basis function $\varphi_T = \mathcal{I}_h(e_T)$ for the proposed method (left) and the original ccG method of [4] (right).

The discrete functional space. The procedure outlined in the previous section allows to reconstruct a piecewise affine field on \mathcal{S}_h which is continuous across interfaces and has continuous diffusive fluxes. Let $\mathbb{P}_d^k(\mathcal{S}_h)$ denote the space of broken polynomial functions on \mathcal{S}_h of total degree at most k , and define the operator $\mathcal{I}_h \in \mathcal{L}(\mathbb{R}^{\mathcal{T}_h}, \mathbb{P}_d^1(\mathcal{S}_h))$ which realizes the mapping $\mathbb{R}^{\mathcal{T}_h} \ni \mathbf{v} \mapsto \mathcal{I}_h(\mathbf{v}) \in \mathbb{P}_d^1(\mathcal{S}_h)$ with

$$\forall T \in \mathcal{T}_h, \forall F \in \mathcal{F}_T, \quad \mathcal{I}_h(\mathbf{v})|_{\mathcal{P}_{T,F}}(x) = v_T + \nabla_{\mathcal{P}_{T,F}} \cdot (x - x_T).$$

The injectivity of the operator follows from an invertibility assumption on the matrices \mathbf{A}_G together with the fact that every $T \in \mathcal{T}_h$ appears in at least one local reconstruction. Drawing on the ideas of [4], we then select the following discrete functional space:

$$V_h \stackrel{\text{def}}{=} \mathcal{I}_h(\mathbb{R}^{\mathcal{T}_h}) \subset \mathbb{P}_d^1(\mathcal{S}_h).$$

Choose $\{\varphi_T \stackrel{\text{def}}{=} \mathcal{I}_h(e_T)\}_{T \in \mathcal{T}_h}$ as basis for V_h (e_T denotes here the T th vector of the canonical basis of $\mathbb{R}^{\mathcal{T}_h}$). The reduced stencil with respect to the original ccG method of [4] is mainly related to the more compact support of the basis functions. Although the actual support of each basis function varies according to the specific choice of the group G_F in the local reconstruction, it is possible to identify minimal and maximal supports (cf. Figure 2 for a comparison with the original ccG method of [4] in $d = 2$). We recall that reducing the support of the basis functions allows to obtain sparser matrices, which results in reduced memory allocation and a more efficient solution of the associated linear system. Let $V_{\dagger} \stackrel{\text{def}}{=} \{v \in V \cap H^2(\mathcal{T}_h) \mid \forall F \subset \partial T_1 \cap \partial T_2, \kappa_{T_1}(\nabla v)|_{T_1} \cdot \mathbf{n}_F = \kappa_{T_2}(\nabla v)|_{T_2} \cdot \mathbf{n}_F\}$ and set $V_{\dagger h} \stackrel{\text{def}}{=} V_{\dagger} + V_h$ (the additional local regularity in V_{\dagger} is required to define the discrete bilinear form on a space containing the exact solution). To alleviate the notation, we denote by ∇_h the piecewise constant broken gradient on \mathcal{S}_h . The discrete bilinear form is defined on $V_{\dagger h} \times V_h$ as follows:

$$\begin{aligned} a_h(w, v_h) \stackrel{\text{def}}{=} & \int_{\Omega} \kappa \nabla_h w \cdot \nabla_h v_h - \sum_{F \in \mathcal{F}_h^b} \int_F (\kappa \nabla w \cdot \mathbf{n}_F v_h + \kappa \nabla v_h \cdot \mathbf{n}_F w) + \sum_{F \in \mathcal{F}_h^b} \int_F \frac{\eta \lambda_F}{h_F} w v_h \\ & - \sum_{T \in \mathcal{T}_h} \sum_{S \in \mathcal{S}_T} \int_S (\kappa \{\nabla_h w\} \cdot \mathbf{n}_S [[v_h]] + \kappa \{\nabla_h v_h\} \cdot \mathbf{n}_S [[w]]) + \sum_{T \in \mathcal{T}_h} \sum_{S \in \mathcal{S}_T} \int_S \frac{\eta \lambda_S}{h_S} [[w]] [[v_h]]. \end{aligned} \quad (3)$$

Owing to the subsurface and boundary terms, two basis functions whose support shares at least one subsurface interact. All integrals in the expression of a_h can be evaluated using a single quadrature node corresponding to the center of mass of the suitable mesh entity. The discrete problem reads

$$\text{Find } u_h \in V_h \text{ s.t. } a_h(u_h, v_h) = \int_{\Omega} f v_h \text{ for all } v_h \in V_h. \quad (4)$$

Basic error estimate. We now identify some relevant properties of the bilinear form a_h . To prove coercivity, we equip the space V_h with the norm

$$\|v_h\|^2 \stackrel{\text{def}}{=} \|\kappa^{1/2} \nabla_h v_h\|_{[L^2(\Omega)]^d}^2 + \sum_{T \in \mathcal{T}_h} \sum_{S \in \mathcal{S}_T} \lambda_S \|[[v_h]]\|_{L^2(S)}^2 + \sum_{F \in \mathcal{F}_h^b} \lambda_F \|v_h\|_{L^2(F)}^2.$$

To prove the boundedness of a_h , we introduce the following extended norm on $V_{\dagger h}$: For all $v \in V_{\dagger h}$,

$$\|v\|_*^2 \stackrel{\text{def}}{=} \|v\|^2 + \sum_{T \in \mathcal{T}_h} \sum_{S \in \mathcal{S}_T} h_S \|\kappa \{\nabla_h v\} \cdot \mathbf{n}_S\|_{L^2(S)}^2 + \sum_{F \in \mathcal{F}_h^b} h_F \|\kappa \nabla v \cdot \mathbf{n}_F\|_{L^2(F)}^2.$$

The $\|\cdot\|$ -norm and the $\|\cdot\|_*$ -norm are uniformly equivalent on V_h . The bilinear form a_h enjoys the following properties: (i) *Consistency*. Let u solve (2) and additionally assume that $u \in V_{\dagger}$. Then, for all $v_h \in V_h$, $a_h(u, v_h) = \int_{\Omega} f v_h$. (ii) *Coercivity*. For all $\eta > \underline{\eta}$ (with $\underline{\eta}$ only depending on the mesh regularity), there exists $C_{\text{sta}} > 0$ independent of both the meshsize h and of the diffusion field κ such that for all $v_h \in V_h$, $a_h(v_h, v_h) \geq C_{\text{sta}} \|v_h\|^2$. (iii) *Boundedness*. There exists C_{bnd} independent of both the meshsize h and of the diffusion field κ such that, for all $(w, v_h) \in V_{\dagger h} \times V_h$, $a_h(w, v_h) \leq C_{\text{bnd}} \|w\|_* \|v_h\|$. Using the above properties, the following error is classically inferred proceeding in the spirit of Céa's Lemma.

Theorem 2 (Error estimate) *Let u solve (2) and additionally assume that $u \in V_{\dagger}$. Then,*

$$\|u - u_h\| \leq (1 + C_{\text{bnd}} C_{\text{sta}}^{-1}) \inf_{w_h \in V_h} \|u - w_h\|_*.$$

The multiplicative constant in the right-hand side does not depend on the diffusion tensor, which renders the method robust with respect to both heterogeneity and anisotropy of κ .

Convergence to minimal regularity solutions. To prove convergence to solutions that are barely in V , let

$$W_h \stackrel{\text{def}}{=} \{w_h \in \mathbb{P}_d^1(\mathcal{S}_h) \mid \forall F \in \mathcal{F}_h, w_h \in C^0(\mathcal{D}_F) \text{ and } \forall F \subset \partial T_1 \cap \partial T_2, (\kappa \nabla w_h)|_{T_1} \cdot \mathbf{n}_F = (\kappa \nabla w_h)|_{T_2} \cdot \mathbf{n}_F\}.$$

The $\|\cdot\|$ -norm satisfies a discrete Poincaré inequality on W_h and, *a fortiori*, on V_h (proceed as in [5]). As a consequence, we have the uniform *a priori* estimate $\|u_h\| \leq \sigma_2 C_{\text{sta}}^{-1} \|f\|_{L^2(\Omega)}$ (σ_2 results from the discrete Poincaré inequality). In the spirit of [3], we define a discrete gradient that is weakly asymptotically consistent by introducing the lifting operators solutions of: For all $T \in \mathcal{T}_h$, all $S \in \mathcal{S}_T$, and all $\varphi \in L^1(S)$,

$$\int_{\Omega} \mathbf{r}_S(\varphi) \cdot \tau_h = \int_S \varphi \{\tau_h\} \cdot \mathbf{n}_S, \quad \forall \tau_h \in [\mathbb{P}_d^0(\mathcal{S}_h)]^d,$$

and, for all $F \in \mathcal{F}_h^b$ and all $\varphi \in L^1(F)$,

$$\int_{\Omega} \mathbf{r}_F(\varphi) \cdot \tau_h = \int_F \varphi \tau_h \cdot \mathbf{n}_F, \quad \forall \tau_h \in [\mathbb{P}_d^0(\mathcal{S}_h)]^d.$$

Let $\mathbf{R}_h(v_h) \stackrel{\text{def}}{=} \sum_{T \in \mathcal{T}_h} \sum_{S \in \mathcal{S}_T} \mathbf{r}_S(\llbracket v_h \rrbracket) + \sum_{F \in \mathcal{F}_h^b} \mathbf{r}_F(v_h)$. The discrete gradient is given by

$$\mathfrak{G}_h(v_h) \stackrel{\text{def}}{=} \nabla_h v_h - \mathbf{R}_h(v_h) \quad \forall v_h \in V_h,$$

The following key property corresponds to the weak asymptotic consistency (cf. [3, Hypothesis 2.4]).

Lemma 3 (Discrete Rellich theorem) *Let $(v_h)_{h \in \mathcal{H}}$ be a sequence in $(V_h)_{h \in \mathcal{H}}$ uniformly bounded in the $\|\cdot\|$ -norm. Then, there exists $v_h \in H_0^1(\Omega)$ such that, up to a subsequence, $v_h \rightarrow v$ strongly in $L^2(\Omega)$ and $\mathfrak{G}_h(v_h) \rightharpoonup \nabla v$ weakly in $[L^2(\Omega)]^d$.*

To prove the strong consistency of \mathfrak{G}_h for regular functions (cf. [3, Hypothesis 2.7]), one can use the space of piecewise smooth functions with continuous diffusive flux introduced in [2, Lemma 3.2]. The convergence proof relies on the following equivalent form for a_h in $V_h \times V_h$:

$$a_h(u_h, v_h) = \int_{\Omega} \kappa \mathfrak{G}_h(u_h) \cdot \mathfrak{G}_h(v_h) + j_h(u_h, v_h),$$

where $j_h(u_h, v_h) = - \int_{\Omega} \kappa \mathbf{R}_h(u_h) \cdot \mathbf{R}_h(v_h) + \sum_{T \in \mathcal{T}_h} \sum_{S \in \mathcal{S}_T} \int_S \frac{\eta \lambda_S}{h_S} \llbracket u_h \rrbracket \llbracket v_h \rrbracket + \sum_{F \in \mathcal{F}_h^b} \int_F \frac{\eta \lambda_F}{h_F} u_h v_h$.

Theorem 4 (Convergence to minimal regularity solutions) *Let $(u_h)_{h \in \mathcal{H}}$ denote the sequence of discrete solutions to (4) on the admissible mesh sequence $(\mathcal{T}_h)_{h \in \mathcal{H}}$ and let $u \in V$ solve (2). Then, $u_h \rightarrow u$ strongly in $L^2(\Omega)$ and $\nabla_h u_h \rightarrow \nabla u$ strongly in $[L^2(\Omega)]^d$.*

Table 1

Comparison of the proposed method (top) with the original ccG method of [4] (bottom) on the heterogeneous test case (5) (left) and on the anisotropic test case (6) (right). To appreciate the stencil reduction, compare the rightmost columns.

card(\mathcal{T}_h) $\ u - u_h\ _{L^2(\Omega)}$ order $\ u - u_h\ $ order stencil					card(\mathcal{T}_h) $\ u - u_h\ _{L^2(\Omega)}$ order $\ u - u_h\ $ order stencil						
proposed method					proposed method						
3584	1.3027e-03	-	4.5746e-02	-	15.03	806	6.2131e-03	-	9.0381e-02	-	14.69
14336	3.1039e-04	2.07	2.2608e-02	1.02	15.16	3162	1.7417e-03	1.83	4.5942e-02	0.98	15.25
57344	8.6435e-05	1.84	1.1581e-02	0.97	15.22	12632	5.7562e-04	1.60	2.2897e-02	1.00	15.38
229376	1.9971e-05	2.11	5.6241e-03	1.04	15.25	50548	1.4492e-04	1.99	1.1341e-02	1.01	15.52
ccG method of [4]					ccG method of [4]						
3584	5.2227e-04	-	2.8501e-02	-	25.26	806	4.0093e-03	-	4.8250e-02	-	25.33
14336	1.2926e-04	2.01	1.4194e-02	1.01	25.73	3162	1.7803e-03	1.17	2.7231e-02	0.83	26.35
57344	3.2545e-05	1.99	7.1126e-03	1.00	26.11	12632	4.9217e-04	1.85	1.3674e-02	0.99	26.85
229376	7.9803e-06	2.03	3.5293e-03	1.01	26.10	50548	1.3945e-04	1.82	6.9776e-03	0.97	27.12

3. Numerical validation

To assess the robustness with respect to the heterogeneity of κ , consider the following pseudo two-dimensional exact solution to (2) on the unit square domain $\Omega = (0, 1)^2$:

$$u = \begin{cases} -\frac{1}{2}x^2 + \frac{3+\epsilon}{4(1+\epsilon)}x & \text{if } x \leq \frac{1}{2}, \\ -\frac{1}{2\epsilon}x^2 + \frac{3+\epsilon}{4\epsilon(1+\epsilon)}x + \frac{\epsilon-1}{4\epsilon(1+\epsilon)} & \text{if } x > \frac{1}{2}. \end{cases} \quad \kappa = \begin{cases} 1 & \text{if } x < \frac{1}{2}, \\ \epsilon & \text{if } x > \frac{1}{2}, \end{cases} \quad f = 1. \quad (5)$$

The parameter ϵ represents here the heterogeneity ratio, and it has been taken equal to 10^{-3} . To evaluate the method with respect to the anisotropy of κ , we consider the following exact solution on $\Omega = (0, 1)^2$:

$$u = \sin(\pi x) \sin(\pi y), \quad \kappa_{xx} = 1, \kappa_{xy} = \kappa_{yx} = 0, \kappa_{yy} = \epsilon, \quad (6)$$

with adapted right-hand side f . The parameter ϵ represents, this time, the anisotropy ratio. Both tests are run on non-structured triangular meshes and the results are collected in Table 1. An inspection of the rightmost columns (which contain an estimate of the stencil based on the average number of non-zero element per row) shows a significant stencil reduction in both cases for the proposed method.

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