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CONTROLLABILITY TO TRAJECTORIES FOR SOME PARABOLIC SYSTEMS OF THREE AND TWO EQUATIONS BY ONE CONTROL FORCE

ASSIA BENABDALLAH ^{*}, MICHEL CRISTOFOL [†], PATRICIA GAITAN [‡], AND LUZ DE TERESA [§]

Abstract. We present a controllability result for a class of linear parabolic systems of 3 equations. To prove the result, we establish a global Carleman estimate for the solutions of a system of 2 coupled parabolic equations with first order terms. We also obtain stability results for the identification of coefficients of the systems are also obtained.

Key words. Control, observability, Carleman estimates, reaction diffusion systems, inverse problems.

AMS subject classifications. 93B05, 93B07, 35K05, 35K55, 35R30.

1. Introduction and notations. The controllability of linear ordinary differential systems is a well understood subject. In particular, if $n, m \in \mathbb{N}$ with $n, m \geq 1$ and $A \in \mathcal{L}(R^n)$ and $C \in \mathcal{L}(R^m, R^n)$, then the linear ordinary differential system $Y' = AY + Cu$ is controllable at time $T > 0$ if and only if the Kalman rank condition

$$(1.1) \quad \text{rank} [A | C] = \text{rank} [A^{n-1}C | A^{n-2}C | \dots | C] = n,$$

is satisfied (see for example [18, Chapter 2, p. 35]).

In the last ten years there has been an increasing interest in the study of null controllability and inverse problems for systems of parabolic equations. To the best of our knowledge most of the existing results in the literature deal with zero order coupling systems or constant coefficients (see for instance [23], [1] [7], [2], [3], [9], [13], [15], [4], [5], [6] and [14]). In these papers, almost all the results have been established for 2×2 systems where the control is exerted on the first equation. The most general results in this context seem to be those in [14], [4] and [5]. In [14], the authors study a *cascade* parabolic system of n equations ($n \geq 2$) controlled with one single distributed control. In [4] and [5], the authors provide necessary and sufficient conditions for the controllability of $n \times n$ parabolic linear systems with constant or time-dependent coefficients. The main goal of our work is to propose sufficient conditions to control 3×3 parabolic linear systems with *non constant* coefficients by *one control*. They generalize the Kalman condition obtained for parabolic systems with constant coefficients in [4]. The main ingredient of the proof is a Carleman estimate for a 2×2 reaction-convection-diffusion system. This Carleman estimate allows us to derive a new controllability result for such systems and a stability estimate for identifications of coefficient of 3×3 or 2×2 systems.

Let $\Omega \subset R^n$, $n \geq 1$ be a bounded connected open set of class C^2 . Let $T > 0$ and let ω be an open non empty subset of Ω . We define $\Omega_T = \Omega \times (0, T)$, $\omega_T = \omega \times (0, T)$ and $\Sigma_T = \partial\Omega \times (0, T)$. Our main objective is to establish new controllability results

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for coupled parabolic equations with *one distributed* control force in ω_T . To be more precise, let us consider second order elliptic self adjoint operators given by

$$(1.2) \quad \operatorname{div}(H_l \nabla) = \sum_{i,j=1}^n \partial_i (h_{ij}^l(x) \partial_j), \quad l = 1, 2, 3,$$

with

$$\begin{cases} h_{ij}^l \in W^{1,\infty}(\Omega), \\ h_{ij}^l(x) = h_{ji}^l(x) \quad \text{a.e. in } \Omega, \end{cases}$$

and the coefficients h_{ij}^l satisfying the uniform elliptic condition

$$(1.3) \quad \sum_{i,j=1}^n h_{ij}^l(x) \xi_i \xi_j \geq h_0 |\xi|^2, \quad \forall \xi \in R^n, \quad \text{a.e. in } \Omega, \quad l = 1, 2, 3,$$

for a positive constant h_0 . Let $(a_{ij})_{1 \leq i,j \leq 3} \in C^4(\overline{\Omega_T})$ (this assumption can be weakened see Remark 3.5). We consider the following 3×3 reaction-diffusion system

$$(1.4) \quad \begin{cases} \partial_t y = (\mathcal{L} + A)y + Cf1_\omega \text{ in } \Omega_T, \\ y = 0 \text{ on } \Sigma_T, \\ y(\cdot, 0) = y_0(\cdot) \text{ in } \Omega, \end{cases}$$

where

$$\mathcal{L} = \begin{pmatrix} \operatorname{div}(H_1 \nabla) & 0 & 0 \\ 0 & \operatorname{div}(H_2 \nabla) & 0 \\ 0 & 0 & \operatorname{div}(H_3 \nabla) \end{pmatrix},$$

$A = (a_{ij})_{1 \leq i,j \leq 3}$, $C = (1, 0, 0)^t \in \mathcal{L}(R, R^3)$, $f \in L^2(\Omega_T)$ and $y_0 = (y_{0,i})_{1 \leq i \leq 3} \in L^2(\Omega)^3$. In (1.4), $y = (y_i)_{1 \leq i \leq 3}$ is the state variable while 1_ω denotes the characteristic function of the open subset ω .

We will say that (1.4) is null controllable if for all initial data in $L^2(\Omega)^3$ there exists $f \in L^2(\Omega_T)$ such that the solution of (1.4) satisfy $y(\cdot, T) = 0$. In [4] it has been proved, among other results, that if all the coefficients (a_{ij}) are *constant* and if there exists $d_1 \in R$ such that $d_1 H_1 = H_2 = H_3$, then System (1.4) is null controllable if and only if the algebraic Kalman condition, $\det[A|C] \neq 0$, is satisfied. Our main interest is to remove the assumption that the coefficients have to be constant.

The main result of this paper is

THEOREM 1.1. *Suppose that a_{21} , a_{31} are time independent, that there exists $j \in \{2, 3\}$ such that $|a_{j1}(x)| \geq C > 0$ for all $x \in \omega$ and that $H_2 = H_3$. Let j be as before. We define $k_j = \frac{6}{j}$,*

$$B_{k_j} := -2H_2 \left(\nabla a_{k_j 1} - \frac{a_{k_j 1}}{a_{j1}} \nabla a_{j1} \right),$$

and

$$b_j = \frac{2H_2 \nabla a_{j1} (\nabla a_{k_j 1} a_{j1} - \nabla a_{j1} a_{k_j 1})}{a_{j1}^2} + \frac{a_{k_j 1} \operatorname{div}(H_2 \nabla a_{j1}) - a_{j1} \operatorname{div}(H_2 \nabla a_{k_j 1})}{a_{j1}} - (-1)^j \frac{\det[A|C]}{a_{j1}}.$$

Assume that either

1. $B_{k_j} = 0$ and $b_j \neq 0$ on ω_T ,

or

2. $\partial\omega \cap \partial\Omega = \gamma$, $|\gamma| \neq 0$, and $B_{k_j} \cdot \nu \neq 0$, on γ .

Then, for all $y_0 \in (L^2(\Omega))^3$, there exists $f \in L^2(\Omega_T)$ such that the corresponding solution of (1.4) satisfies $y_1(\cdot, T) = y_2(\cdot, T) = y_3(\cdot, T) = 0$.

REMARK 1.2.

1. If all the coefficients are constant, $B_{k_j} = 0$ for all $j \in \{2, 3\}$, then assumptions of Theorem 1.1 are reduced to $b_j \neq 0$ and more precisely to $\det[A|C] \neq 0$. So we recover the condition obtained in [5].

2. Furthermore, it is not difficult to construct an example where the algebraic Kalman rank condition is not satisfied but assumption 2 of Theorem 1.1 is verified. For example, take Ω any smooth domain in R^2 containing $\omega = \{y < -1, (x-2)^2 + (y+1)^2 < 1\}$ and $\gamma = [1, 3] \times \{-1\}$. Take now $a_{32} = a_{23} = a_{22} = a_{33} = 0$, $a_{31}(x, y) = -y^2$, $a_{21} = x + y$ and $H_1 = H_2 = H_3 = I_d$. Then $a_{31} \neq 0$ in ω_T , but $\det[A, C] = 0$ in Ω . Moreover, we have

$$\left(\nabla a_{21} - \frac{a_{21}}{a_{31}} \nabla a_{31} \right) \cdot \nu(x) = 2x - 1 > 0, \text{ on } \gamma.$$

So assumption 2 of Theorem 1.1 is satisfied.

The proof of Theorem 1.1 uses in an essential way a new Carleman inequality for two coupled reaction-diffusion-convection equations. This inequality allows us to prove a new controllability result for such a system (see Section 2).

Our controllability result does not need restrictive assumptions such as for *cascade* systems or constant coefficients. In the case of constant coefficients, we recover the Kalman's criterion proved in [4]. In Section 2, we derive a controllability result for a class of 2×2 reaction-diffusion-convection systems. In [15], the author studies the null controllability of systems of two parabolic equations, where the coupling terms are first order space derivatives in one equation and second order space derivatives in the other. The results of [15] cannot be derived as a particular case of ours. The reverse holds true as well. They are independent results, proved with different techniques, both aiming for a better understanding of the complexity of Carleman estimates for coupled equations and their controllability and identification consequences.

The results in this paper are inspired in related research results that can be found in [6] where similar systems are studied under the same conditions. However, we obtain a null controllability result while in [6] the authors obtain an approximate controllability result.

A large area of applications in ecology and biology requires the identification of coefficients for reaction-diffusion systems. Starting with the pioneer work of Bukhgeim-Klibanov [8], Carleman estimates have been successfully used for the uniqueness and stability for determining coefficients. Often, it is difficult to observe all the components for reaction-diffusion systems, thus the increasing interest in reducing the number of observed components. There are very few research papers devoted to this problem. We can refer to [9], [6], for 2×2 parabolic systems and to [17] for Lamé systems. We use our new Carleman estimate for 2×2 parabolic systems with advection terms to prove, for a 3×3 reaction-diffusion system, a stability result for three coefficients (one in each equation). We only observe one component and we need the knowledge of the solution at a fixed time $T' \in (0, T)$ and all the domain Ω and these three coefficients on ω . We generalize for a $n \times n$ parabolic system: a stability result

for n coefficients observing only $n - 2$ components with the partial knowledge of solely three coefficients.

The paper is organized as follows. In Section 2 we prove a new controllability result for a 2×2 reaction-diffusion-convection system. This section contains the main ingredient of this paper: a new Carleman estimate for 2×2 reaction-diffusion-convection systems with only one observation. In Section 3 we prove our main result: Theorem 1.1. To prove this, we establish an observability estimate for the corresponding adjoint system. This Carleman estimate for 3×3 reaction-diffusion systems follows from the previous one for 2×2 reaction-diffusion-convection systems. We conclude in Section 4 with some applications to inverse problems and generalizations for more than 3 equations.

2. Controllability for a 2×2 reaction-diffusion-convection system. In this section we present a null controllability result for coupled parabolic systems. We consider the case of two coupled reaction-diffusion-convection equations and we control one of them. Let us consider

$$(2.1) \quad \begin{cases} \partial_t y_1 = \operatorname{div}(H_1 \nabla y_1) + a_{11} y_1 + a_{12} y_2 + A_{11} \cdot \nabla y_1 + A_{12} \cdot \nabla y_2 + f \chi_\omega & \text{in } \Omega_T, \\ \partial_t y_2 = \operatorname{div}(H_2 \nabla y_2) + a_{21} y_1 + a_{22} y_2 + A_{21} \cdot \nabla y_1 + A_{22} \cdot \nabla y_2 & \text{in } \Omega_T, \\ y_1(\cdot, t) = y_2(\cdot, t) = 0 & \text{on } \Sigma_T, \\ y_1(\cdot, 0) = y_1^0(\cdot), y_2(\cdot, 0) = y_2^0(\cdot) & \text{in } \Omega. \end{cases}$$

with $a_{ij} \in C^4(\overline{\Omega_T})$ and $A_{ij} \in C^1(\overline{\Omega_T})^n$ for $i, j = 1, 2$ (here again this regularity can be weakened see Remark 3.5). We say that system (2.1) is null controllable if for every initial data $(y_1^0, y_2^0) \in (L^2(\Omega))^2$ there exists $f \in L^2(\omega_T)$ such that the corresponding solution of (2.1) satisfies

$$y_1(T) = y_2(T) = 0.$$

The first controllability result we obtain is the following one :

THEOREM 2.1. *Let us assume that A_{21} is time independent on ω_T and that ω of class C^2 , $\omega \subset \Omega$ is such that for some $\gamma \subset \partial\Omega$, $|\gamma| \neq 0$ with $\gamma \subset \partial\omega \cap \partial\Omega$ we have that $|A_{21}(x) \cdot \nu(x)| \neq 0$ for every $x \in \gamma$. Furthermore, assume that $A_{21}|_\omega \in W^{3,\infty}(\omega)^n$. Then, System (2.1) is null controllable at time $T > 0$.*

As in the scalar case the null controllability of (2.1) is equivalent to the obtention of an observability inequality for the adjoint system (2.3) (see e.g [12], [11], [10]). That is, Theorem 2.1 is equivalent to prove the following result:

THEOREM 2.2. *Under the assumptions of Theorem 2.1 there exists $C > 0$ such that*

$$(2.2) \quad \int_{\Omega} (|\varphi_1(0)|^2 + |\varphi_2(0)|^2) dx \leq C \int_0^T \int_{\omega} |\varphi_1|^2 dx dt$$

holds true for any solution of

$$(2.3) \quad \begin{cases} -\partial_t \varphi_1 = \operatorname{div}(H_1 \nabla \varphi_1) + (a_{11} - \nabla \cdot A_{11})\varphi_1 + (a_{21} - \nabla \cdot A_{21})\varphi_2 - A_{11} \cdot \nabla \varphi_1 - A_{21} \cdot \nabla \varphi_2 & \text{in } \Omega_T, \\ -\partial_t \varphi_2 = \operatorname{div}(H_2 \nabla \varphi_2) + (a_{12} - \nabla \cdot A_{12})\varphi_1 + (a_{22} - \nabla \cdot A_{22})\varphi_2 - A_{12} \cdot \nabla \varphi_1 - A_{22} \cdot \nabla \varphi_2 & \text{in } \Omega_T, \\ \varphi_1(\cdot, t) = \varphi_2(\cdot, t) = 0 & \text{on } \Sigma_T, \\ \varphi_1(\cdot, T) = \varphi_1^T(\cdot), \varphi_2(\cdot, T) = \varphi_2^T(\cdot) & \text{in } \Omega. \end{cases}$$

2.1. Proof of the controllability result. We prove Theorem 2.1 assuming that Theorem 2.2 holds true. The proof of Theorem 2.2 uses the Carleman inequality proved in the next subsection. So we first state the Carleman inequality, prove Theorem 2.2, and conclude the section with the proof of the Carleman inequality, i.e Theorem 2.5.

Proof. [Proof of Theorem 2.1]

We now prove Theorem 2.1 using (2.2). There are several ways to prove it. We use the most direct technique. Let $V = L^2(\Omega) \times L^2(\Omega)$, and let G and L be the following linear mappings:

$$L : L^2(\omega_T) \rightarrow V \quad f \mapsto (y_1(T), y_2(T))$$

where $(y_1(\cdot), y_2(\cdot))$ is the corresponding solution of (2.1) with $(y_1^0, y_2^0) = (0, 0)$, and

$$G : V \rightarrow V \quad (y_1^0, y_2^0) \mapsto (y_1(T), y_2(T))$$

where $(y_1(\cdot), y_2(\cdot))$ solves (2.1) with $f = 0$. Then Theorem 2.1 is equivalent to the inclusion

$$(2.4) \quad R(G) \subset R(L).$$

Both G and L are V -valued, bounded linear operators. So (2.4) holds if and only if, for every $(\varphi_1^T, \varphi_2^T) \in V$,

$$(2.5) \quad \|G^*(\varphi_1^T, \varphi_2^T)\|_V \leq C \|L^*(\varphi_1^T, \varphi_2^T)\|_{L^2(\omega_T)}$$

for some constant $C > 0$. A simple computation shows that

$$G^*(\varphi_1^T, \varphi_2^T) = (\varphi_1(x, 0), \varphi_2(x, 0)), \quad L^*(\varphi_1^T, \varphi_2^T) = \varphi_1 1_{\omega_T}$$

where φ_1 and φ_2 solve the adjoint system (2.3). Hence (2.5) is just (2.2) and Theorem 2.1 is proved.

□

2.2. A new Carleman estimate for a 2×2 reaction-diffusion-convection system with one observation. We consider the following reaction-diffusion-convection system:

$$(2.6) \quad \begin{cases} \partial_t u = \operatorname{div}(H_1 \nabla u) + au + bv + A \cdot \nabla u + B \cdot \nabla v + f & \text{in } \Omega_T, \\ \partial_t v = \operatorname{div}(H_2 \nabla v) + cu + dv + C \cdot \nabla u + D \cdot \nabla v + g & \text{in } \Omega_T, \\ u(\cdot, t) = v(\cdot, t) = 0 & \text{on } \Sigma_T, \\ u(\cdot, 0) = u_0(\cdot), v(\cdot, 0) = v_0(\cdot) & \text{in } \Omega. \end{cases}$$

Recall that for $a, b, c, d \in L^\infty(\Omega_T)$ and $A, C, D \in L^\infty(\Omega_T)^n$, $B \in L^\infty(\Omega)^n$ and for $u_0, v_0 \in L^2(\Omega)$, $f, g \in L^2(\Omega_T)$ the reaction-diffusion-convection system (2.6) admits a unique solution $(u, v) \in C([0, T]; L^2(\Omega))^2 \cap L^2(0, T; H_0^1(\Omega))^2$. Moreover, if $u_0, v_0 \in H^2(\Omega) \cap H_0^1(\Omega)$, then $(u, v) \in C([0, T]; H^2(\Omega) \cap H_0^1(\Omega))^2 \cap C^1([0, T]; L^2(\Omega))^2$. This result can be obtained in a classical way, see for example [20].

In order to prove (2.2), our main interest is to derive an observability estimate for (u, v) solutions of (2.6) by *solely* the observation of u in ω_T . First of all let us recall the notations for Carleman's estimates (see [12] and [16]).

Let $\beta \in C^2(\bar{\Omega})$ be the function constructed by Fursikov and Imanuvilov in [12] such that $\beta \geq 0$ in $\bar{\Omega}$, $|\nabla\beta| > 0$ in $\Omega \setminus \bar{\omega}$ and define by

$$(2.7) \quad \begin{cases} \eta(x, t) & := \frac{e^{2\lambda K} - e^{\lambda\beta(x)}}{t(T-t)}, & \forall (x, t) \in \Omega_T, \\ \rho(t) & := \frac{e^{\lambda\beta(x)}}{t(T-t)}, & \forall t \in (0, T), \end{cases}$$

where

$$(2.8) \quad K \geq \|\beta\|_{L^\infty(\Omega)}$$

is a fixed constant whose choice will be specified later. We introduce the functional

$$(2.9) \quad I(\tau, \varphi) = \iint_{\Omega_T} (s\rho)^{\tau-1} e^{-2s\eta} \left(|\varphi_t|^2 + \sum_{1 \leq i \leq j \leq n} |\partial_{x_i x_j}^2 \varphi|^2 + (s\lambda\rho)^2 |\nabla\varphi|^2 + (s\lambda\rho)^4 |\varphi|^2 \right) dxdt.$$

ASSUMPTION 2.3.

1. Let $\omega \subset \Omega$ be a non-empty subdomain of class C^2 with $\partial\omega \cap \partial\Omega = \gamma$ and $|\gamma| \neq 0$,
2. $|B(x) \cdot \nu(x)| \neq 0$, $x \in \gamma$,
3. $H_1|_\omega \in W^{2,\infty}(\omega)^n$, $B|_\omega \in W^{2,\infty}(\omega)^n$, $A|_{\omega_T} \in L^\infty(0, T; W^{1,\infty}(\omega))^n$ and $b|_{\omega_T} \in W^{2,\infty}(\omega_T)$.

We first recall a Carleman inequality for a single parabolic equation. That is, for $a \in L^\infty(\Omega_T)$, $A \in L^\infty(\Omega_T)^n$, $H \in W^{1,\infty}(\Omega)^n$, let $R = \operatorname{div}(H\nabla) + A \cdot \nabla + a$ and $I(\tau, \varphi)$ defined by (2.9) we have the following result

THEOREM 2.4. *Let $\omega \subset \Omega$ open and non empty, $\tau \in \mathbb{R}$. Then, there exist two positive constants s_0, C_0 (which only depend on $\Omega, \omega, T, H, A, a$ and τ) such that for every $\varphi \in L^2(0, T; H_0^1(\Omega))$ with $\partial_t \varphi \pm R\varphi \in L^2(\Omega_T)$, the following Carleman estimate holds*

$$(2.10) \quad I(\tau, \varphi) \leq \tilde{C}_0 \left(\iint_{\Omega_T} (s\rho)^\tau e^{-2s\eta} |\partial_t \varphi \pm R\varphi|^2 dxdt + \lambda^4 \iint_{\omega_T} (s\rho)^{\tau+3} e^{-2s\eta} |\varphi|^2 dxdt \right),$$

for all $s \geq s_0$ and η, ρ defined in (2.7) with $K > 0$ satisfying (2.8). The proof of this result can be found in [16]. The new Carleman estimate for a 2×2 reaction-diffusion-convection system is:

THEOREM 2.5. *Under Assumption 2.3 there exist two positive constants s_0, C (which only depend on $\Omega, \omega, a, b, c, d, A, B, C, D, H_1, H_2, \tau_1$ and τ_2) and a constant K , satisfying (2.30), such that for every $(u_0, v_0) \in L^2(\Omega)^2$ and $|\tau_1 - \tau_2| < 1$, the following Carleman estimate holds*

$$(2.11) \quad \begin{aligned} I(\tau_1, u) + I(\tau_2, v) &\leq C \left(\lambda^8 \iint_{\omega_T} e^{-2s\alpha} (s\rho^*)^{\tau^*} |u|^2 dxdt \right. \\ &\quad \left. + \lambda^4 \iint_{\omega_T} e^{-2s\eta} (s\rho)^{3+\tau_2} |Qf|^2 dxdt + \iint_{\Omega_T} e^{-2s\eta} ((s\rho)^{\tau_1} |f|^2 + (s\rho)^{\tau_2} |g|^2) dxdt \right) \end{aligned}$$

where Q is an appropriate operator defined in the Appendix (see (5.4)), $\eta^* = \max_{\bar{\Omega}} \eta$, $\eta_- = \min_{\bar{\Omega}} \eta$, $\alpha = 4\eta_- - 3\eta^*$, $\rho^* = \max_{\bar{\Omega}} \rho$ and $\tau^* = 4\tau_2 - 3\tau_1 + 15$, for all $s \geq s_0$ and

for all (u, v) solution of (2.6) and η defined by (2.7) .

We first prove Theorem 2.2 assuming that Theorem 2.5 holds true.

Proof. [Proof of Theorem 2.2]

We want to prove (2.2). We define $u(t) = \varphi_1(T - t), v(t) = \varphi_2(T - t)$. Then, u and v are solutions of (2.6) with $a(x, t) = (a_{11}(x, T - t) - \nabla \cdot A_{11}(x, T - t))$, $b(x, t) = (a_{21}(x, T - t) - \nabla \cdot A_{21}(x, T - t))$, $A(x, t) = -A_{11}(x, t - T)$, $B(x, t) = -A_{21}(x, t - T)$, $c(x, t) = (a_{12}(x, T - t) - \nabla \cdot A_{12}(x, T - t))$, $d(x, t) = (a_{22}(x, T - t) - \nabla \cdot A_{22}(x, T - t))$, $C(x, t) = -A_{12}(x, t - T)$, $D(x, t) = A_{22}(x, t - T)$, $f = g = 0$ and initial conditions $u(0) = \varphi_1^T, v(0) = \varphi_2^T$. We can then apply the results of Theorem 2.5 to φ_1 and φ_2 and get

$$\iint_{\Omega_T} e^{-2s\eta} \rho^3 (|\varphi_1|^2 + |\varphi_2|^2) dx dt \leq C \iint_{\omega_T} |\varphi_1|^2 dx dt.$$

Note that $C > 0$ is a generic constant that may change from line to line. We multiply the first equation in (2.3) by φ_1 and the second equation by φ_2 , we add the two equations, we integrate by parts and apply Gronwall inequality, we then get that for any $0 \leq t < \tau \leq T$

$$\int_{\Omega} (|\varphi_1(t)|^2 + |\varphi_2(t)|^2) dx \leq C \int_{\Omega} (|\varphi_1(\tau)|^2 + |\varphi_2(\tau)|^2) dx.$$

This inequality implies on one hand that for $\tau > 0$

$$\int_{\Omega} (|\varphi_1(0)|^2 + |\varphi_2(0)|^2) dx \leq C \int_{\Omega} (|\varphi_1(\tau)|^2 + |\varphi_2(\tau)|^2) dx,$$

and on the other hand that

$$(2.12) \quad \int_{\Omega} (|\varphi_1(0)|^2 + |\varphi_2(0)|^2) dx \leq C(T) \int_{T/4}^{3T/4} \int_{\Omega} (|\varphi_1(\tau)|^2 + |\varphi_2(\tau)|^2) dx d\tau.$$

Observe that, by construction, $\rho^3(t)e^{-2s\eta(t)} \geq C(T)$ for $t \in [T/4, 3T/4]$. This fact combined with (2.12) imply that

$$\int_{\Omega} (|\varphi_1(0)|^2 + |\varphi_2(0)|^2) dx \leq C \iint_{\omega_T} |\varphi_1|^2 dx dt.$$

□

Proof. [Proof of Theorem 2.5]

Consider $\tilde{\omega} \subset \subset \omega$ open and non empty. If $|\tau_1 - \tau_2| < 1$, a direct application of Theorem 2.4 leads to

$$(2.13) \quad \begin{aligned} I(\tau_1, u) + I(\tau_2, v) \leq C & \left(\lambda^4 \iint_{\tilde{\omega}_T} (s\rho)^{\tau_1+3} e^{-2s\eta} |u|^2 dx dt + \lambda^4 \iint_{\tilde{\omega}_T} (s\rho)^{\tau_2+3} e^{-2s\eta} |v|^2 dx dt \right. \\ & \left. + \iint_{\Omega_T} (s\rho)^{\tau_1} e^{-2s\eta} |f|^2 dx dt + \iint_{\Omega_T} (s\rho)^{\tau_2} e^{-2s\eta} |g|^2 dx dt \right). \end{aligned}$$

The main question is to get rid of the term

$$\iint_{\tilde{\omega}_T} (s\rho)^{\tau_2+3} e^{-2s\eta} |v|^2 dx dt .$$

As in ([2], [9]), one will use the first equation in order to derive a *local observability* estimate for v with respect to u . In comparison with the previous results, the main difficulty here is the presence of *first order terms* on v . Roughly speaking, the idea used in ([2], [9]) is to *transform locally* (in $\omega \times (0, T)$) the first equation of (2.6) as $v = Lu$ where L is a *partial differential operator* (first order in time and second order in space). In these cases, $A = B = C = D = 0$ and therefore the main assumption for this solvability is that $b \neq 0$ in an open set $\omega' \times (0, T) \subset \omega_T$. In our case, this condition is replaced by $B \cdot \nu \neq 0$ on $\gamma \times (0, T)$ where γ is a part of the boundary of $\Omega \cap \omega$ and requires ω to be a neighborhood of γ . With these assumptions, one can still transform the first equation of (2.6) *locally* in space as $v = Lu$, but the operator L is not a partial differential operator. With some technical computations, we are able to still deduce a local observability estimate for v with respect to u :

THEOREM 2.6. *Let Assumption 2.3 be satisfied. Suppose that $u \in C([0, T]; H^2(\Omega) \cap H_0^1(\Omega)) \cap C^1([0, T]; L^2(\Omega))$ and v satisfies*

$$\begin{cases} bv + B \cdot \nabla v = \partial_t u - \operatorname{div}(H_1 \nabla u) - au - A \cdot \nabla u - f & \text{in } \omega_T, \\ v(\cdot, t) = 0 & \text{on } \gamma_T. \end{cases}$$

Then, for any $\tilde{\omega} \subset\subset \omega$ and for all $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that

$$(2.14) \quad \begin{aligned} \lambda^4 \iint_{\tilde{\omega}_T} (s\rho)^{\tau_2+3} e^{-2s\eta} |v|^2 dxdt &\leq C_\varepsilon \lambda^{16} \iint_{\omega_T} e^{-2s\alpha} (s\rho^*)^{\tau^*} |u|^2 dxdt \\ &+ \lambda^4 \iint_{\omega_T} e^{-2s\eta} (s\rho)^{3+\tau_2} |Qf|^2 dxdt + \varepsilon I(\tau_2, v), \end{aligned}$$

where Q is an appropriate operator defined in the Appendix (see (5.4)), $\eta^* = \max_{\Omega} \eta$, $\eta_- = \min_{\Omega} \eta$, $\alpha = 4\eta_- - 3\eta^*$, $\rho^* = \max_{\Omega} \rho$ and $\tau^* = 4\tau_2 - 3\tau_1 + 15$, for all $s \geq s_0$ and η defined by (2.7).

Proof. [Proof of Theorem 2.6] In order to make the proof clearer to the reader, we are going to prove Theorem 2.6 in the simplest case where

$$(2.15) \quad \Omega = (0, 1) \times \Omega', \quad \omega = (0, \varepsilon) \times \omega'$$

with $\omega' \subset \Omega' \subset R^{n-1}$, $0 < \varepsilon < 1$, and

$$(2.16) \quad B(x) = (1, 0, \dots, 0).$$

In this case $\gamma = \{0\} \times \omega'$. As we will show in Remark 2.8, the general case comes down to this simplest one. With these assumptions, the first equation of (2.6) has the particular form

$$(2.17) \quad \partial_{x_1} v + bv = \partial_t u - \operatorname{div}(H_1 \cdot \nabla u) - au - A \cdot \nabla u - f,$$

with

$$(2.18) \quad x = (x_1, x').$$

We will denote $\gamma_T := \gamma \times (0, T)$.

The proof of the Theorem will be done in 3 steps

• Step 1 : **An equation for v**

One can define the following operator

$$(2.19) \quad L := \partial_{x_1} + b,$$

with $D(L) = \{v \in L^2(0, T); H^1(\omega)\}; v(0, x', t) = 0$ on γ_T . $(L, D(L))$ is an unbounded invertible operator from $L^2(\omega_T)$ to $L^2(\omega_T)$. For $w \in L^2(\omega_T)$, direct computations give that

$$(2.20) \quad L^{-1}(w)(x, t) = e^{\int_0^{x_1} b(y_1, x', t) dy_1} \int_0^{x_1} e^{-\int_0^{y_1} b(x_1, x', t) dx_1} w(y_1, x', t) dy_1.$$

For, $p, q \in L^\infty(\omega_T)$, let us define

$$(2.21) \quad K(p, q)w(x, t) = p(x, t) \int_0^{x_1} q(y_1, x', t) w(y_1, x', t) dy_1.$$

Note that $K(p, q) \in \mathcal{L}(L^2(\omega_T))$ and $L^{-1} = K(p, q)$ with

$$(2.22) \quad p(x, t) = e^{\int_0^{x_1} b(y_1, x', t) dy_1}, \quad q(x, t) = e^{-\int_0^{x_1} b(y_1, x', t) dy_1}.$$

Moreover under Assumption 2.3, we have $p, q \in W^{2, \infty}(\omega_T)$. As it will be clear in the sequel, we need to compute the effect of the composition of operators as $K(p, q)$ on partial differential operators.

We summarize here these computations:

1. For $p, q, e \in L^\infty(\omega_T)$, $w \in L^2(\omega_T)$

$$(2.23) \quad K(p, q)(ew) = K(p, qe)w.$$

2. For $p, q \in W^{1, \infty}(\omega_T)$ and $E \in L^\infty(0, T; W^{1, \infty}(\omega))$, $2 \leq i \leq n$ and $w \in H^1(\omega_T)$, we have

$$(2.24) \quad \begin{aligned} K(p, q)(E\partial_{x_1} w)(x, t) &= -K(p, \partial_{x_1}(Eq)w)(x, t) + pqEw(x, t) - p(x, t)(qEw)(0, x', t), \\ K(p, q)(E\partial_{x_i} w)(x, t) &= \partial_{x_i} K(p, Eq)w(x, t) - K(\partial_{x_i} p, Eq)w(x, t) - K(p, \partial_{x_i}(Eq))w(x, t), \\ K(p, q)(\partial_t w)(x, t) &= \partial_t K(p, q)w(x, t) - K(\partial_t p, q)w(x, t) - K(p, \partial_t q)w(x, t). \end{aligned}$$

3. For $p, q \in W^{2, \infty}(\omega_T)$, $H \in L^\infty(0, T; W^{1, \infty}(\omega))$, $1 \leq i \leq n$ and $w \in L^2(0, T; H^2(\omega))$, we have

$$K(p, q)(\partial_{x_i}(H\partial_{x_i} w))(x, t) = K(p, q\partial_{x_i} H)\partial_{x_i} w(x, t) + K(p, Hq)\partial_{x_i}^2 w(x, t),$$

$$\begin{aligned} K(p, q)\partial_{x_1}^2 w(x, t) &= K(p, \partial_{x_1}^2 q)w(x, t) - (p\partial_{x_1} q w)(x, t) \\ &\quad + (pq\partial_{x_1} w)(x, t) + p(x, t)(w\partial_{x_1} q - q\partial_{x_1} w)(0, x', t), \end{aligned}$$

$$(2.25) \quad \begin{aligned} K(p, q)(\partial_{x_i}^2 w)(x, t) &= \partial_{x_i}^2 K(p, q)w(x, t) - 2\partial_{x_i} K(\partial_{x_i} p, q)w(x, t) - 2\partial_{x_i} K(p, \partial_{x_i} q)w(x, t) \\ &\quad + K(\partial_{x_i}^2 p, q)w(x, t) + K(p, \partial_{x_i}^2 q)w(x, t) + 2K(\partial_{x_i} p, \partial_{x_i} q)w(x, t). \end{aligned}$$

LEMMA 2.7. *Let H_1, A, b, B satisfy Assumption 2.3, p, q defined in (2.22). Then there exist $(p_i, q_i)_{2 \leq i \leq n} \in W^{2, \infty}(\omega_T)^{2(n-1)}$, $(\tilde{p}_i, \tilde{q}_i)_{1 \leq i \leq n} \in W^{1, \infty}(\omega_T)^{2n}$, $k \in L^\infty(\omega_T)$ such that, for any $u \in C([0, T]; H^2(\Omega) \cap H_0^1(\Omega)) \cap C^1([0, T]; L^2(\Omega))$, the solution v of (2.6) satisfies for every $(x, t) \in \omega_T$:*

$$(2.26) \quad \begin{aligned} v(x, t) &= \partial_t K(\tilde{p}_1, \tilde{q}_1)u(x, t) + \sum_{i=2}^n \partial_{x_i}^2 K(p_i, q_i)u(x, t) + \sum_{i=2}^n \partial_{x_i} K(\tilde{p}_i, \tilde{q}_i)u(x, t) \\ &\quad + K(p, aq)u(x, t) + k(x, t)u(x, t) + K(p, q)f(x, t) + pq(x, t)h_1(0, x', t)\partial_{x_1} u(0, x', t). \end{aligned}$$

Proof. [Proof of the Lemma 2.7] It is a direct consequence of (2.23), (2.24) and (2.25). \square

- Step 2: **An observability inequality for v with two observations: u on ω_T and $\partial_\nu u$ on $\gamma \times (0, T)$.**

Let $\tilde{\omega} \subset\subset \tilde{\omega} \subset\subset \omega$, and $\xi \in C^\infty(\bar{\Omega})$ a cut-off function, such that

$$(2.27) \quad \begin{aligned} \xi(x) &= 1, \forall x \in \tilde{\omega} \\ \xi(x) &= 0, \forall x \notin \omega. \end{aligned}$$

We multiply (2.26) by $(s\rho)^{\tau_2+3}\xi e^{-2s\eta}v$ and we integrate on Ω_T .

$$\begin{aligned} & \iint_{\Omega_T} (s\rho)^{\tau_2+3}\xi e^{-2s\eta}v^2 dxdt = \iint_{\Omega_T} (s\rho)^{\tau_2+3}\xi e^{-2s\eta}v\partial_t K(\tilde{p}_1, \tilde{q}_1)u dxdt \\ & + \iint_{\Omega_T} (s\rho)^{\tau_2+3}\xi e^{-2s\eta}v \sum_{i=2}^n \partial_{x_i}^2 K(p_i, q_i)u dxdt \\ & + \iint_{\Omega_T} (s\rho)^{\tau_2+3}\xi e^{-2s\eta}v \sum_{i=2}^n \partial_{x_i} K(\tilde{p}_i, \tilde{q}_i)u dxdt \\ & + \iint_{\Omega_T} (s\rho)^{\tau_2+3}\xi e^{-2s\eta}v K(p, aq)u dxdt + \iint_{\Omega_T} (s\rho)^{\tau_2+3}\xi e^{-2s\eta}v k u dxdt \\ & + \iint_{\Omega_T} (s\rho)^{\tau_2+3}\xi e^{-2s\eta}v L^{-1} f dxdt + \iint_{\Omega_T} (s\rho)^{\tau_2+3}\xi e^{-2s\eta}v p \partial_{x_1} u(0, x', t) dxdt. \end{aligned}$$

We estimate each term of the right hand side of the previous equality. For example for the first one, using the definition of $I(\tau_2, v)$, Cauchy-Schwartz and Young inequalities, we can write :

$$\lambda^4 \iint_{\Omega_T} (s\rho)^{\tau_2+3}\xi e^{-2s\eta}v\partial_t K(\tilde{p}_1, \tilde{q}_1)u dxdt = \lambda^4 \iint_{\Omega_T} \partial_t ((s\rho)^{\tau_2+3}\xi e^{-2s\eta}v) K(\tilde{p}_1, \tilde{q}_1)u dxdt,$$

then

$$\begin{aligned} & |\lambda^4 \iint_{\Omega_T} \partial_t ((s\rho)^{\tau_2+3}\xi e^{-2s\eta}v) K(\tilde{p}_1, \tilde{q}_1)u dxdt| \leq |\lambda^4 \iint_{\Omega_T} \partial_t ((s\rho)^{\tau_2+3}\xi e^{-2s\eta}) v K(\tilde{p}_1, \tilde{q}_1)u dxdt| \\ & \quad + |\lambda^4 \iint_{\Omega_T} (s\rho)^{\tau_2+3}\xi e^{-2s\eta} \partial_t (v) K(\tilde{p}_1, \tilde{q}_1)u dxdt| \\ & \leq \varepsilon I(\tau_2, v) + \lambda^4 s^{\tau_2+4} \iint_{\Omega_T} (\rho)^{\tau_2+5}\xi e^{-2s\eta} (K(\tilde{p}_1, \tilde{q}_1)u)^2 dxdt \\ & \quad + \lambda^8 s^{\tau_2+7} \iint_{\Omega_T} (\rho)^{\tau_2+7}\xi e^{-2s\eta} (K(\tilde{p}_1, \tilde{q}_1)u)^2 dxdt \\ & \leq C_\varepsilon \lambda^8 s^{\tau_2+7} \|p\|_\infty^2 \|q\|_\infty^2 \int_0^T \int_{\omega'} \xi e^{-2s\eta} (\rho)^{\tau_2+7} \left(\int_0^{x_1} |u|^2(y_1, x', t) dy_1 \right) dx' dt + \varepsilon I(\tau_2, v) \\ & \leq C_\varepsilon \lambda^8 s^{\tau_2+7} \int_0^T \int_{\omega'} \left(\int_0^\varepsilon e^{-2s\eta} (\rho)^{\tau_2+7} dx_1 \right) dx' dt + \varepsilon I(\tau_2, v) \\ & \leq C_\varepsilon \lambda^8 s^{\tau_2+7} \int_0^T \int_{\omega'} \left[\int_0^\varepsilon |u|^2(y_1, x', t) \left(\int_0^1 e^{-2s\eta} (\rho)^{\tau_2+7} dx_1 \right) dy_1 \right] dx' dt + \varepsilon I(\tau_2, v) \\ & \leq \varepsilon I(\tau_2, v) + C_\varepsilon \lambda^8 s^{\tau_2+7} \iint_{\omega_T} M(x', t) |u|^2 dxdt \end{aligned}$$

with $M(x', t) = \int_0^1 \rho^{\tau_2+7} e^{-2s\eta} dx_1$.

After technical calculations, we keep the higher exponents for s and λ , we obtain:

$$(2.28) \quad \begin{aligned} & \lambda^4 \iint_{\tilde{\omega}_T} (s\rho)^{\tau_2+3} e^{-2s\eta} |v|^2 dx dt \leq \varepsilon I(\tau_2, v) \\ & + C_\varepsilon \left(\lambda^8 s^{\tau_2+7} \iint_{\omega_T} M(x', t) |u|^2 dx dt + \lambda^4 \iint_{\omega_T} e^{-2s\eta} (s\rho)^{3+\tau_2} |L^{-1} f|^2 dx dt \right. \\ & \left. + \lambda^4 \int_0^T \int_{\omega'} \int_0^1 (s\rho)^{\tau_2+3} e^{-2s\eta} |\partial_{x_1} u(0, x', t)|^2 dx_1 dx' dt \right). \end{aligned}$$

• **Step 3: Estimate of the boundary term**

Observe that for any f and h in $H^2(\omega)$,

$$\int_{\omega} \partial_{x_1} f(hf) dx = -\frac{1}{2} \int_{\omega} |f|^2 \partial_{x_1}(h) dx + \frac{1}{2} \int_{\omega'} (|f|^2 h)(\varepsilon) dx' - (|f|^2 h)(0).$$

We apply this formula for $f = \partial_{x_1} u$, $f = u$ and h such that $h(\varepsilon) = 0$. If we denote $N(x', t) = \int_0^1 \rho^{\tau_2+3} e^{-2s\eta} dx_1$, $n^* = 2\tau_2 - \tau_1 + 7$ then we have

$$\begin{aligned} & \lambda^4 \iint_{\omega_T} (s\rho)^{\tau_2+3} e^{-2s\eta} |\partial_{x_1} u(0, x', t)|^2 dx dt \\ & = \lambda^4 s^{\tau_2+3} \int_0^T \int_{\omega'} \left(\int_0^1 (\rho)^{\tau_2+3} e^{-2s\eta} dx_1 \right) |\partial_{x_1} u(0, x', t)|^2 dx' dt \\ & = \lambda^4 s^{\tau_2+3} \int_0^T \int_{\omega'} N(x', t) |\partial_{x_1} u(0, x', t)|^2 dx' dt \\ & = -\lambda^4 s^{\tau_2+3} \iint_{\Omega_T} \partial_{x_1}^2 u \partial_{x_1} u \xi N(x', t) dx dt - \frac{1}{2} \lambda^4 \int_{\Omega_T} |\partial_{x_1} u|^2 N(x', t) \partial_{x_1} \xi dx dt \\ & \leq \varepsilon \iint_{\Omega_T} (s\rho)^{\tau_1-1} e^{-2s\eta^*} |\partial_{x_1}^2 u|^2 dx dt + \lambda^8 C_\varepsilon s^{n^*} \iint_{\omega_T} \rho^{-\tau_1+1} e^{2s\eta^*} N^2(x', t) |\partial_{x_1} u|^2 dx dt \end{aligned}$$

For the first term in the right hand side of the previous estimate, we obtain

$$\iint_{\Omega_T} (s\rho)^{\tau_1-1} e^{-2s\eta^*} |\partial_{x_1}^2 u|^2 dx dt \leq I(\tau_1, u).$$

For the second one, we have

$$\begin{aligned} \lambda^8 \iint_{\omega_T} \rho^{-\tau_1+1} s^{n^*} e^{2s\eta^*} N^2(x', t) |\partial_{x_1} u|^2 dx dt & \leq \lambda^8 \iint_{\omega_T} \rho^{-\tau_1+1} s^{n^*} e^{2s\eta^*} N^2(x', t) u \partial_{x_1}^2 u dx dt \\ & + \lambda^8 \iint_{\omega_T} \partial_{x_1} (\rho^{-\tau_1+1} s^{n^*} e^{2s\eta^*} N^2(x', t)) \partial_{x_1} \left(\frac{u^2}{2} \right) dx dt \\ & := P_1 + P_2 \end{aligned}$$

Using Young estimate, for the first term P_1 , we have

$$\begin{aligned} P_1 & \leq \varepsilon I(\tau_1, u) + \lambda^{16} C_\varepsilon \iint_{\omega_T} \rho^{-3\tau_1+3} s^{4\tau_2-3\tau_1+15} e^{6s\eta^*} N^4(x', t) |u|^2 dx dt \\ & \leq \varepsilon I(\tau_1, u) + \lambda^{16} C_\varepsilon \iint_{\omega_T} (s\rho^*)^{4\tau_2-3\tau_1+15} e^{-8s\eta_- + 6s\eta^*} |u|^2 dx dt. \end{aligned}$$

The last term will be bounded if

$$(2.29) \quad -4\eta_- + 3\eta^* < 0.$$

For P_2 , we obtain

$$\begin{aligned} P_2 &\leq \varepsilon I(\tau_1, u) + C_\varepsilon \lambda^8 \iint_{\omega_T} \rho^{-3\tau_1+1} s^{4\tau_2-3\tau_1+13} e^{6s\eta^*} N^4(x', t) |u|^2 dxdt \\ &\leq \varepsilon I(\tau_1, u) + C_\varepsilon \lambda^8 \iint_{\omega_T} (s\rho^*)^{4\tau_2-3\tau_1+13} e^{-8s\eta_-+6s\eta^*} |u|^2 dxdt. \end{aligned}$$

The last term is bounded under the same condition (2.29).
So if we come back to the definition of the weight function

$$\eta(x, t) = \frac{e^{2\lambda K} - e^{\lambda\beta(x)}}{t(T-t)},$$

assumption (2.29) is checked for

$$(2.30) \quad K \geq \max\left\{\frac{2ln2}{\|\beta\|_\infty}, \|\beta\|_\infty\right\}.$$

Finally, we obtain

$$\begin{aligned} (2.31) \quad &\lambda^4 \int_0^T \int_{\omega'} (s\rho)^{\tau_2+3} e^{-2s\eta} |\partial_{x_1} u(0, x', t)|^2 dx' dt \\ &\leq C_\varepsilon \lambda^{16} s^{\tau^*} \iint_{\omega_T} \rho^{3-3\tau_1} e^{6s\eta^*} N^4(x', t) |u|^2 dxdt + \varepsilon I(\tau_1, u). \end{aligned}$$

So, using (2.28) and (2.31), we obtain

$$\begin{aligned} \lambda^4 \iint_{\omega_T} (s\rho)^{\tau_2+3} e^{-2s\eta} |v|^2 dxdt &\leq C_\varepsilon \lambda^{16} s^{\tau^*} \iint_{\omega_T} \rho^{3-3\tau_1} e^{6s\eta^*} N^4(x', t) |u|^2 dxdt, \\ &+ C_\varepsilon \lambda^8 s^{\tau_2+7} \iint_{\omega_T} M(x', t) |u|^2 dxdt, \\ &+ \lambda^4 \iint_{\omega_T} e^{-2s\eta} (s\rho)^{3+\tau_2} |L^{-1}f|^2 dxdt \\ &+ \varepsilon I(\tau_1, u) + \varepsilon I(\tau_2, v). \end{aligned}$$

Moreover

$$\begin{aligned} &\lambda^{16} s^{\tau^*} \iint_{\omega_T} \rho^{3-3\tau_1} e^{6s\eta^*} N^4(x', t) |u|^2 dxdt + \lambda^8 s^{\tau_2+7} \iint_{\omega_T} M(x', t) |u|^2 dxdt \\ &\leq \lambda^{16} s^{\tau^*} \iint_{\omega_T} \rho^{3-3\tau_1} (\rho^*)^{4\tau_2+12} e^{-2s\alpha} |u|^2 dxdt, \end{aligned}$$

and the proof of Theorem 2.6 is complete for the case where $B = (1, 0, \dots, 0)$.

□

REMARK 2.8.

1. Note that for more general vector field B we obtain, by a change of variables, a similar equation:

$$\partial_{x_1} \tilde{v} + b\tilde{v} = \partial_t \tilde{u} - \operatorname{div}(H \cdot \nabla \tilde{u}) - E \cdot \nabla \tilde{u} - e\tilde{u} - \tilde{f}.$$

The proof is given in Appendix.

2. Observe that in the case B and b are such that for some subset $\mathcal{O} \subset \omega$, $B|_{\mathcal{O}} = 0$ and $b|_{\mathcal{O}} \neq 0$ the proof is much more simpler.

Finally using Theorem 2.6 in (2.13), we obtain Theorem 2.5. \square

3. Carleman Estimate for 3×3 Systems.

3.1. Statement of the problem. In this section we prove the main result of this paper, i.e., Theorem 1.1, that is the null controllability under appropriate conditions of the 3×3 reaction-diffusion system

$$(3.1) \quad \begin{cases} \partial_t y = (\mathcal{L} + A)y + Cf1_\omega & \text{in } \Omega_T, \\ y = 0 & \text{on } \Sigma_T, \\ y(\cdot, 0) = y_0(\cdot) & \text{in } \Omega, \end{cases}$$

where,

$$\mathcal{L} = \begin{pmatrix} \operatorname{div}(H_1 \nabla) & 0 & 0 \\ 0 & \operatorname{div}(H_2 \nabla) & 0 \\ 0 & 0 & \operatorname{div}(H_2 \nabla) \end{pmatrix},$$

$A = (a_{ij})_{1 \leq i, j \leq 3}$, $C = (1, 0, 0)^t \in \mathcal{L}(R, R^3)$, $f \in L^2(\Omega_T)$ and $y_0 = (y_{0,i})_{1 \leq i \leq 3} \in L^2(\Omega)^3$.

As in the two dimensional case the result is reduced to proving an observability inequality to the adjoint system to (3.1). That is, consider

$$(3.2) \quad \begin{cases} -\partial_t \varphi_1 = \operatorname{div}(H_1 \nabla \varphi_1) + a_{11} \varphi_1 + a_{21} \varphi_2 + a_{31} \varphi_3 & \text{in } \Omega_T, \\ -\partial_t \varphi_2 = \operatorname{div}(H_2 \nabla \varphi_2) + a_{12} \varphi_1 + a_{22} \varphi_2 + a_{32} \varphi_3 & \text{in } \Omega_T, \\ -\partial_t \varphi_3 = \operatorname{div}(H_2 \nabla \varphi_3) + a_{13} \varphi_1 + a_{23} \varphi_2 + a_{33} \varphi_3 & \text{in } \Omega, \\ \varphi_1 = \varphi_2 = \varphi_3 = 0 & \text{on } \Sigma_T, \\ \varphi_1(\cdot, T) = \varphi_1^T, \varphi_2(\cdot, T) = \varphi_2^T, \varphi_3(\cdot, T) = \varphi_3^T, & \text{in } \Omega \end{cases}$$

Then, Theorem 1.1 is equivalent to the following result:

THEOREM 3.1. *Suppose that $a_{ij} \in C^4(\overline{\Omega_T})$, Suppose that a_{21} , a_{31} are time independent, that there exists $j \in \{2, 3\}$ such that $|a_{j1}(x)| \geq C > 0$ for all $x \in \omega$. For such j we define $k_j = \frac{6}{j}$,*

$$B_{k_j} := -2H_2 \left(\nabla a_{k_j 1} - \frac{a_{k_j 1}}{a_{j1}} \nabla a_{j1} \right),$$

and

$$b_j = \frac{2H_2 \nabla a_{j1} (\nabla a_{k_j 1} a_{j1} - \nabla a_{j1} a_{k_j 1})}{a_{j1}^2} + \frac{a_{k_j 1} \operatorname{div}(H_2 \nabla a_{j1}) - a_{j1} \operatorname{div}(H_2 \nabla a_{k_j 1})}{a_{j1}} - (-1)^j \frac{\det[A|C]}{a_{j1}}.$$

Assume that either

1. $B_{k_j} = 0$ and $b_j \neq 0$ on ω_T ,
or

2. $\partial\omega \cap \partial\Omega = \gamma$, $|\gamma| \neq 0$, and $B_{k_j} \cdot \nu \neq 0$, on γ ,
Then, there exists $C > 0$ such that for every $(\varphi_1^T, \varphi_2^T, \varphi_3^T) \in L^2(\Omega)^3$ the corresponding solution to (3.2) satisfies:

$$(3.3) \quad \int_{\Omega} (|\varphi_1(x, 0)|^2 + |\varphi_2(x, 0)|^2 + |\varphi_3(x, 0)|^2) dx \leq C \iint_{\omega_T} |\varphi_1(x, t)|^2 dx dt.$$

Observe that the case $B_{k_j}(x, t) = 0$ and $b_j(x, t) \neq 0$ was already treated in [3]. The really new result is the case $B_{k_j} \cdot \nu(x) \neq 0$, on γ . Inequality (3.3) will be deduced by an appropriate Carleman estimate (as in the two dimensional case treated below). So, the next subsection is devoted to the proof of this Carleman inequality.

3.2. A new Carleman estimate for a 3×3 reaction-diffusion system with one observation. In view of applications regarding inverse problems we will consider the following 3×3 reaction-diffusion system (which is the adjoint system of system (3.1)) where $(f, g, h) \in L^2(\Omega)^3$.

Let $(a_{ij})_{1 \leq i, j \leq 3} \in C^4(\overline{\Omega_T})$, $H_l = (h_{ij}^l)_{1 \leq i, j \leq 3}$, $1 \leq l \leq 2$ defined by (1.2) and (1.3). We consider the following system:

$$(3.4) \quad \begin{cases} \partial_t u = \operatorname{div}(H_1 \nabla u) + a_{11}u + a_{21}v + a_{31}w + f & \text{in } \Omega_T, \\ \partial_t v = \operatorname{div}(H_2 \nabla v) + a_{12}u + a_{22}v + a_{32}w + g & \text{in } \Omega_T, \\ \partial_t w = \operatorname{div}(H_2 \nabla w) + a_{13}u + a_{23}v + a_{33}w + h & \text{in } \Omega, \\ u = v = w = 0 & \text{on } \Sigma_T, \\ u(\cdot, 0) = u_0, v(\cdot, 0) = v_0, w(\cdot, 0) = w_0, & \text{in } \Omega. \end{cases}$$

Uniqueness, existence and stability results for (3.4) can be proved by classical theory (e.g., [20]). In particular, (3.4) admits a unique solution $(u, v, w) \in C([0, T]; L^2(\Omega))^3 \cap L^2(0, T; H_0^1(\Omega))^3$.

Moreover, if $u_0, v_0, w_0 \in H^2(\Omega) \cap H_0^1(\Omega)$, then $(u, v, w) \in C([0, T]; H^2(\Omega) \cap H_0^1(\Omega))^3 \cap C^1([0, T]; L^2(\Omega))^3$ and we will call it *strong solution*.

The Carleman estimate (2.11) allows us to perform a new Carleman estimate for this 3×3 reaction-diffusion-system. First of all, let us precise the assumptions on the coefficients of system (3.4).

ASSUMPTION 3.2.

1. a_{21} and a_{31} do not depend on time t ,
2. $\omega \subset \Omega$ is a non-empty subdomain of class C^2 with $\partial\omega \cap \partial\Omega = \gamma$ and $|\gamma| \neq 0$,
3. There exists $j \in \{2, 3\}$ such that $|a_{j1}(x)| \geq C > 0$ for all $x \in \omega$ and for

$$k_j = \frac{6}{j}$$

$$|(H_2 \left(\nabla a_{k_j 1} - \frac{a_{k_j 1}}{a_{j1}} \nabla a_{j1} \right)) \cdot \nu(x)| \neq 0, \text{ on } \gamma,$$

4. Let $H_2|_{\omega} \in (W^{3, \infty}(\omega))^{n^2}$.

REMARK 3.3. In this result, we are again assuming that $B_{k_j} \neq 0$ in ω_T . The Carleman inequality is still true in the other case and it was already proved in [3]. We obtain the following theorem :

THEOREM 3.4. Under Assumption 3.2 there exist a positive function $\beta \in C^2(\overline{\Omega})$ (only depending on Ω and ω) and two positive constants s_0, C (which only depend on $\Omega, \omega, (a_{ij})_{(1 \leq i, j \leq 3)}, \tau$), a constant K (see (3.9)) such that for every $(u_0, v_0, w_0) \in$

$L^2(\Omega)^3, (f, g, h) \in L^2(\Omega_T)^3$, the following Carleman estimate holds

$$(3.5) \quad \begin{aligned} I(\tau, u) + I(\tau, v) + I(\tau, w) \leq C & \left(\lambda^{32} \iint_{\omega_T} s^{(\tau+33)} (\rho^*)^{\tau+31} e^{(-4s\alpha+2s\eta)} (|u|^2 + |f|^2) dxdt \right. \\ & + \lambda^4 \iint_{\omega_T} e^{-2s\eta} (s\rho)^{3+\tau} (|Qg|^2 + |Qh|^2) dxdt \\ & \left. + \iint_{\Omega_T} e^{-2s\eta} (s\rho)^\tau (|f|^2 + |g|^2 + |h|^2) dxdt \right). \end{aligned}$$

for $s \geq s_0$ and all (u, v, w) solution of (3.4), η defined by (2.7), and Q defined in (5.4), $\alpha = 4\eta_- - 3\eta^*$, $\eta^* = \max_{\bar{\Omega}} \eta$, $\eta_- = \min_{\bar{\Omega}} \eta$ and $\rho^* = \max_{\bar{\Omega}} \rho$.

Proof. [Proof of Theorem 3.4] The proof is done in three steps:

1. We first prove a Carleman estimate with three observations taking sets $\tilde{\omega} \subset \subset \bar{\omega} \subset \omega$, such that $\tilde{\gamma} = \partial\tilde{\omega} \cap \partial\Omega \subset \gamma$ and $\text{dist}(\partial\tilde{\omega} \setminus \tilde{\gamma}, \partial\omega \setminus \gamma) > 0$ (this is necessary technicality that allows to construct ξ satisfying (2.27). A direct application of Theorem 2.4 leads to

$$\begin{aligned} I(\tau, u) + I(\tau, v) + I(\tau, w) \leq C & \left(\lambda^4 \iint_{\tilde{\omega}_T} (s\rho)^{\tau+3} e^{-2s\eta} |u|^2 dxdt \right. \\ & + \lambda^4 \iint_{\tilde{\omega}_T} (s\rho)^{\tau+3} e^{-2s\eta} |v|^2 dxdt + \lambda^4 \iint_{\tilde{\omega}_T} (s\rho)^{\tau+3} e^{-2s\eta} |w|^2 dxdt \\ & \left. + \iint_{\Omega_T} (s\rho)^\tau e^{-2s\eta} |f|^2 dxdt + \iint_{\Omega_T} (s\rho)^\tau e^{-2s\eta} |g|^2 dxdt + \iint_{\Omega_T} (s\rho)^\tau e^{-2s\eta} |h|^2 dxdt \right), \end{aligned}$$

for any $\tilde{\omega} \subset \subset \Omega$. The main question is to get rid of

$$\iint_{\tilde{\omega}_T} (s\rho)^{\tau+3} e^{-2s\eta} |v|^2 dxdt + \iint_{\tilde{\omega}_T} (s\rho)^{\tau+3} e^{-2s\eta} |w|^2 dxdt.$$

2. We eliminate two observations. Let

$$(3.6) \quad z = a_{21}v + a_{31}w \quad \text{in } \omega_T.$$

Suppose, for example, that Assumption 3.2 is satisfied for $j = 3$. If (u, v, w) is a strong solution of System (3.4), then z defined by (3.6) satisfies:

$$(3.7) \quad \begin{cases} \partial_t u = \text{div}(H_1 \nabla u) + a_{11}u + z + f & \text{in } \omega_T, \\ \partial_t z = \text{div}(H_2 \nabla z) + A \cdot \nabla z + az + eu + B \cdot \nabla v + bv + G & \text{in } \omega_T, \end{cases}$$

with

$$(3.8) \quad \begin{aligned} b &= 2H_2 \nabla a_{31} \left(\frac{\nabla a_{21} a_{31} - \nabla a_{31} a_{21}}{a_{31}^2} \right) + \frac{a_{21} \text{div}(H_2 \nabla a_{31}) - a_{31} \text{div}(H_2 \nabla a_{21})}{a_{31}} \\ &\quad - \frac{a_{21}^2 a_{32} + a_{31} a_{21} a_{33} - a_{31} a_{21} a_{22} - a_{31}^2 a_{23}}{a_{31}}, \\ B &= -2H_2 \left(\nabla a_{21} - \frac{a_{21}}{a_{31}} \nabla a_{31} \right), \end{aligned}$$

$$A = -2H_2 \frac{\nabla a_{31}}{a_{31}}, \quad a = 2 \frac{(H_2 \nabla a_{31}) \cdot \nabla a_{31}}{a_{31}^2} - \frac{\text{div}(H_2 \nabla a_{31})}{a_{31}} + \frac{a_{21} a_{32} + a_{31} a_{33}}{a_{31}},$$

$$e = a_{12}a_{21} + a_{13}a_{31}, \quad G = a_{21}g + a_{31}h.$$

We first use Theorem 2.6 to estimate v by z , u and G . Then with Assumption 3.2 and (3.6), we have $w = \frac{z - a_{21}v}{a_{31}}$. Therefore v and w will be estimated by z (and u, G). It will remain to estimate z by u . Of course, all these estimates are locally (in ω_T). We apply Theorem 2.6 to the second equation of (3.7), we obtain:

$$\begin{aligned} \lambda^4 \iint_{\tilde{\omega}_T} (s\rho)^{\tau+3} e^{-2s\eta} |v|^2 dxdt &\leq C_\varepsilon \lambda^{16} \iint_{\tilde{\omega}_T} e^{-2s\alpha} (s\rho^*)^{\tau+15} |z|^2 dxdt \\ &+ C\lambda^4 \iint_{\omega_T} e^{-2s\eta} (s\rho)^{3+\tau} (|Qu|^2 + |Qg|^2 + |Qh|^2) dxdt \\ &+ \varepsilon I(\tau, v). \end{aligned}$$

Now we are going to estimate the local observation in z by the local observation in u using the first equation of (3.7). For this, we multiply this equation by $\lambda^{16}(s\rho^*)^{\tau+15}\xi e^{-2s\alpha}z$ where ξ is defined in (2.27) and we integrate on ω_T .

$$\begin{aligned} &\lambda^{16} \iint_{\tilde{\omega}_T} (s\rho^*)^{\tau+15} e^{-2s\alpha} |z|^2 dxdt \\ &\leq \lambda^{16} \left(\iint_{\omega_T} (s\rho^*)^{\tau+15} e^{-2s\alpha} z(x,t) \xi (\partial_t u - \operatorname{div}(H_1 \nabla u) - a_{11}u - f) dxdt \right). \end{aligned}$$

Using integrations by part we obtain

$$\begin{aligned} &\lambda^{16} \iint_{\tilde{\omega}_T} e^{-2s\alpha} (s\rho^*)^{\tau+15} |z|^2 dxdt \leq \varepsilon I(\tau, z) \\ &+ C_\varepsilon \lambda^{32} \iint_{\omega_T} s^{(\tau+33)} (\rho^*)^{(\tau+31)} e^{(-4s\alpha+2s\eta)} (|u|^2 + |f|^2) dxdt. \end{aligned}$$

The integral in the right hand side of the previous inequality is bounded if:

$$8\eta_- - 7\eta^* > 0.$$

In others terms we need

$$(3.9) \quad K \geq \max\left\{ \frac{3 \ln 2}{\|\beta\|_\infty}, \|\beta\|_\infty \right\}.$$

Finally we obtain :

$$\begin{aligned} I(\tau, u) + I(\tau, v) + I(\tau, w) &\leq C \left(\lambda^{32} \iint_{\omega_T} s^{(\tau+33)} (\rho^*)^{(\tau+31)} e^{(-4s\alpha(t)+2s\eta)} (|u|^2 + |f|^2) dxdt \right. \\ &+ \lambda^4 \iint_{\omega_T} e^{-2s\eta} (s\rho)^{3+\tau} (|Qg|^2 + |Qh|^2) dxdt \\ &\left. + \iint_{\Omega_T} e^{-2s\eta} (s\rho)^\tau (|f|^2 + |g|^2 + |h|^2) dxdt \right). \end{aligned}$$

□

REMARK 3.5. *Following the proofs presented here it is clear that the regularity assumptions on a_{jk} and A_{jk} are not optimal. We took much more regularity than the necessary for simplicity. In fact, in Theorem 1.1 we can assume that $a_{jk} \in L^\infty(\Omega_T)$ for every $j, k \in \{1, 2, 3\}$, $a_{kj}|_{\omega_T} \in W^{2,\infty}(\omega_T)$ for every $j, k \in \{2, 3\}$ and $a_{k,1}|_\omega \in W^{4,\infty}(\omega)$ for $k \in \{2, 3\}$. In that case, one can consider the following system:*

$$(3.10) \quad \begin{cases} \partial_t y_1 = \operatorname{div}(H_1 \nabla y_1) + F(y_1, y_2, y_3) + \chi_\omega f & \text{in } \Omega_T, \\ \partial_t y_2 = \operatorname{div}(H_2 \nabla y_2) + a_{21}y_1 + a_{22}y_2 + a_{23}y_3 & \text{in } \Omega_T, \\ \partial_t y_3 = \operatorname{div}(H_2 \nabla y_3) + a_{31}y_1 + a_{32}y_2 + a_{33}y_3 & \text{in } \Omega, \\ y_1 = y_2 = y_3 = 0 & \text{on } \Sigma_T, \\ y_1(\cdot, 0) = y_1^0, y_2(\cdot, 0) = y_2^0, y_3(\cdot, 0) = y_3^0, & \text{in } \Omega. \end{cases}$$

If F is assumed to be globally Lipschitz with respect to each variable, and a_{21}, a_{31} and ω satisfy Assumption 3.2, then System (3.10) is null controllable at time $T > 0$.

4. Applications and Generalizations.

4.1. Generalization for $n \times n$ reaction diffusion systems for $n \geq 3$. The results of Section 3 can be generalized to $n \times n$ parabolic systems controlled by $(n-2)$ controls. Consider $A = (a_{lm})_{1 \leq l, m \leq n}$ a matrix of order n with $a_{lm} \in C^4(\overline{\Omega_T})$. For $j \neq k \in \{1, \dots, n\}$ consider $C_{jk} = (\rho_1, \rho_2, \dots, \rho_n)^t$ with $\rho_l = e_l$ (where (e_1, \dots, e_n) is the euclidian basis of R^n) except for the two indexes j, k where $\rho_j = \rho_k = 0$. We assume that j, k can be chosen in such a way that there exists $i \neq j, i \neq k$ such that $|a_{ji}(x, t)| \geq C > 0$ for all $(x, t) \in \omega_T$. We denote $B := -2 \left(\nabla a_{ki} - \frac{a_{ki}}{a_{ji}} \nabla a_{ji} \right)$ and

$$b = \frac{2 \nabla a_{ji} \left(\frac{\nabla a_{ki} a_{ji} - \nabla a_{ji} a_{ki}}{a_{ji}^2} \right) + \frac{a_{ki} \Delta a_{ji} - a_{ji} \Delta a_{ki}}{a_{ji}}}{\frac{a_{ki}^2 a_{ji} + a_{ji} a_{ki} a_{jj} - a_{ji} a_{ki} a_{kk} - a_{ji}^2 a_{kij}}{a_{ji}}}.$$

Furthermore, we assume that $B(x, t) = 0$ and $b(x, t) \neq 0$ on ω_T , or that all the following conditions are checked

1. $a_{m,k} = 0$ for $m \neq k, j, i$.
2. $a_{m,j} = 0$ for $m \neq k, j, i$.
3. a_{ki} and a_{ji} are time independent.
4. $\partial\omega \cap \partial\Omega = \gamma$, $|\gamma| \neq 0$,
5. $B \cdot \nu(x) \neq 0$, on γ .

THEOREM 4.1. *Under the previous assumptions, the following system in $L^2(\Omega \times (0, T))^n$.*

$$(4.1) \quad \begin{cases} \partial_t Y = \Delta Y + AY + C_{jk} U \chi_\omega & \text{in } \Omega_T, \\ Y(\cdot, t) = 0 & \text{on } \Sigma_T, \\ Y(\cdot, 0) = Y_0(\cdot) & \text{in } \Omega. \end{cases}$$

is null controllable.

Proof. Controllability of System (4.1) is equivalent to the following observability estimate:

$$\exists C > 0; \|\Phi(T)\|_{L^2(\Omega)^n}^2 \leq C \sum_{l=1, l \neq j, k}^n \iint_{\omega_T} |\phi_l(t)|^2 dt,$$

for all $\Phi = (\phi_1, \dots, \phi_n)^t$ solution of the adjoint system:

$$(4.2) \quad \begin{cases} \partial_t \Phi = \Delta \Phi + A^* \Phi & \text{in } \Omega_T, \\ \Phi(\cdot, t) = 0 & \text{on } \Sigma_T. \end{cases}$$

By scalar Carleman estimate applied to each equations of System (4.2), we obtain

$$(4.3) \quad \sum_{l=1}^{l=n} I(\tau, \phi_l) \leq C \sum_{l=1}^{l=n} \iint_{\omega_T} (s\rho)^{\tau+3} e^{-2s\eta} |\phi_l(x, t)|^2 dx dt.$$

So (ϕ_i, ϕ_j, ϕ_k) satisfies

$$(4.4) \quad \begin{cases} \partial_t \phi_i = \Delta \phi_i + \sum_{l=1}^n a_{li} \phi_l & \text{in } \Omega_T, \\ \partial_t \phi_j = \Delta \phi_j + a_{ij} \phi_i + a_{jj} \phi_j + a_{kj} \phi_k & \text{in } \Omega_T, \\ \partial_t \phi_k = \Delta \phi_k + a_{ik} \phi_i + a_{jk} \phi_j + a_{kk} \phi_k & \text{in } \Omega_T, \\ \Phi(\cdot, t) = 0 & \text{on } \Sigma_T, \\ \Phi(\cdot, 0) = \Phi_0(\cdot) & \text{in } \Omega, \end{cases}$$

satisfy the following Carleman estimate:

$$(4.5) \quad \begin{aligned} I(\tau, \phi_i) + I(\tau, \phi_j) + I(\tau, \phi_k) \leq C & \left(\lambda^{32} \iint_{\omega_T} s^{(\tau+33)} (\rho^*)^{(\tau+31)} e^{(-4s\alpha(t)+2s\eta)} (|\phi_i|^2 + |f|^2) dx dt \right. \\ & \left. + \iint_{\Omega_T} e^{-2s\eta} (s\rho)^\tau |f|^2 dx dt \right). \end{aligned}$$

where $f = \sum_{l=1, l \neq i, j, k}^n a_{li} \phi_l$. It is now straightforward to see that a combination of (4.3) and (4.5) gives

$$\sum_{l=1}^{l=n} I(\tau, \phi_l) \leq C \sum_{l=1; l \neq k, j}^{l=n} \iint_{\omega_T} (s\rho)^{\tau+3} e^{-2s\eta} |\phi_l(x, t)|^2 dx dt.$$

Classical energy estimates give now the observability inequality and therefore the null controllability result. \square

REMARK 4.2. *Observe that the use of Carleman inequality (1.1) involves terms that include the operator Q defined in (5.4). This imposes the previous conditions 1 and 2 on the coefficients a_{mk} and a_{mj} for $m \neq i, j, k$.*

All the regularity assumptions of System (4.1) can be weakened at it has been pointed out in Remark 3.5.

4.2. Inverse Problems. This subsection is devoted to the question of the identification of coefficients for a reaction-diffusion system of n equations ($n \geq 3$) in a bounded domain, with the main particularity that we observe *only* $(n - 2)$ components of the system. The key ingredient is the stability result for 3×3 reaction diffusion systems and the case of n equations is a direct consequence, so let us focus for $n = 3$.

Consider the following 3×3 reaction-diffusion system where $(a_{ij})_{1 \leq i, j \leq 3} \in C^4(\overline{\Omega})^9$:

$$(4.6) \quad \begin{cases} \partial_t U = \Delta U + a_{11}U + a_{21}V + a_{31}W & \text{in } \Omega_T, \\ \partial_t V = \Delta V + a_{12}U + a_{22}V + a_{32}W & \text{in } \Omega_T, \\ \partial_t W = \Delta W + a_{13}U + a_{23}V + a_{33}W & \text{in } \Omega_T, \\ U = k_1, V = k_2, W = k_3 & \text{on } \Sigma_T, \\ U(\cdot, 0) = U_0 \quad V(\cdot, 0) = V_0 \quad \text{and } W(\cdot, 0) = W_0 & \text{in } \Omega. \end{cases}$$

Let $(\tilde{U}, \tilde{V}, \tilde{W})$ be solution of

$$(4.7) \quad \begin{cases} \partial_t \tilde{U} = \Delta \tilde{U} + a_{11} \tilde{U} + \tilde{a}_{21} \tilde{V} + a_{31} \tilde{W} & \text{in } \Omega_T, \\ \partial_t \tilde{V} = \Delta \tilde{V} + a_{12} \tilde{U} + a_{22} \tilde{V} + \tilde{a}_{32} \tilde{W} & \text{in } \Omega_T, \\ \partial_t \tilde{W} = \Delta \tilde{W} + \tilde{a}_{13} \tilde{U} + a_{23} \tilde{V} + a_{33} \tilde{W} & \text{in } \Omega_T, \\ \tilde{U} = k_1, \tilde{V} = k_2, \tilde{W} = k_3 & \text{on } \Sigma_T, \\ \tilde{U}(\cdot, 0) = U_0, \tilde{V}(\cdot, 0) = V_0 \text{ and } \tilde{W}(\cdot, 0) = W_0 & \text{in } \Omega. \end{cases}$$

with $\tilde{a}_{ij} \in C^4(\bar{\Omega})^9$. Following the method developed in [6] and [9], from Carleman estimate (3.5), we obtain the following identifiability and Lipschitz stability estimate for three coefficients (one in each equation) (e.g. a_{21}, a_{32}, a_{13}) by the observation of only one component on ω assuming the knowledge of these coefficients on ω for any subset ω of Ω .

THEOREM 4.3. *Assume that Assumption 3.2 is checked. Assume that there exists $\varepsilon > 0$ such that $k_1, k_2, k_3 \in H^1(0, T; H^{2+\varepsilon}(\partial\Omega)) \cap H^2(0, T; H^\varepsilon(\partial\Omega))$, $U_0, V_0, W_0 \in H^2(\Omega)$. Suppose that (\tilde{a}_{ij}) are such that there exist $C > 0$ and $T' \in (0, T)$ such that $|\tilde{U}(\cdot, T')| \geq C$ in Ω , $|\tilde{V}(\cdot, T')| \geq C$ in Ω , $|\tilde{W}(\cdot, T')| \geq C$ in Ω . Assume that $a_{ij} = \tilde{a}_{ij}$ on ω for $(i, j) \in \{(2, 1), (3, 2), (1, 3)\}$.*

Then there exists $\kappa > 0$ such that

$$(4.8) \quad \begin{aligned} & \|a_{21} - \tilde{a}_{21}\|_{L^2(\Omega)}^2 + \|a_{32} - \tilde{a}_{32}\|_{L^2(\Omega)}^2 + \|a_{13} - \tilde{a}_{13}\|_{L^2(\Omega)}^2 \leq \kappa \left(\|\partial_t(U - \tilde{U})\|_{L^2(\omega_{T'})}^2 \right. \\ & \left. + \|(U - \tilde{U})(T')\|_{H^2(\Omega)}^2 + \|(V - \tilde{V})(T')\|_{H^2(\Omega)}^2 + \|(W - \tilde{W})(T')\|_{H^2(\Omega)}^2 \right). \end{aligned}$$

REMARK 4.4. *As for subsection 4.1, the previous result can be extended to the identification of n coefficients by $(n - 2)$ locally observations for a $(n \times n)$ reaction diffusion system. We need only the knowledge of 3 coefficients on ω . Indeed, following the previous method for a (3×3) system, consider a $(n \times n)$ reaction diffusion system like (4.1) and fix three components (e.g. y_1, y_2, y_3), the three first associated equations and three coefficients inside to recover. Assuming the knowledge of these three coefficients on the set of observation ω and using the previous Carleman estimate, we derive easily a stability estimate similar to (4.8) for n coefficients, one in each equation, by the observation of only $n - 2$ components (y_1 , and $(y_i)_{i \geq 3}$) without anymore assumption on the coefficients to recover.*

All the regularity assumptions of System (4.1) can be weakened at it has been pointed out in Remark 3.5.

5. Appendix . Let us briefly show how one can transform (2.6) to the form (2.17). This idea is to make a change of variables such that $B \cdot \nabla$ is transformed in ∂_ξ where (ξ, σ) denotes the new variable. From Assumption 2.3, there exists $\Xi > 0$ such that the following map is well defined:

$$\Lambda : \begin{array}{ccc} (0, \Xi) \times \gamma & \rightarrow & \Omega \\ (\xi, \sigma) & \mapsto & \Lambda(\xi, \sigma) = x \end{array}$$

where

$$\frac{dx}{d\xi}(\xi) = B(x(\xi)) \text{ and } x(0) = \sigma,$$

and $\omega_1 := \Lambda((0, \Xi) \times \gamma) \subset \omega$.

One has $\partial\omega_1 \cap \partial\Omega = \partial\omega \cap \partial\Omega = \gamma$. We have to define the derivative with respect to

one point of γ . To do this, we consider a parametering of $\gamma \in C^2$. The new map, still denoted Λ , will be defined on $(0, \Xi) \times \mathcal{O}$, where \mathcal{O} is the open set of R^{n-1} where we define the parametering. We denote (ξ, σ) the new variables where $\sigma = (\sigma_1, \dots, \sigma_{n-1})$. It is easy to check that

$$\partial_\xi \Lambda = B \circ \Lambda.$$

It is not difficult to show that under Assumption 2.3, the map $\Lambda \in W^{2,\infty}(\omega_1)$. It is onto from $(0, \Xi) \times \gamma$ to ω_1 and $\Lambda^{-1} \in W^{2,\infty}(0, \Xi) \times \mathcal{O}$.

If we set

$$\tilde{v} = v \circ \Lambda,$$

we have

$$\partial_\xi \tilde{v} = (\nabla v \circ \Lambda) \partial_\xi \Lambda = (B \cdot \nabla v) \circ \Lambda.$$

Define the operator $(L, D(L))$ with

$$Lv = B \nabla v + bv, \quad D(L) = \{v \in H^1(\omega_1); v|_\gamma = 0\}.$$

From the first equation of (2.6), one has

$$(5.1) \quad Lv = \partial_t u - \operatorname{div}(H_1 \cdot \nabla u) - au - A \cdot \nabla u - f.$$

By the change of variables, L is transformed in

$$(5.2) \quad \tilde{L}\tilde{v} = \partial_\xi \tilde{v} + \tilde{b}\tilde{v},$$

$$D(\tilde{L}) = \{\tilde{v} \in H^1((0, \Xi) \times \gamma); v(0, \sigma) = 0, (\sigma, t) \in \mathcal{O}\}.$$

where $\tilde{b} = b \circ \Lambda$. Denote $\tilde{u} = u \circ \Lambda$, $\tilde{f} = f \circ \Lambda$, it is clear that there exist a $n \times n$ matrix $H = (h_{i,j})$, a vector field $E = (E_i)$, a scalar field e such that (5.1) is transformed into

$$(5.3) \quad \tilde{L}\tilde{v} = \partial_t \tilde{u} - \operatorname{div}(H \cdot \nabla \tilde{u}) + E \cdot \nabla \tilde{u} + e\tilde{u} - \tilde{f}.$$

The equation (5.3) with \tilde{L} defined by (5.2) has the form (2.17). Then we apply \tilde{L}^{-1} to (5.3) and obtain an equation for \tilde{v} similar to (2.26). Then we multiply this last equation by $((s\rho)^{\tau_2+3}\xi e^{-2s\eta v}) \circ \Lambda |Jac(\Lambda)|$, where $Jac(\Lambda)$ denotes the Jacobian of Λ and we integrate on $(0, \Xi) \times \mathcal{O} \times (0, T)$.

Like this, we get

$$\iiint_{(0,\Xi) \times \mathcal{O} \times (0,T)} ((s\rho)^{\tau_2+3}\xi e^{-2s\eta \circ \Lambda}) |\tilde{v}|^2 |Jac(\Lambda)| d\xi d\sigma dt = \iint_{\Omega_T} (s\rho)^{\tau_2+3}\xi e^{-2s\eta} |v|^2 dx dt.$$

Therefore, we obtain (2.14) with

$$(5.4) \quad Qf(x, t) = (\tilde{L}^{-1}\tilde{f}) \circ \Lambda^{-1}(x, t),$$

with \tilde{L} and \tilde{f} defined previously.

REFERENCES

- [1] Ammar Khodja F., Benabdallah A., Dupaix C. et Kostin. I, Controllability to the trajectories of phase-field models by one control force, SIAM J. Control Optim., Vol. 42, n°5, 1661-1689, 2003.
- [2] F. AMMAR KHODJA, A. BENABDALLAH, C. DUPAIX AND I. KOSTINE, *Null controllability of some systems of parabolic type by one control force*, ESAIM:COVC, 11 (2005), 426-448.
- [3] F. AMMAR-KHODJA, A. BENABDALLAH, C. DUPAIX, *Null controllability of some reaction-diffusion systems with one control force*, J. Math. Anal. Appl. 320, no. 2 (2006), 928–943.
- [4] F. AMMAR-KHODJA, A. BENABDALLAH, C. DUPAIX, M. GONZÁLEZ-BURGOS, *A generalization of the Kalman rank condition for time-dependent coupled linear parabolic systems*, Differ. Equ. Appl. 1, no. 3 (2009), 139–151.
- [5] F. AMMAR-KHODJA, A. BENABDALLAH, C. DUPAIX, M. GONZÁLEZ-BURGOS, *A Kalman rank condition for the localized distributed controllability of a class of linear parabolic systems*, J. Evol. Equ. 9, no. 2 (2009), 267–291.
- [6] A. BENABDALLAH, M. CRISTOFOL, P. GAITAN, M. YAMAMOTO. *Inverse problem for a parabolic system with two components by measurements of one component*, Applicable Analysis, 88, no. 5 (2009), 683–710.
- [7] O. BODART, M. GONZÁLEZ-BURGOS, R. PÉREZ-GARCÍA, *Insensitizing controls for a heat equation with a nonlinear term involving the state and the gradient*, Nonlinear Anal. 57, no. 5-6 (2004), 687–711.
- [8] A.L. BUKHGEIM AND M.V. KLIBANOV, *Uniqueness in the large of a class of multidimensional inverse problems*, Soviet Math. Dokl., 17 (1981), 244-247.
- [9] M. CRISTOFOL, P. GAITAN AND H. RAMOUL, *Inverse problems for a 2X2 reaction-diffusion system using a Carleman estimate with one observation*, Inverse Problems, 22 (2006), 1561-1573.
- [10] A. DOUBOVA, E. FERNÁNDEZ-CARA, M. GONZÁLEZ-BURGOS, E. ZUAZUA, *On the controllability of parabolic systems with a nonlinear term involving the state and the gradient*, SIAM J. Control Optim., 41, no. 3 (2002), 798–819.
- [11] E. FERNÁNDEZ-CARA, *Null controllability of the semilinear heat equation*, ESAIM: Control, Optimization and Calculus of Variations, 2 (1997), 87–107.
- [12] A. FURSIKOV, O. YU. IMANUVILOV, *Controllability of Evolution Equations*, Lecture Notes Series 34, Seoul National University, Research Institute of Mathematics, Global Analysis Research Center, Seoul, 1996.
- [13] M. GONZÁLEZ-BURGOS, R. PÉREZ-GARCÍA, *Controllability results for some nonlinear coupled parabolic systems by one control force*. Asymptot. Anal. 46, no. 2 (2006), 123–162.
- [14] M. GONZÁLEZ-BURGOS, L. DE TERESA, *Controllability results for cascade systems of m coupled parabolic PDEs by one control force*, To appear in Portugaliae Mathematica.
- [15] S. GUERRERO, *Null controllability of some systems of two parabolic equations with one control force*, SIAM J. Control Optim. 46, no. 2 (2007), 379–394.
- [16] O. YU. IMANUVILOV, M. YAMAMOTO, *Carleman inequalities for parabolic equations in Sobolev spaces of negative order and exact controllability for semilinear parabolic equations*, Publ. Res. Inst. Math. Sci. 39, no. 2 (2003), 227–274.
- [17] O. YU. IMANUVILOV, V. ISAKOV AND M. YAMAMOTO, *An inverse problem for dynamical Lamé systems with two sets of boundary*, Comm. Pure Appl. Math. LVI (2003), 1366–1382.
- [18] R. E. KALMAN, P. L. FALB AND M. A. ARBIB, *Topics in Mathematical Control Theory*, McGraw-Hill Book Co., New York-Toronto, Ont.-London 1969.
- [19] O. A. LADYZENSKAJA, V.A. SOLONNIKOV AND N.N. URAL'CEVA, *Linear and Quasilinear Equations of Parabolic Type*, AMS, Providence, RI, 1968.
- [20] J.-L. LIONS, E. MAGENES, *Non-homogeneous boundary value problems and applications*, Vol. I, Die Grundlehren der mathematischen Wissenschaften, Band 181, Springer-Verlag, New York-Heidelberg, 1972.
- [21] A.PAZY, *Semigroups of Linear Operators and Applications to Partial Differential Equations*, SPRINGER-VERLAG, BERLIN, 1983.
- [22] H. TANABE, *Equations of Evolution*, PITMAN, LONDON, 1979.
- [23] L. DE TERESA, *Insensitizing controls for a semilinear heat equation*, COMM. PARTIAL DIFFERENTIAL EQUATIONS 25, NO. 1–2 (2000), 39–72.