



Riemann hypothesis proof

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RIEMANN HYPOTHESIS PROOF

MUSTAPHA BEKKHOUCHA

ABSTRACT. The proof that one proposes here for the Riemann hypothesis on the nonreal zeros of the function $\zeta(s)$ comes indirectly :

- in the first stage, via its regular part $\zeta^*(s) = \zeta(s) - 1/(s-1)$, study of which in the upper half-plane, reveals the existence of a double zero of the latter, point which have the following property : the image of the first quadrant of apex $\omega_0 = \Re z_0$ is included in the first quadrant of apex the origin.
- in a second stage, via the function $U(s) = [\ln(1-s)\zeta^*(s)]^{-1}$, thanks to which one shows that $\zeta(s)$ does not have a zero in the same quadrant.
- finally, in a third stage, where one uses successive analytic determinations of $U(s)$, defined in successive sectors of quadrants of apex $\omega < \omega_0$, to show that the nonreal zeros of $\zeta(s)$ cannot be inside any of these sectors.

This result, together with the property of symmetry of the nonreal zeros of $\zeta(s)$ allows us to conclude.

1. INTRODUCTION

This work started from question in connection with the following remark : $\zeta^*(\Delta)$, for Δ a half-line of origin ω , $\frac{1}{2} - \epsilon < \omega < 1$, sweeping the first quadrant of apex ω , image of which at the beginning coincide with the segment $[\zeta^*(\omega), 1]$, with $\zeta^*(\omega) > 0$ and must varies according to the properties of the holomorphic transformations, by which distortions comes infinitely close to the origin, when the half-line tends towards the vertical ? (this was suggested by result of Titchmarsh ([2])).

The answer given to this questions necessitated the intervention of almost all the known properties of $\zeta(s)$, namely :

- Behaviour of $\zeta(s)$ in the half-plane $\Re s > 1$
- Real zeros of $\zeta(s)$ - Existence of an infinity of zeros of $\zeta(s)$ on the axis $\Re s = 1/2$
- Development of $\zeta^*(s)$ in series of $(s-1)$
- Symmetry of the nonreal zeros of $\zeta(s)$ relatively to the axis $\Re s = 1/2$
- Nonexistence (established by computer) of zeros of $\zeta(s)$ in the critical strip up to large ordinate, apart from the axis.

Thus the role of these families (family of functions connected to $\zeta(s)$, family of properties of $\zeta(s)$) is determinating in the proof of Riemann hypothesis. The same goes for its direct consequences :

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- Family of points of the axis $\Re s = 1/2$ associated with each zero of ζ .
- Similarly, some of these families could be associated by the ordinate of the corresponding ζ -zeros. (This is suggested by a theorem of Ingham, in connection with a conjecture of Mertens, see ([1]). In fact, the necessary condition for this, namely the multiplicity of all zeros $= 1$, can be established easily by using the complements in this paper.)

2. POSITION OF THE PROBLEM

In order to locate the complex zeros of the Riemann function $\zeta(z)$, which, as we know, are all situated in the strip $0 < \Re(z) < 1$, we'll first locate the image of the quadrant of apex ω , ω real neighboring $\frac{1}{2}$ at left, by the function $\zeta(z)$, or rather by its regular part

$$\zeta^*(z) = \zeta(z) - \frac{1}{z-1}$$

The quadrant is sweeping by a half-straight Δ_m , of origin ω and slope m , $m \geq 0$, in the direct way. The image $\zeta^*(\Delta_m)$ that start at the point $\zeta^*(\omega)$ on the real positive half axis with a tangent which has the same direction and the same way as Δ_m (because $\zeta'^*(\omega) > 0$, as we shall prove further) finish at the point 1 with a tangent whose direction is symmetrical to Δ_m direction related to the vertical. It must exists a value of m , for which $\zeta^*(\Delta_m)$ goes for the first time through the vertical of $\zeta^*(\omega)$. We know in fact that the origin 0 belong to the closure image of the quadrant by ζ^* : we are basing this statement on a well-know result ([2]), namely: there is a sequence of points z_n which tends to infinity on the vertical $\Re(z) = 1$ so that $\zeta(z_n) \rightarrow 0$. Thus, there is, for m enough large, a vertical tangent to $\zeta^*(\Delta_m)$ situated the most on the left.

3. MAIN RESULT

Proposition 1. *The contact point M (unique or not) of the vertical tangent to $\zeta^*(\Delta_m)$ the most on the left, depends analytically on m .*

Proof. M corresponds to a value of the parameter t on Δ_m , of equation $\sigma - \omega = t/m$, where $\sigma = \Re(z)$, $t = \Im(z)$, so that $\Re(\zeta^*(\omega + s))$ has a derivative related to t , equals to zero: we express by this that the tangent on M is vertical. So we have

$$(3.1) \quad \frac{\partial}{\partial t} \Re(\zeta^*(t, m)) = 0.$$

Let us consider now this equation in \mathbb{C} (complex field), namely with (t, m) complex variables, $\Re(\zeta^*)$ representing the analytical extension of the real part of ζ^* . Let us place on $m = m_1$, positive real. The equation (3.1) has among its roots, a finite number N_1 of real roots $t_j(m_1)$ which corresponds to $\Re(\zeta^*(t_j(m_1), m_1))$ the smallest of the values taken by $\Re(\zeta^*(t, m_1))$ on t real. That is, on one hand the roots are isolated, and on the other hand,

$$\Re(\zeta^*(t, m_1)) \longrightarrow 1 \text{ when } t \longrightarrow \infty$$

Some of these roots can be multiple (then, they are counted as their multiplicity order). So, in a neighborhood of m_1 sufficiently small, let us say a disc centered on m_1 in the complex plan, there among the roots of the equation (3.1), N_1 simple complex roots $t_j(m)$ which tend to $t_j(m_1)$ when $m \rightarrow m_1$. That not means, we have N_1 functions on disk. But, if we consider those of roots which have as imaginary part the smallest in absolute value, and next among these, those for which $\Re(\zeta^*(t_j(m), m))$ (here, it's not that analytical extension of $\Re(\zeta^*(t, m))$, but its the real part of the analytical extension of $\zeta^*(t, m)$) is the smallest, and finally, among these last, those for which $\Im(\zeta^*(t_j(m), m))$ (its about the imaginary part of the analytical extension of $\zeta^*(t, m)$) is the greatest, we really define a function on the disc by setting,

$$\Re(\zeta^*(t_h(m), m)) + i\Im(\zeta^*(t_h(m), m))$$

that is to say $\zeta^*(t_h(m), m)$, where $t_h(m)$ is some root carried after the three successive selection. This function is holomorphic in the disc. It is in fact possible to decide on an unique choice of $t_h(m)$ among the carried roots, by the two following successive selections : we take the $t_h(m)$ which the imaginary part is the smallest, in absolute value and on the whole, and among these last one, those which the real part is the smallest. Then $t_h(m)$ is well an holomorphic function of m , in a neighborhood of each point m of the pointed disk $\Delta(m_1, r)$. In fact, the index h will be the same for all the points of such a neighborhood if it is sufficiently small. Let us place in a neighborhood of m_0 , in the pointed disk $\Delta(m_1, r)$. The different solutions of the equation for each point m , make so much holomorphic function $t_h(m)$ in this neighborhood supposed sufficiently small. Let $t_{h_0}(m)$ one among those which take in m_0 the value associated by the preceding process. We'll prove that h_0 is still the value of h associated to all the points of a neighborhood of m_0 if it is sufficiently small. Let any index $h \neq h_0$:

Whether $\zeta^*(t_h(m_0)) \neq \zeta^*(t_{h_0}(m_0))$, and then, by continuity, it's impossible that h being hold again by the first selections, if m is sufficiently approaching by m_0 . For Instance, if, at first, we have

$$|\Im(\zeta^*(t_h(m_0)))| \neq |\Im(\zeta^*(t_{h_0}(m_0)))|$$

this implies

$$|\Im(\zeta^*(t_h(m_0)))| > |\Im(\zeta^*(t_{h_0}(m_0)))|$$

Then, by continuity, we have also

$$|\Im(\zeta^*(t_h(m)))| > |\Im(\zeta^*(t_{h_0}(m)))|$$

for any m sufficiently near m_0 . It results that the index h cannot be carried, because it not corresponds with minimum of $|\Im(\zeta^*(t_{h'}(m)))|$ when h' varies.

Whether $\zeta^*(t_h(m_0)) = \zeta^*(t_{h_0}(m_0))$, but $t_h(m_0) \neq t_{h_0}(m_0)$, and then, by continuity also, it's impossible that h being hold again by the last selections, if m is sufficiently approached by m_0 . It is clear, by making so for all the h , distinct from h_0 , that we must have $h = h_0$ for all the points of a neighborhood sufficiently small of m_0 . As $t_h(m)$ is holomorphic function in a neighborhood of m , and $\zeta^*(t, m)$ is holomorphic in (t, m) , the composited function $\zeta^*(t_h(m), m)$ is holomorphic in the disk minus the center. Finally, it is extended by

continuity, as an holomorphic function in all the disk. It is clear that its trace on the real axis in a neighborhood of m_1 is the affix of the highest contact point of $\zeta^*(\Delta_m)$ with its most left tangent. In fact, for m real neighboring m_1 , the first selection made on the $t_j(m)$ amounts to carry only those which are real (for, in this case, the equation (3.1) allows real roots), since there is at least those which correspond to the point the most on the left on $\zeta^*(\Delta_m)$. Then, by the second selection, we carry only those which correspond effectively to the point the most on the left, and, finally, we keep only those which correspond to these point the most left, and the highest on $\zeta^*(\Delta_m)$. It results from which proceed, that the highest contact point of $\zeta^*(\Delta_m)$ with its most to the left vertical tangent, depends analytically m . \square

It results from which preced, that the highest contact point of $\zeta^*(\Delta_m)$ with its most to the left vertical tangent, depends analytically m .

It is clear that we can show with the same manner, that the lowest contact point of $\zeta^*(\Delta_m)$ with its most to the left vertical tangent, depends analytically m .

More precisely, we've proved that the two extreme contact points (the highest and the lowest of $\zeta^*(\Delta_m)$ with its most to the left tangent, are holomorphic functions of m in an open neighborhood in the complex plan of certain half real straight $]m_\omega, +\infty[$, whose origin depends on ω .

Moreover, it emerges from the proof that the difference of these two functions is pure imaginary (its real part is zero), and therefore, it is constant, specially on $]m_\omega, +\infty[$.

More generally, we can say that the most on the left vertical tangent, has a fixed number of contact points and their mutual distances are constant. We'll see further that this number reduce to 1.

But, first, we've to prove the following proposition :

Proposition 2. *It is impossible that, from a point $-\lambda$, exterior to $\zeta^*(\Delta_m)$ situated on the most on the left vertical tangent, we could see $\zeta^*(\Delta_m)$ under an angle which exceeds a right angle by a small angle (by cutting on the curve a small arc) when the slope of Δ_m goes from m to $m + dm$.*

Proof. We are leading to introduce the function

$$(3.2) \quad \Phi_{t,d}(s) = [\lambda + \zeta^*(\omega + s) + d \exp(tse^{-i\theta})]^{-1}$$

where d and t are two positive parameters, $s = re^{i\theta}$ represent a Δ_m parametrization of polar angle θ . The logarithmic derivative with respect to $r = |s|$ is

$$(3.3) \quad \frac{\Phi'_{t,d}(s)}{\Phi_{t,d}(s)} = -\frac{e^{i\theta}\zeta'^*(\omega + s) + dte^{rt}}{\lambda + \zeta^*(\omega + s) + de^{rt}}$$

Reasoning by absurd, if we suppose that, going from m to $m + dm$, the angle under which we see from $-\lambda$ the image of Δ_m by ζ^* been lightly more than a right angle, holds too for the image by $z \mapsto \zeta^*(z) + d \exp(tze^{-i\theta})$ if d is choosen sufficiently small.

Moreover, it's a matter of the same right angle for $\zeta^*(\Delta_m)$ because they have the same horizontal side.

Let us designate by s' , s'' the curvilinear abscissa of points on Δ_m which have for images

the extremities of the small arc cut up on the curve image. We'll show that these gives a contradiction, if the parameters t and d are suitably choosen. For this, we are leading to consider the relative arrangement of $\Phi'_{t,d}(s)$ and $\Phi_{t,d}(s)$ when s describes the small segment $[s', s'']$, and thus when the representative point describes the small arc which is outside the right angle.

For the choice of the parameters, it's a matter of satisfying all at once the two following conditions :

- (1) $de^{tr''}$ (where $r'' = |s''|$ is supposed $> r' = |s'|$) could be made as small as we want, in such a way the space difference (horizontally measured) between the image curves of Δ_m by $\zeta^*(\omega + s) + d \exp(tse^{-i\theta})$ been infinitely small in a neighborhood of the contact point of $\zeta^*(\Delta_m)$ with the most of the left vertical tangent.
- (2) de^{tr} could made as large as we want with respect to $|\zeta^{*'}(\omega + s)|$ in such a way that the numerator argument of the second number of the above formula be infinitely small, always for r between r' and r'' .

Let us precise this:

Given ϵ , M two positive numbers, ϵ arbitrary small, M arbitrary large, we've

$$dte^{rt} = de^{r't}.te^{(r-r')t}$$

we suppose t sufficiently large for the second factor to be superior to $M.\epsilon^{-1}$. A such t been fixed, we choose d in such a way that $de^{tr''} = \epsilon$. In these conditions, the angle-space between $\Phi_{t,d}(s)$ and $\Phi'_{t,d}(s)$, space which is given by the argument of $\frac{\Phi'_{t,d}(s)}{\Phi_{t,d}(s)}$ could be made as near as we want the argument of

$$\frac{-1}{\lambda + \zeta^*(\omega + s) + d \exp(tse^{-i\theta})}$$

when $\zeta^*(\omega + s) + d \exp(tse^{-i\theta})$ describes the arc included between the two tangents taken from the point $-\lambda$.

By taking as the origin the horizontal tangent, the argument of

$$\lambda + \zeta^*(\omega + s) + d \exp(tse^{-i\theta})$$

carries from 0 to $\pi/2 + \delta$ in the case where $-\lambda$ is above the contact point with the vertical tangent, and varies from 0 to $-\pi/2 - \delta$ in the case where $-\lambda$ is below. We can obviously confine ourself to the first case.

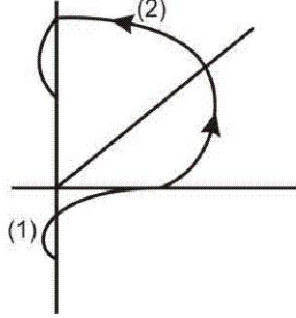


FIGURE 1

We've oversimplified on the Figure 1: The arc (1) described by $\lambda + \zeta^*(\omega + s) + d \exp(tse^{-i\theta})$. The arc (2) described by $(\lambda + \zeta^*(\omega + s) + d \exp(tse^{-i\theta}))^{-1}$ i.e, $\Phi_{t,d}(s)$.

Let us follow the advance of $\Phi_{t,d}(s)$, from the contact point with the real axis, in the decreasing r way. We can distinguish two phases, corresponding respectively to these half quadrant : $\Re(\Phi_{t,d}(s))$ and $\Im(\Phi_{t,d}(s))$ increase for been in accordance with the fact $\Phi'_{t,d}(s)$ been then in the 3rd quadrant, the functions $\Re(\Phi_{t,d}(s))$ and $\Im(\Phi_{t,d}(s))$ are decreasing: since r decrease, $\Re(\Phi_{t,d}(s))$ and $\Im(\Phi_{t,d}(s))$ increase.

On the arc situated in the 2nd half quadrant, $\Re(\Phi_{t,d}(s))$ decrease, while $\Im(\Phi_{t,d}(s))$ continuous to increase, which is still in accordance with the fact that, $\Phi'_{t,d}(s)$ been this time in the 4th quadrant, the function $\Re(\Phi_{t,d}(s))$ is increasing while the function $\Im(\Phi_{t,d}(s))$ is decreasing: since r decrease, $\Re(\Phi_{t,d}(s))$ decrease and $\Im(\Phi_{t,d}(s))$ increase.

That contradiction comes about the arc which gets out the quadrant. That's for $\Phi'_{t,d}(s)$ is so in the 1st quadrant, and then $\Re(\Phi'_{t,d}(s))$ been positive, $\Re(\Phi_{t,d}(s))$ had to don't step decreasing, while obviously, it has at least to come through an episode of increasing. \square

We've seen that it follow from the proposition 1, that the number of the contact points of $\zeta^*(\Delta_m)$ with its most on the left vertical tangent, is fixed when m changes, and that their mutual distances remained constant. Thanks to the proposition 2, this result can be precised by the following manner:

Proposition 3. *The contact point of $\zeta^*(\Delta_m)$ with its most the left tangent, is unique.*

Proof. For that, we note firstly that we've analogous results about the highest horizontal tangent to $\zeta^*(\Delta_m)$. But furthermore, in this case, we easily se that theses results are good for any apex ω of the quadrant as far as it is on the half real positif axis.

But, if we fix m , and we give analogous proof to these given higher, taking for this time as variables (t, ω) , we can establish the analyticity with respect to ω , of the contact point (unique or not) of $\zeta^*(\Delta_m)$ with highest horizontal tangent, and, the unvarying of the number of the contact points and their mutual distances. As the image curve $\zeta^*(\Delta_m)$ tends to be

confused with the point 1, when $\omega \rightarrow \infty$, we conclude that there is only one contact point. Then returning to the situation examined firstly, we observe that the contact point of $\zeta^*(\Delta_m)$ with its most to the left vertical tangent, be infinitely neighboring of the contact point with the highest horizontal tangent. That's because the proposition 2 excluded other contact points. \square

As we've seen above, the contact point of $\zeta^*(\Delta_m)$ with its most on the left vertical tangent is infinitely approaching the contact point of the curve with its highest horizontal tangent, so that we've nearly a retrogression point. It must be supposed moreover that it's under favor of a retrogression point that $\zeta^*(\Delta_m)$ could have for the first time, a most on the left vertical tangent. From this, m increasing, the curve couldn't progress to the left so as to avoid the application of the proposition 2, then, so as the most on the left point and the highest point remained infinitely neighboring.

Proposition 4. *The image by ζ^* of the quadrant of apex ω , $\omega \geq 1/2$ is entirely contained in the upper half plane, unless the quadrant contains a z_0 such as $\zeta^*(z_0)$ is real and $\zeta^{*'}(z_0) = 0$.*

Proof. We suppose $1/2 \leq \omega < 1$, the upper boundary 1 is taken for being in the conditions of the preceding propositions.

But its obvious that all we have to do is to establish the result for all $\omega \geq 1/2$, for the image of the quadrant are included some into others when ω grows.

Let us follow the advance towards the left of the contact point M of $\zeta^*(\Delta_m)$ with the most left vertical tangent. A such tangent surely exist as soon as m is large enough, since $\zeta^*(\Delta_m)$ coming through the neighboring of the origin: as we've pointed out highly, it cuts the vertical $\zeta^*(\omega)$ in at least two points and necessary has a vertical tangent in certain points of the arc bounded by these two points.

We reason by absurd, by supposing that $\zeta^*(\Delta_m)$ comes, for the first time, to touch the real axis in a point, the real axis is then the lowest horizontal tangent to $\zeta^*(\Delta_m)$: we'll note it T_{B_0} . Let $-\lambda_0$ be the point where T_{B_0} meet the most left vertical tangent T_{G_0} . Given to m a growth dm , let T_B be the new position of the lowest horizontal tangent to $\zeta^*(\Delta_{m+dm})$, $-\lambda$ the point where it meets T_{G_0} , T_G the most left tangent to $\zeta^*(\Delta_{m+dm})$ taked from $-\lambda$. (T_B, T_G) is then the angle under which we see the curve $\zeta^*(\Delta_{m+dm})$ from the point $-\lambda$. It exceeds on right angle the angle (T_{G_0}, T_G) which cuts on $\zeta^*(\Delta_{m+dm})$ a little arc, if dm is sufficiently small.

There is however a situation where the preceding proof is not possible: It is when $-\lambda_0$ is precisely the point where $\zeta^*(\Delta_m)$ comes to touch the real axis. That occurs only if it is retrogression point (which corresponds then to a point z_0 of Δ_{m_0} so that $\zeta^*(z_0)$ is real and $\zeta^{*'}(z_0) = 0$).

But as soon as the real axis is crossed by Δ_m images for $m > m_0$, in order to avoid that proposition 2 been applied again, that leads to a contradiction, we have to suppose that $\zeta^*(\Delta_m)$ has, in the inferior half-plan, only one small arc, getting round nearly the point $\zeta^*(z_0)$, and which the lowest point is nearly confused with the most left point. The first one is then the contact point of $\zeta^*(\Delta_m)$ with its lowest horizontal tangent, a unique contact

point as for the highest horizontal tangent. The second one must be supposed infinitely neighboring of the trace on the real axis of the most left vertical tangent. \square

Lemma 1. *The first quadrant which has its image which crosses the real-axis has for apex ω_0 , and ω_0 is the orthogonal projection of a double zero of ζ^* . It's the unique zero of ζ^* in the upper half-plane.*

Proof. (1) By considering at the beginning $-\omega$ being fixed, the first value of m for which the point on the left of $\zeta^*(\Delta_m)$ has an x -coordinate $= \zeta^*(\omega)$, it cannot have a point of $\zeta^*(\Delta_m)$ below the real axis, nor on, (apart from $\zeta^*(\omega)$ and 1). In fact : let us call $z_1(m, \omega)$ the antecedent of the point on the left, $z_2(m, \omega)$ that of the point low, supposed to exist.

It is known that z_1, z_2 are analytical functions (and holomorphic in (m, ω) in the field of complexes. We saw for z_1 in the course of demonstration of proposition 1, and, for z_2 , it is clear that one has a similar demonstration.

Under the particular conditions where one placed itself, we have

$$\begin{cases} \Re(\zeta^*(z_1(m, \omega))) = \zeta^*(\omega) & (1) \\ \zeta^{*'}(z_2(m, \omega)) = 0 & (2) \end{cases}$$

The equation (2) is expressing the fact that the point low of $\zeta^*(\Delta_m)$ is necessary a retrogression point, because the application of proposition 2, interdict that is a regular point.

Let us designate by (m_0, ω_0) a solution of (1). If

$$\frac{\partial}{\partial m} \Re(\zeta^*(z_1(m, \omega)))_0 \neq 0$$

then (1) is solved in the neighborhood of (m_0, ω_0) by $m = m(\omega)$ where $m(\omega)$ is an analytical function, and takes the value m_0 for $\omega = \omega_0$. Now

$$\frac{\partial}{\partial m} \Re(\zeta^*(z_1(m, \omega))) = \frac{\partial}{\partial t} \Re(\zeta^*(z_1(m, \omega))) \cdot \frac{\partial t}{\partial m} + \frac{\partial}{\partial \sigma} \Re(\zeta^*(z_1(m, \omega))) \cdot \frac{\partial \sigma}{\partial m}$$

with $\frac{\partial}{\partial t} \Re(\zeta^*(z_1(m, \omega)))_0 = 0$ (according to equation (3.1)). On the other hand $(\frac{\partial \sigma}{\partial m})_0 \neq 0$ in accordance with the unicity of the value of σ (or t) corresponding to the point of contact with $\zeta^*(\Delta_{\omega_0, m_0})$ of the vertical tangent on the left (cf proof of equation (3.1))

So one has

$$\frac{\partial}{\partial m} \Re(\zeta^*(z_1(m, \omega)))_0 \neq 0$$

iff $\frac{\partial}{\partial \sigma} \Re(\zeta^*(z_1(m, \omega)))_0 \neq 0$ or $\frac{\partial}{\partial t} \Im(\zeta^*(z_1(m, \omega)))_0 \neq 0$ (conditions of Cauchy- Riemann), which, taking into account

$$\frac{\partial}{\partial t} \Re(\zeta^*(z_1(m, \omega)))_0 = 0$$

is equivalent to $\zeta^{*'}(z_1(\omega, m))_0 \neq 0$ ie, the point on the left of $\zeta^*(\Delta_{\omega_0, m_0})$ is not a retrogression point. Thus, let us suppose at first, that one in this case.

Then, as one saw higher, (1) is solved in the neighborhood of ω_0 by $m = m(\omega)$, analytical and such as $m(\omega_0) = m_0$. Consequently, the point low of $\zeta^*(\Delta_{\omega_0, m_0})$ describes a continuous arc, arc which cannot be reduced to a point since wary while remaining nearest to the vertical tangent on the left, which has abscissa $= \zeta^*(\omega)$. Because, by other hand, it is a retrogression point of $\zeta^*(\Delta_{\omega, m(\omega)})$, thus a point where the derivate $= 0$ (condition 2 of the system), one has a contradiction with the fact that zeros of a holomorphic function $\neq 0$, of a single variable, are isolated. One concludes from it, that examine situation doesn't take place.

Let us consider now the case $\zeta^*(z_1(\omega_0, m(\omega_0)))$ is a retrogression point. It will be sufficient to show that one can always find a sequence of points $\omega_n \rightarrow \omega_0$, such as the corresponding sequence $m(\omega_n)$ (it is clear that, in all the cases, it corresponds to any ω , starting from a certain value, a single value of m , such as the equation is verified) has a limit, and for which $\zeta^*(z_1(\omega_n, m(\omega_n)))$ is not a retrogression point of $\zeta^*(\Delta_{\omega_n, m(\omega_n)})$.

In fact, under these conditions, these last curves, not having a point in the lower half-plane, by continuity, it will be similarly for $\zeta^*(\Delta_{\omega_0, m(\omega_0)})$.

For that, one reasons by absurd. Thus, let us suppose that for any sequence $\omega_n \rightarrow \omega_0$ we have $\zeta^*(z_1(\omega_n, m(\omega_n)))$ is a retrogression point of $\zeta^*(\Delta_{\omega_n, m(\omega_n)})$. It is obvious that there is at last one of there sequences for which $m(\omega_n)$ is bounded. One can extract a convergent sequence, $\zeta^*(z_1(\omega_n, m(\omega_n)))$ which corresponds to him, is a convergent sequence where the derivate $= 0$. It is the sought contradiction.

- (2) ω being always fixed, now let us consider the first value of m for which $\zeta^*(\Delta_m)$ crosses for the first time, the real-axis. Let us call $\zeta^*(z_0(m, \omega))$ this point of contact with the real-axis. One will show that $z_0(m, \omega)$ does not depend on ω , and that $\zeta^*(z_0(m, \omega))$ also. Since $\zeta^*(z_0(m, \omega))$ is the point low of $\zeta^*(\Delta_m)$ it depends analytically on (m, ω) . One knows also it is necessarily a retrogression point. $z_0(m, \omega)$ thus verified

$$\begin{cases} \Im \zeta^*(z_0(m, \omega)) = 0 & (1) \\ \zeta^{*'}(z_0(m, \omega)) = 0 & (2) \end{cases}$$

Given $(\omega, m(\omega))$ the general solution of (1) ($m(\omega)$ understood here as the single value of m corresponding to ω , so that $\zeta^*(\Delta_{\omega, m})$ intersect for the first time the real axis) Let us designate $(\omega_0, m(\omega_0))$ a particular solution. One will show that the function $m(\omega)$ is continuous.

- First, it is easy to see that it is an increasing function. Indeed, if $\omega < \omega_0$, the sector limited by Δ_{ω, m_0} because it contains the sector limited by Δ_{ω_0, m_0} , has already points of its image in the lower half plane. It is thus for some $m < m_0$ that $\zeta^*(\Delta_{\omega, m(\omega)})$ intersect for the first time the real axis.
- The function $m(\omega)$ is surjective : In fact, let us designate Z_0 the point of Δ_{ω_0, m_0} whose the ζ^* image is the point of contact of $\zeta^*(\Delta_{\omega_0, m_0})$ with real axis. Supposing $m < m_0$. Let us sweep the higher half plane by half straight of slope fixed m , and origin varying in $]-\infty, \omega_0[$. It is clear that if ω is sufficiently on the left of ω_0 , the sector limited by $\Delta_{\omega, m}$ will going to

contain Z_0 , and consequently its will have already points in the lower half-plane. Necessarily, there was before a position of ω for which $\zeta^*(\Delta_{\omega,m})$ intersect for the first time, the real axis.

- One can now conclude that, $m(\omega)$ being an increasing and surjective function of $] -\infty, .[\rightarrow]0, .[$ is necessarily continuous.

Its results from it that $z_0(m, \omega(\omega))$ is continuous. On account of equation (2), one concludes that, if $z_0(m, \omega(\omega))$ were not fixed, one would have a contradiction with that $\zeta^{*'} \equiv 0$ in all the points of a continuous arc described by $z_0(m, \omega(\omega))$.

- (3) Profiting of the fact that z_0 is independent of ω , let us tend ω towards $-\infty$. That is possible in fact: at first, the equalities $\zeta^*(-2n) = 1/2n + 1 \ \forall n \in \mathbb{N}$, imply that there are intervals as far as we want on the negative real axis in which ζ^* is increasing and $\zeta^{*'} positive. Then, this last property extends at all the points of \mathbb{R} , because $\forall \omega \exists \omega' < \omega$ where $\zeta^{*'}(\omega')$ is positive, and which is thus the apex of a sector of the higher half-plane, of origin-side carried by the real axis and of which the image by ζ^* is very whole in the upper half-plane (provided that the sector does not contain the point z_0) - What implies that $\zeta^{*'}(\omega)$ is positive throughout the real axis. ω thus tending towards $-\infty$, because $\zeta^*(\omega)$ tends toward 0 by decreasing positive values, and $\zeta^*(\Delta_{m,\omega})$ crosses for the first time the real axis only one point on the left of $\zeta^*(\omega)$ (the point $\zeta^*(z_0)$), there is $\zeta^*(z_0) \leq 0$. We show that one has in fact $\zeta^*(z_0) = 0$ (and thus z_0 is double zero of ζ^*).$

For that, one will reason by the absurd by supposing $\zeta^*(z_0) < 0$. One take $\omega < 1/2$. If the image of Δ_m by ζ^* did not surround the origin underneath, for m rather large, it would result from it as we will see it with proposition 5 - then ζ does not have zero in the quadrant of apex ω , which is in contradiction with existence of zeros on the axis $\Re z = 1/2$

It is via its initial arc (which starts in $\zeta^*(\omega)$) that $\zeta^*(\Delta_m)$ passes below the origin while cutting the real-axis obliquely, because, any other manner is not allowed for him under the proposition 2. But, with the limit ($\omega = -\infty$), also this manner of surrounding the origin becomes impossible, because the curve image of horizontal, admitting the origin as initial point, with a tangent of slop varying from 0 to $+\infty$, could surround the origin underneath only while crossing the real-axis on the right of origin, which, once again, is not allowed by proposition 2.

□

We can add the following argument:

Let show that the angle φ whose the extremity radius vector described the image of the sides of the horizontal band $[0, h]$ is \neq multiple of 2π , as it would being if the band contained zero of ζ .

ζ^* -image of origin-side is the segment $[0, 1]$. The image of second side:

-on the one hand, circumvents $\zeta^*(z_0)$ by an arc contained in the lower half-plane. Let us indicate by α , $\alpha < \zeta^*(z_0)$, its left extremity.

-on the other hand, admits the origin as double-point, with tangent of polar angles respectively $0, \theta_0$, $0 < \theta_0 < \pi/2$. One have:

$\varphi = \pi$ (corresponding to the arc between 1 and α in the upper half-plane) $+ \theta_0$ (corresponding to the arc between α and 0) $+ (-\theta_0)$ (corresponding to the final loop of the curve) $= \pi$. One can conclude $\zeta^*(z_0) = 0$.

Before coming to, with proposals 5 and 6, the proof itself of the Riemann hypothesis one introduces below two questions which do not have a direct connection with it, and where the interest will appear only later.

- (1) Disposition of $\zeta^*(\Delta_{\omega, m_0})$ in the neighborhood of the retrogression-point $\zeta^*(z_0)$.

When ω varies from 0 to $-\infty$, the half-straight $z_0\omega$ turn $-\pi/2$. Then, because $\zeta^*(z_0) = 0$, and the first derived of $\zeta^*(z)$ at z_0 , the non-null one, is even, more precisely of order 2 (otherwise, $\zeta^*(\Delta_{\omega, m_0})$ would have points in the lower half-plane, in the neighborhood of the origin), the retrogression tangent turns $2 \times (-\pi/2) = -\pi$ around $\zeta^*(z_0)$.

That does not leave him an other choice only: tend to being directed in the direction of the negative half real-axis when $\omega \rightarrow \omega_0$ and in the opposed direction when $\omega \rightarrow -\infty$. A particular intermediate situation is that where $z_0\omega$ has a polar angle $= -3\pi/4$; in which case, the retrogression tangent is directed as the positive imaginary axis.

- (2) Behavior of $\zeta^*(\Delta_{\omega, m})$ for $m > m_0$

It is clear that for $m > m_0$, the image of the sector of apex ω , with sides having respective slopes 0 and m , contains the image of the sector of same origin-side, and with extremity-side of slope m_0 . One concludes from it that the application of proposition 2 has impossible being for the first values of $m > m_0$. In order to clarify that, best is to place itself in the limit case $\omega = -\infty$. Δ_{ω, m_0} is then the z_0 horizontal $H z_0$. $\zeta^*(H z_0)$ is the limit of $\zeta^*(\Delta_{\omega, m_0})$ when $\omega \rightarrow -\infty$.

Its retrogression-point is exactly at the foot of the vertical tangent on the left. In fact, let us consider the situation for $\omega \neq -\infty$, sufficiently far. Let us call θ_1 the polar angle of the first half-straight of origin ω , having its image crosses the $\zeta^*(\omega)$ vertical. Let us call θ_0 the polar angle of the first half-straight having its image which crosses the realaxis at 0, retrogression-point as one know. It arises from the lemma demonstration that one has $\theta_0 > \theta_1$. Now, the retrogression tangent has a polar angle $= 2\theta_0$, and one saw that this angle $\rightarrow 0$ when $\omega \rightarrow -\infty$. It results from it that, so $\theta_1 \rightarrow 0$.

Therefore, when $\omega \rightarrow -\infty$, the images of the both half straight tend to have their tangents coincided at 0. What's more, their images coincides. In fact :

The contrary implies that the point on the left of $\zeta^*(H z_0)$ is strictly on the left of imaginary axis. So as well, the analogous point of $\zeta^*(\Delta_{\omega, m_0})$ for $\omega < 0$ in the neighborhood of $-\infty$. Then, in accordance of the proposition 2, the point low of the arc going to become the retrogression arc of $\zeta^*(H z_0)$, shouldn't to exist : in fact, the proposition 2 can be easily extended to the points with horizontal tangent and concavity turned upwards. It will be able to apply here, since the point concerned, tending towards 0, is clearly isolated of the left vertical tangent at $\zeta^*(\Delta_{\omega, m_0})$.

From there, ie. for horizontal with ordinates $> \Im z_0$ the low point of their image by ζ^* could by continuity, follow displacement towards the left, of the vertical tangent on the left. It is precisely that which makes impossible the application of proposition 2, the point $-\lambda$ being with the right hand side of the point low (the point $-\lambda$ must

be necessarily placed on the negative imaginary demi-axis, and consequently on the right of the point low of the curve). Let us follow the progression of the ζ^* image of the horizontal when its ordinate increase from that of $H z_0$. It can happen that for a certain position of the horizontal the point low of its image coincide again with the foot of the vertical tangent on the left, on the horizontal tangent lowest. It is then a retrogression-point. Let us call z_1 its antecedent (by analogy with z_0 , the antecedent of the retrogression point of $\zeta^*(H z_0)$). One will see that this analogy goes, in fact, further : $\zeta^*(z_1)$ is the point low on all the $\zeta^*(\Delta_{\omega, z_1})$ curves, $\omega < \omega_1 < \omega_0$, $\omega_1 = \Re z_1$. It is a retrogression point for there curves, with a tangent whose polar angle varies from 0 (for $\omega \rightarrow -\infty$) to π (for $\omega \rightarrow \omega_1$) :

One bases oneself on the fact that the point low of $\zeta^*(\Delta_{\omega, z_1})$ (or $\zeta^*(\Delta_{\omega, m_1})$ by putting $m_1 = \text{slope of } \omega z_1$), depends analytically on ω , which is shown of same way as, for instance, the analytic of the high point of $\zeta^*(\Delta_{\omega, m})$, when, m remaining constant, ω varies. Because, on the other hand, the function holomorphic ζ' take necessarily the value 0 at this point (without what, proposition 2 applies again, and led to a contradiction), its results of what proceeds, that the point concerned can only be fixed.

It's a matter of the same point $\zeta^*(z_1)$, which is the point low of $\zeta^*(H z_1)$, as one see where $\omega \rightarrow -\infty$. The both other assertions result from this, easily, and so the fact that z_1 verifies: $\Re z_1 < \Re z_0$, $\Im z_1 < \Im z_0$.

Clearly, if the ordinate of the horizontal, continue to increase, the preceding processes can be repeated. Unless it would not be a position of the horizontal above $H z_0$, for which the point low of its ζ^* -image coincides with the foot of the vertical tangent on the left, these is a sequence $z_1, z_2, \dots, z_k, \dots$, placed more and more high, and more and more on the left, whose ζ^* -images are the point of $\zeta^*(\Delta_{\omega, m_k})$ ($\omega < \omega_k = \Re z_k$, $m_k = \text{slope of } \omega z_k$) and are retrogression points for these last.

Proposition 5. *If ω , $1/2 < \omega < 1$, is the apex of a quadrant in which ζ^* has no double zero, ζ has no zero in this quadrant.*

Proof. It is clear that the proceeding used for the study of the image of the quadrant by ζ^* , can be applied to any function f , having analogous properties, especially the following ones:

- Derivative strictly positive in the apex ω of the quadrant.
- Existence of images-points on the left of the $f(\omega)$ -vertical.
- Existence of a limit $f(z)$ when $z \rightarrow \infty$ on any half-straight issued from ω , inside the quadrant, and $f(\infty) > f(\omega)$.

Besides, on the real half straight $[\omega, +\infty]$, $\Re f(z) > f(\omega)$, $\Im f(z) \geq 0$. Let us consider in particular the function

$$U(z) = (\log(1 - z)\zeta^*(z))^{-1}$$

defined as follows:

Taking the principal determination of \log in the cutted plane in the negative half real axis, $\log \zeta^*(z)$ has a meaning for z in the quadrant, by virtue of the proposition 4. In other hand,

$\log(1 - z)$ is obviously well defined.

Setting

$$\log((1 - z)\zeta^*(z)) = \log(1 - z) + \log \zeta^*(z)$$

we have an holomorphic function in the quadrant, and then, it's also the case for $U(z)$, outside of the points where $\log((1 - z)\zeta^*(z)) = 0$. The formula

$$U(z) = \frac{\log |(1 - z)\zeta^*(z)| - i \arg((1 - z)\zeta^*(z))}{(\log |(1 - z)\zeta^*(z)|)^2 + (\arg((1 - z)\zeta^*(z)))^2}$$

(where $\arg((1 - z)\zeta^*(z))$ indicates in reality $\arg(1 - z) + \arg(\zeta^*(z))$) shows that there is a discontinuity of $\Im(U(z))$ on the real axis, at the point 1. We'll avoid it by surrounding by a half-circle centered in 1, of arbitrarily small radius, in the upper half-plane.

Thus, we are assured that $\arg(1 - z)$ varies continuously between 0 and $-\pi$, in the closed quadrant except for a small half disc, and, in consequence, $U(z)$ is continuous in this domain and holomorphic inside (except of course for points where $\log((1 - z)\zeta^*(z)) = 0$, i.e at the eventual zeros of $\zeta^*(z)$ in the quadrant).

Let us verify that $U(z)$ satisfied the required conditions:

- We have

$$U'(\omega) = [\log((1 - \omega)\zeta^*(\omega))]^{-2} \cdot \left[\frac{1}{1 - \omega} - \frac{\zeta^{*'}(\omega)}{\zeta^*(\omega)} \right]$$

By using the expansion

$$\zeta^*(z) = \sum_{r \geq 0} \gamma_r (z - 1)^r$$

where the coefficients (reals) verified $\sum_{r \geq 0} |\gamma_r| < 1$ (see [1] p.74, ex 2-4), we have, easily:

$$\zeta^{*'}(\omega) = \sum_{r \geq 1} r \gamma_r (\omega - 1)^{r-1}$$

increased on $[1/2, 1]$ by $\sum_{r \geq 1} r |\gamma_r| (1/2)^{r-1}$ and then by

$$(1 - \gamma_0) \sup_{r \geq 1} (1/2)^{r-1} = 1 - \gamma_0$$

where γ_0 is the Euler constant, and elsewhere $\zeta^*(\omega)$ is bounded by

$$\gamma_0 - (1 - \gamma_0) \sup_{r \geq 1} (1/2)^r = \gamma_0 - (1 - \gamma_0)/2$$

It result that :

$$\frac{1}{1 - \omega} - \frac{\zeta^{*'}(\omega)}{\zeta^*(\omega)} > (1 - 1/2)^{-1} - \frac{1 - \gamma_0}{\gamma_0 - (1 - \gamma_0)/2}$$

is a positive number and consequently $U'(\omega) > 0$. - The expression $\log((1 - z)\zeta^*(z))$ has its real part $\log |(1 - z)\zeta^*(z)|$ which tends to $+\infty$ when $z \rightarrow \infty$ on Δ_m , and in consequence $U(z) \rightarrow 0$, in other word $U(\infty) = 0$. Elsewhere $U(\omega) = \log(1 - \omega)\zeta^*(\omega)$, with $\zeta^*(\omega) < \zeta^*(1)$ ($= \gamma_0$) for the function $\zeta^*(z)$ is increasing on $[\omega, +\infty]$, its derivative being positive on this interval, as it results from the hypothesis, by virtue of the proposition 4: (if in a point of the interval, the derivative was negative, the image on the quadrant by ζ^* , had some points

below the real axis). Thus, we have $U(\omega) < 0$, and then $U(\infty) > U(\omega)$.

- Finally, as to the existence of the images-points at left of the vertical of $U(\omega)$, this contingency will be examined in the proof which follows. \square

We can then proceed to a study of $U(\Delta_m)$, analogous to that one about $\zeta^*(\Delta_m)$, at least in the sector of the quadrant included between the real axis and the first half-straight issued from ω that goes through a singularity of $U(z)$, i.e a zero of ζ . More exactly, to avoid that there was more than one zero of ζ , we'll replace, in case of need, ω by a suitable point, arbitrarily neighboring at its left. Let us call ρ_0 , the unique zero of ζ , be on the half-straight. There are three contingencies when $z \rightarrow \rho_0$, on the half straight: or $\Re U(z) \rightarrow +\infty$, or $\Re U(z) \rightarrow -\infty$, or $\Re U(z) \rightarrow \pm\infty$ that correspond respectively to the case where : $\Re(\log((1-z)\zeta^*(z))) \rightarrow 0_+$, or $\rightarrow 0_-$, or $\rightarrow 0_\pm$

That is to say $|(1-z)\zeta^*(z)| \rightarrow 1$ by superior values, by inferior values or by the both. We still can, in case of need, replacing ω by an appropriate point arbitrarily neighboring its right, choicen in the manner to modify the inclination of the tangent to the half straight image by $(1-z)\zeta^*(z)$ at the point 1, image of ρ_0 , so that we had the third eventualities.

Thus $U(\Delta_{m_0})$ has two branches which went to infinity, one by the left, the other by the right. We will prove that it gives rise to a contradiction. In fact, it results in particular that $U(\Delta_m)$ has its most left point which tends to infinity at left when $m \rightarrow m_0$.

Now, as well as we have seen precedently for $\zeta^*(\Delta_m)$, the most left point and the highest point of $U(\Delta_m)$, for m sufficiently neighboring of m_0 , are infinitely neighboring, they are nearly to form a retrogression point.

Its results that $U(\Delta_{m_0})$ admits in $U(\rho_0)$ a retrogression point, i.e, it has two infinite branches asymptot to an oblique directed to the direction of $U(\rho_0)$. This is in contradiction with the initial hypothesis on the infinite branch of $U(\Delta_{m_0})$. This argument by absurd shows that ζ has no zero in the quadrant.

We'll establish, by an analogous process to whese been used for proposition 5 the :

Proposition 6. *The complex zeros of ζ are all on the axis $\Re(z) = 1/2$.*

Proof. It is naturally to the function $U(z)$ that we call again, as it was the case for the proof of proposition 5. This function already defined in the sector of apex ω , $\omega < \omega_0$, included between the real axis and the half-straight ωz_0 , can be prolonged analytically in an additional sector of sides $\Delta_{m_0}, \Delta_{m_1}$, m_1 being the value of m for that $\zeta^*(\Delta_m)$ come to touch (ie with a contact point) for the first time $\zeta^*(\Delta_{m_0})$ on the arc included between $\zeta^*(\omega)$ and 0, in such a way that it form around the origin a loop composed of one arc of each of the two curves.

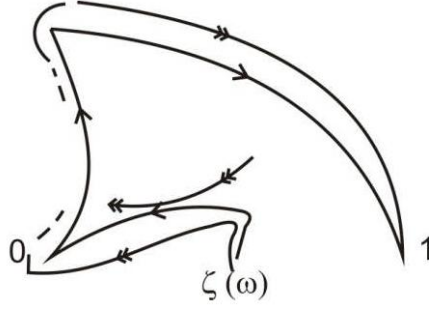


FIGURE 2

Δ_m continuing to turn from Δ_{m_1} , sweeps a new sector $\Delta_{m_1}, \Delta_{m_2}$ where we can define a new holomorphic determination of $U(z) : U_{\omega,1}(z)$, from the new holomorphic determination of $\log \zeta^*(z)$: $\log_{\omega,1}(z) = \log \zeta^*(z) - 2i\pi$, the log of the second member being the principal determination. And so one after another ..., we divide the quadrant of apex ω , into sectors $S_{\omega,k}$, $k \in \mathbb{N}$, in which the successive analytical prolongation of $U(z)$, $U_{\omega,k}(z)$ are defined, by the intermediate of the successive analytical prolongation of $\log \zeta^*(z)$: $\log_{\omega,k}(z) = \log_{\omega,k-1}(z) - 2i\pi$. The formula above imply that the laces which form around the origin by the image of sectors $S_{\omega,k}$, surround the origin once. For the previous formula, it has been supposed that $\zeta^*(\partial S_{\omega,k})$ forme a single cycle. But, it is not certainty. However, we note again $\log_{\omega,k}(z)$ the function $\log \zeta^*(z) - 2ik\pi$, understood that it's not necessary in the k_e sector that this function is defined, but, in a sector with an index, possibly $\neq k$. In fact, the concerned formula is not necessary for the demonstration of the Riemann hypothesis. The preceding construction holds for any ω , $\omega < \omega_0, \omega = -\infty$ included. In this last case, the sectors are replaced by horizontal strips. This operation being made, we'll show analogously as for the quadrant of apex ω_{θ_0} , that none of the interior point to each of the preceding sectors can be a zero of ζ : let ρ be such a point, supposed interior to $S_{\omega,k}$. It is obvious that it is still interior to each of $S_{\omega',k}$ sectors, for ω' sufficiently neighboring of ω .

It is also obvious that, in the same conditions, the diverse prolongations $U_{\omega',k}$ defined a same function in $\bigcup_{\omega'} S_{\omega',k}$. It's to this function, or rather to these deduces from by replacing the $\log_{\omega',k}(\dots)$ by $\log_{\omega',k}(\dots) + 2ik\pi$ that we'll apply the indicated proceeding, as it's for this latest that the eventual zeros of ζ in the considerate domain, arc singularities, the function tending to infinite in their neighboring.

The only points at left of the quadrant of apex ω_0 , which don't give way to the preceding proceeding, and which consequently, could be eventually the zeros of ζ , are the points of intersection of the extremities sides of the sectors $S_{\omega,k}$, for all ω , $\omega < \omega_0$, for any fixed k (but to don't know if in all theses points the derivative of the functions $U_{\omega,k}$ is positive, we'll take only points sufficiently neighboring on the left of $1/2$). In fact, for we cannot apply to

it the indicated proceeding, the point must be, for each fixed ω , on the extremity side of a certain $S_{\omega,k}$, with an index k , which, depending continuously on ω , is locally constant, and independent on ω by connexion, while the corresponding function, by continuity also, have a well defined index (not necessarily $= k$).

Let us show that such points are necessarily on the axis $\Re(z) = 1/2$. Given, in fact, a complex zero p of ζ , which is not on this axis, let us suppose it at its right. We know moreover that the complex zero of ω, ρ_1 , which has the smallest ordinate (positive), is unique and is placed on the axis. The straight which joins the two zeros ρ_1 and p cuts then the real axis in a point ω_1 , at left of $1/2$. We can use ω_1 and the points ω of its neighborhood, for it is sufficiently near $1/2$, p been, as we know, highly placed in the half-vertical strip. Making use of the sectors $S_{\omega,k}$ whose extremities sides pass respectively through ρ_1 and p , we see that we must have: on the one hand $k = k_1$, because ρ_1 and p are the extremity side of the sector of apex ω_1 , but on the other hand, $k \neq k_1$ because the sectors of apex $\omega, \omega \neq \omega_1$, are distinct. \square

4. COMPLEMENTS

One proposes in what follows, to prolong the method used for the demonstration of the Riemann-hypothesis (proposition 6), in order to specify the nature of the network formed by the intersection-points of the sectors $S_{\omega,k}$, for k fixed, ω varying. One saw that some of them are the zeros of the function ζ in the upper half-plane, and that they are the intersection-points of the sides extremity, for ω near to $1/2$, on its left. One will see that it is so for other points. First, one frees oneself from the condition posed initially on ω . For that, it is sufficient to remark that, for instance, the proposition 5 - proof is based on the behavior of the function $U(z) = (\log(1 - z)\zeta^*(z))^{-1}$ in the neighborhood of a possible zero of ζ in the sector $S_{\omega,0}$. Now this behavior is not modified by a rotation of the figure, i.e. by the multiplication of $U(z)$ by a constant. Thus, the property for a point s to be intersection of the extremity sides of the sectors $S_{\omega,k}$, is generalizable, first locally, i.e. in the neighborhood of any point $\omega < \Re(s)$, then, by connexity, on the whole, i.e. for all $\omega < \Re(s)$. Now, one seeks to establish the

Proposition 7. *In addition to the zeros of ζ , are also intersection-points of the sides of sectors $S_{\omega,k}$ for all $\omega < 1/2$ and k fixed:*

- (1) *The zeros of $\zeta^{*'} previously shown, and which are of same nature that z_0 (cf sketch after the previous lemma). One will see that their number is finished, and they are placed on the right of the axis $\Re(z) = 1/2$.$*
- (2) *An infinite sequence of zeros of $\zeta^{*'} , and placed on the axis.$*

Proof. (1) It is sufficient to show that any of these zeros cannot be inside a sector $S_{\omega,k}$. For that one reasons by the absurd supposing, for instance, that the zero z_1 is inside the sector $S_{\omega,k}$, where is defined the holomorphic function

$$U_k(z) = (\log_k(1 - z)\zeta^*(z))^{-1} (:= Z)$$

z_1 is a regular point for $U_k(z)$ because

$$(4.1) \quad \begin{aligned} U'_k(z_1) &= [\log(1 - z_1)\zeta^*(z_1)]^{-2} \cdot \left[\frac{1}{1 - z_1} - \frac{\zeta^{*'}(z_1)}{\zeta^*(z_1)} \right] \\ &= [\log((1 - z_1)\zeta^*(z_1))]^{-2} (1 - z_1)^{-1} \neq 0 \end{aligned}$$

One can thus reverse the relation $U_k(z) = Z$, in the neighborhood of z_1 , by

$$(4.2) \quad z - z_1 = \sum_{n \geq 1} a_n (Z - Z_1)^n$$

or $z = U_k^{-1}(Z)$. But, it is possible also to reverse it in two stages:

- One writes it $(1 - z)\zeta^*(z) = \exp(Z^{-1})$, and substitutes $U_k^{-1}(Z)$ to z in $1 - z$

$$\zeta^*(z) = (1 - U_k^{-1}(Z))^{-1} \cdot \exp(Z^{-1}) := T$$

Then on the one hand $\zeta^*(z) = T$ is reversed by

$$z - z_1 = \sum_{n \geq 1} b_n ((T - T_1)^{1/2})^n$$

with $b_1 \neq 0$ (because $\zeta^{*'}(z_1) = 0$ and $\zeta^{*''}(z_1) \neq 0$ (like for z_0 as one sees it in sketch (after the previous lemma))), in a small disk centered in T_1 , deprived of a ray, in the T -plane.

- On the other hand

$$(1 - U_k^{-1}(Z))^{-1} \cdot \exp(Z^{-1}) = T$$

is expanded as

$$T - T_1 = \sum_{n \geq 1} c_n (Z - Z_1)^n$$

in a small disk centered in Z_1 .

If r is the first integer for which $c_r \neq 0$, the disk is deprived of the r arcs, of origin Z_1 , and extremity on the border. So, in crossing some or other of these arcs, $(T - T_1)^{1/2}$ changes determination. i.e is exchanged for its opposite. Now, in accordance with (4.2)

$$\sum_{n \geq 1} b_n ((T - T_1)^{1/2})^n$$

has to remain unchanged. This implies that

$$\sum_{n \text{ odd}} b_n ((T - T_1)^{1/2})^n \equiv 0$$

on these arcs. So, the holomorphic-function defined by $\sum_{n \text{ odd}} b_n ((T - T_1)^{1/2})^n \equiv 0$ on a continuum, and, consequently $\equiv 0$, so, all its coefficients $= 0$, in particular $b_1 = 0$, hence, the contradiction. The two other properties are obtained by a same argument that used for the demonstration of Riemann-hypothesis (proposition 6):

- They are not on the left of the axis $\Re(z) = 1/2$, because, if such were the case for

z_j , there will be a contradiction in connection with the indices of the sectors $S_{\omega,k}$, having z_j on their extremity-side, containing respectively the zeros ρ_i and $\rho_{i'}$ of ζ .

- They are in a finished number, because if they were an infinity, they would necessarily tend towards ∞ in the vertical half-strip $]1/2, \omega[$, and then, by supposing z_j placed enough high so that the straight joining it to $\rho_i, \rho_{i'}$ intersect the real axis (respectively in $\omega_i, \omega_{i'}$ on the left of $1/2$) one has a contradiction analogous to that of the previous case. Thus, the z_j are placed lower than the first zero ρ_i of ζ . They verifies:

$$\Im(z_0) < \Im(z_1) < \dots < \Im(z_{n_0}) < \Im(\rho_1)$$

with, as one knows

$$\omega_0 = \Re(z) > \Re(z_1) > \dots > \Re(z_{n_0}) > 1/2$$

(2) One needs the following lemma.

□

Lemma 2. *Given a sector $S_{\omega,k}$ $\omega < 1/2$, of which the sides Δ, Δ' , intersect the axis $\Re(z) = 1/2$ respectively in s, s' , which are not retrogression-point for the ζ^* -image of Δ, Δ' (i.e. $\zeta^{*'}(s) \neq 0, \zeta^{*'}(s') \neq 0$). If s' is on the extremity sides of all the sectors $S_{\omega,k}$, $\omega < 1/2$, having the same index k that (Δ, Δ') , then, it is of same for s , towards the origin-side of theses.*

Proof. Let us call p, p' the points respectively on Δ, Δ' which the ζ^* -image is the point of contact by which closes the cycle around the origin, formed by one arc of $\zeta^*(\Delta)$ and one arc of $\zeta^*(\Delta')$. Thus, on crossing p' on the right on Δ' , $\log_k \zeta^*(z)$ changes its determination. That implies that p' coincides with s' , because, if not, there is a contradiction with the fact that $\log_k \zeta^*(z)$ does not change its determination in p' , regarded as a point of the sector $S_{\omega,k}$, of apex ω_1 , near to ω , on its left. It results from it that $\zeta^*(p)(= \zeta^*(p')) = \zeta^*(s')$ and is thus a fixed point (i.e. independent of ω).

Now, p depends continuously on ω . Because the property of the isolated zeros of a holomorphic function of one variable, $\neq 0$; one has necessarily the point p fixed, and consequently, it coincided with s , on the axis $\Re(z) = 1/2$, (here, the argument is similar to that used for the z_j in the 1°) to show that p is not at left of the axis $\Re(s) = 1/2$, on the one hand, and, on the other hand, it is not, also, in the right.

Reasoning by absurd, let us tend ω towards $1/2$, it results from hypothesis that $\log \zeta^*(z)$ has the same determination on all the points of the vertical half axis $\Re(z) = 1/2$, above of $s'(= p')$.

Now, clearly, this determination on these points varies : it is the $\log_k \zeta^*(z)$ determination for k arbitrary large, for example, these corresponding to ζ -zeros with large ordinate.

□

Let us return to the demonstration of proposition 7, point 2. Given a ζ -zero ρ , one identifies ρ with the s' of the lemma because $\zeta^{*'}(\rho)$ is $\neq 0$ (in fact, if there where $\zeta^{*'}(\rho) = 0$, the rebroussement tangent to the curve image $\zeta^*(\omega\rho)$ would make a turn when ω described

$] -\infty, 1/2[$, and could take, thus a vertical position. There would be, then an image curve neighbouring with horizontal tangent and concavity turned to the bottom, near the contact point, which is impossible, according to the proposition 2 extended (cf remark made in connection with a similar situation p 12) Then, one thing or the other:

- $\zeta^{*'}(s)$ is also $\neq 0$, and then, according to the lemma 2, s is the intersection point of the origin side of all the sectors $S_{\omega,k}$, k fixed, and it satisfies $\zeta^*(s) = \zeta^*(\rho)$ (let us recall that $\zeta^*(\rho) \in C(1,1)$). This process can continue with the secteur $S_{\omega,k-1}$ if it satisfies with the lemma conditions, and so on...

But it is clear that it must cease, before coming at the sector with extremity-side passing through the zero ρ' witch precedes, otherwise, one would lead $\zeta^*(\rho') = \zeta^*(\rho)$ thus to $\frac{1}{\rho'-1} = \frac{1}{\rho-1}$ or $\rho' = \rho$ (!)

- That leads us to consider the other alternative ; namely the sector $S_{\omega,k}$ satisfies to $\zeta^{*'}(s') \neq 0$, but $\zeta^{*'}(s) = 0$ There the demonstration of proposition 7 is finished.

We give below some details about the new situation. In fact, the sector $S_{\omega,k}$ does not have this only configuration. This is always accompanied by another (at least) whose the origin-side passes by some point p , with $\Re(p) \geq 1/2$ independent on ω . That which one distinguished higher is such as p coincided with s .

To the various configurations of $S_{\omega,k}$, it correspond a partition of the interval $] -\infty, 1/2[$, ω describing such or such partial interval, according to the configuration considered. Thus, that which one distinguished has, among the partial intervals who are reserved for him, the interval with end at $1/2$. In fact, it is this possibility for ω to tend towards $1/2$, which makes that the point p coincided with s (situation similar to that treated in the lemma).

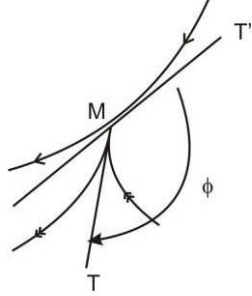


FIGURE 3

Particular case

Let us suppose that the number of configurations of $S_{\omega,k}$, is 2 and that $p_1(=s_1)$, p_2 the zeros correspondents of $\zeta^{*'}$ are with multiplicity = 1. Then, the sector $S_{\omega,k}$, has its origin-side which passes alternatively by p_1 and p_2 , when its apex ω passes from interval to another, of a some infinite partition of the interval $]-\infty, 1/2[$.

Proof. Naturally the first partial interval $[\omega_1, 1/2[$ corresponds to s_1 . Let us call ϕ the angle (MT', MT) in $M = \zeta^*(s_1)(= \zeta^*(s'))$ of the curves image by ζ^* of the half lines $\omega s', \omega s_1$.

If ϕ_0 is its initial value (that which corresponds to $\omega = 1/2$) and $\Delta\theta$, $\Delta\theta'$, the respective variation of the polars angles of $\omega\rho$ and $\omega\rho'$, one have $\phi = \phi_0 + (2\Delta\theta - \Delta\theta')$.

The first phase finishes when ω comes in ω_1 , such as $\phi = \phi_1 = -\pi$. It is then with the turn of p_2 to intervene, ω decreasing from ω_1 to ω_2 . ω_2 is $\neq -\infty$. In fact :

Let us call ψ , $\Delta\tau$, $\Delta\tau'$ the quantities analogous to ϕ , $\Delta\theta$, $\Delta\theta'$ whose intervened in this second phase. One have : $\psi = \psi_1 + (2\Delta\tau - \Delta\tau')$ with $-\pi < \psi < 0$. At the same time $\Delta\phi$ have to remain $< -\pi$, $\phi = -\pi + \beta$ with $\beta < 0$. One have then $0 > \psi > -\pi > \phi = -\pi + \beta$, and therefore $\psi - \phi > -\beta$. If $\omega_2 = -\infty$,

$$|\beta| = |2\Delta\theta - \Delta\theta'| = 2 \left| \Delta\theta - \frac{\Delta\theta'}{2} \right|$$

is > 0 because s_1 nearest to s' than the foot of the bissectrix of the angle $\frac{1}{2}\omega_1 s'$ on the $[1/2, s']$.

In fact, when $\omega \rightarrow -\infty$, the foot of the bissectrix tends about the middle of the side $[1/2, s']$ which is clearly below the zero ρ' of ζ , taking into account the law of well known distribution of the zeros of on the axis $\Re(z) = 1/2$. But, $\psi - \phi = \text{angle of curves images of } s'\omega \text{ and } p_2\omega - \text{angle of curves images of } s'\omega \text{ and } s_1\omega = \text{angle of curves-image of } s_1\omega \text{ and } p_2\omega = 2\widehat{p_2\omega s_1}$ which $\rightarrow 0$ when $\omega \rightarrow -\infty$. Here the contradiction. One have similar arguments to show that this process continues indefinitely. \square

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