HAL open science

Fractional smoothness and applications in Finance

Stefan Geiss, Emmanuel Gobet

To cite this version:

Stefan Geiss, Emmanuel Gobet. Fractional smoothness and applications in Finance. Giulia Di Nunno and Bernt Øksendal. Advanced Mathematical Methods for Finance, Springer, pp.313-331, 2011, 978-3-642-18411-6. 10.1007/978-3-642-18412-3_12 . hal-00474803

HAL Id: hal-00474803

https://hal.science/hal-00474803

Submitted on 20 Apr 2010

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

1

Fractional smoothness and applications in Finance

Stefan Geiss and Emmanuel Gobet

[1] Department of Mathematics
University of Innsbruck
Technikerstraße 13/7
A-6020 Innsbruck
Austria
stefan.geiss@uibk.ac.at

[2] Laboratoire Jean Kuntzmann
Université de Grenoble and CNRS
BP 53, 38041 Grenoble Cedex 9
France
emmanuel.gobet@imag.fr

Abstract. This overview article concerns the notion of fractional smoothness of random variables of the form $g(X_T)$, where $X = (X_t)_{t\in[0,T]}$ is a certain diffusion process. We review the connection to the real interpolation theory, give examples and applications of this concept. The applications in stochastic finance mainly concern the analysis of discrete time hedging errors. We close the review by indicating some further developments.

1.1 Introduction

From the practitioners one learns that hedging an option which payoff is discontinuous is more difficult compared to the case of smooth payoffs: this feature appears for instance for digital options or barrier options (we refer the reader to [Tal97] among others). On the one hand, for such options the number of assets (i.e. the delta) to incorporate in the hedging portfolio is unbounded, and it may become larger and larger when one gets close to the singularity (i.e. the maturity and the strike for digital options, or the trigger level for barrier options). On the other hand, the numerical estimation of this delta becomes less and less accurate, leading to global stability issues. These heuristic observations are the starting point for deeper mathematical investigations about the concept of irregular payoffs, in order to formalize it and to quantify the payoff irregularity (with the notion of *fractional smoothness*). In

the current contribution, we aim to give an overview of this concept and some applications in stochastic finance. Actually, the applications go beyond the financial framework and more generally, they concern the theory of stochastic differential equations and their approximations.

The discrete time hedging error as an important application.

Since most of the results presented here are applied to the aforementioned example of hedging possibly irregular options, we start with a brief presentation of this problem, in order to emphasize the issues to handle and to raise some natural questions. Take for instance an European-style option exercised at maturity $T > 0$, with a payoff of the form $h(S_T)$ where $S_t := [S_t^1, \cdots, S_t^d]$ denotes the price of a d-dimensional underlying asset at time $0 \leq t \leq T$. Sometimes, we will use the notation $X_t^i = \log(S_t^i)$ for the log-asset, and $g(x) = h(e^{x_1}, \cdots, e^{x_d})$ for the payoff in the logarithmic variables. In what follows, we assume a Markovian dynamics without jumps for the asset (solution to a SDE defined below), we suppose that the interest rate is equal to 0 (to simplify the presentation) and that the market is complete (for details about this standard framework, see [KS98]). Thus, under some regularity assumptions, the payoff $h(S_T)$ can be replicated perfectly by a continuous time strategy, where $\delta_t^S = \nabla_x H(t, S_t)$ defines the vector of number of assets to hold at time t. Here, H is the fair price of the option, that is

$$H(t, x) = \mathbb{E}_{\mathbb{Q}}(h(S_T)|S_t = x)$$

where $\mathbb{Q}$ is the (unique) risk-neutral measure. In practice, only discrete-time hedging is possible at some times $\tau = (t_i)_{i=0}^n$ with $0 = t_0 < t_1 < \cdots < t_{n-1} < t_n = T$. Thus, at time $t \in [0, T]$ the option seller is left with the tracking error

$$\begin{aligned} C_t(h(S_T), \tau) &= H(t, S_t) - H(0, S_0) - \sum_i \nabla_x H(t_i, S_{t_i}) \cdot (S_{t_{i+1} \wedge t} - S_{t_i \wedge t}) \\ &= \int_0^t (\nabla_x H(s, S_s) - \nabla_x H(\phi(s), S_{\phi(s)})) \cdot dS_s \end{aligned} \tag{1.1}$$

with $\phi(s) = t_i$ when $t_i < s \leq t_{i+1}$. We expect the tracking error (1.1) to converge to 0 as the number n of re-balancing dates goes to infinity. With the above formulation (1.1), the tracking error is naturally associated to the problem of approximation of a stochastic integral using piece-wise constant integrated processes. But the delta process $(\nabla_x H(s, S_s))_{s \in [0,T)}$ may exhibit very different behaviors from payoff to payoff: if the payoff is smooth enough, then the delta might be *bounded* as time goes to maturity, while an irregular payoff usually yields an *exploding* delta as $s \to T$. This gives rise to the first question.

(Q1) Is there an intrinsic way to relate the growth rate (as $s \to T$) of the derivatives of H to the irregularity of the payoff h?

The answer will be *yes* via the notion of *fractional smoothness* introduced below, see Theorems 1 and 2.

The estimation of stochastic integrals is usually performed with L_2-norms, but in our financial setting, both measures $\mathbb{P}$ and $\mathbb{Q}$ can be considered. For practitioners, errors under the historical probability $\mathbb{P}$ are presumably more relevant, while the mathematical treatment under the risk-neutral measure $\mathbb{Q}$ is simpler in our context (because the tracking error process (1.1) is a $\mathbb{Q}$-local martingale).

(Q2) Is the definition of fractional smoothness affected by the choice of a specific measure? Do the L_2-convergence rates depend on the choice of the probability measures $\mathbb{P}$ or $\mathbb{Q}$?

In the context we consider the answer concerning the fractional smoothness is usually *no* in the sense of the comments after Theorem 2. Concerning the approximation rates the same is checked for examples so far (see the remarks after Theorem 9).

Beyond the approach to measure tracking errors in L_2, we could alternatively identify the weak limit of the re-normalized tracking error.

(Q3) Do the weak convergence rates coincide with those in the L_2 sense?

The answer is *not necessarily*, as there are counter-examples in which the convergence in L_2 and in distribution hold at different rates, see Section 1.5.

Finally, through an efficient choice of re-balancing dates τ, one can expect to reduce tracking errors and improve the risk management of options.

(Q4) Which time-nets $\tau = (t_i)_{i=0}^n$ lead to optimal convergence rates? And how to relate them to the *fractional smoothness* of the payoff?

As answer we get that according to the index of fractional smoothness of the payoff, one can define explicitly re-balancing times achieving the optimal convergence rates, see Section 1.5.

These preliminary questions serve as references for the reader when reading the next sections.

Organization of the paper.

First, we define the probabilistic framework and the assumptions used throughout this work. Then in Section 1.2, we define the *fractional smoothness* and provide basic properties: we choose a presentation that is quite illuminating regarding the previous preliminary questions. In Section 1.3, we take another view on *fractional smoothness* using the interpolation theory. In Section 1.4, we consider examples of terminal conditions and identify their fractional smoothness. Then, in Section 1.5, we go back to the analysis of discrete time hedging errors and state the main results. We close by further developments and applications of the *fractional smoothness* in Section 1.6.

Assumptions.

Let us define the probabilistic setting used in the following. We fix a d-dimensional Brownian motion $W = (W_t)_{t\in[0,T]}$ defined on a complete probability space $(\Omega, \mathcal{F}_T, \mathbb{P})$ and we let $(\mathcal{F}_t)_{t\in[0,T]}$ be the augmentation of the natural filtration of W. The log-asset X is the solution of the d-dimensional forward diffusion

$$X_t = x_0 + \int_0^t b(s, X_s)ds + \int_0^t \sigma(s, X_s)dW_s.$$

To state the results, we mainly consider two types of assumptions:

(SDE) $d \geq 1$ and $b, \sigma \in C_b^\infty([0,T] \times \mathbb{R}^d)$ and $\sigma\sigma^* \geq \delta I_{\mathbb{R}^d}$ for some $\delta > 0$.
(GBM) $d = 1$ and $X_t = \ln(S_t) = W_t - (t/2)$.

The smoothness conditions in (SDE) are too strong, they are chosen to simplify the presentation. Whenever useful to simplify even more, we may consider the very simple case of the geometric Brownian motion (GBM) (here, the asset is a martingale, meaning that $\mathbb{P} = \mathbb{Q}$). The reader is referred to the corresponding original papers for the possible weaker conditions.

In the following $|\cdot|$ stands for the Euclidean norm and $A \sim_c B$ for $A/c \leq B \leq cA$ if $c \geq 1$ and $A, B \geq 0$. Expectations and conditional expectations under $\mathbb{P}$ are simply denoted by $\mathbb{E}(.)$ and $\mathbb{E}(.|\mathcal{F}_t)$, while under $\mathbb{Q}$, we indicate explicitly the dependency w.r.t. the probability measure by writing $\mathbb{E}_{\mathbb{Q}}(.)$ and $\mathbb{E}_{\mathbb{Q}}(.|\mathcal{F}_t)$.

1.2 Definition of fractional smoothness and basic properties

Fractional smoothness on the Wiener space can be defined in various ways, see [Wat93, Hir99]. Our approach is motivated by the questions discussed in Section 1.1. Since we consider only random variables of the form $Z = g(X_T) = h(S_T)$ (a function of the process at maturity T), the time T plays a specific role in our definition. It would be necessary to modify our definition for more general dependencies like $Z = g(X_{t_1}, \cdots, X_{t_n})$, see [GGG10].

Definition 1. Assume that $Z \in L_2(\mathbb{P})$.

(i) For $0 < \theta \leq 1$ we let $Z \in \widetilde{\mathbb{B}}^\theta_{2,\infty}$ provided that, for all $0 \leq t < T$,

$$\|Z - \mathbb{E}(Z|\mathcal{F}_t)\|_{L_2(\mathbb{P})} \leq c(T-t)^{\frac{\theta}{2}}.$$

(ii) For $0 < \theta < 1$ we let $Z \in \widetilde{\mathbb{B}}^\theta_{2,2}$ provided that

$$\int_0^T (T-t)^{-1-\theta}\|Z - \mathbb{E}(Z|\mathcal{F}_t)\|^2_{L_2(\mathbb{P})}dt < \infty.$$

The spaces $\widetilde{\mathbb{B}}^{\theta}_{2,q}$ above will always be obtained by the conditional expectation and the L_2-norm under the measure $\mathbb{P}$. Therefore we omit the dependency on $\mathbb{P}$ in the notation.

The following properties follow straight from the definition:

Proposition 1. *For $0 < \theta < \eta < 1$ and $p, q \in \{2, \infty\}$ we have that*

$$\widetilde{\mathbb{B}}^{1}_{2,\infty} \subseteq \widetilde{\mathbb{B}}^{\eta}_{2,p} \subseteq \widetilde{\mathbb{B}}^{\theta}_{2,q} \quad \textit{and} \quad \widetilde{\mathbb{B}}^{\theta}_{2,2} \subseteq \widetilde{\mathbb{B}}^{\theta}_{2,\infty}.$$

Given a bounded [3] measurable $g : \mathbb{R}^d \to \mathbb{R}$ and $(t, x) \in [0, T) \times \mathbb{R}^d$, we let

$$u(t,x) := \mathbb{E}(g(X_T)|X_t = x),$$
$$D^2u(t,x) := \left(\frac{\partial^2 u}{\partial x_i \partial x_j}(t,x)\right)_{i,j=1}^{d}.$$

The following equivalences are useful to exploit properties of $\widetilde{\mathbb{B}}^{\theta}_{2,2}$ and $\widetilde{\mathbb{B}}^{\theta}_{2,\infty}$.

Theorem 1 ([GM10a, Proposition 4]). *Under the condition* (SDE)*, for $0 < \theta < 1$ and a bounded g, the following assertions are equivalent:*

(i) $g(X_T) \in \widetilde{\mathbb{B}}^{\theta}_{2,2}$.
(ii) $\int_0^T (T-t)^{-\theta} \mathbb{E} \left|\nabla_x u(t, X_t)\right|^2 dt < \infty$.
(iii) $\int_0^T (T-t)^{1-\theta} \mathbb{E} \left|D^2 u(t, X_t)\right|^2 dt < \infty$.

Theorem 2 ([GM10a, Lemma 6]). *Under the condition* (SDE)*, for $0 < \theta \leq 1$ and a bounded g, the following assertions are equivalent:*

(i) $g(X_T) \in \widetilde{\mathbb{B}}^{\theta}_{2,\infty}$.
(ii) $\sup_{t\in[0,T)} (T-t)^{1-\theta} \mathbb{E} \left|\nabla_x u(t, X_t)\right|^2 < \infty$.
(iii) *For $0 < \theta < 1$ we have that* $\sup_{t\in[0,T)} (T-t)^{2-\theta} \mathbb{E} \left|D^2 u(t, X_t)\right|^2 < \infty$.

Theorems 1 and 2 generalize results obtained in [GG04] and [GH07]. We see that the fractional smoothness index θ measures exactly the growth rate of the derivatives of the associated PDE solved by u (see question (Q1) in the introduction).

The two above theorems are also valid if u is computed using the risk-neutral measure $\mathbb{Q}$ (i.e. $u_{\mathbb{Q}}(t,x) = \mathbb{E}_{\mathbb{Q}}(g(X_T)|X_t = x)$), while the other L_2-norms are computed under $\mathbb{P}$. For instance, for $0 < \theta < 1$ the equivalence of (i) and (ii) of Theorem 1 becomes $g(X_T) \in \widetilde{\mathbb{B}}^{\theta,\mathbb{P}}_{2,2}$ if and only if $\int_0^T (T -$

[3] Here again, the boundedness assumptions on g can be weakened and we refer to the original papers.

$t)^{-\theta}\mathbb{E}_{\mathbb{P}}\left|\nabla_x u_{\mathbb{Q}}(t,X_t)\right|^2 dt < \infty$ where we have indicated explicitly if the L_2-norms or conditional expectations are computed under $\mathbb{P}$ or $\mathbb{Q}$. This property can be established following [GM10a] and the proof of [GM10b, Lemma 7]. This accommodates well the fact that the price functions are usually computed under the risk-neutral measure, while hedging is made under the historical probability (see question (Q2) in the introduction).

Proof (Simplified proof of Theorem 2).

We sketch the proof in the simple case where $X = W$ is a linear Brownian motion, $d = 1$ and $\theta \in (0,1)$. First, $(u(t,W_t) = \mathbb{E}(g(W_T)|\mathcal{F}_t))_{t\le T}$ is a martingale in $L_2(\mathbb{P})$. In addition, for any fixed $0 < \delta < T$ the processes $(\nabla_x u(t,W_t))_{t\le T-\delta}$ and $(D^2u(t,W_t))_{t\le T-\delta}$ are $L_2(\mathbb{P})$-martingales. This property is obtained by checking that $\nabla_x u$ and D^2u both solve the parabolic heat equation and that certain integrability assumptions are satisfied. Then by Itô's formula, one obtains for $0 \le s \le t < T$ that

$$g(W_T) - u(t,W_t) = \int_t^T \nabla_x u(s,W_s)dW_s, \tag{1.2}$$

$$\nabla_x u(t,W_t) - \nabla_x u(s,W_s) = \int_s^t D^2u(r,W_r)dW_r. \tag{1.3}$$

From the Itô isometry, one deduces from (1.2) that $\mathbb{E}|g(W_T) - u(t,W_t)|^2 = \int_t^T \mathbb{E}|\nabla_x u(s,W_s)|^2 ds$ and it follows that $(ii) \Rightarrow (i)$. Similarly from (1.3) one obtains

$$\mathbb{E}|\nabla_x u(t,W_t)|^2 \le 2\mathbb{E}|\nabla_x u(0,W_0)|^2 + 2\int_0^t \mathbb{E}|D^2u(r,W_r)|^2 dr$$

which proves $(iii) \Rightarrow (ii)$. Finally, we show $(i) \Rightarrow (iii)$. Standard computations give that

$$\begin{aligned}
(D^2u)(t,W_t) &= D_z^2 \int_{\mathbb{R}} g(x)\frac{e^{-\frac{(x-z)^2}{2(T-t)}}}{\sqrt{2\pi(T-t)}}dx\Big|_{z=W_t} \\
&= \int_{\mathbb{R}} g(x)\frac{(x-z)^2-(T-t)}{(T-t)^2}\frac{e^{-\frac{(x-z)^2}{2(T-t)}}}{\sqrt{2\pi(T-t)}}dx\Big|_{z=W_t} \\
&= \mathbb{E}\left(g(W_T)\frac{(W_T-W_t)^2-(T-t)}{(T-t)^2}|\mathcal{F}_t\right) \\
&= \mathbb{E}\left([g(W_T)-\mathbb{E}(g(W_T)|\mathcal{F}_t)]\frac{(W_T-W_t)^2-(T-t)}{(T-t)^2}|\mathcal{F}_t\right)
\end{aligned}$$

which implies that

$$\|D^2u(t,W_t)\|_{L_2(\mathbb{P})} \le \frac{\|W_1^2-1\|_{L_2(\mathbb{P})}}{T-t}\|g(W_T)-\mathbb{E}(g(W_T)|\mathcal{F}_t)\|_{L_2(\mathbb{P})}$$

so that we are done.

1.3 Connection to real interpolation

Let us connect Definition 1 to the classical notion of fractional smoothness which also explains the notation we have used. In particular, this connection will make clear the difference between $\widetilde{\mathbb{B}}^{\theta}_{2,\infty}$ and $\widetilde{\mathbb{B}}^{\theta}_{2,2}$.

Definition 2 ([BL76, BS88]). *Assume a couple of Banach spaces (E_0, E_1) so that E_1 is continuously embedded into E_0. Given $x \in E_0$ and $0 < \lambda < \infty$, the K-functional is given by*

$$K(x, \lambda; E_0, E_1) := \inf\{\|x_0\|_{E_0} + \lambda\|x_1\|_{E_1} : x = x_0 + x_1\}.$$

Moreover, given $0 < \theta < 1$ and $1 \leq q \leq \infty$ we define the real interpolation norm

$$\|x\|_{\theta,q} := \left\|\lambda^{-\theta} K(x, \lambda; E_0, E_1)\right\|_{L_q\left((0,\infty), \frac{d\lambda}{\lambda}\right)}$$

and the space $(E_0, E_1)_{\theta,q} := \{x \in E_0 : \|x\|_{\theta,q} < \infty\}$.

With our setting (E_1 is continuously embedded into E_0) we obtain the following lexicographical ordering of the interpolation spaces:

$$E_1 \subseteq (E_0, E_1)_{\theta,p} \subseteq (E_0, E_1)_{\theta,q} \subseteq (E_0, E_1)_{\eta,r} \subseteq E_0$$

for all $0 < \eta < \theta < 1$, $1 \leq p \leq q \leq \infty$ and all $1 \leq r \leq \infty$.

We apply this concept to the analysis on the Wiener space, which needs to introduce some standard notation (see [Nu06, Sections 1.1 and 1.2]). Let H be a separable real Hilbert space with the scalar product denoted by $\langle ., . \rangle_H$ and $(\mathcal{M}, \Sigma, \mu)$ be a complete probability space. We assume an isonormal family $g = \{g_h : h \in H\}$ of centered Gaussian random variables, i.e.

$$\mathbb{E}_\mu(g_h \; g_k) = \langle h, k \rangle_H \quad \text{for all } h, k \in H,$$

and that Σ is the completed σ-field generated by the random variables $\{g_h : h \in H\}$.

For each $n \geq 1$, we denote by $\mathcal{H}_n$ the closed linear subspace of $L_2(\mu)$ generated by the random variables $\{H_n(g_h) : h \in H, \|h\|_H = 1\}$ where

$$H_n(x) = \frac{(-1)^n}{\sqrt{n!}} e^{\frac{x^2}{2}} \frac{d^n}{dx^n}(e^{-\frac{x^2}{2}}), \tag{1.4}$$

i.e. the n-th Hermite polynomial. $\mathcal{H}_0$ is the set of constants. $\mathcal{H}_n$ is the so-called Wiener chaos of order n and we define by $J_n : L_2(\mu) \to L_2(\mu)$ the orthogonal projection onto $\mathcal{H}_n$. The following orthogonal decomposition is known as the Wiener chaos decomposition:

$$L_2(\mu) = \bigoplus_{n=0}^{\infty} \mathcal{H}_n.$$

Now, we are in a position to define the Malliavin Sobolev space and Malliavin Besov space.

Definition 3. *The Malliavin Sobolev space* $\mathbb{D}_{1,2}(\mu) \subseteq L_2(\mu)$ *is given by*

$$\mathbb{D}_{1,2}(\mu) := \left\{ Z \in L_2(\mu) : \|Z\|_{\mathbb{D}_{1,2}(\mu)} := \left(\sum_{n=0}^{\infty} (n+1) \|J_n Z\|^2_{L_2(\mu)} \right)^{\frac{1}{2}} < \infty \right\}.$$

Moreover, given $0 < \theta < 1$ *and* $1 \leq q \leq \infty$*, we define the Malliavin Besov space*

$$\mathbb{B}^{\theta}_{2,q}(\mu) := (L_2(\mu), \mathbb{D}_{1,2}(\mu))_{\theta,q}$$

of fractional smoothness θ *with fine parameter* q*.*

We use this construction in the case that $H = \ell_2^d$ and $\mathcal{M} = \mathbb{R}^d$, Σ is the completion of the Borel σ-algebra on $\mathbb{R}^d$ and $\mu = \gamma_d$ is the d-dimensional standard Gaussian measure. The family of Gaussian random variables is given by

$$g_h(x) := \langle x, h \rangle \quad \text{for} \quad x \in \mathcal{M} = \mathbb{R}^d \quad \text{and} \quad h \in H = \ell_2^d.$$

To make the connection between the definitions of $\widetilde{\mathbb{B}}^{\theta}_{2,q}$ and $\mathbb{B}^{\theta}_{2,q}(\gamma_d)$ for $q \in \{2, \infty\}$ we let, as before, $(W_t)_{t\in[0,1]}$ be the standard d-dimensional Brownian motion on $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t\in[0,1]})$. Then we have

Theorem 3 ([GH07, Corollary 2.3]). *For* $0 < \theta < 1$*,* $1 \leq q \leq \infty$*, and* $g \in L_2(\gamma_d)$ *one has*

$$\|g\|_{\mathbb{B}^{\theta}_{2,q}(\gamma_d)} \sim_c \|g\|_{L_2(\gamma_d)} + \left\| (1-t)^{-\frac{\theta}{2}} \|M_1 - M_t\|_{L_2(\mathbb{P})} \right\|_{L_q([0,1), \frac{dt}{1-t})}$$

where $M_t := \mathbb{E}\left(g(W_1)|\mathcal{F}_t\right)$ *and* $c \geq 1$ *depends on* (θ, q) *only.*

Applying this theorem to $q = \infty$ gives that

$$\|g\|_{\mathbb{B}^{\theta}_{2,\infty}(\gamma_d)} \sim_c \|g\|_{L_2(\gamma_d)} + \sup_{0\leq t\leq 1} (1-t)^{-\frac{\theta}{2}} \|M_1 - M_t\|_{L_2(\mathbb{P})},$$

whereas $q = 2$ gives that

$$\|g\|_{\mathbb{B}^{\theta}_{2,2}(\gamma_d)} \sim_c \|g\|_{L_2(\gamma_d)} + \left(\int_0^1 (1-t)^{-1-\theta} \|M_1 - M_t\|^2_{L_2(\mathbb{P})} \, dt \right)^{\frac{1}{2}}$$

which brings us back to Definition 1.

Multi-dimensional Black-Scholes-Samuelson model.

This is a log-normal model which dynamics on the price and the log-price can be written as

$$dS_t^i = S_t^i \left(\sum_{j=1}^d \sigma_{ij} dW_t^j + \mu_i dt \right), \quad 1 \le i \le d,$$

$$X_t^i = \log(s_0^i) + \sum_{j=1}^d \sigma_{ij} W_t^j + (\mu_i - \frac{1}{2}\sigma_i^2)t,$$

where $\sigma_i := \sqrt{\sum_j \sigma_{ij}^2}$. Assume that $(\sigma_{ij})_{i,j=1}^d$ is invertible. To the payoff function $S \mapsto h(S)$, we associate $g(x_1, ..., x_d) := h\left(\left(s_0^i e^{\sum_{j=1}^d \sigma_{ij} x_j + \mu_i - \frac{\sigma_i^2}{2}}\right)_{i=1}^d\right)$.

From this we see that

$$g \in \mathbb{B}_{2,q}^\theta(\gamma_d) \quad \text{if and only if} \quad h(S_1) \in \widetilde{\mathbb{B}}_{2,q}^\theta$$

for $q \in \{2, \infty\}$ and $g \in L_2(\gamma_d)$.

Remark 1. In the case $\theta = 1$ we get that

$$g \in \mathbb{D}_{1,2}(\gamma_d) \quad \text{if and only if} \quad h(S_1) \in \widetilde{\mathbb{B}}_{2,\infty}^1$$

for all $g \in L_2(\gamma_d)$. This can be checked by using arguments from the proof of [GH07, Corollary 2.3].

1.4 Examples

In this section, we provide examples of random variables $Z = g(X_T)$ for which we determine the fractional smoothness.

Example 1 (Lipschitz function). The case, where the fractional smoothness is obvious, is the Lipschitz case. Assume a Lipschitz function $g : \mathbb{R}^d \to \mathbb{R}$ with constant $L \ge 0$, i.e. $|g(x) - g(y)| \le L|x - y|$ and assume (SDE). Then one has that

$$\begin{aligned} \mathbb{E}\,|g(X_T) - \mathbb{E}(g(X_T)|\mathcal{F}_t)|^2 &\le \mathbb{E}\,|g(X_T) - g(X_t)|^2 \\ &\le L^2 \mathbb{E}\,|X_T - X_t|^2 \\ &\le L^2 c^2 (T - t), \end{aligned}$$

using standard estimates on the increments of X. Hence, $g(X_T) \in \widetilde{\mathbb{B}}_{2,\infty}^1$. This example includes call and put payoffs, i.e. $g(x) = (x - K)^+$ or $g(x) = (K - x)^+$.

Exactly the same argument as above yields for θ-Hölder functions g with $\theta \in (0,1)$ that $g(X_T) \in \widetilde{\mathbb{B}}^{\theta}_{2,\infty}$. But the situation is here not as clear as one expects as shown by

Example 2. Assume the setting (GBM) and that

$$h_\theta(x) := (x-K)^\theta_+$$

for some $K > 0$ and $0 < \theta < 1/2$. Then it is shown in [GT01, Lemma 2] (under more general assumptions) that $\mathbb{E}|D^2 u(t,X_t)|^2 \le c(T-t)^{-3/2+\theta}$ so that Theorem 2 gives that

$$h_\theta(S_T) \in \widetilde{\mathbb{B}}^{\theta+\frac{1}{2}}_{2,\infty}.$$

For $1/2 < \theta < 1$ one gets $h_\theta(S_T) \in \widetilde{\mathbb{B}}^{1}_{2,\infty}$.

Example 3 (Binary option). Generally, indicator functions yield to a fractional smoothness of order $\frac{1}{2}$. In the case $X = W$, $d = 1$ and $g(x) = \mathbf{1}_{[L,\infty)}(x)$ with $L \in \mathbb{R}$ one has

$$\begin{aligned} u(t,x) &= \mathbb{P}(x + W_T - W_t \ge L) = \mathcal{N}\left(\frac{x-L}{\sqrt{T-t}}\right), \\ \nabla_x u(t,x) &= \frac{1}{\sqrt{2\pi(T-t)}} \exp\left(-\frac{(x-L)^2}{2(T-t)}\right), \end{aligned}$$

so that

$$\mathbb{E}|\nabla_x u(t,W_t)|^2 \sim_c \frac{1}{\sqrt{T-t}}$$

and $g(W_T) \in \widetilde{\mathbb{B}}^{\frac{1}{2}}_{2,\infty}$ because of Theorem 2. This can be extended to the (SDE) case as follows: Our assumption guarantees that X has a transition density Γ such that

$$\Gamma(s,x;t,y) \le \sqrt{\frac{\kappa}{2\pi(t-s)}} e^{-\frac{1}{2}\frac{(x-y)^2}{\kappa(t-s)}} = \kappa \gamma_{\kappa(t-s)}(x-y)$$

for some $\kappa > 0$ and all $0 \le s < t \le T$, where γ_t is the Gaussian density with zero expectation and variance t (see [Fr64]). Then we can compute that

$$\begin{aligned} &\mathbb{E}\left|\mathbf{1}_{[L,\infty)}(X_T) - \mathbb{E}\left(\mathbf{1}_{[L,\infty)}(X_T)|\mathcal{F}_t\right)\right|^2 \\ &\le \mathbb{E}\left|\mathbf{1}_{[L,\infty)}(X_T) - \mathbf{1}_{[L,\infty)}(X_t)\right| \\ &= \mathbb{P}(X_T < L \le X_t) + \mathbb{P}(X_t < L \le X_T) \\ &\le \kappa^2[\mathbb{P}(W_{\kappa T} < L - x_0 \le W_{\kappa t}) + \mathbb{P}(W_{\kappa t} < L - x_0 \le W_{\kappa T})] \\ &\le c\sqrt{T-t} \end{aligned}$$

where $X_0 = x_0$ so that $\mathbf{1}_{[L,\infty)}(X_T) \in \widetilde{\mathbb{B}}^{\frac{1}{2}}_{2,\infty}$. The application in financial mathematics is done via $S_t = e^{X_t}$ which gives, for a positive strike $K > 0$,

$$\mathbf{1}_{\{S_T \geq K\}} = \mathbf{1}_{\{X_T \geq \log K\}} \in \widetilde{\mathbb{B}}^{\frac{1}{2}}_{2,\infty}.$$

In our context the fractional smoothness of jump functions (under different assumptions) was considered in [GT01, Gei02, GG04]. In certain multidimensional settings one can deduce for $g(x) = \mathbf{1}_{\{x_1 \geq K_1, \cdots, x_d \geq K_d\}}$ (or variants of it) the same fractional smoothness from the 1-dimensional case. Finally, the indicator function $g(x) = \mathbf{1}_D(x)$ of a $\mathcal{C}^2$-domain D also leads to $g(X_T) \in \widetilde{\mathbb{B}}^{\frac{1}{2}}_{2,\infty}$ (see [GM05, Proposition 1.2]).

Example 4 (an extreme case). By the choice of the previous examples, we emphasize that random variables $g(X_T) = h(S_T)$, usually used in financial applications, belong to a space $\widetilde{\mathbb{B}}^{\theta}_{2,\infty}$ for some $\theta \in (0,1]$. However, it is not true that $\cup_{\theta \in (0,1]} \widetilde{\mathbb{B}}^{\theta}_{2,\infty} = L_2(\mathbb{P})$. The following result gives a way to construct $g(W_1)$ belonging to $L_2(\mathbb{P})$ (here W is the linear Brownian motion) but $g(W_1) \notin \widetilde{\mathbb{B}}^{\theta}_{2,\infty}$ for all $\theta \in (0,1]$:

Proposition 2 ([GH07]). *Let $0 < \theta < 1$, $g = \sum_{k=0}^{\infty} \alpha_k H_k \in L_2(\gamma_1)$, where $(H_k)_{k \geq 0}$ is the orthogonal basis of Hermite polynomials defined in* (1.4)*. Then*

$$g(W_1) \in \widetilde{\mathbb{B}}^{\theta}_{2,\infty} \quad \textit{if and only if} \quad \sup_{0 \leq t < 1} (1-t)^{1-\theta} \sum_{k=1}^{\infty} k t^{k-1} \alpha_k^2 < \infty.$$

Approximation properties as described in Section 1.5.2 for g with $g(W_1) \in L_2(\mathbb{P}) \setminus \bigcup_{0<\theta\leq 1} \widetilde{\mathbb{B}}^{\theta}_{2,\infty}$ were studied in [Huj06] and [Sep08].

1.5 Applications

In this section we discuss some applications in stochastic finance which lead us to the fractional smoothness as introduced above. As mentioned at the beginning, a central role is played by the tracking error that arises when discrete time hedging is used, instead of a continuous time strategy. For the sake of convenience, we briefly recall the notation:

- the option payoff at maturity T is $Z = h(S_T)$;
- the fair price function is $H(t,x) = \mathbb{E}_{\mathbb{Q}}(h(S_T)|S_t = x)$;
- the n re-balancing dates are defined by a deterministic time-net $\tau = (t_i)_{i=0}^n$ with $0 = t_0 < t_1 < \cdots < t_{n-1} < t_n = T$;
- the resulting tracking error process $C(Z,\tau) = (C_t(Z,\tau))_{t \in [0,T]}$ is given by

$$C_t(Z,\tau) := \mathbb{E}_{\mathbb{Q}}(Z|\mathcal{F}_t) - \mathbb{E}_{\mathbb{Q}} Z - \sum_{i=0}^{n-1} \nabla_x H(t_i, S_{t_i}) \cdot (S_{t_{i+1} \wedge t} - S_{t_i \wedge t}).$$

1.5.1 Weak limits of error processes

Weak limits of stochastic processes have been intensively studied in the literature; see, for instance, [KP91, Jac97, JS03]. For the particular problem of the weak convergence of the tracking error the reader is referred to [Roo80, GT01, HM05, GT09]. To formulate our results, we let $\widetilde{W} = (\widetilde{W}_t)_{t\geq 0}$ be a standard Brownian motion starting at zero defined on some auxiliary probability space, where we may and do assume that all paths are continuous. In the following $\Longrightarrow_{C[0,s]}$ stands for the weak convergence in $C[0,s]$ for some $s > 0$.

In this paragraph we assume that $T = 1$ [4] and that S is the standard geometric Brownian motion, i.e. the setting of (GBM) and $\mathbb{P} = \mathbb{Q}$. The following result is the starting point of this section:

Theorem 4 ([GT01]). *Let $\tau_n = (i/n)_{i=0}^n$ be the equidistant time-nets and let $Z := \mathbf{1}_{[K,\infty)}(S_1)$ be the payoff of a digital option with strike price $K > 0$. Then one has that*

$$\sqrt{n}C_1(Z,\tau_n) \Longrightarrow \widetilde{W}_{\frac{1}{2}\int_0^1 \left|S_t^2 \frac{\partial^2 H}{\partial x^2}(t,S_t)\right|^2 dt}$$

where $\Longrightarrow$ denotes the weak convergence as n goes to infinity.

The remarkable fact is that the weak limit is not square-integrable. In the following we describe a way to increase the integrability of the weak limit. This is of particular interest for risk management purposes, as a higher integrability gives better tail-estimates. The idea is to use adapted time-nets that are more concentrated close to maturity. They are defined as follows: Given a parameter $\theta \in (0,1]$, we define the nets $\tau^{n,\theta}$ by

$$t_k^{n,\theta} := 1 - \left(1 - \frac{k}{n}\right)^{\frac{1}{\theta}}.$$

For $\theta = 1$ we have the equidistant time-nets, i.e. $t_k^{n,1} = \frac{k}{n}$. Now we have

Theorem 5 ([GT09]). *Let $0 < \theta \leq 1$, $Z = h(S_1) \in L_2(\mathbb{P})$ and $0 \leq s < 1$. Then*

$$(\sqrt{n}C_t(Z,\tau^{n,\theta}))_{t\in[0,s]} \Longrightarrow_{C[0,s]} \left(\widetilde{W}_{\int_0^t \frac{(1-r)^{1-\theta}}{2\theta}\left|S_r^2 \frac{\partial^2 H}{\partial x^2}(r,S_r)\right|^2 dr}\right)_{t\in[0,s]}.$$

Moreover, the following assertions are equivalent:

[4] With $T = 1$ we are in accordance with the quoted literature that used Hermite polynomials. Of course, we could do a re-scaling to $T > 0$ afterwards.

(i) *One has* $h(S_1) \in \widetilde{\mathbb{B}}^{\theta}_{2,2}$ *for* $0 < \theta < 1$ *or* $h(S_1) \in \widetilde{\mathbb{B}}^{1}_{2,\infty}$ *for* $\theta = 1$.
(ii) *On some stochastic basis there exists a continuous square-integrable martingale* $M = (M_t)_{t\in[0,1]}$ *such that* $\sqrt{n}C(Z, \tau^{n,\theta}) \Longrightarrow_{C[0,1]} M$.
(iii) *For*

$$A := \int_0^1 \frac{(1-t)^{1-\theta}}{2\theta} \left| S_t^2 \frac{\partial^2 H}{\partial x^2}(t, S_t) \right|^2 dt$$

one has that $\mathbb{E}A < \infty$ *and*

$$\sqrt{n}C(Z, \tau^{n,\theta}) \Longrightarrow_{C[0,1]} \left(\widetilde{W}_{\mathbf{1}_{\{A<\infty\}} \int_0^t \frac{(1-r)^{1-\theta}}{2\theta} \left| S_r^2 \frac{\partial^2 H}{\partial x^2}(r, S_r) \right|^2 dr} \right)_{t\in[0,1]}.$$

The theorem above gives us one way to consider the L_p-setting for $2 \leq p < \infty$. Given a differentiable function $\psi : (0, \infty) \to \mathbb{R}$ we let

$$(A\psi)(x) := x\psi'(x) - \psi(x).$$

In the following $AH(t, x)$ means that A acts on the x-variable of the function $H(t, x)$.

Definition 4. For $h(S_1) \in L_2(\mathbb{P})$, $0 < \theta < 1$, and $0 \leq t < 1$ we let

$$D_t^{S,\theta} h(S_1) := \frac{1-\theta}{2} \int_0^1 (1-u)^{-\frac{1+\theta}{2}} [AH(u \wedge t, S_{u\wedge t}) - AH(0, S_0)] du.$$

For $\theta = 1$ and $t \in [0, 1)$ we let $D_t^{S,1} h(S_1) := AH(t, S_t) - AH(0, S_0)$.

The process $D^{S,\theta} h(S_1) = (D_t^{\theta} h(S_1))_{t\in[0,1)}$ is a quadratic integrable martingale on the half open time interval $[0, 1)$. Using the Riemann-Liouville operator of partial integration the process $D^{S,\theta} h(S_1)$ can be interpreted as a fractional differentiation of order θ in x (see [GT09]). The point of the construction of $D^{S,\theta} h(S_1)$ is that we may have L_p-singularities of $S_t \frac{\partial H}{\partial x}(t, S_t)$ as $t \uparrow 1$ whereas $D^{S,\theta} h(S_1)$ remains L_p-bounded.

Theorem 6 ([GT09]). *For* $2 \leq p < \infty$, $0 < \theta \leq 1$, *and* $Z = h(S_1) \in L_2(\mathbb{P})$ *the following assertions are equivalent:*

(i) *On some stochastic basis there exists a continuous* $L_p(\mathbb{P})$*-integrable martingale* M *such that* $\sqrt{n}C(Z, \tau^{n,\theta}) \Longrightarrow_{C[0,1]} M$.
(ii) *The martingale* $D^{S,\theta} h(S_1)$ *is bounded in* $L_p(\mathbb{P})$.

1.5.2 L_2-estimates of the tracking error

In this section we work in the 1-dimensional martingale case assuming (SDE) with $\sigma(t,x) = \sigma(x)$ and $b(t,x) = -\frac{1}{2}\sigma^2(x)$ (meaning $\mathbb{P} = \mathbb{Q}$). The payoff function h is polynomially bounded and the option maturity is $T > 0$. We remind the reader about the time-nets $\tau^{n,\theta}$ given by

$$t_k^{n,\theta} := T\left(1 - \left(1 - \frac{k}{n}\right)^{\frac{1}{\theta}}\right)$$

and that for $\theta = 1$ we obtain the equidistant nets. Let us first check what quadratic hedging error one can expect at all if the portfolio is re-balanced n-times. The answer is the rate $1/\sqrt{n}$ as shown by

Theorem 7 ([GG04, Theorem 2.5]). *Assume that there are no constants $c_0, c_1 \in \mathbb{R}$ such that $h(S_T) = c_0 + c_1 S_T$ a.s. Then*

$$\inf_{\substack{n=1,2,\dots \\ 0=t_0<\dots<t_n=T}} n^{\frac{1}{2}} \|C(h(S_T),(t_k)_{k=0}^n)\|_{L_2(\mathbb{P})} > 0$$

where the infimum is taken over deterministic time-nets.

This was extended to the case of random time-nets in [GG06] in the case of the geometric Brownian motion.

Now we continue with the case of equidistant time-nets which are often used in discretizations.

Equidistant time-nets.

Here a starting point is the following result of Zhang:

Theorem 8 ([Zha99, Theorem 2.4.1]). *Assume that $h : \mathbb{R} \to \mathbb{R}$ is a Lipschitz function. Then we have that*

$$\lim_n n^{\frac{1}{2}} \|C(h(S_T),\tau^{n,1})\|_{L_2(\mathbb{P})} \in [0,\infty).$$

This is the result one would expect: Given a Lipschitz payoff, the L_2-rate of the error is $1/2$ for equidistant nets. But this is not the case in general as shown in

Theorem 9 ([GT01, Theorem 1]). *For $h(x) = \mathbf{1}_{[K,\infty)}(x)$ for some $K > 0$ we have that*

$$\lim_n n^{\frac{1}{4}} \|C(h(S_T),\tau^{n,1})\|_{L_2(\mathbb{P})} \in (0,\infty).$$

This means that the L_2-approximation rate for the binary option is $n^{1/4}$ if one uses equidistant nets. The two above results also hold true for appropriate $\mathbb{Q} \neq \mathbb{P}$ (i.e. S is not martingale) where the outer L_2-norm is computed w.r.t. the historical probability $\mathbb{P}$ (cf. the remarks after Theorem 2).

Theorems 8 and 9 lead naturally to two questions: What is the reason for the rate 1/4 and, secondly, can one improve the rate 1/4? Both questions can be answered by the usage of the concept of fractional smoothness.

Theorem 10 ([GG04, Theorems 2.3 and 2.8]). *For $0 < \theta \leq 1$ and a polynomially bounded $h : (0, \infty) \to \mathbb{R}$ the following assertions are equivalent:*

(i) $h(S_T) \in \widetilde{\mathbb{B}}^{\theta}_{2,\infty}$.
(ii) $\sup_n n^{\frac{\theta}{2}} \|C(h(S_T), \tau^{n,1})\|_{L_2(\mathbb{P})} < \infty$.

In particular, it turns out that $h(S_T) \in \mathbb{D}_{1,2}$ if and only if

$$\sup_n n^{\frac{1}{2}} \|C(h(S_T), \tau^{n,1})\|_{L_2(\mathbb{P})} < \infty,$$

see [GG04, Theorem 2.6], where $\mathbb{D}_{1,2}$ is the Malliavin Sobolev space obtained from the construction in Section 1.3 with $H = L_2[0, T]$ and $g_h := \int_0^T h(t) dW_t$.

For the binary option one has in Theorem 10 that $\theta = 1/2$ (cf. Example 3 in Section 1.4). This recovers the rate 1/4 obtained in Theorem 9.

Non equidistant time-nets.

Next we show how to obtain the optimal rate $n^{1/2}$ by a suitable choice of the trading dates (see question (Q4) in Section 1.1). We can combine [GG04, Lemmas 3.2 and 5.3] and [GH07, Lemma 3.8] to get

Theorem 11. *For $0 < \theta \leq 1$ and a polynomially bounded $h : (0, \infty) \to \mathbb{R}$ the following assertions are equivalent:*

(i) $\int_0^T (T - t)^{1-\theta} \mathbb{E} \left| S_t^2 \frac{\partial^2 H}{\partial x^2}(t, S_t) \right|^2 dt < \infty$.
(ii) $\sup_n n^{\frac{1}{2}} \|C(h(S_T), \tau^{n,\theta})\|_{L_2(\mathbb{P})} < \infty$.

For $0 < \theta < 1$ (and at least a bounded h) the condition Theorem 11(i) is equivalent to

(i') $h(S_T) \in \widetilde{\mathbb{B}}^{\theta}_{2,2}$

which can be checked by using Theorem 1. For the binary option this gives that

$$\sup_n n^{\frac{1}{2}} \|C(\mathbf{1}_{[K,\infty)}(S_T), \tau^{n,\eta})\|_{L_2(\mathbb{P})} < \infty$$

for any strike $K > 0$ and $0 < \eta < 1/2$.

For the next two theorems we assume that $T = 1$, that $S_t = e^{W_t - \frac{t}{2}}$ and that h might be general, i.e. not polynomially bounded. The formulation of Theorem 11 in the language of the interpolation spaces introduced in Section 1.3 gives

Theorem 12 ([GH07, Theorem 3.2]). *For $0 < \theta \leq 1$ and $h(S_1) \in L_2(\mathbb{P})$ the following assertions are equivalent:*

(i) $h(e^{\cdot - (1/2)}) \in \mathbb{B}^{\theta}_{2,2}(\gamma_1)$ *if* $0 < \theta < 1$ *and* $h(e^{\cdot - (1/2)}) \in \mathbb{D}_{1,2}(\gamma_1)$ *if* $\theta = 1$.
(ii) $\sup_n n^{\frac{1}{2}} \|C(h(S_1), \tau^{n,\theta})\|_{L_2(\mathbb{P})} < \infty$.

And Theorem 10 can be extended in this context to the full scale of real interpolation spaces as

Theorem 13 ([GH07, Theorem 3.5]). *For $1 \leq q \leq \infty$, $0 < \theta < 1$ and $h(S_1) \in L_2(\mathbb{P})$ the following assertions are equivalent:*

(i) $h(e^{\cdot - (1/2)}) \in \mathbb{B}^{\theta}_{2,q}(\gamma_1)$.
(ii) $\left\| \left(n^{\frac{\theta}{2} - \frac{1}{q}} a_n \right)_{n=1}^{\infty} \right\|_{\ell_q} < \infty$ *for* $a_n := \|C(h(S_1), \tau^{n,1})\|_{L_2(\mathbb{P})}$.

Concluding remarks

(i) The higher dimensional case for X was considered in the literature as well. Roughly speaking, one can analogously obtain upper bounds, however precise lower bounds as in the one-dimensional case are still missing. This is due to the fact that a characterization of the L_2-error proved in [Gei02, Theorem 4.4] and [GG04, Lemma 3.2] is missing for higher dimensions. However, after Zhang [Zha99] started with the regular case, Temam [Tem03] extended results from [GT01] to higher dimensions and Hujo [Huj05] used non-uniform time-nets to improve the approximation rates for certain irregular payoffs to the optimal rate $1/\sqrt{n}$ in this setting.
(ii) Seppälä [Sep08] found a criterion to characterize under certain conditions that there is a constant $c > 0$ such that

$$\inf_{\substack{\tau = (t_i)_{i=0}^n \\ 0 = t_0 < \cdots < t_n = 1}} \|C(h(S_1), \tau)\|_{L_2(\mathbb{P})} \leq \frac{c}{\sqrt{n}}$$

where deterministic time-nets are taken. It should be noted that one has a *non-linear* approximation problem as the time-nets may change for fixed n from payoff to payoff h.
(iii) In the above discussion, the time-nets τ are deterministic. Alternatively, one can allow the time-nets to be stochastic and adapted. This issue has been handled by [MP99] using optimal stopping tools. The estimation of convergence rates is an open question. However, it was shown in [GG06]

that the random time-nets do not improve the best possible approximation rate $1/\sqrt{n}$ in the case (GBM) when in the n-th approximation a sequence of n stopping times is used.

(iv) Similar studies can be performed when studying the Delta-Gamma hedging strategies. Instead of hedging the payoff using only the asset, we use other traded options written on the same asset. For a one-dimensional asset, if the price of the additional option is $(P(t,S_t))_{0\le t\le T}$, the numbers of options P and assets to hold at time t_i are respectively equal to

$$\delta^P_{t_i} := \frac{\partial^2_S H(t_i,S_{t_i})}{\partial^2_S P(t_i,S_{t_i})} \quad \text{and} \quad \delta^S_{t_i} := \partial_S H t_i, S_{t_i}) - \frac{\partial^2_S H(t_i,S_{t_i})}{\partial^2_S P(t_i,S_{t_i})}\partial_S P(t_i,S_{t_i}).$$

In [GM10b, Theorem 6], considering a multi-dimensional Black-Scholes model, it is established that for an exponentially bounded payoff such that $g(X_T) \in \widetilde{\mathbb{B}}^{\theta}_{2,\infty}$ for some $0<\theta<1$, the use of equidistant time-nets leads to the same convergence rate $1/n^{\theta/2}$ as for the delta hedging strategy. On the contrary, the use of non equidistant time-nets $\tau^{n,\eta}$ with $0<\eta<\theta/2$ enables us to obtain the improved convergence rate $1/n$.

1.6 Further developments

1.6.1 Backward stochastic differential equations

Makhlouf and the second author applied in [GM10a] the concept of fractional smoothness to backward stochastic differential equations of the type

$$Y_t = g(X_T) + \int_t^T f(s,X_s,Y_s,Z_s)ds - \int_t^T Z_s dW_s$$

where $X=(X_t)_{t\in[0,T]}$ is our forward diffusion and the generator f is continuous in its four arguments, continuously differentiable in (x,y,z) with uniformly bounded derivatives. These equations are particularly useful in stochastic finance, since they allow to take into account market frictions and constraints (we refer to [EPQ97] for a more complete account on this subject).

Solving numerically this type of equation is a challenging issue since it concerns a non-linear problem (due to the generator f), generally defined in a multi-dimensional setting. One possible approach consists in approximating the BSDE using a discrete-time dynamic programming equation (see [Zha04, BT04, LGW06] among others). One of the main error contribution is related to the L_2-regularity on Z, defined by

$$\mathcal{E}(Z,\tau) = \sum_{i=1}^n \int_{t_{i-1}}^{t_i} \|Z_t - Z_{t_{i-1}}\|^2_{L_2(\mathbb{P})} dt.$$

If f were equal to 0, then the Z-component is given by $z_t = \nabla_x u(t, X_t)\sigma(t, X_t)$ where $u(t,x) = \mathbb{E}(g(X_T)|X_t = x)$. Studying the L_2-regularity of z is thus very similar to the analysis of the tracking error presented in Section 1.5. Additionally, using BSDE techniques, one can prove explicit upper bounds for the difference $Z - z$

Theorem 14 ([GM10a, Corollary 14]). *Assume* (SDE) *and* $g(X_T) \in \widetilde{\mathbb{B}}^{\theta}_{2,\infty}$ *for* $0 < \theta \leq 1$. *Then, for some* $c > 0$, *one has that*

$$|Z_t - z_t| \leq c \int_t^T \frac{\sqrt{\mathbb{E}\left[|g(X_T) - \mathbb{E}(g(X_T)|\mathcal{F}_s)|^2 \,|\mathcal{F}_t\right]}}{T-s} ds + c(T-t),$$
$$\mathbb{E}\,|Z_t - z_t|^2 \leq c(T-t)^{\theta}.$$

Taking advantage of this approximation result close to the time singularity, we can prove that the estimate of $\mathcal{E}(z,\tau)$ (linear case) transfers to $\mathcal{E}(Z,\tau)$ (non-linear case) and get

Theorem 15 ([GM10a, Theorem 21]). *Assume* (SDE), $g(X_T) \in \widetilde{\mathbb{B}}^{\theta}_{2,\infty}$ *and that* $0 < \eta < \theta < 1$ *or* $\eta = \theta = 1$. *Then one has that*

$$\mathcal{E}(Z, \tau^{n,\eta}) \leq \frac{c}{n}.$$

In [GGG10], extensions of the above in different directions are discussed.

1.6.2 Lévy processes

An extension of the results of [GH07] to Lévy Processes is done by C. Geiss and Laukkarinen in [GL10]. Moreover, Tankov and Brodén proved in [TB09] results along the line of [GT01].

1.6.3 Multigrid Monte-Carlo Methods

In the context of Multigrid Monte-Carlo Methods it turned out that the concept of fractional smoothness is useful as well. The reader is referred to the papers of Avikainen [Av09a, Av09b].

References

[Av09a] R. Avikainen. On irregular functionals of SDEs and the Euler scheme. *Finance and Stochastics*, 13:381–401, 2009.

[Av09b] R. Avikainen. On generalized bounded variation and approximation of SDEs. *Preprint, Department of Mathematics and Statistics, University of Jyväskylä*, 2009. Available at `www.math.jyu.fi/research/pspdf/383.pdf`.

[BS88] C. Bennet and R. Sharpley. *Interpolation of operators.* Academic Press, 1988.

[BL76] J. Bergh and J. Lőfstrőm. *Interpolation spaces. An introduction.* Springer, 1976.

[BT04] B. Bouchard and N. Touzi. Discrete time approximation and Monte Carlo simulation of backward stochastic differential equations. *Stochastic Processes and their Applications*, 111:175–206, 2004.

[Fr64] A. Friedman. *Partial Differential Equations of Parabolic Type.* Prentice-Hall, 1964.

[GG04] C. Geiss and S. Geiss. On approximation of a class of stochastic integrals and interpolation. *Stoch. Stoch. Rep.*, 76(4):339–362, 2004.

[GG06] C. Geiss and S. Geiss. On an approximation problem for stochastic integrals where random time nets do not help. *Stoch. Proc. Appl.*, 116:407–422, 2006.

[GGG10] C. Geiss, S. Geiss, and E. Gobet. Generalized fractional smoothness and L_p-variation of BSDEs with non-Lipschitz terminal condition. *In preparation*, 2010.

[GL10] C. Geiss and E. Laukkarinen. A note on Malliavin fractional smoothness and discrete time hedging in the Lévy model. *Preprint, Department of Mathematics and Statistics, University of Jyväskylä*, 2010.

[Gei02] S. Geiss. Quantitative approximation of certain stochastic integrals. *Stoch. Stoch. Rep.*, 73(3-4):241–270, 2002.

[GH07] S. Geiss and M. Hujo. Interpolation and approximation in $L_2(\gamma)$. *Journal of Approximation Theory*, 144:213–232, 2007.

[GT09] S. Geiss and A. Toivola. Weak convergence of error processes in discretizations of stochastic integrals and Besov spaces. *Bernoulli*, 15(4):925–954, 2009.

[GM10a] E. Gobet and A. Makhlouf. L_2-time regularity of BSDEs with irregular terminal functions. *To appear in Stochastic Processes and their Applications*, 2010.

[GM10b] E. Gobet and A. Makhlouf. The tracking error rate of the Delta-Gamma hedging strategy. *To appear in Mathematical Finance*, 2010.

[GM05] E. Gobet and R. Munos. Sensitivity analysis using Itô-Malliavin calculus and martingales. Application to stochastic control problem. *SIAM Journal of Control and Optimization*, 43:5:1676–1713, 2005.

[GT01] E. Gobet and E. Temam. Discrete time hedging errors for options with irregular pay-offs. *Finance and Stochastics*, 5(3):357–367, 2001.

[HM05] T. Hayashi and P. Mykland. Evaluating hedging errors: an asymptotic approach. *Mathematical Finance*, 15(2):309–343, 2005.

[Hir99] F. Hirsch. Lipschitz functions and fractional Sobolev spaces. *Potential Anal.*, 11(4):415–429, 1999.

[Huj05] M. Hujo. On discrete time hedging in d-dimensional option pricing models. *Preprint 317, Department of Mathematics and Statistics, University of Jyväskylä*, 2005. Available at `www.math.jyu.fi/research/pspdf/317.pdf`.

[Huj06] M. Hujo. Is the approximation rate for European pay-offs in the Black-Scholes model always $1/\sqrt{n}$? *J. Theoret. Probab.* 19:190–203, 2006.

[Jac97] J. Jacod. On continuous conditional Gaussian martingales and stable convergence in law. *Séminaire de Probabiliés XXXI - Lecture Notes in Mathematics 1655, Springer Verlag*, 1997.

[JS03] J. Jacod and A.N. Shiryaev. *Limit theorems for stochastic processes.* Springer-Verlag, 2003.

[KS98] I. Karatzas and S.E. Shreve. *Methods of mathematical finance.* Springer-Verlag, New York, 1998.

[EPQ97] N. El Karoui, S.G. Peng, and M.C. Quenez. Backward stochastic differential equations in finance. *Math. Finance*, 7(1):1–71, 1997.

[KP91] T.G. Kurtz and P. Protter. Weak limit theorems for stochastic integrals and stochastic differential equations. *Ann. Probab.*, 19(3):1035–1070, 1991.

[LGW06] J.P. Lemor, E. Gobet, and X. Warin. Rate of convergence of an empirical regression method for solving generalized backward stochastic differential equations. *Bernoulli*, 12(5):889–916, 2006.

[MP99] C. Martini and C. Patry. Variance optimal hedging in the Black-Scholes model for a given number of transactions, 1999. Available at `hal.inria.fr/docs/00/07/28/95/PDF/RR-3767.pdf`.

[Nu06] D. Nualart. *Malliavin calculus and related topics.* Springer-Verlag, Berlin, 2006.

[Roo80] H. Rootzen. Limit distributions for the error in approximations of stochastic integrals. *The Annals of Probability*, 8(2):241–251, 1980.

[Sep08] H. Seppälä. Optimal approximation rate of certain stochastic integrals. *In revision for J. Approx. Theory, 2010. Preprint 375, Department of Mathematics and Statistics, University of Jyväskylä*, 2008. Available at `www.math.jyu.fi/research/pspdf/375.pdf`.

[Tal97] N. Taleb. *Dynamic Hedging: Managing Vanilla and Exotic Options.* Wiley Finance Editions, 1997.

[TB09] P. Tankov and M. Brodén. Errors from discrete hedging in exponential Lévy models: the L_2 approach. *Preprint*, 2009.

[Tem03] E. Temam. Analysis of error with Malliavin calculus: application to hedging. *Mathematical Finance*, 13(1):201–214, 2003.

[Wat93] S. Watanabe. Fractional order Sobolev spaces on Wiener space. *Probab. Theory Related Fields*, 95(2):175–198, 1993.

[Zha99] R. Zhang. *Couverture approchée des options Européennes.* PhD thesis, Ecole Nationale des Ponts et Chaussées, 1999. Available at `cermics.enpc.fr/theses/99/zhang-ruotao.ps.gz`.

[Zha04] J. Zhang. A numerical scheme for BSDEs. *Ann. Appl. Probab.*, 14(1):459–488, 2004.