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# Motivic decompositions of projective homogeneous varieties and change of coefficients

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## Abstract

We prove that under some assumptions on an algebraic group  $G$ , indecomposable direct summands of the motive of a projective  $G$ -homogeneous variety with coefficients in  $\mathbb{F}_p$  remain indecomposable if the ring of coefficients is any field of characteristic  $p$ . In particular for any projective  $G$ -homogeneous variety  $X$ , the decomposition of the motive of  $X$  in a direct sum of indecomposable motives with coefficients in any finite field of characteristic  $p$  corresponds to the decomposition of the motive of  $X$  with coefficients in  $\mathbb{F}_p$ . We also construct a counterexample to this result in the case where  $G$  is arbitrary.

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## Résumé

**Décompositions motiviques des variétés projectives homogènes et changement des coefficients.** Nous prouvons que sous certaines hypothèses sur un groupe algébrique  $G$ , tout facteur direct indécomposable du motif associé à une variété projective  $G$ -homogène à coefficients dans  $\mathbb{F}_p$  demeure indécomposable si l'anneau des coefficients est un corps de caractéristique  $p$ . En particulier pour toute variété projective  $G$ -homogène  $X$ , la décomposition du motif de  $X$  comme somme directe de motifs indécomposables à coefficients dans tout corps fini de caractéristique  $p$  correspond à la décomposition du motif de  $X$  à coefficients dans  $\mathbb{F}_p$ . Nous exhibons de plus un contre-exemple à ce résultat dans le cas où le groupe  $G$  est quelconque.

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## Introduction

Let  $F$  be a field,  $\Lambda$  be a commutative ring,  $CM(F; \Lambda)$  be the category of *Grothendieck Chow motives* with coefficients in  $\Lambda$ ,  $G$  a semi-simple affine algebraic group and  $X$  a projective  $G$ -homogeneous  $F$ -

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variety. The purpose of this note is to study the behaviour of the complete motivic decomposition (in a direct sum of indecomposable motives) of  $X \in CM(F; \Lambda)$  when changing the ring of coefficients. In the first part we prove some very elementary results in non-commutative algebra and find sufficient conditions for the tensor product of two connected rings to be connected. In the second part we show that under some assumptions on  $G$ , indecomposable direct summands of  $X$  in  $CM(F; \mathbb{F}_p)$  remain indecomposable if the ring of coefficients is any field of characteristic  $p$  (Theorem 2.1), since these conditions hold for the *reduced endomorphism ring* of indecomposable direct summands of  $X$ . In particular theorem 2.1 implies that the complete decomposition of the motive of  $X$  with coefficients in any finite field of characteristic  $p$  corresponds to the complete decomposition of the motive of  $X$  with coefficients in  $\mathbb{F}_p$ . Finally we show that theorem 2.1 doesn't hold for arbitrary  $G$  by producing a counterexample.

Let  $\Lambda$  be a commutative ring. Given a field  $F$ , an  $F$ -variety will be understood as a separated scheme of finite type over  $F$ . Given such  $\Lambda$  and an  $F$ -variety  $X$ , we can consider  $CH_i(X; \Lambda)$ , the Chow group of  $i$ -dimensional cycles on  $X$  modulo rational equivalence with coefficients in  $\Lambda$ , defined as  $CH_i(X) \otimes_{\mathbb{Z}} \Lambda$ . These groups are the first step in the construction of the category  $CM(F; \Lambda)$  of *Grothendieck Chow motives* with coefficients in  $\Lambda$ . This category is constructed as the *pseudo-abelian envelope* of the category  $CR(F; \Lambda)$  of *correspondences* with coefficients in  $\Lambda$ . Our main reference for the construction and the main properties of these categories is [2]. For a field extension  $E/F$  and any correspondence  $\alpha \in CH(X \times Y; \Lambda)$  we denote by  $\alpha_E$  the pull-back of  $\alpha$  along the natural morphism  $(X \times Y)_E \rightarrow X \times Y$ . Considering a morphism of commutative rings  $\varphi : \Lambda \rightarrow \Lambda'$  we define the two following functors. The *change of base field* functor is the additive functor  $res_{E/F} : CM(F; \Lambda) \rightarrow CM(E; \Lambda)$  which maps any summand  $(X, \pi)[i] \in CM(F; \Lambda)$  to  $(X_E, \pi_E)[i]$  and any morphism  $\alpha : (X, \pi)[i] \rightarrow (Y, \rho)[j]$  to  $\alpha_E$ . The *change of coefficients* functor is the additive functor  $coef_{\Lambda'/\Lambda} : CM(F; \Lambda) \rightarrow CM(F; \Lambda')$  which maps any summand  $(X, \pi)[i]$  to  $(X, (id \otimes \varphi)(\pi))[i]$  and any morphism  $\alpha : (X, \pi)[i] \rightarrow (Y, \rho)[j]$  to  $(id \otimes \varphi)(\alpha)$ .

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## 1. On the tensor product of connected rings

Recall that a ring  $A$  is *connected* if there are no idempotents in  $A$  besides 0 and 1.

**Proposition 1.1** *Let  $A$  be a finite and connected ring. Then any element  $a$  in  $A$  is either nilpotent or invertible. The set  $\mathcal{N}$  of nilpotent elements in  $A$  is a two-sided and nilpotent ideal.*

In order to prove Proposition 1.1 we will need the following elementary lemma.

**Lemma 1.2** *Let  $A$  be a finite ring. An appropriate power of any element  $a$  of  $A$  is idempotent.*

Proof. For any  $a \in A$ , the set  $\{a^n, n \in \mathbb{N}\}$  is finite, hence there is a couple  $(p, k) \in \mathbb{N}^2$  (with  $k$  non-zero) such that  $a^p = a^{p+k}$ . The sequence  $(a^n)_{n \geq p}$  is  $k$ -periodic and for example if  $s$  is the lowest integer such that  $p < sk$ ,  $a^{sk}$  is idempotent.

Proof of Proposition 1.1. For any  $a \in A$ , an appropriate power of  $a$  is an idempotent by lemma 1.2. Since  $A$  is connected, this power is either 0 or 1, that is to say  $a$  is either nilpotent or invertible.

We now show that the set  $\mathcal{N}$  of nilpotent elements in  $A$  is a two-sided ideal. First if  $a$  is nilpotent in  $A$ , then for any  $b$  in  $A$ ,  $ab$  and  $ba$  are not invertible, hence  $ab$  and  $ba$  belong to  $\mathcal{N}$ .

It remains to show that the sum of two nilpotent elements in  $A$  is nilpotent. Setting  $\nu$  for the number of nilpotent elements in  $A$ , we claim that for any sequence  $a_1, \dots, a_\nu$  in  $\mathcal{N}$ ,  $a_1 \dots a_\nu = 0$ . Indeed if  $a_{\nu+1}$  is any nilpotent in  $A$  the finite sequence  $\Pi_1 = a_1, \Pi_2 = a_1 a_2, \dots, \Pi_{\nu+1} = a_1 a_2 \dots a_{\nu+1}$  consists of nilpotents and by the pigeon-hole principle  $\Pi_k = \Pi_s$ , for some  $k$  and  $s$  satisfying  $1 \leq k < s \leq \nu + 1$ . Therefore  $\Pi_s = \Pi_k a_{k+1} \dots a_s = \Pi_k$  which implies that  $\Pi_k(1 - a_{k+1} \dots a_s) = 0$  and  $\Pi_k = 0$  since  $1 - a_{k+1} \dots a_s$

invertible. With this in hand it is clear that for any  $a$  and  $b$  in  $\mathcal{N}$ ,  $(a + b)^\nu = 0$ . Furthermore  $\mathcal{N}^\nu = 0$  and  $\mathcal{N}$  is nilpotent.

**Corollary 1.3** *Let  $A$  be a finite and connected  $\mathbb{F}_p$ -algebra endowed with a ring morphism  $\varphi : A \rightarrow \mathbb{F}_p$ . Then the set  $\mathcal{N}$  of nilpotent elements in  $A$  is precisely  $\ker(\varphi)$ . Furthermore for any connected  $\mathbb{F}_p$ -algebra  $E$ ,  $A \otimes_{\mathbb{F}_p} E$  is connected.*

Proof. For any  $a \in \mathcal{N}$  and  $n \in \mathbb{N}^*$  such that  $a^n = 0$ ,  $0 = \varphi(a^n) = \varphi(a)^n$ , hence  $a$  lies in the kernel of  $\varphi$ . On the other hand if  $\varphi(a) = 0$ ,  $a$  is not invertible thus  $a$  is nilpotent and  $\mathcal{N} = \ker(\varphi)$ . Since  $\mathcal{N}$  is nilpotent,  $\mathcal{N} \otimes E$  is also nilpotent. The sequence

$$0 \longrightarrow \mathcal{N} \otimes E \longrightarrow A \otimes E \xrightarrow{\psi} E \longrightarrow 0$$

is exact and we want to show that any idempotent  $P$  in  $A \otimes_{\mathbb{F}_p} E$  is either 0 or 1. Since  $E$  is connected,  $\psi(P)$  is either 0 or 1. We may replace  $P$  by  $1 - P$  and so assume that  $P$  lies in the kernel of  $\psi$ , which implies that the idempotent  $P$  is nilpotent, hence  $P = 0$ .

## 2. Application to motivic decompositions of projective homogeneous varieties

For any semi-simple affine algebraic group  $G$ , the full subcategory of  $CM(F; \Lambda)$  whose objects are finite direct sums of twists of direct summands of the motives of projective  $G$ -homogeneous  $F$ -varieties will be denoted  $CM_G(F; \Lambda)$ . We now use corollary 1.3 to study how motivic decompositions in  $CM_G(F; \Lambda)$  behave when extending the ring of coefficients. A pseudo-abelian category  $\mathcal{C}$  satisfies the *Krull-Schmidt principle* if the monoid  $(\mathfrak{C}, \oplus)$  is free, where  $\mathfrak{C}$  denotes the set of the isomorphism classes of objects of  $\mathcal{C}$ .

In the sequel  $\Lambda$  will be a connected ring and  $X$  an  $F$ -variety. A field extension  $E/F$  is a *splitting field* of  $X$  if the  $E$ -motive  $X_E$  is isomorphic to a finite direct sum of twists of Tate motives. The  $F$ -variety  $X$  is *geometrically split* if  $X$  splits over an extension of  $F$ , and  $X$  satisfies the *nilpotence principle*, if for any field extension  $E/F$  the kernel of the morphism  $res_{E/F} : End(M(X)) \rightarrow End(M(X_E))$  consists of nilpotents. Any projective homogeneous variety (under the action of a semi-simple affine algebraic group) is geometrically split and satisfies the nilpotence principle (see [1]), therefore if  $\Lambda$  is finite the Krull-Schmidt principle holds for  $CM_G(F; \Lambda)$  by [5, Corollary 3.6], and we can serenely deal with motivic decompositions in  $CM_G(F; \Lambda)$ .

Let  $G$  be a semi-simple affine adjoint algebraic group over  $F$  and  $p$  a prime. The absolute Galois group  $Gal(F_{sep}/F)$  acts on the Dynkin diagram of  $G$  and we say that  $G$  is of *inner type* if this action is trivial. By [1] the subfield  $F_G$  of  $F_{sep}$  corresponding to the kernel of this action is a finite Galois extension of  $F$ , and we will say that  $G$  is *p-inner* if  $[F_G : F]$  is a power of  $p$ . We now state the main result.

**Theorem 2.1** *Let  $G$  be a semi-simple affine adjoint p-inner algebraic group and  $M \in CM_G(F; \mathbb{F}_p)$ . Then for any field  $L$  of characteristic  $p$ ,  $M$  is indecomposable if and only if  $coeff_{L/\mathbb{F}_p}(M)$  is indecomposable.*

If  $X$  is geometrically split the image of any correspondence  $\alpha \in CH_{\dim(X)}(X \times X; \Lambda)$  by the *change of base field* functor  $res_{E/F}$  to a splitting field  $E/F$  of  $X$  will be denoted  $\bar{\alpha}$ . The *reduced endomorphism ring* of any direct summand  $(X, \pi)$  is defined as  $res_{E/F}(End_{CM(F; \Lambda)}((X, \pi)))$  and denoted by  $\overline{End}((X, \pi))$ .

Let  $X$  be a complete and irreducible  $F$ -variety. The pull-back of the natural morphism  $Spec(F(X)) \times X \rightarrow X \times X$  gives rise to  $mult : CH_{\dim(X)}(X \times X; \Lambda) \rightarrow CH_0(X_{F(X)}; \Lambda) \rightarrow \Lambda$  (where the second map is the usual *degree* morphism). For any correspondence  $\alpha \in CH_{\dim(X)}(X \times X; \Lambda)$ ,  $mult(\alpha)$  is called the *multiplicity* of  $\alpha$  and we say that a direct summand  $(X, \pi)$  given by a projector  $\pi \in CH_{\dim(X)}(X \times X; \Lambda)$  is *upper* if  $mult(\pi) = 1$ . If  $(X, \pi)$  is an upper direct summand of a complete and irreducible  $F$ -variety, the multiplicity  $mult : End_{CM(F; \Lambda)}((X, \pi)) \rightarrow \Lambda$  is a morphism of rings by [4, Corollary 1.7].

**Proposition 2.1** *Let  $G$  be a semi-simple affine algebraic group and  $M = (X, \pi) \in CM(F; \mathbb{F}_p)$  the upper direct summand of the motive of an irreducible and projective  $G$ -homogeneous  $F$ -variety. Then for any*

field  $L$  of characteristic  $p$ ,  $M$  is indecomposable if and only if  $\text{coeff}_{L/\mathbb{F}_p}(M)$  is indecomposable.

Proof. Since the change of coefficients functor is additive and maps any non-zero projector to a non-zero projector, it is clear that if  $\text{coeff}_{L/\mathbb{F}_p}(M)$  is indecomposable,  $M$  is also indecomposable. Considering a splitting field  $E$  of  $X$ , the reduced endomorphism ring  $\overline{\text{End}}(M) := \overline{\pi} \circ \overline{\text{End}}(X) \circ \overline{\pi}$  is connected since  $M$  is indecomposable and finite. Corollary 1.3, with  $A = \overline{\text{End}}(M)$ ,  $E = L$  and  $\varphi = \text{mult}$  implies that  $\overline{\text{End}}(M) \otimes L = \overline{\text{End}}(\text{coeff}_{L/\mathbb{F}_p}(M))$  is connected, therefore by the nilpotence principle  $\text{End}(\text{coeff}_{L/\mathbb{F}_p}(M))$  is also connected, that is to say  $\text{coeff}_{L/\mathbb{F}_p}(M)$  is indecomposable.

Proof of Theorem 2.1. Recall that  $G$  is a semi-simple affine adjoint  $p$ -inner algebraic group and consider a projective  $G$ -homogeneous  $F$ -variety  $X$ . By [6, Theorem 1.1], any indecomposable direct summand  $M$  of  $X$  is a twist of the upper summand of the motive of an irreducible and projective  $G$ -homogeneous  $F$ -variety  $Y$ , thus we can apply proposition 2.1 to each indecomposable direct summand of  $X$ .

**Remark :** If  $\Lambda$  is a finite, commutative and connected ring, complete motivic decompositions in  $CM(F; \Lambda)$  remain complete when the coefficients are extended to the residue field of  $\Lambda$  by [7, Corollary 2.6], hence the study of motivic decompositions in  $CM_G(F; \Lambda)$ , where  $\Lambda$  is any finite connected ring whose residue field is of characteristic  $p$ , is reduced to the study motivic decompositions in  $CM_G(F; \mathbb{F}_p)$ .

We now produce a counterexample to Theorem 2.1 in the case where the algebraic group  $G$  doesn't satisfy the needed assumptions. Let  $L/F$  be a Galois extension of degree 3. By [1, Section 7], the endomorphism ring  $\text{End}(M(\text{Spec}(L)))$  of the motive associated with the  $F$ -variety  $\text{Spec}(L)$  with coefficients in  $\mathbb{F}_2$  is the  $\mathbb{F}_2$ -algebra of  $\text{Gal}(L/F)$ , i.e.  $\frac{\mathbb{F}_2[X]}{(X^3-1)} \simeq \mathbb{F}_2 \times \mathbb{F}_4$ , hence  $M(\text{Spec}(L)) = M \oplus N$ , with  $\text{End}(N) = \mathbb{F}_4$  and both  $M$  and  $N$  are indecomposable. Now  $\text{End}(\text{res}_{\mathbb{F}_4/\mathbb{F}_2}(N)) = \mathbb{F}_4 \otimes \mathbb{F}_4$  is not connected since  $1 \otimes \alpha + \alpha \otimes 1$  is a non-trivial idempotent for any  $\alpha \in \mathbb{F}_4 \setminus \mathbb{F}_2$ , hence  $\text{res}_{\mathbb{F}_4/\mathbb{F}_2}(N)$  is decomposable.

Consider the  $(PGL_2)_L$ -homogeneous  $L$ -variety  $\mathbb{P}_L^1$ . The Weil restriction  $\mathcal{R}(\mathbb{P}_L^1)$  is a projective homogeneous  $F$ -variety under the action of the Weil restriction of  $(PGL_2)_L$ , and the minimal extension such that  $\mathcal{R}((PGL_2)_L)$  is of inner type is  $L$ . By [3, Example 4.8], the motive with coefficients in  $\mathbb{F}_2$  of  $\mathcal{R}(\mathbb{P}_L^1)$  contains two twists of  $\text{Spec}(L)$  as direct summands, therefore at least two indecomposable direct summands of  $\mathcal{R}(\mathbb{P}_L^1)$  split off over  $\mathbb{F}_4$ .

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