

TYPICAL RECURRENCE FOR THE EHRENFEST WIND-TREE MODEL

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ABSTRACT. We show that the typical wind-tree model, in the sense of Baire, is recurrent and has a dense set of periodic orbits. The recurrence result also holds for the Lorentz gas : the typical Lorentz gas, in the sense of Baire, is recurrent. These Lorentz gases need not be of finite horizon!

In 1912 Paul and Tatiana Ehrenfest proposed the wind-tree model of diffusion in order to study the statistical interpretation of the second law of thermodynamics and the applicability of the Boltzmann equation [EhEh]. In the Ehrenfest wind-tree model, a point particle (the “wind”) moves freely on the plane and collides with the usual law of geometric optics with randomly placed fixed square scatterers (the “trees”). The notion of “randomness” was not made precise, in fact it would have been impossible to do so before Kolmogorov laid the foundations of probability theory in the 1930s. We will call the subset of the plane obtained by removing the obstacles the billiard table, and the the motion of the point the billiard flow.

From the mathematical rigorous point of view, there have been two results on recurrence for wind-tree models, both on a periodic version where the scatterers are identical rectangular obstacles located periodically along a square lattice on the plane, one obstacle centered at each lattice point. Hardy and Weber [HaWe] proved recurrence and abnormal diffusion of the billiard flow for special dimensions of the obstacles and for very special directions, using results on skew products above rotations. More recently Hubert, Lelièvre, and Troubetzkoy have studied the general full occupancy periodic case [HuLeTr]. They proved that, if the lengths of the sides of the rectangles belong to a certain dense G_δ subset \mathcal{E}' , then the dynamics is recurrent and they gave a lower bound on the diffusion rate. The recurrence was proven by analysis of the case when the lengths of the sides are rational and

belong to the set

$$\mathcal{E} = \left\{ (a, b) = (p/q, r/s) \in \mathbb{Q} \times \mathbb{Q} : \right. \\ \left. (p, q) = (r, s) = 1, \quad 0 < p < q, \quad 0 < r < s, \right. \\ \left. p, r \text{ odd, } q, s \text{ even} \right\}.$$

Another set of rational lengths was used to prove the diffusion result. The set \mathcal{E}' consists of dimensions which are sufficiently well approximable by both of these rational sets.

In this article we will prove the recurrence of random wind-tree models. We consider the following model. Fix a finite or countable set of dimensions of obstacles $\mathcal{F} \subset (0, 1)^2 \cup \{(0, 0)\}$ such that $\mathcal{F} \cap (\mathcal{E} \cup \mathcal{E}') \neq \emptyset$. Let e be in this intersection and let W_e denote the billiard table with identical obstacle e at each lattice site. Consider the set of all wind-tree models $\mathcal{F}^{\mathbb{Z}^2}$ with the product topology on $\mathcal{F}^{\mathbb{Z}^2}$. A lattice site with an obstacle of dimension $(0, 0)$ will be interpreted as a lattice site without obstacle.

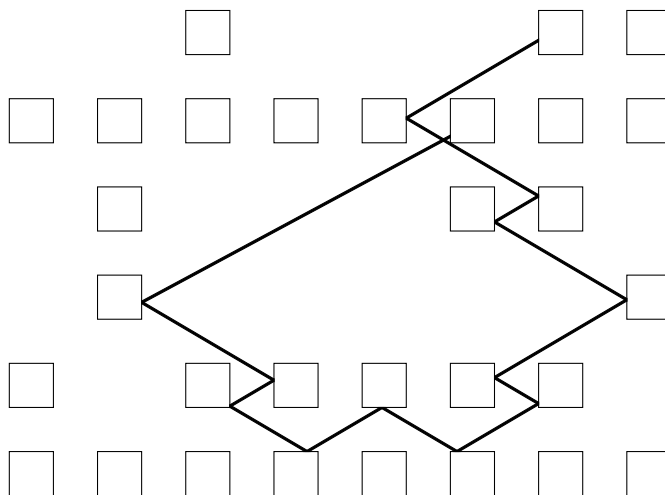


FIGURE 1. A random wind-tree model with obstacles of sizes $(1/2, 1/2)$ and $(0, 0)$.

Consider a flow Φ on a measured topological space (Ω, μ) . A point $x \in \Omega$ is called recurrent for Φ if for every neighborhood U of x and any $T_0 > 0$, there is a time $T > T_0$ such that $\Phi_T(x) \in U$; the flow Φ itself is *recurrent* if μ -almost every point is recurrent. In our setting, the billiard flow ϕ is the flow at constant unit speed bouncing off at equal angles upon hitting the rectangular obstacles. This flow preserve the natural phase volume.

Our first result is that recurrence satisfies a 0-1 law:

Proposition 1. *For each ergodic shift invariant measure on $\mathcal{F}^{\mathbb{Z}^2}$, the wind-tree models in $\mathcal{F}^{\mathbb{Z}^2}$ are almost surely recurrent or almost surely non recurrent.*

The following topological result gives evidence that wind-tree models are almost surely recurrent.

Theorem 2. *There is a dense G_δ subset G of $\mathcal{F}^{\mathbb{Z}^2}$ such that the billiard flow is recurrent for every billiard table in G with respect to the natural phase volume.*

A direction $\theta \in \mathbb{S}^1$ is called *purely periodic* if all regular orbits are periodic in this direction are periodic. The tables in $\mathcal{F}^{\mathbb{Z}^2}$ are called *square tiled* if \mathcal{F} is a finite subset of \mathcal{E} . In this case there is a positive integer Q , the least common multiple of the denominators of the dimensions of the obstacles, such that each table $W \in \mathcal{F}^{\mathbb{Z}^2}$ can be tiled in the standard way (checkerboard tiling) by squares with side length $1/Q$.

Theorem 3. *If $\mathcal{F}^{\mathbb{Z}^2}$ is square tiled, then there is a dense G_δ subset G of $\mathcal{F}^{\mathbb{Z}^2}$ such that the billiard flow is recurrent for every billiard table in G with respect to the natural phase volume and for every billiard table in G there is a dense set of purely periodic directions $\theta \subset \mathbb{S}^1$.*

In the proofs we will prove the recurrence of certain first return maps. For this purpose, consider a map F on a measured topological space (Ω, μ) . A point $x \in \Omega$ is called recurrent for F if for every neighborhood U of x there is a time $N > 0$ such that $F^N(x) \in U$; the map F itself is *recurrent* if μ -almost every point is recurrent.

Next we introduce the first return maps we will use. Fix a table $W \in \mathcal{F}^{\mathbb{Z}^2}$ and suppose that $N \geq 1$. Let $B_N := \{(i, j) \in \mathbb{Z}^2 : |i| + |j| < N\}$ and $A_{N, N_1} := \{(i, j) \in \mathbb{Z}^2 : N \leq |i| + |j| < N_1\}$. Consider the continuous simple curve $D_N = D_N(W)$ in the billiard table W consisting of the segments of $|x| + |y| = N$ which are in the interior of the table (not in the obstacles) and the “outer” part of the boundary of the obstacles e with centers (i, j) satisfying $|i| + |j| = N$ (see figure 1). This curve separates the table into two parts, the finite (or inner) part and the infinite (or outer) part. Let \hat{D}_N^- consist of the unit vectors with base point in D_N pointing into the finite part and \hat{D}_N^+ pointing into the infinite part of the table. Let $\hat{D}_N := \hat{D}_N^- \cup \hat{D}_N^+$ and $\hat{D} = \cup_{N \geq 1} \hat{D}_N$. Consider the first return maps $f_N : \hat{D}_N \rightarrow \hat{D}_N$ (wherever they are well defined) and $f : \hat{D} \rightarrow \hat{D}$. Clearly this map preserve the phase area if they are well defined. Note that the singular points (points whose

image, or preimage hits a corner or is tangent to a side of an obstacle) are of measure 0.

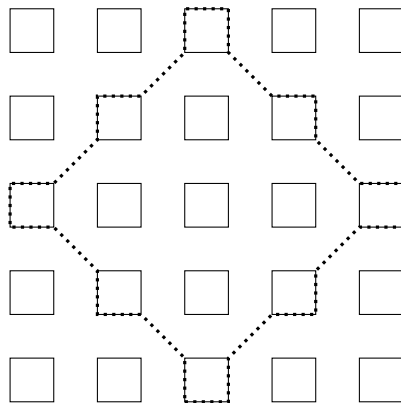


FIGURE 2. The curve D_2 as seen on the table W_e consisting of the obstacle e at such lattice point.

The proofs of the results rely on the following lemma.

Lemma 4. *The following statements are equivalent.*

- (1) *The wind-tree model W is recurrent*
- (2) *$f : \hat{D} \rightarrow \hat{D}$ is recurrent*
- (3) *there is a positive sequence $\varepsilon_n \searrow 0$ and a sequence $N_n \rightarrow \infty$ such that f_{N_n} is well defined for at least $(1 - \varepsilon_n)\%$ of the points in \hat{D}_{N_n} .*

Proof. Clearly (1) implies (2) and (2) implies (3). We will now show that (3) implies (2). We claim that if $N_1 < N_2$ and f_{N_2} is well defined almost everywhere then f_{N_1} is also well defined almost everywhere. Simply consider the map f induced on the set $\cup_{N \leq N_2} \hat{D}_N$. This map is well defined almost everywhere since f_{N_2} is. Thus f_{N_1} is recurrent by the Poincaré recurrence theorem. Thus to show that each f_N is actually defined almost everywhere and thus f is recurrent it suffices to show that f_N is for infinitely many N .

Note that the map f is invertible. Consider a set $U \subset \hat{D}_{N_n}$ which never recurs to \hat{D}_{N_n} . We claim that $f^j U \cap f^k U = \emptyset$ for all $j > k \geq 0$. If not then by the invertibility of f we would have $f^{j-k} U \cap U \neq \emptyset$, i.e. the points in U recur to $U \subset \hat{D}_{N_n}$, a contradiction. This implies that since the set \hat{D}_{N_n} are of finite measure, almost every point in U can visit each set \hat{D}_{N_n} only a finite number of times. Thus we can define for almost every $x \in U$ a (finite time) $m_n(x)$ be the last time the orbit of x visits \hat{D}_{N_n} . The map $F(x) = f^{m_n(x)}(x)$ is a measure preserving map

of U into the set of nonrecurrent points in \hat{D}_{N_n} . This set has measure at most ε_n . Since this holds for arbitrarily large n we conclude that U is of measure 0, i.e. f_N is well defined almost everywhere and thus f is recurrent.

Finally we need to show that (2) implies (1). Consider any small open ball B in the phase space. Flow each (non-singular) point in this ball until it hits the set \hat{D} . Since the ball is open, it has positive phase volume, and its image on the set \hat{D} also has positive phase area. Almost every of these points is f recurrent.

Fix a nonsingular $x \in B$ such that $x_N := \Phi_t(x) \in \hat{D}_N$. Note that by transversality and the Fubini theorem almost every $x \in B$ corresponds to a f -recurrent x_N . To conclude the proof we suppose that x_N is f -recurrent and we will show that this implies that x is ϕ -recurrent. Choose a open neighborhood U of x small enough that for each $y \in U$ there is a $t(y)$ very close to t such that $\Phi_{t(y)}(y) \in \hat{D}_N$. Let $U' = \{\Phi_{t(y)}(y) : y \in U\}$. This is a small neighborhood of $\Phi_t(x)$, and by the above results there is an (arbitrarily large) n such that $f^n x_N \cap U' \neq \emptyset$. Thus $f^n x_N = \Phi_s(x_N) = \Phi_{s+t}(x) \in U'$ for some large s . Since this point $\Phi_{s+t}(x)$ is in U' is the image $\Phi_{t(y_0)} y_0$ of some $y_0 \in U$. Thus $\Phi_{s+t-t(y_0)}(x) = y \in U$ and we conclude that x is recurrent. \square

Proof of Theorem 2. The idea of the proof is simple. A table in our dense G_δ will have infinitely many large annuli for which the table has the obstacle e at all lattice sites in the annuli. The widths of these annuli will increase sufficiently quickly to guarantee the recurrence.

Fix $\varepsilon > 0$ and $N \geq 1$. Fix a cylinder set in $C = C_{k,N} \in \mathcal{F}^{B_N}$, i.e. C is given by specifying the rectangle size (or absence of rectangle) at all the lattice points in B_N (the index k enumerates the finite (or countable) collection of all such cylinder sets). We consider an $N_1 \gg N$ and the cylinder set $C' = C'_{k,N,N_1}$ such that $C' \subset C$ and for each $c \in C'$ $c_{(i,j)} = e$ for all $(i,j) \in A_{N,N_1}$. Consider the table W_e . Since it is recurrent, for each fixed N , we can choose $N_1 = N_1(N, \varepsilon)$ sufficiently large so that on this table $(1 - \varepsilon)\%$ of the points in \hat{D}_N^+ recur to \hat{D}_N before leaving A_{N,N_1} . The dynamics for any table in the cylinder C' is identical to the dynamics on the table W_e as long as it stays in the annulus A_{N,N_1} . Since the wind-tree tables in the cylinder C' agree with W_e on A_{N,N_1} the $(1 - \varepsilon)\%$ -recurrence holds for all these tables.

Consider the set $O_\varepsilon := \cup_{N \geq 1} \cup_k C'_{k,N,N_1(N,\varepsilon)}$. Since cylinder sets are open this set is open. Since the union is taken over all cylinder sets it is dense. Now fix a sequence $\varepsilon_n \searrow 0$ and let $G := \cap_n O_{\varepsilon_n}$. Clearly G is a dense G_δ set. For each table $W \in G$ and for all n there exists

$N_n = N_n(W)$ and $k(N_n)$ such that $W \in C''_{k(N_n), N_n, N_1(N_n, \varepsilon_n)}$. This means that for each $n \geq 1$ at least $(1 - \varepsilon_n)\%$ of the points in the set \hat{D}_{N_n} recur to \hat{D}_{N_n} . We apply Lemma 4 to conclude the recurrence. \square

For the proof it is very important that in the annuli the tables agree with a recurrent full occupancy table. On the other hand for the recurrence the shape of the table in between the annuli is not at all important. Instead of taking rectangular scatterers, we could choose circular scatterers, or more byzantine ones (as long as we can define billiard dynamics which preserve the phase volume, for example if they are piecewise C^1), disjoint and finitely many in any compact region.

Actually we can also relax the fact that the tables agree with a recurrent full occupancy table on the annuli. We can replace this by an almost agreement in the following sense. The centers of the obstacles could be assumed to be uniformly distributed in a small open ball around each lattice point. For a cylinder set in which the restriction to an annuli (or another finite set) the centers are very close to the lattice, the points in \hat{D}_N^+ are almost recurrent. To write the details is technically more complicated since the measure spaces vary in a more dramatic way than they do in the presented proof.

Proof of Theorem 3. The recurrence is a special case of Theorem 2. A purely periodic direction is called *strongly parabolic* if the phase space decomposes into an infinite number of cylinders isometric to each other. In [HuLeTr] it was shown that for the table W_e the set of strongly parabolic directions θ are dense in \mathbb{S}^1 . Furthermore these directions have rational slope ($\pm p/q$ with p and q depending on θ). We will show that for any strongly parabolic direction θ for W_e , for any table $W \in G$ a.e. orbit on this table in the direction θ is periodic.

Fix θ a strongly parabolic direction for W_e and let M be the common (geometric) length of the cylinders. Let n be so large that $N_1(N_n, \varepsilon_n) - N_n > 2M$ and let $P_n := (N_n + N_1(N_n, \varepsilon_n))/2$. On the table W_e all the cylinders crossing D_{P_n} stay in in $A_{N_n, N_1(N_n, \varepsilon_n)}$. Since the table W coincides with W_e on this set all regular orbits in W starting on D_{P_n} are periodic.

Now consider any phase point in W with regular orbit starting strictly inside B_{P_n} . First of all since D_{P_n} consists completely of periodic orbits this orbit can not reach D_{P_n} without being one of these periodic orbits. If the orbit does not reach D_{P_n} then it stays completely inside B_{P_n} . Since the orbit's slope is rational, bounded and the billiard table is square tiled, the y coordinate can only take a finite number

of values when crossing the lines $x = n/Q$ with $n \in \mathbb{Z}$. Thus it must visit some point $(n_0/Q, y_0)$ twice with the same direction. Since the dynamics is invertible the orbit is periodic. \square

Proof of Proposition 1. We use Lemma 4, a wind-tree model $W \in \mathcal{F}^{\mathbb{Z}^2}$ is recurrent if and only if there are sequences $\varepsilon_n \searrow 0$ and $N_n \rightarrow \infty$ such that $(1 - \varepsilon_n)\%$ of all points in \hat{D}_{N_n} recur to \hat{D}_{N_n} . If a wind-tree model is recurrent with the sequences as above, then there exists $M_n > N_n$ such that $(1 - C\varepsilon_n)\%$ of all points in \hat{D}_{N_n} recur to \hat{D}_{N_n} before hitting \hat{D}_{M_n} . Applying Lemma 4 shows that the converse is also true.

For each recurrent table, by choosing a subsequence we can suppose that $N_n \geq n$. Fix a sequence $\varepsilon'_n \searrow 0$. We can assume furthermore, that it satisfies the above condition for this fixed sequence since if $\varepsilon_n \rightarrow 0$ faster than ε'_n , then it satisfies the condition with ε'_n , and if not then we can choose a subsequence for which it goes faster. Summarizing a wind-tree model is recurrent iff there exists $M_n > N_n \geq n$ such that $(1 - C\varepsilon'_n)\%$ of all points in \hat{D}_{N_n} recur to \hat{D}_{N_n} before hitting \hat{D}_{M_n} .

One needs to specify only a finite part of the table to check this property at a fixed stage n , i.e. if a table $W \in \mathcal{F}^{\mathbb{Z}^2}$ is recurrent, then all the tables in the cylinder set with the obstacles specified to be those of W at lattice points (i, j) with $|i| + |j| \leq M_n$ satisfy this property for this fixed n . Let O_{N_n, M_n} denote the (finite) union of all cylinder sets such that this happens at stage n . This is an open set. Thus by the above characterization, the set of recurrent wind-tree models can be written as $\bigcap_{n=1}^{\infty} \bigcup_{M_n > N_n \geq n} O_{N_n, M_n}$ is a Baire measurable set.

The notion of recurrence is shift invariant (in the space $\mathcal{F}^{\mathbb{Z}^2}$). Thus since the set of recurrent wind-tree models is measurable and invariant it is of measure 0 or 1 for any invariant measure which is ergodic for the \mathbb{Z}^2 shift. \square

LORENTZ GAS

A Lorentz gas is similar to the Ehrenfest wind-tree model, with the rectangular obstacles replaced by strictly convex (C^3) obstacles. A Lorentz gas is said to verify the finite horizon condition if the minimal distance between obstacles is strictly positive and any infinite line intersects infinitely many obstacles with bounded gaps between the intersections. The long standing conjecture that periodic Lorentz gases are recurrent was independently resolved by Conze [Co] and Schmidt [Sc] in the 1990's building on previous results on the hyperbolic structure. Using the hyperbolic structure models, Lenci has shown that

Baire typical Lorentz gases with finite horizon are recurrent [Le2]. Here we prove that Baire typical Lorentz gases are recurrent. Our typical gas will satisfy a weaker property which we call locally finite horizon: the distance between obstacles is still strictly bounded away from 0 and every infinite line intersects infinitely many obstacles, however the gaps between the intersections are not necessarily bounded.

Rather than try to state a general result, we give two examples. Generalizations to other situations should be clear.

1) Let \mathbb{T} denote the triangular lattice. Fix e and convex open set with C^3 boundary (for example a ball), such that if we place the obstacle e at each lattice site, then the corresponding infinite table satisfies the finite horizon condition. Let 0 denote the absence of an obstacle. Then the set of Lorentz billiard tables we consider is $\{0, e\}^{\mathbb{T}}$, note that since we are allowing empty cells, these tables do not have finite horizon, nor a priori locally finite horizon.

2) Consider the \mathbb{Z}^2 lattice. Here we let e denote the obstacle consisting of the union of 5 convex open sets with C^3 boundary (again for examples balls). The 5 convex sets are chosen so the table consisting of the obstacle e at each lattice site is of finite horizon. Again 0 denotes the absence of an obstacle. We consider the set of Lorentz gases $\{0, e\}^{\mathbb{Z}^2}$. These tables, like those above, do not necessarily have finite horizon, nor locally finite horizon.

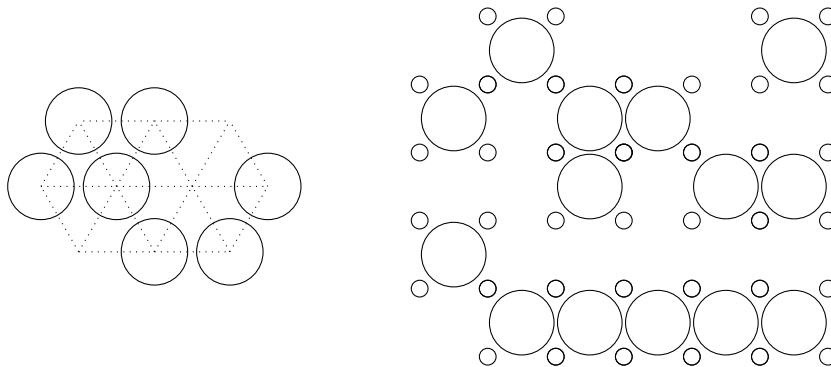


FIGURE 3. Random Lorentz gases with triangular and square lattices

The proof of the following theorem is essentially identical to the proof of Theorem 2 and will be omitted.

Theorem 5. *There is a dense G_δ subset G of each of the above two examples such that the billiard flow is recurrent for every billiard table in G with respect to the natural phase volume.*

All the tables in the dense G_δ will have a locally finite horizon. It would be interesting to investigate if this implies ergodicity like in the finite horizon case (see [Le1],[Le2]).

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