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# Motivic decompositions of projective homogeneous varieties and change of coefficients

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## Abstract

We prove that under some assumptions on an algebraic group  $G$ , indecomposable direct summands of the motive of a projective  $G$ -homogeneous variety with coefficients in  $\mathbb{F}_p$  remain indecomposable if the ring of coefficients is any field of characteristic  $p$ . In particular for any projective  $G$ -homogeneous variety  $X$ , the decomposition of the motive of  $X$  in a direct sum of indecomposable motives with coefficients in any finite field of characteristic  $p$  corresponds to the decomposition of the motive of  $X$  with coefficients in  $\mathbb{F}_p$ . We also construct a counterexample to this result in the case where  $G$  is arbitrary.

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## Résumé

**Décompositions motiviques des variétés projectives homogènes et changement des coefficients.** Nous prouvons que sous certaines hypothèses sur un groupe algébrique  $G$ , tout sommand direct indécomposable du motif associé à une variété projective  $G$ -homogène à coefficients dans  $\mathbb{F}_p$  demeure indécomposable si l'anneau des coefficients est un corps de caractéristique  $p$ . En particulier pour toute variété projective  $G$ -homogène  $X$ , la décomposition du motif de  $X$  comme somme directe de motifs indécomposables à coefficients dans tout corps fini de caractéristique  $p$  correspond à la décomposition du motif de  $X$  à coefficients dans  $\mathbb{F}_p$ . Nous exhibons de plus un contre-exemple à ce résultat dans le cas où le groupe  $G$  est quelconque.

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## Introduction

Let  $F$  be a field,  $\Lambda$  be a commutative ring,  $CM(F; \Lambda)$  be the category of *Grothendieck Chow motives* with coefficients in  $\Lambda$ ,  $G$  a semi-simple affine algebraic group and  $X$  a projective  $G$ -homogeneous  $F$ -

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variety. The purpose of this note is to study the behaviour of the complete motivic decomposition (in a direct sum indecomposable motives) of  $X \in CM(F; \Lambda)$  when changing the ring of coefficients. In the first part we prove some very elementary results in non-commutative algebra and find sufficient conditions for the tensor product of two connected rings to be connected. In the second part we show that under some assumptions on  $G$ , indecomposable direct summands of  $X$  in  $CM(F; \mathbb{F}_p)$  remain indecomposable if the ring of coefficients is any field of characteristic  $p$  (Theorem 2.2), since these conditions hold for the *reduced endomorphism ring* of indecomposable direct summands of  $X$ . In particular theorem 2.2 implies that the complete decomposition of the motive of  $X$  with coefficients in any finite field of characteristic  $p$  corresponds to the complete decomposition of the motive of  $X$  with coefficients in  $\mathbb{F}_p$ . Finally we show that theorem 2.2 doesn't hold for arbitrary  $G$  by producing a counterexample.

Let  $\Lambda$  be a commutative ring. Given a field  $F$ , an  $F$ -variety will be understood as a separated scheme of finite type over  $F$ . Given such  $\Lambda$  and an  $F$ -variety  $X$ , we can consider  $CH_i(X; \Lambda)$ , the Chow group of  $i$ -dimensional cycles on  $X$  modulo rational equivalence with coefficients in  $\Lambda$ , defined as  $CH_i(X) \otimes_{\mathbb{Z}} \Lambda$ . These groups are the first step in the construction of the category  $CM(F; \Lambda)$  of *Grothendieck Chow motives* with coefficients in  $\Lambda$ . This category is constructed as the *pseudo-abelian envelope* of the category  $CR(F; \Lambda)$  of *correspondences* with coefficients in  $\Lambda$ . Our main reference for the construction and the main properties of these categories is [2]. For a field extension  $E/F$  and any correspondence  $\alpha \in CH(X \times Y; \Lambda)$  we denote by  $\alpha_E$  the pull-back of  $\alpha$  along the natural morphism  $(X \times Y)_E \rightarrow X \times Y$ . Considering a morphism of commutative rings  $\varphi : \Lambda \rightarrow \Lambda'$  we define the two following functors. The *change of base field* functor is the additive functor  $res_{E/F} : CM(F; \Lambda) \rightarrow CM(E; \Lambda)$  which maps any summand  $(X, \pi)[i] \in CM(F; \Lambda)$  to  $(X_E, \pi_E)[i]$  and any morphism  $\alpha : (X, \pi)[i] \rightarrow (Y, \rho)[j]$  to  $\alpha_E$ . The *change of coefficients* functor is the additive functor  $coeff_{\Lambda'/\Lambda} : CM(F; \Lambda) \rightarrow CM(F; \Lambda')$  which maps any summand  $(X, \pi)[i]$  to  $(X, (id \otimes \varphi)(\pi))[i]$  and any morphism  $\alpha : (X, \pi)[i] \rightarrow (Y, \rho)[j]$  to  $(id \otimes \varphi)(\alpha)$ .

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## 1. On the tensor product of connected rings

Throughout this section  $F$  is a field. Recall that a ring  $A$  is *connected* if there are no idempotents in  $A$  besides 0 and 1.

**Lemma 1.1** *Let  $A$  be a finite ring. Then an appropriate power of any element  $a$  of  $A$  is idempotent.*

*Proof.* For any  $a \in A$ , the set  $\{a^n, n \in \mathbb{N}\}$  is finite, hence there is a couple  $(p, k) \in \mathbb{N}^2$  such that  $a^p = a^{p+k}$ . The sequence  $(a^n)_{n \geq p}$  is  $k$ -periodic and for example  $(a^{pk})^2 = a^{pk+pk} = a^{pk}$ .

**Definition 1.2** *A ring  $A$  is local if the sum of any two non-invertible elements is not equal to 1.*

**Proposition 1.3** *Any finite connected ring  $A$  is local.*

*Proof.* Indeed by lemma 1.1 an appropriate power of any  $a \in A$  is idempotent, hence any element of  $A$  is either nilpotent or invertible. The result holds since the sum of two nilpotent elements cannot be 1.

**Definition 1.4** *A ring  $A$  is of type  $\mathcal{N}$  if the product of two nilpotents remains nilpotent.*

A finite connected ring is clearly of type  $\mathcal{N}$ , since the product of two nilpotent elements is not invertible.

**Lemma 1.5** *Let  $A$  be a finite ring of type  $\mathcal{N}$ . Let  $\mathcal{N}_A$  be the set of nilpotents in  $A$  and  $\nu = \text{Card}(\mathcal{N}_A)$ . Then for any  $(a_1, \dots, a_{\nu+1}) \in \mathcal{N}_A^{\nu+1}$ ,  $a_1 \dots a_{\nu+1} = 0$ .*

*Proof.* The finite sequence  $\Pi_1 = a_1, \Pi_2 = a_1 a_2, \dots, \Pi_{\nu+1} = a_1 a_2 \dots a_{\nu+1}$  consists of nilpotents since  $A$  is of type  $\mathcal{N}$ . By the pigeon-hole principle  $\Pi_k = \Pi_s$ , for some  $k$  and  $s$  satisfying  $1 \leq k < s \leq \nu + 1$  therefore  $\Pi_s = \Pi_k a_{k+1} \dots a_s = \Pi_k$ . Consequently  $\Pi_k(1 - a_{k+1} \dots a_s) = 0$ , which implies that  $\Pi_k = 0$  since  $1 - a_{k+1} \dots a_s$  is invertible.

**Corollary 1.6** *Let  $A$  and  $E$  be two  $F$ -algebras, with  $A$  finite and connected. Then for any sequence of nilpotents  $(a_1, \dots, a_n) \in A^n$  and for any  $(b_1, \dots, b_n) \in E^n$ ,  $\sum_{i=1}^n a_i \otimes b_i$  is nilpotent in  $A \otimes_F E$ .*

*Proof.* By lemma 1.5  $(\sum_{i=1}^n a_i \otimes b_i)^{\nu+1} = 0$ , where  $\nu$  is the number of nilpotents in  $A$ .

**Theorem 1.7** *Let  $A$  and  $E$  be two connected  $F$ -algebras, with  $A$  finite. Suppose that  $A$  is endowed with a morphism of algebras  $\phi : A \rightarrow F$ . Then  $A \otimes_F E$  is connected.*

*Proof.* Consider the morphism  $\varphi : A \otimes E \xrightarrow{\phi \otimes id} F \otimes E \xrightarrow{\simeq} E$  and an idempotent  $P$  in  $A \otimes_F E$ . Since  $\varphi$  is a morphism of algebras and  $E$  is connected,  $\varphi(P) = 0$  or  $\varphi(P) = 1$ . Let  $\{1, (e_i)_{i \in I}\}$  be an  $F$ -basis of  $E$  containing 1 and  $P = p \otimes 1 + \sum_{k=1}^n p_{i_k} \otimes e_{i_k}$ . If  $\varphi(P) = \phi(p) \cdot 1 + \sum_{k=1}^n \phi(p_{i_k}) \cdot e_{i_k} = 0$ , then  $\phi(p) = 0$  and for any  $k \leq n$ ,  $\phi(p_{i_k}) = 0$ . By lemma 1.1, an appropriate power of  $p$  and each  $p_{i_k}$  is an idempotent of  $A$  which belongs to the kernel of  $\phi$ , hence  $p$  and each  $p_{i_k}$  are nilpotent since  $A$  is connected. By corollary 1.6, the idempotent  $P$  is nilpotent, hence  $P = 0$ . Now if  $\varphi(P) = \phi(p) \cdot 1 + \sum_{k=1}^n \phi(p_{i_k}) \cdot e_{i_k} = 1$ , then by lemma 1.1 an appropriate power of  $p$  is an idempotent  $q$  in  $A$  satisfying  $\phi(q) = 1$ , hence  $q = 1$  and  $p$  is invertible. At the same time each  $p_{i_k}$  is nilpotent since an appropriate power of each  $p_{i_k}$  is an idempotent in  $A$  lying in the kernel of  $\phi$ . It follows that  $P = (p \otimes 1) (1 \otimes 1 + \sum_{k=1}^n p^{-1} p_{i_k} \otimes e_{i_k})$ , hence  $P$  is invertible because  $\sum_{k=1}^n p^{-1} p_{i_k} \otimes e_{i_k}$  is nilpotent (using corollary 1.6 again). In this case  $P = P \cdot 1 = P \cdot P \cdot P^{-1} = P \cdot P^{-1} = 1$ , consequently any idempotent in  $A \otimes_F E$  is either 0 or 1 and  $A \otimes_F E$  is connected.

## 2. Application to motivic decompositions of projective homogeneous varieties

For any semi-simple affine algebraic group  $G$ , the full subcategory of  $CM(F; \Lambda)$  whose objects are finite direct sums of twists of direct summands of projective  $G$ -homogeneous  $F$ -varieties will be denoted  $CM_G(F; \Lambda)$ . We now use theorem 1.7 to study how motivic decompositions in  $CM_G(F; \Lambda)$  behave when extending the ring of coefficients. We will say that a pseudo-abelian category  $\mathcal{C}$  satisfies the *Krull-Schmidt principle* if any object of  $\mathcal{C}$  decomposes in a unique way as a direct sum of indecomposable objects (up to permutation of these indecomposable summands).

In the sequel  $\Lambda$  will be a connected ring and  $X$  an  $F$ -variety. A field extension  $E/F$  is a *splitting field* of  $X$  if the  $E$ -motive  $X_E$  is isomorphic to a finite direct sum of twists of Tate motives. The  $F$ -variety  $X$  is *geometrically split* if  $X$  splits over an extension of  $F$ , and  $X$  satisfies the *nilpotence principle*, if for any field extension  $E/F$  the kernel of the morphism  $res_{E/F} : End(M(X)) \rightarrow End(M(X_E))$  consists of nilpotents.

Any projective variety (under the action of a semi-simple affine algebraic group) is geometrically split and satisfies the nilpotence principle (see [1]), therefore if  $\Lambda$  is finite the Krull-Schmidt principle holds for  $CM_G(F; \Lambda)$  by [5, Corollary 3.6], and we can serenely deal with motivic decompositions in  $CM_G(F; \Lambda)$ . If  $X$  is geometrically split the image of any correspondence  $\alpha \in CH_{\dim(X)}(X \times X; \Lambda)$  by the *change of base field* functor  $res_{E/F}$  to a splitting field  $E/F$  of  $X$  will be denoted  $\bar{\alpha}$ . The *reduced endomorphism ring* of any direct summand  $(X, \pi)$ , defined as  $res_{E/F}(End_{CM(F; \Lambda)}((X, \pi)))$  and denoted by  $\overline{End}((X, \pi))$ , will be one of the key-ingredients in the proof of theorem 2.2.

Let  $X$  be a smooth, complete and irreducible  $F$ -variety. The pull-back of the natural morphism  $Spec(F(X)) \times X \rightarrow X \times X$  gives rise to  $mult : CH_{\dim(X)}(X \times X; \Lambda) \rightarrow CH_0(X_{F(X)}; \Lambda) \rightarrow \Lambda$  (where the second map is the usual *degree* morphism). For any correspondence  $\alpha \in CH_{\dim(X)}(X \times X; \Lambda)$ ,  $mult(\alpha)$  is called the *multiplicity* of  $\alpha$  and we say that a direct summand  $(X, \pi)$  given by a projector  $\pi \in CH_{\dim(X)}(X \times X; \Lambda)$  is *upper* if  $mult(\pi) = 1$ . If  $X$  is a smooth complete and irreducible  $F$ -variety, the multiplicity  $mult : End_{CM(F; \Lambda)}(X) \rightarrow \Lambda$  is a morphism of rings by [4, Corollary 1.7]. We now state the needed restriction on the algebraic group  $G$  to prove theorem 2.2.

**Definition 2.1** *Let  $p$  be a prime. A semi-simple affine algebraic group  $G$  is  $p$ -inner if the degree of the minimal extension  $E/F$  such that  $G_E$  is of inner type is a power of  $p$ .*

Using properties of the *reduced endomorphism ring* of indecomposable motives in  $CM_G(F; \Lambda)$  (where  $G$  is a semi-simple affine  $p$ -inner algebraic group) and theorem 1.7, we can prove the following result.

**Theorem 2.2** *Let  $G$  be a semi-simple affine  $p$ -inner algebraic group and  $M \in CM_G(F; \mathbb{F}_p)$ . Then for any field  $L$  of characteristic  $p$ ,  $M$  is indecomposable if and only if  $\text{coeff}_{L/\mathbb{F}_p}(M)$  is indecomposable.*

Proof. Since the change of coefficients functor is additive and maps any non-zero projector to a non-zero projector, it is clear that if  $\text{coeff}_{L/\mathbb{F}_p}(M)$  is indecomposable,  $M$  is also indecomposable.

Let  $M$  be an indecomposable direct summand of the motive of a projective  $G$ -homogeneous  $F$ -variety  $X$  in  $CM(F; \mathbb{F}_p)$ . By [6, Theorem 1.1],  $M$  is a twist of the upper summand of an irreducible projective  $G$ -homogeneous  $F$ -variety  $Y$  (say,  $M = (Y, \pi)[k]$  where  $\pi$  is a projector in  $CH_{\dim(Y)}(Y \times Y; \mathbb{F}_p)$  and  $k \in \mathbb{Z}$ ). Considering a splitting field  $E$  of  $X$ , the reduced endomorphism ring  $\overline{End}(M) := \overline{\pi} \circ \overline{End}(Y) \circ \overline{\pi}$  is connected since  $M$  is indecomposable and finite.

Theorem 1.7, with  $A = \overline{End}(M)$ ,  $E = L$ ,  $F = \mathbb{F}_p$  and  $\phi = \text{mult}$  implies that  $\overline{End}(M) \otimes L = \overline{End}(\text{coeff}_{L/\mathbb{F}_p}(M))$  is connected, therefore by the nilpotence principle  $End(\text{coeff}_{L/\mathbb{F}_p}(M))$  is also connected, that is to say  $\text{coeff}_{L/\mathbb{F}_p}(M)$  is indecomposable.

**Corollary 2.3** *Let  $G$  be a semi-simple affine  $p$ -inner algebraic group and  $X$  a projective  $G$ -homogeneous  $F$ -variety. The complete motivic decomposition of  $X \in CM(F; \mathbb{F}_p)$  remain complete when the coefficients are extended to any finite field of characteristic  $p$ .*

**Remark :** If  $\Lambda$  is a finite and connected ring, complete motivic decompositions in  $CM(F; \Lambda)$  remain complete when the coefficients are extended to the residue field of  $\Lambda$  by [7, Corollary 2.6], hence the study of motivic decompositions in  $CM_G(F; \Lambda)$ , where  $\Lambda$  is any finite connected ring whose residue field is of characteristic  $p$ , is reduced to the study motivic decompositions in  $CM_G(F; \mathbb{F}_p)$ .

We now produce a counterexample to theorem 2.2 in the case where the algebraic group  $G$  doesn't satisfy the needed assumptions. Let  $L/F$  be a Galois extension of degree 3. By [1, Section 7], the endomorphism ring  $End(M(\text{Spec}(L)))$  of the motive associated with the  $F$ -variety  $\text{Spec}(L)$  with coefficients in  $\mathbb{F}_2$  is the  $\mathbb{F}_2$ -algebra of  $Gal(L/F)$ , i.e.  $\frac{\mathbb{F}_2[X]}{(X^3-1)} \simeq \mathbb{F}_2 \times \mathbb{F}_4$ , hence  $M(\text{Spec}(L)) = M \oplus N$ , with  $End(N) = \mathbb{F}_4$  and both  $M$  and  $N$  are indecomposable. Now  $End(\text{res}_{\mathbb{F}_4/\mathbb{F}_2}(N)) = \mathbb{F}_4 \otimes \mathbb{F}_4$  is not connected since  $1 \otimes \alpha + \alpha \otimes 1$  is a non-trivial idempotent for any  $\alpha \in \mathbb{F}_4 \setminus \mathbb{F}_2$ , hence  $\text{res}_{\mathbb{F}_4/\mathbb{F}_2}(N)$  is decomposable.

Consider the  $(PGL_2)_L$ -homogeneous  $L$ -variety  $\mathbb{P}_L^1$ . The *Weil restriction*  $\mathcal{R}(\mathbb{P}_L^1)$  is a projective homogeneous  $F$ -variety under the action of the Weil restriction of  $(PGL_2)_L$ , and the minimal extension such that  $\mathcal{R}((PGL_2)_L)$  is of inner type is  $L$ . By [3, Example 4.8], the motive with coefficients in  $\mathbb{F}_2$  of  $\mathcal{R}(\mathbb{P}_L^1)$  contains two twists of  $\text{Spec}(L)$  as direct summands, therefore at least two indecomposable direct summands of  $\mathcal{R}(\mathbb{P}_L^1)$  split off over  $\mathbb{F}_4$ .

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