From open quantum systems to open quantum maps

Stéphane Nonnenmacher, Johannes Sjoestrand, Maciej Zworski

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1. Introduction and statement of the results

In this paper we show that for a class of open quantum systems satisfying a natural dynamical assumption (see §2.2) the study of the resolvent, and hence of scattering, and of resonances, can be reduced, in the semiclassical limit, to the study of open quantum maps, that is of finite dimensional quantizations of canonical relations obtained by truncation of symplectomorphisms derived from the classical Hamiltonian flow (Poincaré return maps).

We first explain the result in a simplified setting. For that consider the Schrödinger operator

$$P(h) = -h^2\Delta + V(x) - 1\,, \quad V \in \mathcal{C}^\infty_{\rm c}(\mathbb{R}^n)\,, \tag{1.1}$$

and let Φ^t be the corresponding classical flow on $T^*\mathbb{R}^n \ni (x,\xi)$:

$$\Phi^t(x,\xi) \stackrel{\rm def}{=} (x(t),\xi(t))\,,$$
$$x'(t) = 2\xi(t)\,, \quad \xi'(t) = -dV(x(t))\,, \quad x(0) = x\,, \quad \xi(0) = \xi\,.$$

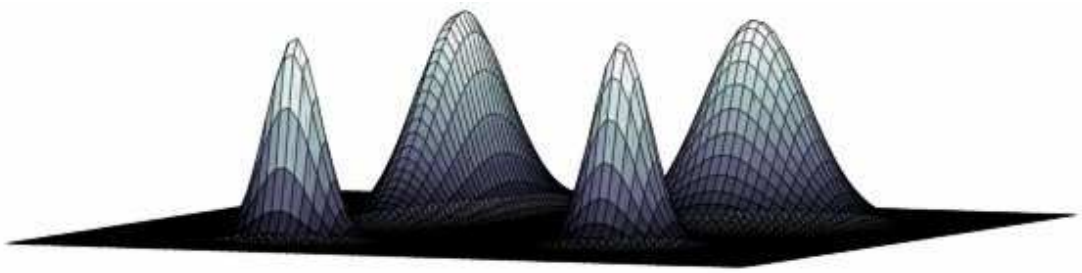

Figure 1. An example of a potential, $V \in \mathcal{C}^\infty_{\rm c}(\mathbb{R}^2)$, to which the results apply: the Hamiltonian flow is hyperbolic on the trapped set in a range of energies – see [38, Appendix c]. In this example each energy surface $p^{-1}(E)$ is three dimensional, so the Poincaré section is two dimensional as shown in Fig. 2.

Equivalently, this flow is generated by the Hamilton vector field

$$H_p(x,\xi) = \sum_{j=1}^{n} \frac{\partial p}{\partial \xi_j}\frac{\partial}{\partial x_j} - \frac{\partial p}{\partial x_j}\frac{\partial}{\partial \xi_j} \tag{1.2}$$

associated with the classical Hamiltonian

$$p(x,\xi) = |\xi|^2 + V(x) - 1\,. \tag{1.3}$$

The energy shift by -1 allows us to focus on the quantum and classical dynamics near the energy $E = 0$, which will make our notations easier[1]. We assume that the Hamiltonian flow has no fixed point at this energy: $dp\restriction_{p^{-1}(0)} \neq 0$.

The trapped set at any energy E is defined as

$$K_E \stackrel{\rm def}{=} \{(x,\xi) \in T^*\mathbb{R}^n \ :\ p(x,\xi) = E\,,\ \Phi^t(x,\xi) \text{ remains bounded for all } t \in \mathbb{R}\}\,. \tag{1.4}$$

The information about spectral and scattering properties of $P = P(h)$ in (1.1) can be obtained by analyzing the resolvent of P,

$$R(z) = (P - z)^{-1}\,, \quad \operatorname{Im} z > 0\,,$$

and its meromorphic continuation – see for instance [33] and references given there. More recently semiclassical properties of the resolvent have been used to obtain local smoothing and Strichartz estimates, leading to applications to nonlinear evolution equations – see [14] for a recent result and for pointers to the literature. In the physics literature the Schwartz kernel of $R(z)$ is referred to as Green's function of the potential V.

The operator P has absolutely continuous spectrum on the interval $[-1,\infty)$; nevertheless, its resolvent $R(z)$ continues meromorphically from $\operatorname{Im} z > 0$ to the disk $D(0,1)$, in the sense that $\chi R(z)\chi$, $\chi \in \mathcal{C}^\infty_{\rm c}(\mathbb{R}^n)$, is a meromorphic family of operators, with poles independent of the choice of $\chi \not\equiv 0$ (see for instance [41, Section 3] and [39, Section 5]).

The multiplicity of the pole $z \in D(0,1)$ is given by

$$m_R(z) \stackrel{\rm def}{=} \operatorname{rank} \oint_z \chi R(w)\chi dw\,,$$

where the integral runs over a sufficiently small circle around z.

We now assume that at energy $E = 0$, the flow Φ^t is *hyperbolic* on the trapped set K_0 and that this set is *topologically one dimensional*. Hyperbolicity means [24, Def. 17.4.1] that at any point $\rho = (x,\xi) \in K_0$ the tangent space to the energy surface splits into the neutral ($\mathbb{R}H_p(\rho)$), stable (E^-_ρ), and unstable (E^+_ρ) directions:

$$T_\rho p^{-1}(0) = \mathbb{R}H_p(\rho) \oplus E^-_\rho \oplus E^+_\rho\,, \tag{1.5}$$

[1]There is no loss of generality in this choice: the dynamics of the Hamiltonian $\xi^2 + \tilde V(x)$ at some energy $E > 0$ is equivalent with that of $\xi^2 + \tilde V/E - 1$ at energy 0, up to a time reparametrization by a factor $\sqrt{E}$. The same rescaling holds at the quantum level.

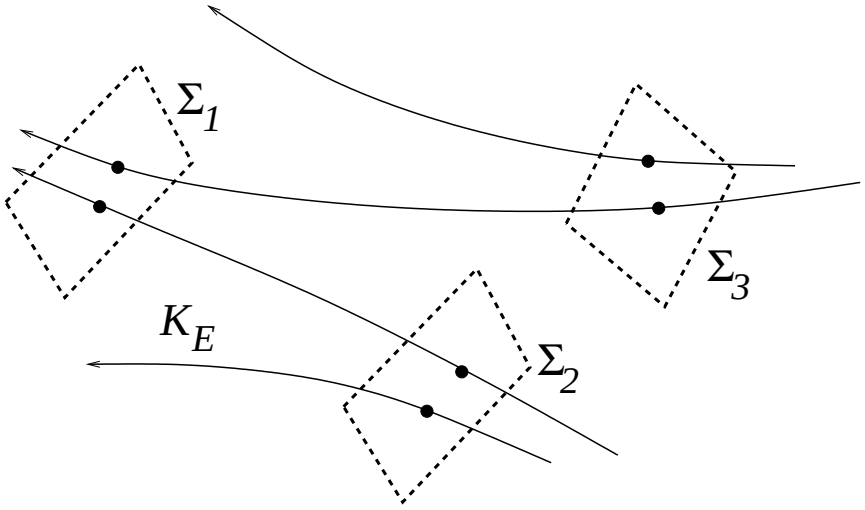

FIGURE 2. A schematic view of a Poincaré section $\mathbf{\Sigma} = \sqcup_j \Sigma_j$ for K_E inside $p^{-1}(E)$. The flow near K_E can be described by an ensemble of symplectomorphisms between different components Σ_j – see §2.2 for abstract assumptions and a discussion why they are satisfied when the flow is hyperbolic on K_E and K_E has topological dimension one. The latter condition simply means that the intersections of K_E with Σ_j's are totally disconnected.

this decomposition is preserved through the flow, and is characterized by the following properties:

$$\exists\, C > 0,\ \exists\, \lambda > 0, \quad \|d \exp tH_p(\rho) v\| \leq C\, e^{-\lambda |t|} \|v\|\,, \quad \forall\, v \in E^{\mp}_{\rho},\ \pm t > 0\,. \tag{1.6}$$

When K_0 is topologically one dimensional we can find a Poincaré section which reduces the flow near K_0 to a combination of symplectic transformations, called the Poincaré map F: see Fig.2 for a schematic illustration and §2.2 for a precise mathematical formulation. The *structural stability* of hyperbolic flows [24, Thm. 18.2.3] implies that the above properties will also hold for any energy E in a sufficientlys short interval $[-\delta, \delta]$ around $E = 0$, in particular the flow near K_E can be described through a Poincaré map F_E.

Under these assumptions, we are interested in semiclassically locating the resonances of the operator $P(h)$ in a neighbourhood of this energy interval:

$$\mathcal{R}(\delta, M_0, h) \stackrel{\text{def}}{=} [-\delta, \delta] + i[-M_0 h \log(1/h), M_0 h \log(1/h)]\,,$$

where δ, M_0 are independent of $h \in (0, 1]$. Here the $h \log(1/h)$-size neighbourhood is natural in view of results on resonance free regions in case of no trapping – see [26].

To characterize the resonances in $\mathcal{R}(\delta, M_0, h)$ we introduce a family of "quantum propagators" quantizing the Poincaré maps F_E.

Theorem 1. *Suppose that Φ^t is hyperbolic on K_0 and that K_0 is topologically one dimensional. More generally, suppose that $P(h)$ and Φ^t satisfy the assumptions of §2.1-§2.2.*

Then, for any $\delta > 0$ small enough and any $M_0 > 0$, there exists $h_0 > 0$ such that there exists a family of matrices,

$$\{M(z,h),\ z \in \mathcal{R}(\delta, M_0, h),\ h \in (0, h_0]\}\,,$$

holomorphic in the variable z, and satisfying

$$h^{-n+1}/C_0 \leq \operatorname{rank} M(z,h) \leq C_0 h^{-n+1}\,,\quad C_0 > 1\,,$$

such that for any $h \in (0, h_0]$, the zeros of

$$\zeta(z,h) \stackrel{\text{def}}{=} \det(I - M(z,h))\,,$$

give the resonances of $P(h)$ in $\mathcal{R}(\delta, M_0, h)$, with correct multiplicities.

The matrices $M(z,h)$ are open quantum maps associated with the Poincaré maps $F_{\mathrm{Re}\, z}$ described above: for any $L > 0$, there exist a family of h-Fourier integral operators, $\{\mathcal{M}(z,h)\}$, quantizing the Poincaré maps $F_{\mathrm{Re}\, z}$ (see §2.3.2 and §3.3), and projections Π_h (see §5.2.2) of ranks

$$h^{-n+1}/C_0 \leq \operatorname{rank} \Pi_h \leq C_0 h^{-n+1}\,,$$

such that

$$M(z,h) = \Pi_h \mathcal{M}(z,h) \Pi_h + \mathcal{O}(h^L)\,. \tag{1.7}$$

The statement about the multiplicities in the theorem says that

$$\begin{aligned} m_R(z) &= \frac{1}{2\pi i} \oint_z \frac{\zeta'(w)}{\zeta(w)} dw \\ &= -\frac{1}{2\pi i} \operatorname{tr} \oint_z (I - M(w))^{-1} M'(w) dw\,. \end{aligned} \tag{1.8}$$

A more precise version of Theorem 1, involving complex scaling and microlocally deformed spaces (see §3.4 and §3.5 respectively), will be given in Theorem 2 in §5.4. In particular Theorem 2 gives us a full control over both the cutoff resolvent of P, $\chi R(z)\chi$, and the full resolvent $(P_{\theta,R} - z)^{-1}$ of the complex scaled operator $P_{\theta,R}$, in terms of the family of matrices $M(z,h)$; for this reason, the latter is often called an *effective Hamiltonian* for P.

The mathematical applications of Theorem 1 and its refined version below include simpler proofs of fractal Weyl laws [43] and of the existence of resonance free strips [31]. The advantage lies in eliminating flows and reducing the dynamical analysis to that of maps. That provides an implicit second microlocalization without any technical complication (see [43, §5]). The key is a detailed understanding of the operators $\mathcal{M}(z,h)$ stated in the theorem.

Relation to semiclassical trace formulæ. The notation $\zeta(z,h)$ in the above theorem hints at the resemblance between this determinant and a *semiclassical zeta function*. Various such

functions have been introduced in the physics literature, to provide approximate ways of computing eigenvalues and resonances of quantum chaotic systems – see [47, 20, 10].

These semiclassical zeta functions are defined through formal manipulations starting from the Gutzwiller trace formula – see [42] for a mathematical treatment and references. They are given by sums, or Euler products, over periodic orbits where each term, or factor is an asymptotic series in powers of h. Most studies have concentrated on the zeta function defined by the principal term, without h-corrections, which strongly resembles the Selberg zeta function defined for surfaces of constant negative curvature. However, unlike the case of the Selberg zeta function, there is no known rigorous connection between the zeroes of the semiclassical zeta function and the exact eigenvalues or resonances of the quantum system, even in the semiclassical limit. Nevertheless, numerical studies have indicated that the semiclassical zeta function admits a strip of holomorphy beyond the axis of absolute convergence, and that its zeroes there are close to actual resonances [10, 48].

The traces of $M(z,h)^k$, $k \in \mathbb{N}$ admit semiclassical expressions as sums over periodic points, which leads to a *formal* representation of

$$\zeta(z,h) = \exp\Big\{ -\sum_{k=1}^{\infty} \frac{\operatorname{tr} M(z,h)^k}{k} \Big\}$$

as a product over periodic points. That gives it the same form as the semiclassical zeta functions in the physics literature. In this sense, the function $\zeta(z,h)$ is a resummation of these formal expressions. As will become clear from its construction below, the operator $M(z,h)$ is not unique: it depends on many choices which affect the remainder term $\mathcal{O}(h^L)$ in (1.7). However, the zeroes of $\zeta(z,h)$ in $\mathcal{R}(\delta, M_0, h)$ are the exact resonances of the quantum Hamiltonian.

Comments on quantum maps in the physics literature. Similar methods of analysis have been introduced in the theoretical physics literature devoted to quantum chaos. The classical case involves a reduction to the boundary for obstacle problems: when the obstacle consists of several strictly convex bodies, none of which intersects a convex hull of any other two bodies, the flow on the trapped set is hyperbolic. The reduction can then be made to boundaries of the convex bodies, resulting with operators quantization Poincaré maps – see Gaspard and Rice [16], and for a mathematical treatment Gérard [18], in the case of two convex bodies, and [28, §5.1], for the general case. Fig.3 illustrates the trapped set in the case of three discs. The semiclassical analogue of the two convex obstacle, a system with one closed hyperbolic orbit, was treated by Gérard and the second author in [19]. The approach of that paper was also based on the quantization of the Poincaré map near this orbit.

A reduction of a more complicated quantum system to a quantized Poincaré map was proposed in the physics literature. Bogomolny [4] studied a Schrödinger operator $P(h)$ with discrete spectrum, and constructed a family of energy dependent quantum transfer operators $T(E,h)$, which are integral operators acting on a hypersurface in the configuration

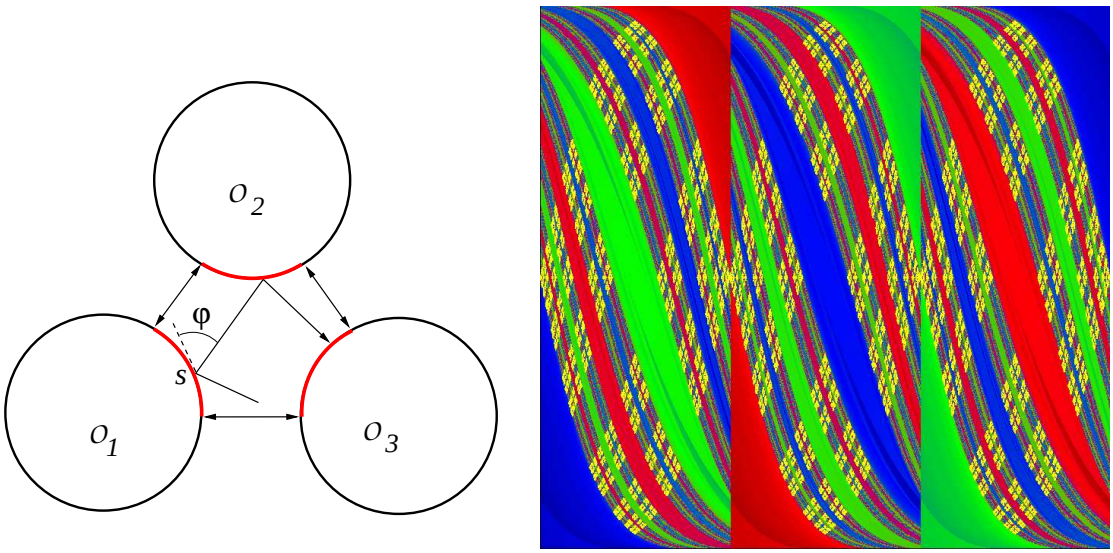

FIGURE 3. This figure, taken from [34], shows the case of symmetric three disc scattering problem (left), and the associated Poincaré section (right). The section is the union of the three coball bundles of circle arcs (in red) parametrized by s (the length parameter on the circle, horizontal axis), and $\cos\varphi$ (vertical axis), where φ is the angle between the velocity after impact and the tangent to the circle. Green, blue,red strips correspond to different regions of forward escape; they are bounded by components of the stable manifold. The trapped set, $\mathcal{T}$, shown in yellow, is the intersection of the latter with the unstable manifold.

space. These transfer operators are asymptotically unitary as $h \to 0$. The eigenvalues of $P(h)$ are then obtained, in the semiclassical limit, as the roots of the equation $\det(1 - T(E)) = 0$. Smilansky and co-workers derived a similar equation in the case of closed Euclidean 2-dimensional billiards [13], replacing $T(E)$ by a (unitary) scattering matrix $S(E)$ associated with the dual scattering problem. Prosen [35] generalized Bogomolny's approach to a nonsemiclassical setting. Bogomolny's method was also extended to study quantum scattering situations [17, 32].

Open quantum maps have first been defined in the quantum chaos literature as toy models for open quantized chaotic systems (see [29, §2.2], [30, §4.3] and references given there). They generalized the unitary quantum maps used to mimic bound chaotic systems [11]. Some examples of open quantum maps on the 2-dimensional torus or the cylinder, have been used as models in various physical settings: Chirikov's quantum standard map (or quantum kicked rotator) was first defined in the context of plasma physics, but then used as well to study ionization of atoms or molecules [9], as well as transport properties in mesoscopic quantum dots [46]. Other maps, like the open baker's map, were introduced

as clean model systems, for which the classical dynamics is well understood [36, 30]. The popularity of quantum maps mostly stems from the much simplified numerical study they offer, both at the quantum and classical levels, compared with the case of Hamiltonian flows or the corresponding Schrödinger operators. For instance, the distribution of resonances and resonant modes has proven to be much easier to study numerically for open quantum maps, than for realistic flows [7, 37, 29, 25, 27]. Precise mathematical definitions of quantum maps on the torus phase space are given in [29, §4.3-4.5].

Organization of the paper. In the remainder of this section we give assumptions on the operator P and on the corresponding classical dynamical system, in particular we introduce a Poincaré section $\mathbf{\Sigma}$ and map associated with the classical flow. We refer to results of Bowen and Walters [8] to show that these assumptions are satisfied if the trapped set supports a hyperbolic flow, and is topologically one dimensional, which is the case considered in Theorem 1.

In §3 we recall various tools needed in our proof: pseudodifferential calculus, the concept of semiclassical microlocalization, local h-Fourier integral operators associated to canonical tranformation (these appear in Theorem 1), complex scaling (used to define resonances as eigenvalues of nonselfadjoint Fredholm operators), microlocally deformed spaces, and Grushin problems used to define the effective Hamiltonians.

In §4 we follow a modified strategy of [42] and construct a microlocal Grushin problem associated with the Poincaré map on $\mathbf{\Sigma}$. No knowledge of that paper is a prerequisite but the self-contained discussion of the problem for the explicit case of S^1 given in [42, §2] can illuminate the complicated procedure presented here. In [42, §2] one finds the proof of the classical Poisson formula using a Grushin problem approach used here.

Because of the hyperbolic nature of the flow the microlocal Grushin problem cannot directly be made into a globally well-posed problem – see the remark at the end of §4. This serious difficulty is overcome in §5 by adding microlocal weights adapted to the flow. This and suitably chosen finite dimensional projections lead to a well posed Grushin problem, with an effective Hamiltonian essentially given by a quantization of the Poincaré map: this fact is summarized in Theorem 2, from which Theorem 1 is a simple corollary.

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2. Assumptions on the operator and on classical dynamics

Here we carefully state the needed assumptions on quantum and classical levels.

2.1. **Assumptions on the quantum Hamiltonian $P(h)$.** Our results apply to operators $P(h)$ satisfying general assumptions given in [31, §3.2] and [43, (1.5),(1.6)]. In particular, they apply to certain elliptic differential operators on manifolds X of the form

$$X = X_R \sqcup \bigsqcup_{j=1}^{J} \left(\mathbb{R}^n \setminus \overline{B_{\mathbb{R}^n}(0,R)} \right) ,$$

where $R > 0$ is large and X_R is a compact subset of X. The reader interested in this higher generality should consult those papers.

Here we will recall these assumptions only in the (physical) case of differential operators on $X = \mathbb{R}^n$. We assume that

$$P(h) = \sum_{|\alpha| \leq 2} a_\alpha(x,h)(hD_x)^\alpha , \tag{2.1}$$

where $a_\alpha(x,h)$ are bounded in $\mathcal{C}^\infty(\mathbb{R}^n)$, $a_\alpha(x,h) = a_\alpha^0(x) + \mathcal{O}(h)$ in $\mathcal{C}^\infty$, and $a_\alpha(x,h) = a_\alpha(x)$ is independent of h for $|\alpha| = 2$. Furthermore, for some $C_0 > 0$ the functions $a_\alpha(x,h)$ have holomorphic extensions to

$$\{x \in \mathbb{C}^n \ : \ |\operatorname{Re} x| > C_0 , \quad |\operatorname{Im} x| < |\operatorname{Re} x|/C_0\} , \tag{2.2}$$

they are bounded uniformly with respect to h, and $a_\alpha(x,h) = a_\alpha^0(x) + \mathcal{O}(h)$ on that set.

Let $P(x,\xi)$ denote the (full) Weyl symbol of the operator P, so that $P = P^w(x; hD; h)$, and assume

$$P(x,\xi;h) \to \xi^2 - 1 \tag{2.3}$$

when $x \to \infty$ in the set (2.2), uniformly with respect to $(\xi,h) \in K \times]0,1]$ for any compact set $K \Subset \mathbb{R}^n$ (here, and below, $\Subset$ means that the set on the left is a pre-compact subset of the set on the right). We also assume that P is classically elliptic:

$$p_2(x,\xi) \stackrel{\text{def}}{=} \sum_{|\alpha|=2} a_\alpha(x)\xi^\alpha \neq 0 \text{ on } T^*\mathbb{R}^n \setminus \{0\}, \tag{2.4}$$

and that P is self-adjoint on $L^2(\mathbb{R}^n)$ with domain $H^2(\mathbb{R}^n)$. The Schrödinger operator (1.1) corresponds to the choices $\sum_{|\alpha|=2} a_\alpha \xi^\alpha = |\xi|^2$, $a_\alpha \equiv 0$ for $|\alpha| = 1$, and $a_0(x) = V(x) - 1$. The assumption (2.3) show that we can also consider a slowly decaying potential, as long as it admits a holomorphic extension in (2.2).

2.2. **Dynamical Assumptions.** The dynamical assumptions we need roughly mean that the flow Φ^t on the energy shell $p^{-1}(0) \subset T^*X$ can be encoded by a Poincaré section, the boundary of which does not intersect the trapped set K_0. The abstract assumptions below are satisfied when the flow is hyperbolic on the trapped set which is assumed to be topologically one dimensional – see Proposition 2.1.

To state the assumption precisely, we notice that

$$p(x,\xi) = \sum_{|\alpha|\leq 2} a^0_\alpha(x)\xi^\alpha \tag{2.5}$$

is the semi-classical principal symbol of the operator $P(x, hD; h)$. We assume that the characteristic set of p (that is, the energy surface $p^{-1}(0)$) is a simple hypersurface:

$$dp \neq 0 \text{ on } p^{-1}(0). \tag{2.6}$$

Like in the introduction, we denote by

$$\Phi^t \stackrel{\text{def}}{=} \exp(tH_p) : T^*X \to T^*X$$

the flow generated by the Hamilton vector field H_p (see (1.2)).

Our assumptions on $p(x,\xi)$ ensure that, for E close to 0, we still have no fixed point in $p^{-1}(E)$, and the trapped set K_E (defined in (1.4)) is a compact subset of $p^{-1}(E)$.

We now assume that there exists a "nice" Poincaré section for the flow near K_0, namely finitely many compact contractible smooth hypersurfaces $\Sigma_k \subset p^{-1}(0)$, $k = 1, 2, \ldots, N$ with smooth boundaries, such that

$$\partial\Sigma_k \cap K_0 = \emptyset\,, \quad \Sigma_k \cap \Sigma_{k'} = \emptyset, \quad k \neq k', \tag{2.7}$$

$$H_p \text{ is transversal to } \Sigma_k \text{ uniformly up to the boundary,} \tag{2.8}$$

$$\begin{gathered}\text{For every } \rho \in K_0, \text{ there exist } \rho_- \in \Sigma_{j_-(\rho)}\,, \quad \rho_+ \in \Sigma_{j_+(\rho)} \\ \text{of the form } \rho_\pm = \Phi^{\pm t_\pm(\rho)}(\rho), \text{ with } 0 < t_\pm(\rho) \leq t_{\max} < \infty, \text{ such that} \\ \{\Phi^t(\rho);\, -t_-(\rho) < t < t_+(\rho),\ t \neq 0\} \cap \Sigma_k = \emptyset\,, \quad \forall\, k\,.\end{gathered} \tag{2.9}$$

We call Poincaré section the disjoint union

$$\mathbf{\Sigma} \stackrel{\text{def}}{=} \sqcup_{k=1}^N \Sigma_k\,.$$

The functions $\rho \mapsto \rho_\pm(\rho)$, $\rho \mapsto t_\pm(\rho)$ are uniquely defined ($\rho_\pm(\rho)$ will be called respectively the successor and predecessor of ρ). They remain well-defined for ρ in some neighbourhood of K_0 in $p^{-1}(0)$) and, in such a neighbourhood, depend smoothly on ρ away from $\mathbf{\Sigma}$. In order to simplify the presentation we also assume the successor of a point $\rho \in \Sigma_k$ belongs to a different component:

$$\text{If } \rho \in \Sigma_k \cap K_0 \text{ for some } k, \text{ then } \rho_+(\rho) \in \Sigma_\ell \cap K_0 \text{ for some } \ell \neq k. \tag{2.10}$$

The section can always be enlarged to guarantee that this condition is satisfied. For instance, for K_0 consisting of one closed orbit we only need one transversal component to have (2.7)-(2.8); to fulfill (2.10) a second component has to be added.

We recall that hypersurfaces in $p^{-1}(0)$ that are transversal to H_p are symplectic. In fact, a local application of Darboux's theorem (see for instance [23, §21.1]) shows that we can make a symplectic change of variables in which $p = \xi_n$ and $H_p = \partial_{x_n}$. If $\Sigma \subset \{\xi_n = 0\}$ is transversal to ∂_{x_n}, then $(x_1, \cdots x_{n-1}; \xi_1, \cdots, \xi_{n-1})$ can be chosen as coordinates on Σ. Since $\omega\restriction_{p^{-1}(0)} = \sum_{j=1}^{n-1} d\xi_j \wedge dx_j$, that means that $\omega\restriction_\Sigma$ is nondegenerate. The local normal form $p = \xi_n$ will be used further in the paper (in its quantized form).

The final assumption guarantees the absence of topological or symplectic peculiarities:

(2.11) There exists a set $\widetilde{\Sigma}_k \Subset T^*\mathbb{R}^{n-1}$ with smooth boundary, and a symplectic diffeomorphism $\kappa_k : \widetilde{\Sigma}_k \to \Sigma_k$ which is smooth up the boundary together with its inverse. We assume that κ_k extends to a neighbourhood of $\widetilde{\Sigma}_k$ in $T^*\mathbb{R}^n$.

In other words, there exist symplectic coordinate charts on Σ_k, taking values in $\widetilde{\Sigma}_k$.

The following result, due to Bowen and Walters [8], shows that our assumptions are realized in the case of 1-dimensional hyperbolic trapped sets.

Proposition 2.1. *Suppose that the assumptions of §2.1 hold, and that the flow $\Phi^t\restriction_{K_0}$ is hyperbolic in the standard sense of Eqs. (1.5,1.6). Then the existence of $\mathbf{\Sigma}$ satisfying* (2.7)-(2.11) *is equivalent with K_0 being topologically one dimensional.*

Remark. Bowen and Walters [8] show more, namely the fact that the sets $\{\Sigma_k\}$ can be chosen of small diameter, and constructed such that $\mathbf{\Sigma} \cap K_0$ forms a *Markov partition* for the Poincaré map. Small diameters ensures that (2.11) holds, while, as mentioned before, (2.10) can always be realized by adding some more components.

Proposition 2.1 shows that the assumptions of Theorem 1 imply the dynamical assumptions made in this section. The proof of [38, Appendix c] shows that the following example of "three-bumps potential",

$$P = -h^2\Delta + V(x) - 1\,, \quad x \in \mathbb{R}^2\,, \quad V(x) = 2\sum_{k=1}^{3} \exp(-R(x - x_k)^2)\,,$$
$$x_k = (\cos(2\pi k/3), \sin(2\pi k/3))\,,$$

satisfies our assumptions as long as $R > 1$ is large enough (see Fig. 1).

2.3. **The Poincaré map.** Here we will analyze the Poincaré map associated with the Poincaré section discussed in §2.2, and its semiclassical quantization.

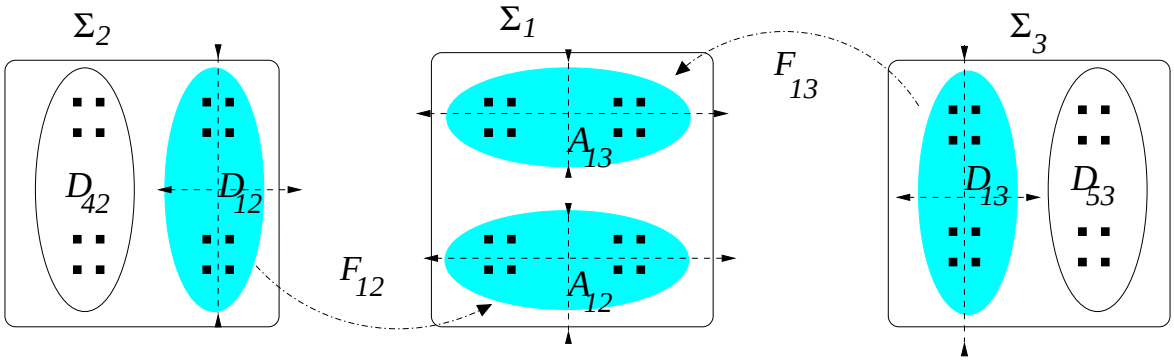

FIGURE 4. Schematic representation of the components F_{ik} of the Poincaré map between the sets D_{ik} and A_{ik} (horizontal/vertical ellipses). The reduced trapped set $\mathcal{T}_i$ is represented by the black squares. The unstable/stable directions of the map are the horizontal/vertical dashed lines.

2.3.1. *Classical analysis.* The assumptions in §2.2 imply the existence of a *symplectic relation*, the so-called Poincaré map on $\mathbf{\Sigma}$.

More precisely, let us identify Σ_k's with $\widetilde{\Sigma}_k$ using κ_k given in (2.11), so that the Poincaré section

$$\mathbf{\Sigma} = \bigsqcup_{k=1}^{N} \Sigma_k \simeq \bigsqcup_{k=1}^{N} \widetilde{\Sigma}_k \subset \bigsqcup_{k=1}^{N} T^*\mathbb{R}^{n-1}\,.$$

Let us call

$$\mathcal{T} \stackrel{\text{def}}{=} K_0 \cap \mathbf{\Sigma} = \bigsqcup_k \mathcal{T}_k \quad \text{the reduced trapped set.}$$

The map

$$f \;:\; \mathcal{T} \longrightarrow \mathcal{T}\,, \quad \rho \longmapsto f(\rho) \stackrel{\text{def}}{=} \rho_+(\rho)$$

(see the notation of (2.9)) is the Poincaré map for $\Phi^t{\restriction}_{K_0}$. It is a Lipschitz bijection. The decomposition $\mathcal{T} = \bigsqcup_k \mathcal{T}_k$ allows us to define the *arrival* and *departure* subsets of $\mathcal{T}$:

$$\mathcal{D}_{ik} \stackrel{\text{def}}{=} \{\rho \in \mathcal{T}_k \subset \Sigma_k \;:\; \rho_+(\rho) \in \mathcal{T}_i\} = \mathcal{T}_k \cap f^{-1}(\mathcal{T}_i)\,,$$

$$\mathcal{A}_{ik} \stackrel{\text{def}}{=} \{\rho \in \mathcal{T}_i \subset \Sigma_i \;:\; \rho_-(\rho) \in \mathcal{T}_k\} = \mathcal{T}_i \cap f(\mathcal{T}_k) = f(\mathcal{D}_{ik})\,,$$

For each k we call $J_+(k) \subset \{1, \dots, N\}$ the set of indices i such that $\mathcal{D}_{ik}$ is not empty (that is, for which $\mathcal{T}_i$ is a successor of $\mathcal{T}_k$). Conversely, the set $J_-(i)$ refers to the predecessors of $\mathcal{T}_i$.

Using this notation, the map f obviously decomposes into a family of Lipschitz bijections $f_{ik} : \mathcal{D}_{ik} \to \mathcal{A}_{ik}$. Similarly to the maps $\rho_\pm$, each f_{ik} can be extended to a neighbourhood of $\mathcal{D}_{ik}$, to form a family of local smooth symplectomorphisms

$$F_{ik} \;:\; D_{ik} \longrightarrow\; F_{ik}(D_{ik}) \stackrel{\text{def}}{=} A_{ik}\,,$$

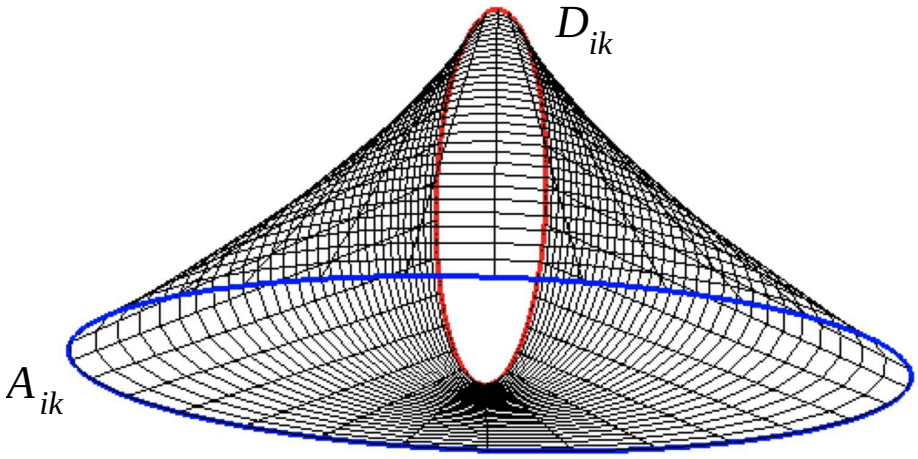

FIGURE 5. Trajectories linking the boundaries of the departure set $D_{ik} \subset \Sigma_k$ and the arrival set $A_{ik} \subset \Sigma_i$. Note the stretching and contraction implied by hyperbolicity. These trajectories and $D_{ik} \cup A_{ik}$ form the boundary of the tube T_{ik} defined by (2.12).

where D_{ik} (resp. A_{ik}) is a neighbourhood of $\mathcal{D}_{ik}$ in Σ_k (resp. a neighbourhood of $\mathcal{A}_{ik}$ in Σ_i). Since our assumption on K_0 is equivalent with the fact that the reduced trapped set $\mathcal{T}$ is *totally disconnected*, we may assume that the sets $\{D_{ik}\}_{i\in J_+(k)}$ (resp. the sets $\{A_{ik}\}_{k\in J_-(i)}$) are mutually disjoint. We will call

$$D_k \stackrel{\text{def}}{=} \sqcup_{i\in J_+(k)} D_{ik}, \qquad A_i \stackrel{\text{def}}{=} \sqcup_{k\in J_-(k)} A_{ik}\,.$$

Notice that, for any index i, the sets D_i, A_i both contain the set $\mathcal{T}_i$, so they are not disjoint.

We will also define the *tubes* $T_{ik} \subset T^*X$ containing the trajectories between D_{ik} and A_{ik}:

$$T_{ik} \stackrel{\text{def}}{=} \{\Phi^t(\rho),\, :\, \rho \in D_{ik},\ 0 \le t \le t_+(\rho)\}\,. \tag{2.12}$$

See Fig. 4 for a sketch of these definitions, and Fig. 5 for an artistic view of T_{ik}

The maps F_{ik} will be grouped into the symplectic bijection F between $\bigsqcup_k D_k$ and $\bigsqcup_k A_k$. We will also call F the Poincaré map, which can be viewed as a symplectic relation on $\mathbf{\Sigma}$. We will sometimes identify the map F_{ik} with its action on subsets of $T^*\mathbb{R}^{n-1}$.

$$\widetilde{F}_{ik} = \kappa_i^{-1} \circ F_{ik} \circ \kappa_k : \widetilde{D}_{ik} \longrightarrow \widetilde{A}_{ik}\,, \quad \widetilde{D}_{ik} \stackrel{\text{def}}{=} \kappa_k^{-1}(D_{ik})\,, \quad \widetilde{A}_{ik} \stackrel{\text{def}}{=} \kappa_i^{-1}(A_{ik})\,.$$

Using the continuity of the flow Φ^t, we will show in §4.1.1 that the above structure can be continuously extended to a small energy interval $z \in [-\delta, \delta]$. The Poincaré map for the flow in $p^{-1}(z)$ will be denoted by $F_z = (F_{ik,z})_{1\le i,k\le N}$ (see §4.1.1 for details).

In the case of K_0 supporting a hyperbolic flow, a structural stability of K_z holds in a stronger sense: the flows $\Phi^t\restriction_{K_z}$ and $\Phi^t\restriction_{K_0}$ are actually orbit-conjugate (that is, conjugate up to time reparametrization) by a homeomorphism close to the identity. [24, Thm. 18.2.3].

2.3.2. *Quantization of the Poincaré map.* In this section we make more explicit the operator $\mathcal{M}(z,h)$ used in Theorem 1. The semiclassical tools we are using will be recalled in §3.

Let us first focus on a single component $F_{ik} : D_{ik} \to A_{ik}$ of the Poincaré map. A quantization of the symplectomorphism F_{ik} (more precisely, of its pullback $\widetilde{F}_{ik}$) is a semiclassical (or h-) Fourier integral operator, that is a family of operators $\mathcal{M}_{ik}(h) : L^2(\mathbb{R}^{n-1}) \to L^2(\mathbb{R}^{n-1})$, $h \in (0,1]$, whose semiclassical wavefront set satisfies

$$\mathrm{WF}'_h(\mathcal{M}_{ik}) \Subset \widetilde{A}_{ik} \times \widetilde{D}_{ik}\,, \tag{2.13}$$

and which is associated with the symplectomorphism $\widetilde{F}_{ik}$. (h-FIOs are defined in §3.3, and WF'_h is defined in (3.9) below).

Being associated to the symplectic map $\widetilde{F}_{ik}$ means the following thing: for any $a \in \mathcal{C}^\infty_{\mathrm{c}}(\widetilde{A}_{ik})$, the quantum operator $\mathrm{Op}^w_h(a)$ transforms as follows when conjugated by $\mathcal{M}_{ik}(h)$

$$\mathcal{M}_{ik}(h)^* \mathrm{Op}^w_h(a) \mathcal{M}_{ik}(h) = \mathrm{Op}^w_h(\alpha_{ik} \widetilde{F}_{ik}^* a) + h^{1-2\delta}\,\mathrm{Op}^w_h(b)\,, \tag{2.14}$$

where the symbol $\alpha_{ik} \in S_\delta(T^*\mathbb{R}^{n-1})$ is independent of a, $\alpha_{ik} = 1$ on some neighbourhood of $\mathcal{T}_k$ in Σ_k, and $b \in S_\delta(T^*\mathbb{R}^{n-1})$, for every $\delta > 0$. Here Op^w_h denotes the semiclassical Weyl quantization on $\mathbb{R}^{2(n-1)}$ (see eq.(3.1)), and $S_\delta(T^*\mathbb{R}^{n-1})$ is the symbol class defined in §3.1. The necessity to have $\delta > 0$ in (2.14) comes from the slightly exotic nature of our Fourier integral operator, due to the presence of some mild exponential weights – see §3.5 below.

The property (2.14), which is a form of *Egorov's theorem*, characterizes $\mathcal{M}_{ik}(h)$ as a semiclassical Fourier integral operator associated with $\widetilde{F}_{ik}$ (see [42, Lemma 2] and [15, §10.2] for that characterization).

We can then group together the $\mathcal{M}_{ik}(h)$ into a single operator-valued matrix (setting $\mathcal{M}_{ik}(h) = 0$ when $i \notin J_+(k)$):

$$\mathcal{M}(h)\ :\ L^2(\mathbb{R}^{n-1})^N \longrightarrow L^2(\mathbb{R}^{n-1})^N\,, \quad \mathcal{M}(h) = \big(\mathcal{M}_{ik}(h)\big)_{1\leq i,k\leq N}\,.$$

We call this $\mathcal{M}(h)$ a quantization of the Poincaré map F.

The operators $\mathcal{M}(z,h)$ in Theorem 1 will also holomorphically depend on $z \in \mathcal{R}(\delta, M_0, h)$, such that for each $z \in \mathcal{R}(\delta, M_0, h) \cap \mathbb{R}$ the family $(\mathcal{M}(z,h))_{h\in(0,1]}$ is an h-Fourier integral operator of the above sense.

Comment on notation. Most of the estimates in this paper include error terms of the type $\mathcal{O}(h^\infty)$, which is natural in all microlocal statements. To simplify the notation we

adopt the following convention (except in places where it could lead to confusion):

$$\begin{aligned} u \equiv v \iff \|u-v\| = \mathcal{O}(h^\infty)\|u\|\,, \\ \|Su\| \lesssim \|Tu\| + \|v\| \iff \|Su\| \le \mathcal{O}(1)(\|Tu\| + \|v\|) + \mathcal{O}(h^\infty)\|u\|\,, \end{aligned} \tag{2.15}$$

with norms appropriate to context. Since most estimates involve functions u microlocalized to compact sets, in the sense that, $u - \chi(x,hD)u \in h^\infty \mathscr{S}(\mathbb{R}^n)$, for some $\chi \in \mathcal{C}^\infty_{\rm c}(T^*\mathbb{R}^n)$, the norms are almost exclusively L^2 norms, possibly with microlocal weights described in §3.5.

The notation $u = \mathcal{O}_V(f)$ means that $\|u\|_V = \mathcal{O}(f)$, and the notation $T = \mathcal{O}_{V\to W}(f)$ means that $\|Tu\|_W = \mathcal{O}(f)\|u\|_V$. Also, the notation

$$\mathrm{neigh}(A,B) \quad \text{for } A \subset B,$$

means an open neighbourhood of the set A inside the set B.

Starting with §4, we denote the Weyl quantization of a symbol a by the same letter $a = a^w(x,hD)$. This makes the notation less cumbersome and should be clear from the context.

Finally, we warn the reader that from §4 onwards the original operator P is replaced by the complex scaled operator $P_{\theta,R}$, whose construction is recalled in §3.4. Because of the formula (3.13), that does not affect the results formulated in this section.

3. Preliminaries

In this section we present background material and references needed for the proofs of the theorems.

3.1. **Semiclassical pseudodifferential calculus.**

We start by defining a rather general class of symbols (that is, h-dependent functions) on the phase space $T^*\mathbb{R}^d$. For any $\delta \in [0,1/2]$ and $m,k \in \mathbb{R}$, let

$$\begin{aligned} S^{m,k}_\delta(T^*\mathbb{R}^d) = \big\{ a \in \mathcal{C}^\infty(T^*\mathbb{R}^d \times (0,1]) : \forall\, \alpha \in \mathbb{N}^d\,,\ \beta \in \mathbb{N}^d\,,\ \exists\, C_{\alpha\beta} > 0\,, \\ |\partial_x^\alpha \partial_\xi^\beta a(x,\xi;h)| \le C_{\alpha\beta} h^{-k-\delta(|\alpha|+|\beta|)} \langle \xi \rangle^{m-|\beta|} \big\}\,. \end{aligned}$$

where $\langle \xi \rangle \stackrel{\rm def}{=} (1+|\xi|^2)^{\frac12}$.

Most of the time we will use the class with $\delta = 0$ in which case we drop the subscript. When $m = k = 0$, we simply write $S(T^*\mathbb{R}^d)$ or S for the class of symbols. In the paper $d = n$ (the dimension of the physical space) or $d = n-1$ (half the dimension of the Poincaré section), and occasionally (as in (2.13)) $d = 2n-2$, depending on the context.

The quantization map, in its different notational guises, is defined as follows

$$\begin{aligned} a^w u = \mathrm{Op}_h^w(a)u(x) &= a^w(x,hD)u(x) \\ &\stackrel{\mathrm{def}}{=} \frac{1}{(2\pi h)^d}\int\!\!\int a\big(\frac{x+y}{2},\xi\big)e^{i\langle x-y,\xi\rangle/h}u(y)dyd\xi\,, \end{aligned} \tag{3.1}$$

and we refer to [12, Chapter 7] for a detailed discussion of semiclassical quantization (see also [40, Appendix]), and to [15, Appendix D.2] for the semiclassical calculus for the symbol classes given above.

We denote by $\Psi_\delta^{m,k}(\mathbb{R}^d)$ or $\Psi^{m,k}(\mathbb{R}^d)$ the corresponding classes of pseudodifferential operators. The quantization formula (3.1) is bijective: each operator $A \in \Psi_\delta^{m,k}(\mathbb{R}^d)$ is exactly represented by a unique (full) symbol $a(x,\xi;h)$. It is useful to consider only certain equivalence classes of this full symbol, thus defining a *principal symbol map* – see [15, Chapter 8]:

$$\sigma_h \;:\; \Psi_\delta^{m,k}(\mathbb{R}^d) \;\longrightarrow S_\delta^{m,k}(T^*\mathbb{R}^d)/S_\delta^{m-1,k-1+2\delta}(T^*\mathbb{R}^d)\,.$$

The combination $\sigma_h \circ \mathrm{Op}_h^w$ is the natural projection from $S_\delta^{m,k}$ onto $S_\delta^{m,k}/S_\delta^{m-1,k-1+2\delta}$. The main property of this principal symbol map is to "restore commutativity":

$$\sigma_h(A\circ B) = \sigma_h(A)\sigma_h(B)\,.$$

Certain symbols in $S^{m,0}(T^*\mathbb{R}^d)$ admit an asymptotic expansion in powers of h,

$$a(x,\xi;h) \sim \sum_{j\geq 0} h^j\, a_j(x,\xi), \qquad a_j \in S^{m-j,0} \text{ independent of } h\,, \tag{3.2}$$

such symbols (or the corresponding operator) are called *classical*, and make up the subclass $S_{cl}^{m,0}(T^*\mathbb{R}^d)$ (the corresponding operator class is denoted by $\Psi_{cl}^{m,0}(\mathbb{R}^d)$). For any operator $A \in \Psi_{cl}^{m,0}(\mathbb{R}^d)$, its principal symbol $\sigma_h(A)$ admits as representative the h-independent function $a_0(x,\xi)$, first term in (3.2). The latter is also usually called the principal symbol of a.

In §3.5 we will introduce a slightly different notion of *leading symbol*, adapted to a subclass of symbols in $S(T^*\mathbb{R})$ larger than $S_{cl}(T^*\mathbb{R}^d)$.

The semiclassical Sobolev spaces, $H_h^s(\mathbb{R}^d)$ are defined using the semiclassical Fourier transform, $\mathcal{F}_h$:

$$\|u\|_{H_h^s}^2 \stackrel{\mathrm{def}}{=} \int_{\mathbb{R}^d}\langle\xi\rangle^{2s}|\mathcal{F}_h u(\xi)|^2 d\xi\,, \quad \mathcal{F}_h u(\xi) \stackrel{\mathrm{def}}{=} \frac{1}{(2\pi h)^{d/2}}\int_{\mathbb{R}^d} u(x)e^{-i\langle x,\xi\rangle/h}dx\,. \tag{3.3}$$

Unless otherwise stated all norms in this paper, $\|\bullet\|$, are L^2 norms.

We recall that the operators in $\Psi(\mathbb{R}^d)$ are bounded on L^2 uniformly in h, and that they can be characterized using commutators by Beals's Lemma (see [12, Chapter 8] and [43, Lemma 3.5] for the S_δ case):

$$A \in \Psi_\delta(X) \iff \left\{ \begin{array}{l} \|\operatorname{ad}_{\ell_N}\cdots\operatorname{ad}_{\ell_1} A\|_{L^2\to L^2} = \mathcal{O}(h^{(1-\delta)N}) \\ \text{for linear functions } \ell_j(x,\xi) \text{ on } \mathbb{R}^d\times\mathbb{R}^d, \end{array}\right. \tag{3.4}$$

where $\mathrm{ad}_B A = [B, A]$.

For a given symbol $a \in S(T^*\mathbb{R}^d)$ we follow [42] and say that the *essential support* is contained in a given compact set $K \Subset T^*\mathbb{R}^d$,

$$\text{ess-supp}_h\, a \subset K \Subset T^*\mathbb{R}^d\,,$$

if and only if

$$\forall\, \chi \in S(T^*\mathbb{R}^d)\,,\ \mathrm{supp}\,\chi \cap K = \emptyset \implies \chi\, a \in h^\infty \mathscr{S}(T^*\mathbb{R}^d)\,.$$

The essential support is then the intersection of all such K's.

Here $\mathscr{S}$ denotes the Schwartz space. For $A \in \Psi(\mathbb{R}^d)$, $A = \mathrm{Op}_h^w(a)$, we call

$$\mathrm{WF}_h(A) = \text{ess-supp}_h\, a\,. \tag{3.5}$$

the semiclassical wavefront set of A. (In this paper we are concerned with a purely semiclassical theory and will only need to deal with *compact* subsets of $T^*\mathbb{R}^d$. Hence, we won't need to define noncompact essential supports).

3.2. **Microlocalization.** We will also consider spaces of L^2 functions (strictly speaking, of h-dependent families of functions) which are *microlocally concentrated* in an open set $V \Subset T^*\mathbb{R}^d$:

$$\begin{aligned} H(V) \stackrel{\mathrm{def}}{=} \{u = (u(h) \in L^2(\mathbb{R}^d))_{h\in(0,1]},&\ \text{ such that} \\ \exists\, C_u > 0\,,\ \forall\, h \in (0,1]\,,&\quad \|u(h)\|_{L^2(\mathbb{R}^d)} \le C_u\,, \\ \exists\, \chi \in \mathcal{C}_{\mathrm{c}}^\infty(V)\,,&\quad \chi^w(x, hD_x)\, u(h) = u(h) + \mathcal{O}_{\mathscr{S}}(h^\infty)\}\,. \end{aligned} \tag{3.6}$$

The semiclassical wave front set of $u \in H(V)$ is defined as:

$$\mathrm{WF}_h(u) = T^*\mathbb{R}^d \setminus \big\{(x,\xi) \in T^*\mathbb{R}^d\ :\ \exists\, a \in S(T^*\mathbb{R}^d)\,,\ \ a(x,\xi) = 1\,,\ \|a^w\, u\|_{L^2} = \mathcal{O}(h^\infty)\big\}\,. \tag{3.7}$$

The condition (3.7) can be equivalently replaced with $a^w\, u = \mathcal{O}_{\mathscr{S}}(h^\infty)$, since we may always take $a \in \mathscr{S}(T^*\mathbb{R}^d)$. This set obviously satisfies $\mathrm{WF}_h(u) \Subset V$. Notice that the condition does not characterize the individual functions $u(h)$, but the full sequence as $h \to 0$.

We will say that an h-dependent family of operators $T = (T(h))_{h\in(0,1]} : \mathscr{S}(\mathbb{R}^d) \to \mathscr{S}'(\mathbb{R}^k)$ is *semiclassically tempered* if there exists $L \ge 0$ such that

$$\|\langle x\rangle^{-L} T(h) u\|_{H_h^{-L}} \le C\, h^{-L} \|\langle x\rangle^L u\|_{H_h^L}\,, \quad h \in (0,1)\,, \quad \langle x\rangle \stackrel{\mathrm{def}}{=} (1+x^2)^{1/2}\,.$$

Such a family of operators is *microlocally defined* on V if one only specifies (or considers) its action on states $u \in H(V)$, modulo $\mathcal{O}_{\mathscr{S}'\to\mathscr{S}}(h^\infty)$. For instance, T is said to be asymptotically uniformly bounded on $H(V)$ if

$$\exists\, C_T > 0\ \forall\, u \in H(V)\ \exists\, h_{T,u} > 0\,,\ \forall\, h \in (0, h_{T,u})\,, \quad \|T(h)u(h)\|_{L^2(\mathbb{R}^k)} \le C_T\, C_u\,. \tag{3.8}$$

Two tempered operators T, T' are said to be microlocally equivalent on V, iff for any $u \in H(V)$ they satisfy $\|(T - T')u\|_{L^2(\mathbb{R}^k)} = \mathcal{O}(h^\infty)$; equivalently, for any $\chi \in \mathcal{C}_{\mathrm{c}}^\infty(V)$, $\|(T - T')\chi^w\|_{L^2\to L^2} = \mathcal{O}(h^\infty)$.

If there exists an open subset $W \Subset T^*\mathbb{R}^k$ and $L \in \mathbb{R}$ such that T maps any $u \in H(V)$ into a state $Tu \in h^{-L} H(W)$, then we will write

$$T = T(h) \ : \ H(V) \longrightarrow H(W)\,,$$

and we say that T is defined microlocally in $W \times V$.

For such operators, we may define only the part of the (twisted) wavefront set which is inside $W \times V$:

(3.9)

$$\begin{aligned}\mathrm{WF}'_h(T) \cap (W \times V) \stackrel{\mathrm{def}}{=} (W \times V) \setminus \{(\rho', \rho) \in W \times V \ : \ \exists\, a \in S(T^*\mathbb{R}^d),\ b \in S(T^*\mathbb{R}^k)\,,\\ a(\rho) = 1\,, b(\rho') = 1\,, \quad b^w\, T\, a^w = \mathcal{O}_{L^2 \to L^2}(h^\infty)\}\,.\end{aligned}$$

If $\mathrm{WF}'_h(T) \cap (W \times V) \Subset W \times V$, there exists a family of tempered operators $\widetilde{T}(h) : L^2 \to L^2$, such that T and $\widetilde{T}$ are microlocally equivalent on V, while $\widetilde{T}$ is $\mathcal{O}_{\mathscr{S}' \to \mathscr{S}}(h^\infty)$ outside V, that is

$$\widetilde{T} \circ a^w = \mathcal{O}(h^\infty) : \mathscr{S}'(\mathbb{R}^d) \to \mathscr{S}(\mathbb{R}^k)\,,$$

for all $a \in S(T^*\mathbb{R}^d)$ such that $\operatorname{supp} a \cap V = \emptyset$. This family, which is unique modulo $\mathcal{O}_{\mathscr{S}' \to \mathscr{S}}(h^\infty)$, is an extension of the microlocally defined $T(h)$, see [15, Chapter 10].

3.3. **Local h-Fourier integral operators.** We first present a a class of globally defined h-Fourier integral operators following [42] and [15, Chapter 10]. This global definition will then be used to define Fourier integral operators microlocally.

Let $(A(t))_{t \in [-1,1]}$ be a smooth family of selfadjoint pseudodifferential operators,

$$\forall t \in [-1,1], \quad A(t) = \mathrm{Op}^w_h(a(t))\,, \quad a(t) \in S_{cl}(T^*\mathbb{R}^d; \mathbb{R})\,,$$

where the dependence on t is smooth, and $\mathrm{WF}_h(A(t)) \subset \Omega \Subset T^*\mathbb{R}^d$, in the sense of (3.5). We then define a family of operators

$$\begin{aligned} U(t) \ : \ L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)\,,\\ hD_t U(t) + U(t)A(t) = 0\,. \ \ U(0) = Id\,.\end{aligned} \tag{3.10}$$

An example is given by $A(t) = A = a^w$, independent of t, in which case $U(t) = \exp(-itA/h)$.

The family $(U(t))_{t \in [-1,1]}$ is an example of a family of unitary h-Fourier integral operators, associated to the family of canonical transformations $\kappa(t)$ generated by the (time-dependent) Hamilton vector fields $H_{a_0(t)}$. Here the real valued function $a_0(t)$ is the principal symbol of $A(t)$ (see (3.2)), and the canonical transformations $\kappa(t)$ are defined through

$$\frac{d}{dt}\kappa(t)(\rho) = (\kappa(t))_*(H_{a_0(t)}(\rho))\,, \quad \kappa(0)(\rho) = \rho\,, \quad \rho \in T^*\mathbb{R}^d\,.$$

If $U = U(1)$, say, and the graph of $\kappa(1)$ is denoted by C, we conform to the usual notation and write

$$U \in I^0_h(\mathbb{R}^d \times \mathbb{R}^d; C')\,, \quad \text{where} \quad C' = \{(x, \xi; y, -\eta) \ : \ (x, \xi) = \kappa(y, \eta)\}\,.$$

Here the twisted graph C' is a Lagrangian submanifold of $T^*(\mathbb{R}^d \times \mathbb{R}^d)$.

In words, U is a unitary h-Fourier integral operator associated to the canonical graph C (or the symplectomorphism $\kappa(1)$ defined by this graph). Locally all unitary h-Fourier integral operators associated to canonical graphs are of the form $U(1)$, since each local canonical transformation with a fixed point can be deformed to the identity, see [42, Lemma 3.2]. For any $\chi \in S(T^*\mathbb{R}^d)$, the operator $U(1)\,\chi^w$, with $\chi \in S(T^*\mathbb{R}^d)$ is still a (nonunitary) h-Fourier integral operator associated with C. The class formed by these operators, which are said to "quantize" the symplectomorphism $\kappa = \kappa(1)$, depends only on κ, and not on the deformation path from the identity to κ. This can be seen from the Egorov characterization of Fourier integral operators – see [42, Lemma 2] or [15, §10.2].

Let us assume that a symplectomorphism κ is defined only near the origin, which is a fixed point. It is always possible to *locally* deform κ to the identity, that is construct a family of symplectomorphisms $\kappa(t)$ on $T^*\mathbb{R}^d$, such that $\kappa(1)$ coincides with κ in some neighbourhood V of the origin [42, Lemma 3.2]. If we apply the above construction to get the unitary operator $U(1)$, and use a cutoff $\chi \in S(T^*\mathbb{R}^d)$, $\mathrm{supp}\,\chi \Subset V$, then the operator $U(1)\chi^w$ is an h-Fourier integral operator associated with the local symplectomorphism $\kappa\restriction V$. Furthermore, if there exists a neighbourhood $V' \Subset V$ such that $\chi\restriction V' \equiv 1$, then $U(1)\chi^w$ is microlocally unitary inside V'.

For an open set $V \Subset \mathbb{R}^d$ and κ a symplectomorphism defined in a neighbourhood $\widetilde{V}$ of V, we say that a tempered operator T satisfying

$$T \,:\, H(\widetilde{V}) \longrightarrow H(\kappa(\widetilde{V}))\,,$$

is a microlololocally defined *unitary* h-Fourier integral operator in V, if any point $\rho \in V$ has a neighbourhood $V_\rho \subset V$ such that

$$T \,:\, H(V_\rho) \longrightarrow H(\kappa(V_\rho))$$

is equivalent to a unitary h-Fourier integral operator associated with $\kappa\restriction V_\rho$, as defined by the above procedure. The microlocally defined operators can also be obtained by oscillatory integral constructions — see for instance [31, §4.1] for a brief self-contained presentation.

An example which will be used in §4.1 is given by the standard conjugation result, see [42, Proposition 3.5] or [15, Chapter 10] for self-contained proofs. Suppose that $P \in \Psi^{m,0}_{cl}(\mathbb{R}^d)$ is a semi-classical *real principal type operator*, namely its principal symbol $p = \sigma_h(P)$ is real, independent of h, and the Hamilton flow it generates has no fixed point at energy zero: $p = 0 \Longrightarrow dp \neq 0$. Then for any $\rho_0 \in p^{-1}(0)$, there exists a canonical transformation, κ, mapping $V = \mathrm{neigh}((0,0), T^*\mathbb{R}^d)$ to $\kappa(V) = \mathrm{neigh}(\rho_0, T^*\mathbb{R}^d)$, with $\kappa(0,0) = \rho_0$ and

$$p \circ \kappa(\rho) = \xi_n(\rho) \quad \rho \in V\,,$$

and a unitary microlocal h-Fourier integral operator $U : H(V) \to H(\kappa(V))$ associated to κ, such that

$$U^* P U \equiv h D_{x_n} : H(V) \to H(V)\,.$$

While ξ_n is the (classical) normal form for the Hamiltonian p in V, the operator hD_{x_n} is the quantum normal form for P, microlocally in V.

The definition of h-Fourier integral operators can be generalized to graphs C associated with certain *relations* between phase spaces of possibly different dimensions. Namely, if a relation $C \subset T^*\mathbb{R}^d \times T^*\mathbb{R}^k$ is such that its twist

$$C' = \{(x,\xi;y,-\eta)\,;\ (x,\xi;y,-\eta') \in C\}$$

is a Lagrangian submanifold of $T^*(\mathbb{R}^d \times \mathbb{R}^k)$, then one can associate with this relation (microlocally in some neighbourhood) a family of h-Fourier integral operators $T : L^2(\mathbb{R}^k) \mapsto L^2(\mathbb{R}^d)$ [2, Definition 4.2]. This class of operators is denoted by $I^r_h(\mathbb{R}^d \times \mathbb{R}^k; C')$, with $r \in \mathbb{R}$. The important property of these operators is that their composition is still a Fourier integral operator associated with the composed relations.

3.4. **Complex scaling.** We briefly recall the complex scaling method of Aguilar-Combes [1] – see [41],[39], and references given there. In most of this section, this scaling is independent of h, and allows to obtain the resonances (in a certain sector) for all operators $P(h)$, $h \in (0,1]$, where $P(h)$ satisfies the assumptions of §2.1.

For any $0 \le \theta \le \theta_0$ and $R > 0$, we define $\Gamma_{\theta,R} \subset \mathbb{C}^n$ to be a totally real deformation of $\mathbb{R}^n$, with the following properties:

$$\begin{aligned} \Gamma_\theta \cap B_{\mathbb{C}^n}(0,R) &= B_{\mathbb{R}^n}(0,R)\,, \\ \Gamma_\theta \cap \mathbb{C}^n \setminus B_{\mathbb{C}^n}(0,2R) &= e^{i\theta}\mathbb{R}^n \cap \mathbb{C}^n \setminus B_{\mathbb{C}^n}(0,2R)\,, \\ \Gamma_\theta = \{x + i f_{\theta,R}(x)\ :\ x \in \mathbb{R}^n\}\,,&\quad \partial_x^\alpha f_{\theta,R}(x) = \mathcal{O}_\alpha(\theta)\,. \end{aligned} \tag{3.11}$$

If R is large enough, the coefficients of P continue analytically outside of $B(0,R)$, and we can define a dilated operator:

$$P_{\theta,R} \stackrel{\text{def}}{=} \widetilde{P}\restriction_{\Gamma_{\theta,R}}\,,\quad P_{\theta,R}u = \widetilde{P}(\tilde{u})\restriction_{\Gamma_{\theta,R}}\,,$$

where $\widetilde{P}$ is the holomorphic continuation of the operator P, and $\tilde{u}$ is an almost analytic extension of $u \in \mathcal{C}^\infty_c(\Gamma_{\theta,R})$ from the totally real submanifold $\Gamma_{\theta,R}$ to $\mathrm{neigh}(\Gamma_{\theta,R},\mathbb{C}^n)$.

The operator $P_{\theta,R} - z$ is a Fredholm operator for $2\theta > \arg(z+1) > -2\theta$. That means that the resolvent, $(P_{\theta,R}-z)^{-1}$, is meromorphic in that region, the spectrum of $P_{\theta,R}$ in that region is independent of θ and R, and consists of the *quantum resonances* of P.

To simplify notations we identify $\Gamma_{\theta,R}$ with $\mathbb{R}^n$ using the map, $S_{\theta,R} : \Gamma_{\theta,R} \to \mathbb{R}^n$,

$$\Gamma_{\theta,R} \ni x \longmapsto \mathrm{Re}\, x \in \mathbb{R}^n\,, \tag{3.12}$$

and using this identification, consider $P_{\theta,R}$ as an operator on $\mathbb{R}^n$, defined by $(S^{-1}_{\theta,R})^* P_{\theta,R} S^*_{\theta,R}$ (here S^* means the pullback through S) We note that this identificaton satisfies

$$C^{-1}\,\|u\|_{L^2(\mathbb{R}^n)} \le \|S^*_{\theta,R}u\|_{L^2(\Gamma_{\theta,R})} \le C\,\|u\|_{L^2(\mathbb{R}^n)}\,,$$

with C independent of θ if $0 \le \theta \le \theta_0$.

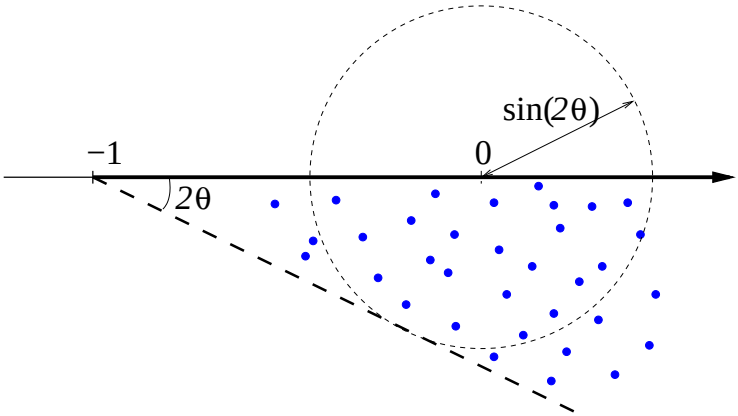

FIGURE 6. The complex scaling in the z-plane used in this paper.

The identification of the eigenvalues of $P_{\theta,R}$ with the poles of the meromorphic continuation of

$$(P-z)^{-1} \,:\, \mathcal{C}^\infty_{\rm c}(\mathbb{R}^n) \;\longrightarrow\; \mathcal{C}^\infty(\mathbb{R}^n)$$

from $\{\operatorname{Im} z > 0\}$ to $D(0,\sin(2\theta))$, and in fact, the existence of such a continuation, follows from the following formula (implicit in [39], and discussed in [45]): if $\chi \in \mathcal{C}^\infty_{\rm c}(\mathbb{R}^n)$, $\operatorname{supp}\chi \Subset B(0,R)$, then

$$\chi(P_{\theta,R}-z)^{-1}\chi = \chi(P-z)^{-1}\chi\,. \tag{3.13}$$

This is initially valid for $\operatorname{Im} z > 0$ so that the right hand side is well defined, and then by analytic continuation in the region where the left hand side is meromorphic. The reason for the Fredholm property of $(P_{\theta,R}-z)$ in $D(0,\sin(2\theta))$ comes from the properties of the principal symbol of $P_{\theta,R}$ – see Fig. 6. Here for convenience, and for applications to our setting, we consider $P_{\theta,R}$ as an operator on $L^2(\mathbb{R}^n)$ using the identification $S_{\theta,R}$ above. Its principal symbol is given by

$$p_{\theta,R}(x,\xi) = p(x+if_{\theta,R}(x), [(1+i df_{\theta,R}(x))^t]^{-1}\xi)\,, \quad (x,\xi)\in T^*\mathbb{R}^d\,, \tag{3.14}$$

where the complex arguments are allowed due to the analyticity of $p(x,\xi)$ outside of a compact set — see §2.1. We have the following properties

$$\begin{aligned} &\operatorname{Re} p_{\theta,R}(x,\xi) = p(x,\xi) + \mathcal{O}(\theta^2)\langle\xi\rangle^2\,,\\ &\operatorname{Im} p_{\theta,R}(x,\xi) = -d_\xi p(x,\xi)[df_{\theta,R}(x)^t\xi] + d_x p(x,\xi)[f_{\theta,R}(x)] + \mathcal{O}(\theta^2)\langle\xi\rangle^2\,. \end{aligned} \tag{3.15}$$

This implies, for R large enough,

$$|p(x,\xi)|\le\delta\,,\quad |x|\ge 2\,R \implies \operatorname{Im} p_{\theta,R}(x,\xi)\le -C\theta\,. \tag{3.16}$$

For our future aims, it will prove convenient to actually let the angle θ explicitly depend on h: as long as $\theta > ch\log(1/h)$, the estimates above guarantee the Fredholm property

of $(P_{\theta,R} - z)$ for $z \in D(0, \theta/C)$, by providing approximate inverses near infinity. We will indeed take θ of the order of $h\log(1/h)$, see (3.31).

3.5. **Microlocally deformed spaces.** Microlocal deformations using exponential weights have played an important role in the theory of resonances since [21]. Here we take an intermediate point of view [26, 43] by combining compactly supported weights with the complex scaling described above. We should stress however that the full power of [21] would allow more general behaviours of $p(x,\xi)$ at infinity, for instance potentials growing in some directions at infinity.

Let us consider an h-independent real valued function $G_0 \in \mathcal{C}^\infty_{\rm c}(T^*\mathbb{R}^d;\mathbb{R})$, and rescale it in an h-dependent way:

$$G(x,\xi) = Mh\log(1/h)G_0(x,\xi)\,, \quad M>0 \text{ fixed.} \tag{3.17}$$

For $A \in \Psi^{m,0}(\mathbb{R}^d)$, we consider the conjugated operator

$$\begin{aligned} e^{-G^w(x,hD)/h} A e^{G^w(x,hD)/h} &= e^{-\operatorname{ad}_{G^w(x,hD)}/h} A \\ &= \sum_{\ell=0}^{L-1} \frac{(-1)^\ell}{\ell!} \left(\frac{1}{h}\operatorname{ad}_{G^w(x,hD)}\right)^\ell A + R_L\,, \end{aligned} \tag{3.18}$$

where

$$R_L = \frac{(-1)^L}{L!}\int_0^1 e^{-tG^w(x,hD)} \left(\frac{1}{h}\operatorname{ad}_{G^w(x,hD)}\right)^L A e^{tG^w(x,hD)} dt\,.$$

The semiclassical calculus of pseudodifferential operators [12, Chapter 7],[15, Chapter 4, Appendix D.2] and (3.17) show that

$$\left(\frac{1}{h}\operatorname{ad}_{G^w(x,hD)}\right)^\ell A = (M\log(1/h))^\ell \,(\operatorname{ad}_{G_0^w(x,hD)})^\ell A \in (Mh\log(1/h))^\ell\, \Psi_h^{-\infty,0}(\mathbb{R}^d)\,, \quad \forall \ell > 0\,.$$

Since $\|G_0^w\|_{L^2\to L^2} \le C_0$, functional calculus of bounded self-adjoint operators shows that

$$\|\exp(\pm tG^w(x,hD))\| \le h^{-tC_0M}\,,$$

so we obtain the bound,

$$R_L = \mathcal{O}_{L^2\to L^2}(\log(1/h)^L\, h^{L-2tC_0M}) = \mathcal{O}_{L^2\to L^2}(h^{L-2tC_0M-L\delta})\,,$$

with $\delta > 0$ arbitrary small. Applying this bound,we may write (3.18) as

$$e^{-G^w(x,hD)/h} A e^{G^w(x,hD)/h} \sim \sum_{\ell=0}^{\infty} \frac{(-1)^\ell}{\ell!} \left(\frac{1}{h}\operatorname{ad}_{G^w(x,hD)}\right)^\ell A \in \Psi^{m,0}(\mathbb{R}^d)\,. \tag{3.19}$$

In turn, this expansion, combined with Beals's characterization of pseudodifferential operators (3.4), implies that the exponentiated weight is a pseudodifferential operator:

$$\exp(G^w(x,hD)/h) \in \Psi_\delta^{0,C_0M}(\mathbb{R}^d)\,, \quad \forall \delta > 0\,. \tag{3.20}$$

Using the weight function G, we can now define our weighted spaces. Let $H^k_h(\mathbb{R}^d)$ be the semiclassical Sobolev spaces defined in (3.3). We put

$$H^k_G(\mathbb{R}^d) = e^{G^w(x,hD)/h} H^k_h(\mathbb{R}^d)\,, \quad \|u\|_{H^k_G} \stackrel{\rm def}{=} \|e^{-G^w(x,hD)/h}u\|_{H^k_h}\,, \tag{3.21}$$

and

$$\langle u, v\rangle_{H^k_G} = \langle e^{-G^w(x,hD)/h}u, e^{-G^w(x,hD)/h}v\rangle_{H^k_h}\,.$$

As a vector space, $H^k_G(\mathbb{R}^d)$ is identical with $H^k_h(\mathbb{R}^d)$, but the Hilbert norms are different. In the case of L^2, that is of $k=0$, we simply put $H^0_G = H_G$.

The mapping properties of $P = p^w(x,hD)$ on $H_G(\mathbb{R}^d)$ are equivalent with those of $P_G \stackrel{\rm def}{=} e^{-G^w/h} P\, e^{G^w/h}$ on $L^2(\mathbb{R}^d)$, which are governed by the properties of the (full) symbol p_G of P_G: formula (3.19) shows that

$$p_G = p - iH_pG + \mathcal{O}(h^2\log^2(1/h))\,. \tag{3.22}$$

At this moment it is convenient to introduce a notion of *leading symbol*, which is adapted to the study of conjugated operators such as P_G. For a given $Q \in S(T^*\mathbb{R}^d)$, we say that $q \in S(T^*\mathbb{R}^d)$ is a leading symbol of $Q^w(x,hD)$, if

$$\forall \gamma \in (0,1)\,, \forall \alpha,\, \beta \in \mathbb{N}^d\,, \quad h^{-\gamma}\partial_x^\alpha \partial_\xi^\beta (Q-q) = \mathcal{O}_{\alpha,\beta}(\langle\xi\rangle^{-|\beta|})\,, \tag{3.23}$$

that is, $(Q-q) \in S^{0,-\gamma}(T^*\mathbb{R}^d)$ for any $\gamma \in (0,1)$. This property is obviously an equivalence relation inside $S(T^*\mathbb{R}^d)$, which is weaker than the equivalence relation defining the principal symbol map on Ψ_h (see §3.1). In particular, terms of the size $h\log(1/h)$ are "invisible" to the leading symbol. For example, the leading symbols of p_G and p are the same. If we can find q independent of h, then it is unique.

For future use we record the following:

Lemma 3.1. *Suppose*

$$Q^w(x,hD)\ :\ H_G(\mathbb{R}^d) \longrightarrow H_G(\mathbb{R}^d)\,, \quad Q \in S(T^*\mathbb{R}^d)\,,$$

is self-adjoint (with respect to the Hilbert norm on H_G). Then this operator admits a real leading symbol. Conversely, if $q \in S(T^*\mathbb{R}^d)$ is real, then there exists $Q \in S(T^*\mathbb{R}^d)$ with leading symbol q, such that $Q^w(x,hD)$ is self-adjoint on $H_G(\mathbb{R}^d)$.

Proof. This follows from noting that

$$Q^w_G \stackrel{\rm def}{=} e^{-G^w/h} Q^w(x,hD) e^{G^w/h}\,,$$

has the same leading symbol as $Q^w(x,hD)$, and that self-adjointness of Q^w on H_G is equivalent to self-adjointness of Q^w_G on L^2: the definition of H_G in (3.21) (the case of $k=0$) gives

$$\langle Q^w u, v\rangle_{H_G} = \langle e^{-G^w/h}Q^w u, e^{-G^w/h}v\rangle_{L^2} = \langle Q^w_G(e^{-G^w/h}u), e^{-G^w/h}v\rangle_{L^2}\,.$$

□

The weighted spaces can also be microlocalized in the sense of §3.2: for $V \Subset T^*\mathbb{R}^d$, we define the space

$$H_G(V) \stackrel{\text{def}}{=} \{u = u(h) \in H_G(\mathbb{R}^d),\ : \exists C_u > 0\,,\ \forall h \in (0,1],\ \|u(h)\|_{H_G(\mathbb{R}^d)} \leq C_u \\ \exists\, \chi \in \mathcal{C}^\infty_{\rm c}(V)\,, \quad \chi^w u = u + \mathcal{O}_{\mathscr{S}}(h^\infty)\}\,. \tag{3.24}$$

In other words, $H_G(V) = e^{G^w(x,hD)/h} H(V)$. This definition depends only on the values of the weight G in the open set V.

For future reference we state the following

Lemma 3.2. *Suppose $T : H(V) \to H(\kappa(V))$ is an h-Fourier integral operator associated to a symplectomorphism κ (in the sense of §3.3), and is asymptotically uniformly bounded (in the sense of (3.8)). Take $G_0 \in \mathcal{C}^\infty_{\rm c}(\mathrm{neigh}(\kappa(V)))$, $G = Mh\log(1/h)G_0$.*
Then the operator

$$T\ :\ H_{\kappa^*G}(V) \to H_G(\kappa(V)) \tag{3.25}$$

is also asymptotically uniformly bounded with respect to the deformed norms.

Proof. Since the statement is microlocal we can assume that V is small enough, so that $T \equiv T_0 A$ in V, where T_0 is unitary on $L^2(\mathbb{R}^d)$ and $A \in \Psi_h$. As in the proof of Lemma 3.1 the boundedness of (3.25) is equivalent to considering the boundedness of

$$e^{-G^w(x,hD)} T_0 e^{(\kappa^*G)^w(x,hD)/h} A_{\kappa^*G}\ :\ L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)\,,$$

where

$$A_{\kappa^*G} \stackrel{\text{def}}{=} e^{-(\kappa^*G)^w(x,hD)/h} A e^{(\kappa^*G)^w(x,hD)/h}\,.$$

Because of (3.19), we have uniform boundedness of A_{κ^*G} on L^2. Unitarity of T_0 means that it is sufficient to show the uniform boundedness of

$$T_0^{-1} e^{-G^w(x,hD)/h} T_0 e^{(\kappa^*G)^w(x,hD)/h} = e^{-M\log(1/h)(T_0^{-1}G_0^w(x,hD)T_0)} e^{M\log(1/h)(\kappa^*G_0)^w(x,hD)}$$

on L^2. Egorov's theorem (see [15, §10.2]) shows that

$$T_0^{-1} G_0^w(x,hD) T_0 = G_\kappa(x,hD)\,, \quad G_\kappa - \kappa^* G_0 \in \Psi_h^{-\infty,-1}(\mathbb{R}^d)\,.$$

Since $[G_\kappa^w, \kappa^* G_0^2] = h^2 B$, $B \in \Psi_h^{-\infty,0}(\mathbb{R}^d)$, the Baker-Campbell-Hausdorff formula for bounded operators shows[†] that

$$\begin{aligned} T_0^{-1} e^{-G^w(x,hD)/h} T_0 e^{(\kappa^*G)^w(x,hD)/h} &= e^{-M\log(1/h)G_\kappa^w(x,hD)}\, e^{M\log(1/h)(\kappa^*G_0)^w(x,hD)} \\ &= e^{M\log(1/h)(-G_\kappa^w(x,hD)+\kappa^*G_0)^w(x,hD))+\mathcal{O}_{L^2\to L^2}(\log(1/h)^2h^2)} \\ &= \exp \mathcal{O}_{L^2\to L^2}(h\log(1/h)) \\ &= \mathrm{Id} + \mathcal{O}_{L^2\to L^2}(h\log(1/h))\,. \end{aligned}$$

[†]Alternatively, we can compare $\exp(M\log(1/h)G_\kappa^w)$ with $(\exp(M\log(1/h)G_\kappa))^w$ and use product formulæ for pseudodifferential operators – see [43, Appendix] or [15, Section 8.2].

This proves uniform bounded of globally defined operators $T_0 A$, and the asymptotic uniformly boundedness in the sense of (3.8) of T on spaces of microlocally localized functions. □

3.6. **Escape function away from the trapped set.** In this section we recall the construction of the specific weight function G which, up to some further small modifications, will be used to prove Theorems 1 and 2.

Let $K_E \subset p^{-1}(E)$ be the trapped set on the E-energy surface, see (1.4), and define

$$\widehat{K} = \widehat{K}_\delta \stackrel{\rm def}{=} \bigcup_{|E|\leq \delta} K_E \,. \tag{3.26}$$

The construction of the weight function is based on the following result of [19, Appendix]: for any open neighbourhoods U, V of $\widehat{K}$, $\overline{U} \subset V$, there exists $G_1 \in \mathcal{C}^\infty(T^*X)$, such that

$$G_1\restriction_U \equiv 0\,, \quad H_p G_1 \geq 0\,, \quad H_p G_1\restriction_{p^{-1}([-2\delta,2\delta])} \leq C\,, \quad H_p G_1\restriction_{p^{-1}([-\delta,\delta])\setminus V} \geq 1\,. \tag{3.27}$$

These properties mean that G_1 is an *escape function*: it increases along the flow, and *strictly increases* along the flow on $p^{-1}([-\delta,\delta])$ away from $\widehat{K}$ (as specified by the neighbourhood V). Furthermore, $H_p G$ is bounded in a neighbourhood of $p^{-1}(0)$.

Since such a function G_1 is necessarily of unbounded support, we need to modify it to be able to use H_G-norms defined in §3.5 (otherwise methods of [21] could be used and that alternative would allow more general behaviours at infinity, for instance a wide class of polynomial potentials). For that we follow [43, §§4.1,4.2,7.3] and [31, §6.1]: G_1 is modified to a compactly supported G_2 in a way allowing complex scaling estimates (3.16) to compensate for the wrong sign of $H_p G_2$. Specifically, [31, Lemma 6.1] states that for any large $R > 0$ and $\delta_0 \in (0, 1/2)$ we can construct G_2 with the following properties: $G_2 \in \mathcal{C}^\infty_{\rm c}(T^*X)$ and

$$\begin{aligned} &H_p G_2 \geq 0 \qquad \text{on } T^*_{B(0,3R)}X, \\ &H_p G_2 \geq 1 \qquad \text{on } T^*_{B(0,3R)}X \cap (p^{-1}([-\delta,\delta]) \setminus V), \\ &H_p G_2 \geq -\delta_0 \qquad \text{on } T^*X. \end{aligned} \tag{3.28}$$

Let

$$G \stackrel{\rm def}{=} Mh\log(1/h) G_2\,, \quad \text{with } M > 0 \text{ a fixed constant.}$$

Then, in the notations of §3.5, we will be interested in the complex-scaled operator

$$P_{\theta,R} \,:\, H^2_G(\mathbb{R}^n) \longrightarrow H_G(\mathbb{R}^n)\,,$$

for a scaling angle depending on h:

$$\theta = \theta(h) = M_1\, h\log(1/h), \quad M_1 > 0 \text{ fixed.} \tag{3.29}$$

Inserting the above estimates in (3.22), we get

$$|\operatorname{Re} p_{\theta,R,G}(\rho)| < \delta/2\,, \quad \operatorname{Re}\rho \notin V\,, \implies \operatorname{Im} p_{\theta,R,G}(\rho) \leq -\theta/C_1\,, \tag{3.30}$$

provided that we choose [31, §6.1]

$$\frac{M}{C} \geq M_1 \geq \frac{\delta_0 M}{C}, \qquad \text{for some } C > 0\,, \tag{3.31}$$

Assuming that the constant M_0 appearing in the statement of Theorem 1 satisfies

$$0 < M_0 \leq M_1$$

for $\delta > 0$ and $h > 0$ small enough, the rectangle $\mathcal{R}(\delta, M_0, h)$ is contained in the uncovered region in Fig. 6, hence the scaling by the angle (3.29) gives us access to the resonance spectrum in the rectangle $\mathcal{R}(\delta, M_0, h)$. In §5.3 we will need to further adjust M_0 with respect to M_1.

3.7. **Grushin problems.** In this section we recall some linear algebra facts related to the Schur complement formula, which are at the origin of the Grushin method we will use to analyze the operator $P_{\theta,R}$.

For any invertible square matrix decomposed into 4 blocks, we have

$$\begin{pmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{pmatrix}^{-1} = \begin{pmatrix} q_{11} & q_{12} \\ q_{21} & q_{22} \end{pmatrix} \implies p_{11}^{-1} = q_{11} - q_{12}\, q_{22}^{-1} q_{21}\,,$$

provided that q_{22}^{-1} exists (which implies that q_{22}, and hence p_{11}, are square matrices). We have the analogous formula for q_{22}^{-1}:

$$q_{22}^{-1} = p_{22} - p_{21} p_{11}^{-1} p_{12}\,.$$

One way to see these simple facts is to apply gaussian elimination to

$$\mathcal{P} = \begin{pmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{pmatrix}$$

so that, if p_{11} is invertible, we have an upper-lower triangular factorization:

$$\mathcal{P} = \begin{pmatrix} p_{11} & 0 \\ p_{21} & 1 \end{pmatrix} \begin{pmatrix} 1 & p_{11}^{-1} p_{12} \\ 0 & p_{22} - p_{21} p_{11}^{-1} p_{12} \end{pmatrix}\,. \tag{3.32}$$

The formula for the inverse of p_{11} leads to the construction of *effective Hamiltonians* for operators (quantum Hamiltonians) $P : \mathcal{H}_1 \to \mathcal{H}_2$. We first search for auxiliary spaces $\mathcal{H}_\pm$ and operators $R_\pm$ for which the matrix of operators

$$\begin{pmatrix} P - z & R_- \\ R_+ & 0 \end{pmatrix} \,:\, \mathcal{H}_1 \oplus \mathcal{H}_- \longrightarrow \mathcal{H}_2 \oplus \mathcal{H}_+\,,$$

is invertible for z running in some domain of $\mathbb{C}$. Such a matrix is called a *Grushin problem*, and when invertible the problem is said to be *well posed*.

When successful this procedure reduces the spectral problem for P to a nonlinear spectral problem of lower dimension. Indeed, if $\dim \mathcal{H}_- = \dim \mathcal{H}_+ < \infty$, we write

$$\begin{pmatrix} P-z & R_- \\ R_+ & 0 \end{pmatrix}^{-1} = \begin{pmatrix} E(z) & E_+(z) \\ E_-(z) & E_{-+}(z) \end{pmatrix},$$

and the invertibility of $(P-z) : \mathcal{H}_1 \to \mathcal{H}_2$ is equivalent to the invertibility of the finite dimensional matrix $E_{-+}(z)$. The zeros of $\det E_{-+}(z)$ coincide with the eigenvalues of P (even when P is not self-adjoint) because of the following formula:

$$\operatorname{tr} \oint_z (P-w)^{-1} dw = -\operatorname{tr} \oint_z E_{-+}(w)^{-1} E'_{-+}(w)\, dw\,, \tag{3.33}$$

valid when the integral on the left hand side is of trace class – see [44, Proposition 4.1] or verify it using the factorization (3.32). Here $\oint_z$ denotes an integral over a small circle centered at z. The above formula shows that $\dim \ker(P-z) = \dim \ker E_{-+}(z)$.

The matrix $E_{-+}(z)$ is often called an *effective Hamiltonian* for the original Hamiltonian P – see [44] for a review of this formalism and many examples. In the physics literature, this reduction is usually called the Feshbach method.

We illustrate the use of Grushin problems with a simple lemma which will be useful later in §5.3.

Lemma 3.3. *Suppose that*

$$\mathcal{P} \stackrel{\text{def}}{=} \begin{pmatrix} P & R_- \\ R_+ & 0 \end{pmatrix} : \mathcal{H}_1 \oplus \mathcal{H}_- \longrightarrow \mathcal{H}_2 \oplus \mathcal{H}_+\,,$$

where $\mathcal{H}_j$ and $\mathcal{H}_\pm$ are Banach spaces. If $P^{-1} : \mathcal{H}_2 \to \mathcal{H}_1$ exists then

$$\mathcal{P} \text{ is a Fredholm operator } \iff R_+P^{-1}R_- : \mathcal{H}_- \to \mathcal{H}_+ \text{ is a Fredholm operator}\,,$$

and

$$\operatorname{ind} \mathcal{P} = \operatorname{ind} R_+P^{-1}R_-\,.$$

Proof. We apply the factorization (3.32) with $p_{11} = P$, $p_{12} = R_-$, $p_{21} = R_+$, $p_{22} = 0$. Since the first factor is invertible we only need to check the the Fredhold property and the index of the second factor:

$$\begin{pmatrix} 1 & P^{-1}R_- \\ 0 & -R_+P^{-1}R_- \end{pmatrix},$$

and the lemma is immediate. □

4. A microlocal Grushin problem

In this section we recall and extend the analysis of [42] to treat a Poincaré section $\mathbf{\Sigma} \subset p^{-1}(0)$ for a flow satisfying the assumptions in §2.2. In [42] a Poincaré section associated to a single closed orbit was considered. The results presented here are purely microlocal in the sense of §3.2, first near a given component Σ_k of the section, then near the trapped set

K_0. In this section P is the original differential operator, but it could be replaced by its complex scaled version $P_{\theta,R}$, since the complex deformation described in §3.4 takes place far away from K_0. Also, when no confusion is likely to occur, we will often denote the Weyl quantization χ^w of a symbol $\chi \in S(T^*\mathbb{R}^d)$ by the same letter: $\chi = \chi^w$.

4.1. **Microlocal study near Σ_k.** First we focus on a single component Σ_k of the Poincaré section, for some arbitrary $k \in \{1, ..., N\}$. Most of the time we will then drop the subscript k. Our aim is to construct a microlocal Grushin problem for the operator

$$\frac{i}{h}(P - z)\,,$$

near $\Sigma = \Sigma_k$, where $|\operatorname{Re} z| \leq \delta$, $|\operatorname{Im} z| \leq M_0 h \log(1/h)$, and δ will be chosen small enough so that the flow on $\Phi^t\restriction_{K_{\operatorname{Re} z}}$ is a small perturbation of $\Phi^t\restriction_{K_0}$.

4.1.1. *A normal form near Σ_k.* Using the assumption (2.11) and a version of Darboux's theorem (see for instance [23, Theorem 21.2.3]), we may extend the map $\kappa_k = \kappa : \widetilde{\Sigma}_k \to \Sigma_k$ to a canonical transformation $\widetilde{\kappa}_k$ defined in a neighbourhood of $\widetilde{\Sigma}_k$ in $T^*\mathbb{R}^n$,

$$\widetilde{\Omega}_k \stackrel{\text{def}}{=} \{(x, \xi) \in T^*\mathbb{R}^n; (x', \xi') \in \widetilde{\Sigma}_k,\ \ |x_n| \leq \epsilon,\ |\xi_n| \leq \delta\}\,,$$

such that

$$\widetilde{\kappa}_k(x', 0, \xi', 0) = \kappa_k(x', \xi') \in \Sigma_k\,, \qquad p \circ \widetilde{\kappa}_k = \xi_n\,. \tag{4.1}$$

We call $\Omega_k = \widetilde{\kappa}_k(\widetilde{\Omega}_k)$ the neighbourhood of Σ_k in T^*X in the range of $\widetilde{\kappa}_k$. The "width along the flow" $\epsilon > 0$ is taken small enough, so that the sets $\{\Omega_k,\ k = 1, \dots, N\}$ are mutually disjoint, and it takes at least a time 20ϵ for a point to travel between any Ω_k and its successors.

The symplectic maps $\widetilde{\kappa}_k$ allow us to extend the Poincaré section $\mathbf{\Sigma}$ to the neighbouring energy layers $p^{-1}(z)$, $z \in [-\delta, \delta]$. Let us call

$$\kappa_{k,z} \stackrel{\text{def}}{=} \widetilde{\kappa}_k\restriction (\widetilde{\Omega}_k \cap \{\xi_n = z\})\,.$$

Then, if $\delta > 0$ is taken small enough, for $z \in [-\delta, \delta]$ the hypersurfaces

$$\Sigma_k(z) = \kappa_{k,z}(\widetilde{\Sigma}_k) = \{\widetilde{\kappa}(x', 0; \xi', z),\ \ (x', \xi') \in \widetilde{\Sigma}_k\}$$

are still transversal to the flow in $p^{-1}(z)$. Using this extension we may continuously deform the departure sets D_{jk} into $D_{jk}(z) \stackrel{\text{def}}{=} \kappa_{k,z}(\widetilde{D}_{jk}) \subset \Sigma_k(z)$, and by consequence the tubes T_{jk} into tubes $T_{jk}(z) \subset p^{-1}(z)$ through a direct generalization of (2.12). The tube $T_{jk}(z)$ intersects $\Sigma_j(z)$ on the arrival set $A_{jk}(z) \subset \Sigma_j(z)$; notice that for $z \neq 0$, the latter is in general different from $\kappa_{j,z}(\widetilde{A}_{jk})$ (equivalently $\widetilde{A}_{jk}(z) = \kappa_{j,z}^{-1}(A_{jk}(z))$ is generally different from $\widetilde{A}_{jk}(0)$). These tubes induce a Poincaré map $F_{jk,z}$ bijectively relating $D_{jk}(z)$ with $A_{jk}(z)$.

The following Lemma, announced at the end of §2.3.1, shows that for $|z|$ small enough the interesting dynamics still takes place inside these tubes: the trapped set is stable with respect to variations of the energy.

Lemma 4.1. *Provided $\delta > 0$ is small enough, for any $z \in [-\delta, \delta]$ the trapped set $K_z \Subset \sqcup_{jk} T_{jk}(z)$.*

As a consequence, in this energy range the Poincaré map associated with $\mathbf{\Sigma}(z)$ fully describes the dynamics on K_z.

Proof. From our assumption in §2.1, there exists a ball $B(0,R)$ (the "interaction region") such that, for any $E \in [-1/2, 1/2]$, the trapped set K_E must be contained inside $T^*_{B(0,R)}X$. If R is large enough, any point $\rho \in p^{-1}(z) \setminus T^*_{B(0,R)}X$, $z \approx 0$, will "escape fast" in the past or in the future, because the Hamilton vector field is close to the one corresponding to free motion, $2\sum_j \xi_j \partial_{x_j}$. Hence we only need to study the behaviour of points in $p^{-1}(z) \cap T^*_{B(0,R)}\mathbb{R}^n$.

Let us define the escape time from the interaction region $T^*_{B(0,R)}X$: for any $\rho \in T^*_{B(0,R)}X$,

$$t_{esc}(\rho) \stackrel{\text{def}}{=} \inf\{t > 0,\ \max(|\pi_x \Phi^t(\rho)|, |\pi_x \Phi^{-t}(\rho)|) \geq R\}\,,$$

For any $E \in [-1/2, 1/2]$, the trapped set K_E can be defined as the set of points in $p^{-1}(E)$ for which $t_{esc}(\rho) = \infty$. Let us consider the neighbourhood of K_0 formed by the interior of the union of tubes, $(\sqcup T_{ik})^\circ$. By compactness, the escape time is bounded from above outside this neighbourhood, that is in $p^{-1}(0) \cap T^*_{B(0,R)}X \setminus (\sqcup T_{ik})^\circ$, by some finite $t_1 > 0$. By continuity of the flow Φ^t, for $\delta > 0$ small enough, the escape time in the deformed neighbourhood $p^{-1}(z) \cap T^*_{B(0,R)}X \setminus (\sqcup T_{ik}(z))^\circ$ will still be bounded from above by $2t_1$: this proves that $K_z \Subset \sqcup T_{ik}(z)$. □

A direct consequence is that the reduced trapped sets $\mathcal{T}_j(z) \stackrel{\text{def}}{=} \Sigma(z) \cap K_z$ are contained inside $D_j(z)$.

For any set $S(z)$ depending on the energy in the interval $z \in [-\delta, \delta]$, we use the notation

$$\widehat{S} \stackrel{\text{def}}{=} \bigcup_{|z| \leq \delta} S(z)\,. \tag{4.2}$$

We will extend the notation to complex values of the parameter $z \in \mathcal{R}(\delta, M_0, h)$, identifying $S(z)$ with $S(\operatorname{Re} z)$.

4.1.2. *Microlocal solutions near Σ.* Let us now restrict ourselves to the neighbourhood of Σ_k, and drop the index k. The canonical transformation $\widetilde{\kappa}$ can be locally quantized using the procedure reviewed in §3.3, resulting in a microlocally defined unitary Fourier integral operator

$$U\ :\ H(\widetilde{\Omega}) \longrightarrow H(\Omega)\,,\quad U^*\,P\,U \equiv hD_{x_n}\,,\ \text{microlocally in } \widetilde{\Omega}. \tag{4.3}$$

For $z \in \mathcal{R}(\delta, M_0, h)$, we consider the microlocal Poisson operator

$$\mathrm{K}(z) : L^2(\mathbb{R}^{n-1}) \to L^2_{\mathrm{loc}}(\mathbb{R}^n)\,, \quad [\mathrm{K}(z)\, v_+](x', x_n) = e^{ix_n z/h}\, v_+(x')\,, \tag{4.4}$$

which obviously satisfies the equation $(hD_{x_n} - z)\,\mathrm{K}(z)\, v_+ = 0$.

For v_+ microlocally concentrated in a compact set, the wavefront set of $\mathrm{K}(z)\, v_+$ is not localized in the flow direction. On the other hand, the Fourier integral operator U is well-defined and unitary only from $\widetilde{\Omega}$ to Ω. Therefore, we use a smooth cutoff function χ_Ω, $\chi_\Omega = 1$ in Ω, $\chi_\Omega = 0$ outside Ω' a small open neighbourhood of Ω (say, such that $|x_n| \leq 2\epsilon$ inside $\widetilde{\Omega}'$), and define the Poisson operator

$$K(z) \stackrel{\mathrm{def}}{=} \chi^w_\Omega\, U\, \mathrm{K}(z) : H(\widetilde{\Sigma}) \to H(\Omega')\,.$$

This operator maps any state $v_+ \in H(\widetilde{\Sigma}) \subset L^2(\mathbb{R}^{n-1})$, to a microlocal solution of the equation $(P - z)u = 0$ in Ω, with $u \in H(\Omega')$. As we will see below, the converse holds: each microlocal solution in Ω is parametrized by a function $v_+ \in H(\widetilde{\Sigma})$.

In a sense, the solution $u = K(z)v_+$ is an extension along the flow of the *transverse data* v_+. More precisely, $K(z)$ is a microlocally defined Fourier integral operator associated with the graph

$$C_- = \{(\widetilde{\kappa}(x', x_n, \xi', \mathrm{Re}\, z); x', \xi'), \quad (x', \xi') \in \widetilde{\Sigma},\ |x_n| \leq \epsilon\} \subset T^*(X \times \mathbb{R}^{n-1})\,. \tag{4.5}$$

Equivalently, this relation associates to each point $(x', \xi') \in \widetilde{\Sigma}$ a short trajectory segment through the point $\widetilde{\kappa}(x', 0; \xi', \mathrm{Re}\, z) \in \Sigma(\mathrm{Re}\, z)$. We use the notation C_- since this relation is associated with the operator R_- defined in (4.13) below.

Back to the normal form hD_{x_n}, let us consider a smoothed out step function,

$$\chi_0 \in \mathcal{C}^\infty(\mathbb{R}_{x_n}), \quad \chi_0(x_n) = 0 \text{ for } x_n \leq -\epsilon/2, \quad \chi_0(x_n) = 1 \text{ for } x_n \geq \epsilon/2\,.$$

We notice that the commutator $(i/h)[hD_{x_n}, \chi_0] = \chi_0'(x_n)$ is localized in the region of the step and integrates to 1: this implies the normalization property

$$\langle (i/h)[hD_{x_n}, \chi_0]\mathrm{K}(z)v_+, \mathrm{K}(\bar{z})v_+\rangle = \|v_+\|^2_{L^2(\mathbb{R}^{n-1})}\,, \tag{4.6}$$

where $\langle \bullet, \bullet \rangle$ is the usual Hermitian inner product on $L^2(\mathbb{R}^n)$. Notice that the right hand side is independent of the precise choice of χ_0.

We now bring this expression to the neighbourhood of Σ through the Fourier integral operator $\chi^w_\Omega U$. This implies that the Poisson operator $K(z)$ satisfies:

$$\langle (i/h)[P, \chi^w]K(z)v_+, K(\bar{z})v_+\rangle \equiv \|v_+\|^2 \quad \text{ for any } v_+ \in H(\widetilde{\Sigma})\,. \tag{4.7}$$

Here the symbol χ is such that $\chi^w \equiv U\, \chi_0^w\, U^*$ inside Ω, so χ is equal to 0 before $\Phi^{-\epsilon}(\Sigma)$ and equal to 1 after $\Phi^\epsilon(\Sigma)$ (in the following we will often use this time-like terminology referring to the flow Φ^t). In (4.7), we are only concerned with $[P, \chi^w]$ microlocally near Ω, since the operator $\chi^w_\Omega U$ is microlocalized in $\Omega' \times \widetilde{\Omega}'$. Hence, at this stage we can ignore the properties of the symbol χ outside Ω'.

The expression (4.7) can be written

$$K(\bar z)^*\,[(i/h)P,\chi^w]K(z) = Id\ :\ H(\widetilde\Sigma)\to H(\widetilde\Sigma)\,. \tag{4.8}$$

Fixing a function χ with properties described after (4.7) and writing $\chi=\chi_f$ (where f is for *forward*), we define the operator

$$R_+(z) \stackrel{\rm def}{=} K(\bar z)^*\,[(i/h)P,\chi_f] = \mathrm{K}(\bar z)^*\,U^*\chi_\Omega^w\,[(i/h)P,\chi_f] \tag{4.9}$$

(from here on we denote $\chi=\chi^w$ in similar expressions). This operator "projects" any $u\in H(\Omega)$ to a certain transversal function $v_+\in H(\widetilde\Sigma)$. But it is important to notice that $R_+(z)$ is also well-defined on states u microlocalized in a small neighbourhood of the full trapped set $\widehat K$: the operator $\chi_\Omega^w\,[(i/h)P,\chi_f]$ cuts off the components of u outside Ω. Hence, we may write

$$R_+(z) : H(\mathrm{neigh}(\widehat K))\to H(\widetilde\Sigma)\,.$$

The equation (4.8) shows that this projection is compatible with the above extension of the transversal function:

$$R_+(z)\,K(z) = Id\ :\ H(\widetilde\Sigma)\to H(\widetilde\Sigma)\,. \tag{4.10}$$

This shows that transversal functions $v_+\in H(\widetilde\Sigma)$ and microlocal solutions to $(P-z)u=0$ are bijectively related. Since $|\operatorname{Im} z|\le M_0 h\log(1/h)$ and $|x_n|\le 2\epsilon$ inside $\widetilde\Omega'$ (resp. $|x_n|\le\epsilon$ inside $\widetilde\Omega$), we have the bounds

$$\|K(z)\|_{L^2\to L^2} = \mathcal{O}(h^{-2\epsilon M_0})\,,\quad \|R_+(z)\|_{L^2\to L^2} = \mathcal{O}(h^{-\epsilon M_0})\,.$$

Just as $K(\bar z)^*$, $R_+(z)$ is a microlocally defined Fourier integral operator associated with the relation

$$C_+ = \{x',\xi';(\widetilde\kappa(x',x_n,\xi',\operatorname{Re} z)),\quad (x',x_n,\xi',\operatorname{Re} z)\in\widetilde\Omega\}\subset T^*(\mathbb{R}^{n-1}\times X)\,, \tag{4.11}$$

namely the inverse of C_- given in (4.5). In words, this relation consists of taking any $\rho\in\Omega\cap p^{-1}(\operatorname{Re} z)$ and projecting it along the flow on the section $\Sigma(z)$.

We now select a second cutoff function χ_b with properties similar with χ_f, and satisfying also the nesting property

$$\chi_b = 1\ \text{ in a neighbourhood of }\ \operatorname{supp}\chi_f\,. \tag{4.12}$$

With this new cutoff, we define the operator

$$R_-(z)u_- = [(i/h)P,\chi_b]\,K(z)\ :\ H(\widetilde\Sigma)\to H(\Omega)\,. \tag{4.13}$$

Starting from a transversal data $u_-\in H(\widetilde\Sigma)$, this operator creates a microlocal solution in Ω and truncates by applying a pseudodifferential operator with symbol $H_p\chi_b$. Like $K(z)$, it is a microlocally defined Fourier integral operator associated with the graph C_-. its norm is bounded by $\|R_-(z)\|_{L^2\to L^2}=\mathcal{O}(h^{-\epsilon M_0})$.

4.1.3. *Solving a Grushin problem.* We are now equipped to define our microlocal Grushin problem in Ω. Given $v \in H(\Omega)$, $v_+ \in H(\widetilde{\Sigma})$, we want to solve the system

$$\begin{cases} (i/h)(P-z)u + R_-(z)u_- &= v, \\ R_+(z)u &= v_+\,, \end{cases} \tag{4.14}$$

with $u \in L^2(X)$ a forward solution, and $u_- \in H(\widetilde{\Sigma})$.

Let us show how to solve this problem. First let $\widetilde{u}$ be the forward solution of $(i/h)(P-z)\widetilde{u} = v$, microlocally in Ω. That solution can be obtained using the Fourier integral operator U in (4.3) and the easy solution for hD_{x_n}. We can also proceed using the propagator to define a forward parametrix:

$$\widetilde{u} \stackrel{\text{def}}{=} E(z)\, v, \qquad E(z) \stackrel{\text{def}}{=} \int_0^T e^{-it(P-z)/h}\, dt. \tag{4.15}$$

The time T is such that $\Phi^T(\Omega) \cap \Omega = \emptyset$ (from the above assumption on the separation between the Ω_k we may take $T = 5\epsilon$). By using the model operator hD_{x_n}, one checks that the parametrix $E(z)$ transports the wavefront set of v as follows:

$$\mathrm{WF}_h(E(z)v) \subset \mathrm{WF}_h(v) \cup \Phi^T(\mathrm{WF}_h(v)) \cup \bigcup_{0\le t\le T} \Phi^t(\mathrm{WF}_h(v) \cap p^{-1}(\mathrm{Re}\, z))\,. \tag{4.16}$$

In general, $\widetilde{u}$ does not satisfy $R_+(z)\widetilde{u} = v_+$, so we need to correct it. For this aim, we solve the system

$$\begin{cases} (i/h)(P-z)\widehat{u} + R_-(z)u_- &\equiv 0\,, \\ R_+(z)\widehat{u} &\equiv v_+ - R_+(z)\widetilde{u} \end{cases} \tag{4.17}$$

through the Ansatz

$$\begin{cases} u_- &= -v_+ + R_+(z)\widetilde{u}\,, \\ \widehat{u} &= -\chi_b\, K(z)\, u_-\,. \end{cases} \tag{4.18}$$

Indeed, the property $(P-z)\,K(z) \equiv 0$ ensures that $(i/h)(P-z)\widehat{u} = -R_-(z)u_-$. We then obtain the identities

$$\begin{aligned} R_+(z)\widehat{u} &= -K(\bar{z})^*\,[(i/h)P, \chi_f]\,\chi_b\, K(z)\, u_- \\ &\equiv -K(\bar{z})^*\,[(i/h)P, \chi_f]\, K(z)\, u_- \\ &\equiv -u_-\,. \end{aligned}$$

The second identity uses the nesting assumption $(H_p\chi_f)\chi_b = H_p\chi_f$, and the last one results from (4.8). This shows that the Ansatz (4.18) solves the system (4.17). Finally, $(u = \widetilde{u}+\widehat{u}, u_-)$ solves (4.14) microlocally in $\Omega\times\widetilde{\Sigma}$, for $v \in H(\Omega)$ and $v_+ \in H(\widetilde{\Sigma})$ respectively. Furthermore, these solutions satisfy the norm estimate

$$\|u\| + \|u_-\| \lesssim h^{-5M_0\epsilon}(\|v\| + \|v_+\|)\,. \tag{4.19}$$

The form of the microlocal construction in this section is an important preparation for the construction of our Grushin problem in the next section. In itself, it only states that, for v microlocalized near Σ, $(i/h)(P-z)u = v$ can be solved microlocally near Σ in the forward direction.

4.2. **Microlocal solution near $\widehat{K}$.** We will now extend the construction of the Grushin problem near each Σ_k, described in §4.1, to obtain a microlocal Grushin problem near the full trapped set $\widehat{K}$. This will be achieved by relating the construction near Σ_k to the one near the successor sections Σ_j. We now need to restore all indices $k \in \{1, \dots, N\}$ in our notations.

4.2.1. *Setting up the Grushin problem.* We recall that $H(\widetilde{\Sigma}_k) \subset L^2(\mathbb{R}^{n-1})$ is the space of functions microlocally concentrated in $\widetilde{\Sigma}_k$ (see (3.6)). For $u \in L^2(X)$ microlocally concentrated in $\mathrm{neigh}(\widehat{K}, T^*X)$, we define

$$R_+(z)u = (R_+^1(z)u, ..., R_+^N(z)u) \in H(\widetilde{\Sigma}_1) \times ... \times H(\widetilde{\Sigma}_N)\,, \tag{4.20}$$

where each $R_+^k(z) : H(\mathrm{neigh}(\widehat{K})) \to H(\widetilde{\Sigma}_k)$ was defined in §4.1 using a cutoff $\chi_f^k \in \mathcal{C}_{\mathrm{c}}^\infty(T^*X)$ realizing a smoothed-out step from 0 to 1 along the flow near Σ_k.

Similarly, we define

$$\begin{aligned} R_-(z) \,&: H(\widetilde{\Sigma}_1) \times \ldots \times H(\widetilde{\Sigma}_N) \to H(\cup_{k=1}^N \Omega_k), \\ R_-(z)u_- &= \sum_1^N R_-^j(z)u_-^j, \qquad u_- = (u_-^1, ..., u_-^N). \end{aligned} \tag{4.21}$$

Each $R_-^k(z)$ was defined in (4.13) in terms of a cutoff function $\chi_b^k \in \mathcal{C}_{\mathrm{c}}^\infty(T^*X)$ which also changes from 0 to 1 along the flow near Σ_k, and does so before χ_f^k. Below we will impose more restrictions on the cutoffs χ_b^k.

With these choices, we now consider the microlocal Grushin problem

$$\begin{cases} (i/h)(P-z)u + R_-(z)u_- & \equiv v\,, \\ R_+(z)u & \equiv v_+\,. \end{cases} \tag{4.22}$$

The aim of this section is to construct a solution (u, u_-) microlocally concentrated in a small neighbourhood of

$$K_0 \times \kappa_1^{-1}(\mathcal{T}_1) \times ... \times \kappa_N^{-1}(\mathcal{T}_N)\,,$$

provided (v, v_+) is concentrated in a sufficiently small neighbourhood of the same set.

To achieve this aim we need to put more constraints on the cutoffs χ_b^k. We assume that each $\chi_b^k \in \mathcal{C}_{\mathrm{c}}^\infty(T^*X)$ is supported near the direct outflow of $\mathcal{T}_k$. To give a precise condition,

let us slightly modify the energy-thick tubes $\widehat{T}_{jk}$ (see (2.12), (4.2)) by removing or adding some parts near their ends:

$$\widehat{T}_{jk}^{s_1 s_2} \stackrel{\text{def}}{=} \{\Phi^t(\rho) \,:\, \rho \in \widehat{D}_{jk},\ -s_2\, 2\epsilon < t < t_+(\rho) + s_1\, 2\epsilon\}\,, \qquad s_i = \pm\,.$$

With this definition, the short tubes $\widehat{T}_{jk}^{--}$ do not intersect the neighbourhoods Ω_k, Ω_j, while the long tubes $\widehat{T}_{jk}^{++}$ intersect both (see Fig. 7).

We then assume that

$$\chi_b^k(\rho) = 1 \quad \text{for} \quad \rho \in \bigcup_{j\in J_+(k)} \widehat{T}_{jk}^{--}\,, \tag{4.23}$$

and $\operatorname{supp}\chi_b^k$ is contained in a small neighbourhood of that set. Furthermore, we want the cutoffs $\{\chi_b^k\}_{k=1,\dots,N}$ to form a *microlocal partition of unity* near K_0: there exists a neighbourhood V_0 of $\widehat{K}$ containing all long tubes:

$$V_0 \supset \bigcup_{k,j} \widehat{T}_{jk}^{++}\,, \tag{4.24}$$

and such that

$$\sum_{k=1}^{N} \chi_b^k(\rho) \equiv 1 \quad \text{for} \quad \rho \in V_0\,. \tag{4.25}$$

These conditions on χ_b^k can be fulfilled thanks to the assumption (2.10) on the section $\mathbf{\Sigma}$. A schematic representation of these sets and cutoffs is shown in Fig. 7.

4.2.2. *Solving the homogeneous Grushin problem.* Let us first solve (4.22) when $v \equiv 0$. The wavefront set $\mathrm{WF}_h(v_+^k) \subset \widetilde{\Sigma}_k$ is mapped through $\kappa_{k,z}$ to a subset of $\Sigma_k(z)$. The microlocal solution $K_k(z)v_+^k$, initially concentrated inside the neighbourhood Ω_k', can be extended along the flow to a larger set Ω_k^+, which intersects the successors $\Sigma_j(z)$ of $\Sigma_k(z)$ and contains the union of tubes $\bigcup_{j\in J_+(k)} \widehat{T}_{jk}^{++}$ (we remind that $j \neq k$ according to assumption (2.10)). This can be done by extending the symplectomorphism $\widetilde{\kappa}_k$, the associated unitary Fourier integral operator U_k, and replace the cutoff function χ_{Ω_k} by a function $\chi_{\Omega_k^+}$ supported in the set Ω_k^+; we can then define the extended Poisson operator as:

$$K_k^+(z) = \chi_{\Omega_k^+}^w\, U_k\, \mathrm{K}(z) : H(\widetilde{\Sigma}) \to H(\Omega_k^+)\,.$$

Assuming $\kappa_{k,z}(\mathrm{WF}_h(v_+^k))$ is contained in the departure set $D_k(z) \subset \Sigma_k(z)$, the extended microlocal solution $K_k^+(z)v_+^k$ is concentrated in the union of tubes $\cup_{j\in J_+(k)} T_{jk}^{++}(z)$. In that case, we take as our Ansatz

$$u_k \stackrel{\text{def}}{=} \chi_b^k\, K_k^+(z)\, v_+^k\,. \tag{4.26}$$

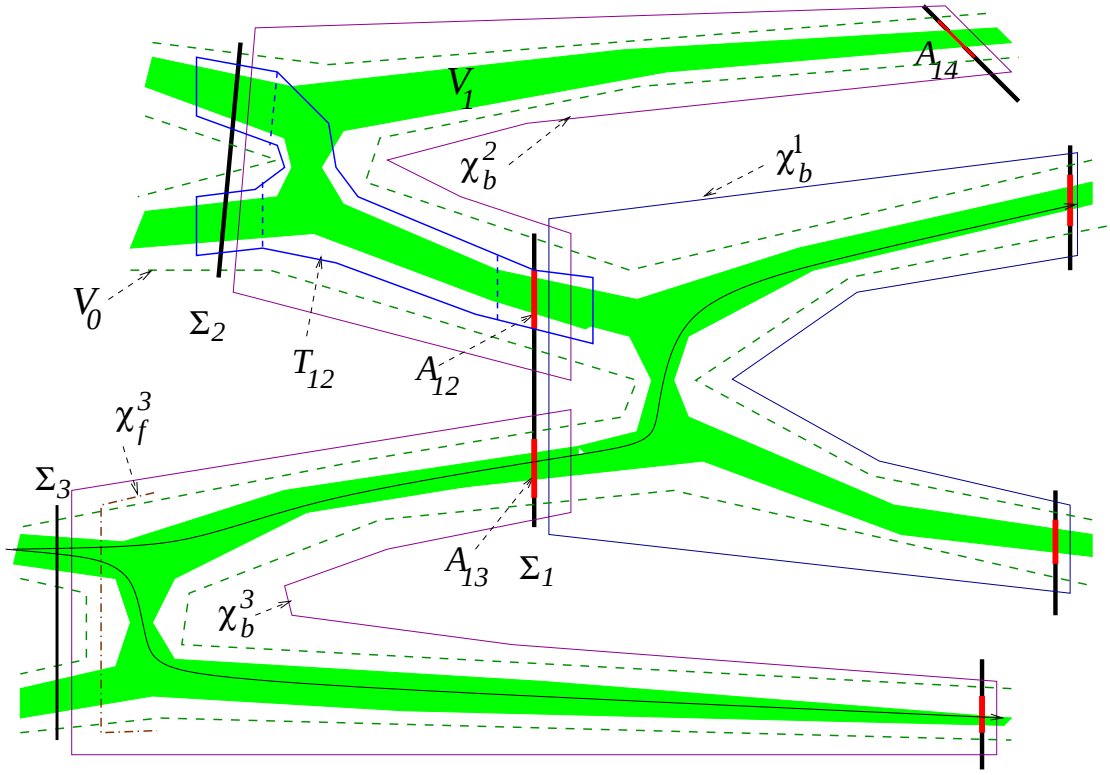

FIGURE 7. Schematic representation of (part of) the neighbourhoods $V_1 \subset V_0$ of K_0 (resp. green shade and green dashed contour), some sections Σ_k (thick black) and arrival sets $A_{kj} \subset \Sigma_k$ (red). We also show the tubes $T_{12}^{\pm\pm}$ connecting Σ_2 with A_{12} (the dashed lines indicate the boundaries of T_{12}^{--}), the supports of the cutoffs χ_b^k and χ_f^3 (dot-dashed line), and two trajectories in K_0 (full lines inside V_1).

Set

$$t_{\max} \stackrel{\text{def}}{=} \max\{t_+(\rho),\ \rho \in \sqcup_k D_k(z),\ |\operatorname{Re} z| \le \delta\} \tag{4.27}$$

the maximal return time for our Poincaré map. Then the above Ansatz satisfies the estimate

$$\|u_k\|_{L^2} \lesssim h^{-M_0(t_{\max}+\epsilon)}\, \|v_+\|_{H(\widetilde{D}_k)}\,. \tag{4.28}$$

Due to the assumption (4.23), the cutoff χ_b^k effectively truncates the solution only near the sections $\Sigma_k(z)$ and $\Sigma_j(z)$, $j \in J_+(k)$, but not on the "sides" of $\operatorname{supp}\chi_b^k$. Hence, the expression

$$(i/h)(P-z)u_k \equiv [(i/h)P, \chi_b^k]\, K_k^+(z)\, v_+^k \tag{4.29}$$

can be decomposed into one component $R^k_-(z)v^k_+$ supported near $D_k(z)$, and other components supported near the arrival sets $A_{jk}(z) \subset \Omega_j$, due to the "step down" of χ^k_b near $A_{jk}(z)$. The assumption (4.25) ensures that

$$[(i/h)P, \chi^k_b] \equiv -[(i/h)P, \chi^j_b] \quad \text{microlocally near } A_{jk}(z)\,, \tag{4.30}$$

so the expression in (4.29) reads

$$(i/h)(P-z)u_k \equiv R^k_-(z)v^k_+ - \sum_{j\in J_+(k)} [(i/h)P, \chi^j_b]\, K^+_k(z)\, v^k_+\,. \tag{4.31}$$

Now, for each $j \in J_+(k)$ we notice that $K^+_k(z)\, v^k_+$ is a solution of $(P-z)u = 0$ near $A_{jk}(z)$, so this solution can also be parametrized by some transversal data "living" on the section $\Sigma_j(z)$ (see the discussion before (4.5)). This data obviously depends linearly on v^k_+, which defines the *monodromy operator* $\mathcal{M}_{jk}(z)$:

$$K^+_k(z)v^k_+ \equiv K_j(z)\,\mathcal{M}_{jk}(z)v^k_+\,, \quad \text{microlocally near } A_{jk}(z)\,. \tag{4.32}$$

The operators $\mathcal{M}_{jk}(z)$ are microlocally defined from $\widetilde{D}_k \subset \widetilde{\Sigma}_k$ to $\widetilde{A}_{jk}(z) \subset \widetilde{\Sigma}_j$, they are zero on $H(\widetilde{D}_{\ell k})$ for $\ell \neq j$. The identity (4.8) provides an explicit formula:

$$\mathcal{M}_{jk}(z) = K_j(\bar{z})^*\,[(i/h)P, \chi^j_f]K^+_k(z) = R^j_+(z)K^+_k(z)\,. \tag{4.33}$$

Before further describing these operators, let us complete the solution of our Grushin problem. Combining (4.31) with (4.32), we obtain

$$(i/h)(P-z)u_k \equiv R^k_-(z)v^k_+ - \sum_{j\in J_+(k)} R^j_-(z)\mathcal{M}_{jk}(z)v^k_+\,. \tag{4.34}$$

This shows that the problem (4.22) in the case $v = 0$ and a single v^+_k, $\mathrm{WF}_h(v^k_+) \subset \widetilde{D}_k$ is solved by

$$u \equiv \chi^k_b\, K^+_k(z)\, v^k_+, \quad u^k_- = -v^k_+, \quad u^j_- = \mathcal{M}_{jk}(z)v^k_+, \quad j \in J_+(k)\,.$$

We now consider the Grushin problem with $v = 0$, $v_+ = (v^1_+, \dots, v^N_+)$ with each v^k_+ microlocalized in $\widetilde{D}_k$. By linearity, this problem is solved by

$$\begin{aligned} u &\equiv \sum_k \chi^k_b\, K^+_k(z)\, v^k_+, \\ u^j_- &\equiv -v^j_+ + \sum_{k\in J_-(j)} \mathcal{M}_{jk}(z)v^k_+\,. \end{aligned} \tag{4.35}$$

From the above discussion, u is microlocalized in the neighbourhood V_0 of $\widehat{K}$, while u^j_- is microlocalized in $\widetilde{D}_j \cup \widetilde{A}_j(z)$.

Let us now come back to the monodromy operators. The expression (4.33) shows that $\mathcal{M}_{jk}(z)$ is a microlocal Fourier integral operator. Since we have extended the solution

$K_k(z)\, v_+^k$ beyond Ω_k, the relation associated with the restriction of $K_k^+(z)$ on $H(\widetilde{D}_{jk})$ is a modification of (4.5), of the form

$$C_-^{jk} = \{(\Phi^t(\widetilde{\kappa}_{k,z}(\rho)); \rho),\ \rho \in \widetilde{D}_{jk},\ \ -\epsilon \le t \le t_{\max} + \epsilon\}\,,$$

such that the trajectories cross Σ_j. On the other hand, the relation C_+ associated with $R_+^j(z)$ is identical with (4.11). By the composition rules, the relation associated with $\mathcal{M}_{jk}(z)$ is

$$C^{jk} = \{(\rho', \rho),\ \ \rho \in \widetilde{D}_{jk},\ \rho' = \kappa_{j,z}^{-1} \circ F_{jk,z} \circ \kappa_{k,z}(\rho) = \widetilde{F}_{jk,z}(\rho)\}\,.$$

This is exactly the graph of the Poincaré map $F_{jk,z} : D_{jk}(z) \to A_{jk}(z)$, seen through the coordinates charts $\kappa_{k,z}$, $\kappa_{j,z}$.

When z is real, the identity (4.8) implies that $\mathcal{M}_{jk}(z) : H(\widetilde{D}_{jk}) \to H(\widetilde{A}_{jk}(z))$ is microlocally unitary. Also, the definition (4.33) shows that this operator depends holomorphically of z in the rectangle $\mathcal{R}(\delta, M_0, h)$. To lowest order, the z-dependence takes the form

$$\mathcal{M}_{jk}(z) = \mathcal{M}_{jk}(0)\, \mathrm{Op}_h^w(\exp(iz\tilde{t}_+/h))\, \big(1 + \mathcal{O}(h\log(1/h))\big)$$

where $\tilde{t}_+ \in \mathcal{C}_{\mathrm{c}}^\infty(\mathbb{R}^{n-1}; \mathbb{R}_+)$ is an extension of the return time associated with the map $\widetilde{F}_{jk,z}$ on $\widetilde{D}_{jk}$. For $z \in \mathcal{R}(\delta, M_0, h)$, this operator satisfies the asymptotic bound

$$\|\mathcal{M}_{jk}(z)\|_{H(\widetilde{D}_k) \to H(\widetilde{A}_j(z))} = \mathcal{O}(h^{-M_0 t_{\max}})\,. \tag{4.36}$$

4.2.3. *Solving the inhomogeneous Grushin problem.* It remains to discuss the inhomogeneous problem

$$(i/h)(P - z)u + R_- u_- \equiv v\,, \tag{4.37}$$

for v microlocalized in a neighbourhood V_1 of $\widehat{K}$, which satistfies

$$V_1 \subset \bigcup_{j,k} \widehat{T}_{jk}^{-+}\,. \tag{4.38}$$

(each tube $\widehat{T}_{jk}^{-+}$ intersects Ω_k only near $\widehat{D}_k$, see figure 7).

Let us first assume that v is microlocally concentrated inside a short tube $\widehat{T}_{jk}^{--}$. We use the forward parametrix $E(z)$ of $(i/h)(P - z)$ given in (4.15) with the time

$$T = t_{\max} + 5\epsilon\,, \tag{4.39}$$

and consider the Ansatz

$$u \stackrel{\mathrm{def}}{=} \chi_b^k\, E(z)\, v\,. \tag{4.40}$$

According to the transport property (4.16), $E(z)v$ is microlocalized in the outflow of $\widehat{T}_{jk}^{--}$, so the cutoff χ_b^k effectively truncates $E(z)v$ only near $A_{jk}(z) \subset \Omega_j$. The partition of unity (4.25) then implies that

$$(i/h)(P - z)u \equiv v + [(i/h)P, \chi_b^k]\, E(z)\, v\ \equiv v - [(i/h)P, \chi_b^j]\, E(z)\, v\,.$$

Also, the choice of the time T ensures that $E(z)v$ is a microlocal solution of $(P-z)u=0$ near $A_{jk}(z)$, so

$$E(z)v \equiv K_j(z)R_+^j(z)E(z)v \quad \text{microlocally near } A_{jk}(z)\,.$$

Thus, we can solve (4.37) by taking

$$u_-^j \equiv R_+^j(z)E(z)v\,, \quad u_-^\ell = 0, \quad \ell \neq j\,.$$

The propagation of wavefront sets given in (4.16) shows that $u_-^j \in H(\widetilde{A}_{jk}(z))$, and that $\mathrm{WF}_h(u) \subset \widehat{T}_{jk}^{+-}$ does not intersect the "step up" region of the forward cutoffs χ_f^ℓ, so that $R_+^\ell(z)u \equiv 0$ for all $\ell = 1,\dots,N$.

If v is microlocally concentrated in $V_1 \cap \cup_{|t|\leq\epsilon}\Phi^t(\widehat{D}_k)$, we can replace the cutoff χ_b^k in (4.40) by

$$\chi_b^k + \sum_{\ell\in J_-(k)} \chi_b^\ell\,,$$

and apply the same construction. The only notable difference is the fact that $R_+^k(z)u$ may be a nontrivial state concentrated in $\cup_{|t|\leq\epsilon}\widehat{D}_k$.

In both cases, we see that $\|u\| + \|u_-\| \lesssim h^{-M_0(t_{\max}+2\epsilon)}\|v\|$. By linearity, the above procedure allows to solve (4.37) for any v microlocalized inside the neighbourhood V_0.

This solution produces a term R_+u, which can be solved away using the procedure of §4.2.2. Notice that $\|R_+u\| \lesssim h^{-M_0(t_{\max}+\epsilon)}\|v\|$.

We summarize the construction of our microlocal Grushin problem in the following

Proposition 4.2. *For $\delta > 0$ small enough, there exist neighbourhoods of $\widehat{K} = \widehat{K}_\delta$ in T^*X, V_+ and V_-, and neighbourhoods of $\widetilde{\kappa}_j^{-1}(\widehat{\mathcal{T}}_j)$ in $\widetilde{\Sigma}_j$, V_+^j, and V_-^j, $j = 1,\cdots N$, such that for any*

$$(v, v_+) \in H(V_+) \times H(V_+^1) \times \cdots H(V_+^N)\,,$$

we can find

$$(u, u_-) \in H(V_-) \times H(V_-^1) \times \cdots H(V_-^N)\,,$$

satisfying

$$\frac{i}{h}(P-z)u + R_-(z)u_- \equiv v\,, \qquad R_+(z)u \equiv v_+ \quad \textit{microlocally in } V_+ \times V_+^1 \times \cdots V_N^+\,.$$

Here $R_\pm(z)$ are given by (4.20) *and* (4.21)*. Furthermore, the solutions satisfy the norm estimates*

$$\|u\| + \|u_-\| \lesssim h^{-M_0(2t_{\max}+2\epsilon)}(\|v\| + \|v_+\|)\,,$$

where $t_{\max}$ is the maximal return time defined in (4.27)*.*

One possible choice for the above sets is

$$V_+ = V_1, \quad V_- \overset{\text{def}}{=} V_0, \quad V_+^k = \widetilde{D}_k, \quad V_-^k = \widetilde{D}_k \cup \bigcup_{|\operatorname{Re} z|\leq\delta} \widetilde{A}_k(z)\,.$$

Proof. Take $v \in H(V_1)$, and call $(\widetilde{u}, \widetilde{u}_-)$ the solution for the inhomogeneous problem (4.37). Then the propagation estimate (4.16) implies that $\widetilde{u}$ is concentrated inside the larger neighbourhood $V_0 \subset \cup_{j,k} \widehat{T}^{++}_{jk}$ (see (4.24)), while $\widetilde{u}^j_- \in H(\widetilde{A}_j(z))$.

We have $R^k_+(z)\widetilde{u} \in H(\widetilde{D}_k)$ so, provided the data satisfies $v^k_+ \in \widetilde{D}_k$, the computations of §4.2.2 show how to solve the homogeneous problem with data $(v_+ - R_+(z)\widetilde{u})$. That solves the full problem. The expressions (4.35) show that the solutions to the homogeneous problem $(\widehat{u}, \widehat{u}^k_-)$ are microlocalized, respectively, in V_0 and in $\widetilde{D}_k \cup \widetilde{A}_k(z)$. □

Remark. The proof of the proposition shows that the neighbourhoods V^k_+ and V^k_- are different. For given data (v, v^+), the solutions (u, u_-) will not in general be concentrated in the same small set as the initial data. This, of course, reflects the fact that a neighbourhood V of K_0 is not invariant under the forward flow, but escapes along the unstable direction. In order to transform the microlocal Grushin problem described in this proposition into a well-posed problem, we need to take care of this escape phenomenon. This will be done using escape functions in order to deform the norms on the spaces $L^2(X)$ (as described in §3.5), but also on the auxiliary spaces $L^2(\mathbb{R}^{n-1})$.

5. A well posed Grushin problem

The difficulty described in the remark at the end of §4 will be resolved by modifying the norms on the space $L^2(X) \times L^2(\mathbb{R}^{n-1})^N$, through the use of exponential weight functions as described in §3.5. These weight functions will be based on the construction described in §3.6.

In most of this section we will consider the scaled operator $P_{\theta,R}$ globally, so we cannot replace it by P any longer. To alleviate notation, we will write this operator

$$P = P_{\theta,R}\,, \quad \theta = M_1 h \log(1/h)\,, \quad R \gg C_0\,, \tag{5.1}$$

where C_0 is the constant appearing in (2.2), and $M_1 > 0$ is a constant (it will be required to satisfy (3.31) once we fix the weight G, and is larger than M_0 appearing in Theorem 1).

We will first discuss the local construction near each Σ_k and then, as in the previous section, adapt it to construct a global Grushin problem.

Our first task is still microlocal: we explain how a deformation of the norm on $L^2(X)$ by a suitable weight function G can be used to deform the norms on the N auxiliary spaces $L^2(\mathbb{R}^{n-1})$, microlocally near $\widetilde{\Sigma}_k$.

5.1. **Exponential weights near Σ_k.** As in §4.1, in this subsection we work microlocally in the neighbourhood Ω_k of one component Σ_k (Ω_k is the neighbourhood described in §4.1); we drop the index k in our notations. Notice that the complex scaling has no effect in this region, so $P \equiv P_{\theta,R}$. We will impose a constraint on the weight function G near Σ, and

construct a weight functions g on $\widetilde{\Sigma}$. The construction of the local solution performed in §4.1 will then be studied in these deformed spaces.

Take a function $g^0 \in \mathcal{C}^\infty_{\rm c}(\mathbb{R}^{n-1})$, and use it to define $\widetilde{G}_0 \in \mathcal{C}^\infty(T^*\mathbb{R}^n)$, so that

$$\widetilde{G}_0(x', x_n, \xi', \xi_n) = g^0(x', \xi') \quad \text{in } \widetilde{\Omega}'.$$

Then, using the Fourier integral operator U given in (4.3), one can construct a weight function $G_0 \in S(T^*X)$ such that

$$G_0^w \equiv U\,(\widetilde{G}_0)^w\,U^* \quad \text{microlocally near } \Omega.$$

Notice that G_0 now depends on h through an asymptotic expansion

$$G_0(h) \sim \sum_{j\geq 0} h^j\,G_{0,j}\,, \quad G_{0,j} \in \mathcal{C}^\infty_{\rm c}(T^*X) \text{ independent of } h\,. \tag{5.2}$$

This weight satisfies $G_{0,0} = \widetilde{G}_0 \circ \widetilde{\kappa}^{-1}$ in Ω, and the invariance property

$$[P(h), G_0^w(x, hD)] \equiv 0 \quad \text{microlocally in } \Omega\,. \tag{5.3}$$

As in §3.5, we rescale these weight functions by

$$G \stackrel{\rm def}{=} Mh\log(1/h)\,G_0, \qquad g \stackrel{\rm def}{=} Mh\log(1/h)\,g^0\,. \tag{5.4}$$

Still using the model hD_{x_n}, one can easily check the intertwining property

$$\begin{aligned} &G^w(x, hD_x; h)\,K(z) \equiv K(z)\,g^w(x', hD_{x'}; h) : H(\widetilde{\Sigma}) \to H(\Omega')\,,\\ &e^{-G^w(x,hD_x;h)/h}\,K(z) \equiv K(z)\,e^{-g^w(x',hD_{x'};h)/h} : H(\widetilde{\Sigma}) \to H(\Omega')\,. \end{aligned} \tag{5.5}$$

Using the weights G and g we define the microlocal Hilbert spaces $H_G(\Omega')$ and $H_g(\widetilde{\Sigma})$ by the method of §3.5. We need to check that the construction of a microlocal solution performed in §4.1.2 remains under control with respect to these new norms.

Lemma 5.1. *The operators*

$$K(z)\ :\ H_g(\widetilde{\Sigma}) \to H_G(\Omega')\,, \quad z \in \mathcal{R}(\delta, M_0, h)$$

satisfy the analogue of (4.7)*. Namely, taking a cutoff χ jumping from 0 to 1 near Σ as in §4.1.2, then any $v_+ \in H_g(\widetilde{\Sigma})$ will satisfy*

$$\langle [(i/h)P, \chi^w]\,K(z)\,v_+, K(\bar{z})\,v_+\rangle_{H_G} \equiv \|v\|^2_{H_g}\,. \tag{5.6}$$

Proof. From the cutoff χ we define the deformed symbol χ_G through

$$\chi_G^w(x, hD) \stackrel{\rm def}{=} e^{-G^w(x,hD)/h}\,\chi^w(x, hD)\,e^{G^w(x,hD)/h}\,.$$

The symbol calculus of §3.5 shows that χ_G also jumps from 0 to 1 near Σ, so that (returning to the convention of using χ for χ^w)

$$\begin{aligned}\langle [(i/h)P,\chi]K(z)v_+, K(\bar z)v_+\rangle_{H_G} &\equiv \langle e^{-G/h}[(i/h)P,\chi]K(z)v_+, e^{-G/h}K(\bar z)v_+\rangle_{L^2}\\ &\equiv \langle K(\bar z)^*[(i/h)P_G,\chi_G]\,K(z)\,e^{-g/h}\,v_+, e^{-g/h}\,v_+\rangle_{L^2}\\ &\equiv \langle K(\bar z)^*[(i/h)P,\chi_G]\,K(z)\,e^{-g/h}\,v_+, e^{-g/h}\,v_+\rangle_{L^2}\\ &\equiv \|e^{-g/h}\,v_+\|^2 \equiv \|v_+\|^2_{H_g}\,.\end{aligned}$$

In the second line we used (5.5), the third line results from $P\equiv P_G$, due to (5.3), and the last one from (4.7) applied to χ_G. □

Equation (5.5) shows that, for $z\in\mathcal{R}(\delta,M_0,h)$, the operators $K(z)$ and $R_\pm(z)$ defined respectively in (4.9) and (4.13), satisfy the same norm estimates with respect to the new norms as for the L^2 norms:

$$\|K(z)\|_{H_g(\widetilde\Sigma)\to H_G(\Omega)} = \mathcal{O}(h^{-M_0\epsilon})\,, \tag{5.7}$$

$$\|R_+(z)\|_{H_G(\Omega)\to H_g(\widetilde\Sigma)} = \mathcal{O}(h^{-M_0\epsilon})\,,\qquad \|R_-(z)\|_{H_g(\widetilde\Sigma)\to H_G(\Omega)} = \mathcal{O}(h^{-M_0\epsilon})\,. \tag{5.8}$$

The arguments presented in §4.1 carry over to the weighted spaces, and the microlocal solution to the problem (4.14) constructed in §4.1.3 satisfies the norm estimates

$$\|u\|_{H_G} + \|u_-\|_{H_g} \lesssim h^{-5M_0\epsilon}\big(\|v\|_{H_G} + \|v_+\|_{H_g}\big)\,. \tag{5.9}$$

Given a function $G_{0,0}(x,\xi)$ satisfying $H_pG_{0,0}=0$ in Ω, one can iteratively construct a full symbol G_0 of the form (5.2), such that (5.3) holds. Now, the lower order terms in G_0 may change the norms only by factors $\big(1+\mathcal{O}(Mh\log(1/h))\big)$, so the same norm estimates hold if we replace G_0 by its principal symbol $G_{0,0}$ in the definition of the new norms. As a result, we get the following

Proposition 5.2. *Take $g^0\in\mathcal{C}^\infty_{\rm c}(T^*\mathbb{R}^{n-1})$, $\widetilde G_0(x',x_n,\xi',\xi_n)=g^0(x',\xi')$, $G_0\in\mathcal{C}^\infty_{\rm c}(X)$ satisfying $G_0=\widetilde G_0\circ\widetilde\kappa$ in Ω, and*

$$G = Mh\log(1/h)\,G_0,\qquad g = Mh\log(1/h)\,\widetilde G_0\,.$$

Then, the estimates (5.7–5.9) hold in the spaces $H_G(\Omega)$, $H_g(\widetilde\Sigma)$.

5.2. **Globally defined operators and finite rank weighted spaces.** In this section we transform our microlocal Grushin problem into a globally defined one. This will require transforming all the microlocally defined operators $(R_\pm(z),\,\mathcal{M}_{jk}(z))$ into globally defined operators acting on $L^2(X)$ or $L^2(\mathbb{R}^{n-1})$. Because our analysis took place near the trapped set K_0, we will need to restrict our auxiliary operators to some subspaces of $L^2(\mathbb{R}^{n-1})$ obtained as images of some finite rank projectors. These subspaces are composed of functions microlocalized near K_0. To show that the resulting Grushin problem is well-posed (invertible), the above construction must be performed using appropriately deformed norms on the spaces $L^2(X)$, $L^2(\mathbb{R}^{n-1})$, obtained by using globally defined weight functions G, g_j.

Our first task is thus to complete the constructions of these global weights, building on §3.6 and §5.1.

5.2.1. *Global weight functions.* We will now construct global weight functions $G \in \mathcal{C}^\infty_{\rm c}(X)$, $g_j \in \mathcal{C}^\infty_{\rm c}(T^*\mathbb{R}^{n-1})$ (one for each section Σ_j). For this, we will use the construction of an escape function away from K_0 presented in §3.6, and modify it near the Poincaré section so that it takes the form required in Proposition 5.2, and allows us to define auxiliary escape functions g_j. These weight functions will allow us to to define *finite rank* realizations of the microlocally defined operators $R_\pm(z)$ and $\mathcal{M}(z)$.

Our escape function $G_0 \in S(T^*X)$ is obtained through a slight modification of the weight $G_2(x,\xi)$ described in (3.28). The modification only takes place near the trapped set $\widehat{K}$, and in particular near the sections Σ_j. The following lemma is easy to verify.

Lemma 5.3. *Let $\{\Omega_j,\}_{j=1,\dots,K}$ be the neighbourhoods of Σ_j described in §4.1.1, Ω'_j and Ω''_j be small neighbourhoods of Ω_j, $\Omega_j \Subset \Omega'_j \Subset \Omega''_j$, and let V be a small neighbourhood of $\widehat{K}_\delta$ (see (3.26)). Then there exists $G_0 \in \mathcal{C}^\infty_{\rm c}(T^*X)$ such that*

$$
\begin{aligned}
&H_pG_0 \geq 1 \quad \text{on} \quad T^*_{B(0,3R)}X \cap p^{-1}([-\delta,\delta]) \setminus W, \quad W \stackrel{\rm def}{=} V \cup \bigcup_{j=1}^N \Omega''_j\,,\\
&H_pG_0 = 0 \quad \text{on} \ \ \Omega'_j,\\
&H_pG_0 \geq 0 \quad \text{on} \quad T^*_{B(0,3R)}X,\\
&H_pG_0 \geq -\delta_0 \quad \text{on } T^*X.
\end{aligned}
\tag{5.10}
$$

Besides, using the coordinate charts $\widetilde{\kappa}_j : \widetilde{\Omega}'_j \to \Omega'_j$ (see §4.1.1), we can construct G_0 such that $G_0 \circ \widetilde{\kappa}_j\restriction \widetilde{\Omega}'_j$ is independent of the energy variable $\xi_n \in [-\delta,\delta]$.

The last assumption (local independence on ξ_n) is not strictly necessary, but it simplifies our construction below, making the auxiliary functions g_j independent of z — see Proposition 5.2.

For the set V we assume that $V \Subset V_1$, where V_1 is the set defined in (4.38). As a consequence, there exists a set V'_1, with $V \Subset V'_1 \Subset V_1$ with the following property. Consider the the parametrix $E(z)$ (4.15) with the time $T = t_{\max} + 5\epsilon$. Then there exists $t_1 > 0$ such that, for any $\rho \in p^{-1}([-\delta,\delta]) \setminus V'_1$, the trajectory segment $\{\Phi^t(\rho),\ 0 \leq t \leq T\}$ spends a time $t \geq t_1$ *outside of* W. The main consequence of this property is a strict increase of the weight along the flow outside V'_1:

$$
\forall \rho \in T^*_{B(0,2R)}X \cap p^{-1}([-\delta,\delta]) \setminus V'_1\,, \qquad G_0(\Phi^T(\rho)) - G_0(\rho) \geq t_1\,.
\tag{5.11}
$$

(Here we use the fact that T is small enough, so that a particle of energy $z \approx 0$ starting inside $T^*_{B(0,2R)}$ at $t = 0$ will remain inside $T^*_{B(0,3R)}$ up to $t = T$.) The set V will be further characterized in the next subsection.

From now on, we will take for weight function $G = Mh\log h\, G_0$ with such a function G_0, and use it to define a global Hilbert norm $\| \bullet \|_{H^k_G(X)}$ as in (3.21). As in Proposition 5.2, we define, for each $j = 1, \dots, N$, the auxiliary weight

$$g_j(x',\xi') \stackrel{\rm def}{=} Mh\log(1/h)\, G_0 \circ \widetilde\kappa_j(x',0,\xi',0)\,, \quad (x',\xi') \in \widetilde\Sigma_j\,, \tag{5.12}$$

and extend it to an element of $\mathcal{C}^\infty_{\rm c}(T^*\mathbb{R}^{(n-1)})$, so that the deformed Hilbert norm

$$\|v\|_{H_{g_j}} = \|e^{-g_j^w(x',hD'_x)/h}\, v\|_{L^2(\mathbb{R}^{n-1})}$$

is globally well-defined. Proposition 5.2 shows that our microlocal construction near Σ_j satisfies nice norm estimates with respect to the spaces $H_G(X)$, H_{g_j}.

To see the advantages of having weights which are escape functions we state the following lemma which results from applying Lemma 3.2 to the Fourier integral operator $\exp(-itP/h)$:

Lemma 5.4. *Suppose that $\rho_1 = \Phi^t(\rho_0)$ for some $t > 0$, and that*

$$\Delta \stackrel{\rm def}{=} G_0(\rho_1) - G_0(\rho_0) > 0\,.$$

Suppose also that $\chi_j \in \mathcal{C}^\infty_{\rm c}(T^*X)$, $j = 0, 1$, have their supports in small neighbourhoods of ρ_j's. Then for h small enough we have

$$\|e^{-itP/h}\,\chi_0^w\|_{H_G\to H_G} \le h^{M\Delta/2}\,, \qquad \|\chi_1^w\, e^{-itP/h}\|_{H_G\to H_G} \le h^{M\Delta/2}\,. \tag{5.13}$$

5.2.2. *Finite dimensional projections.* We want to construct a *finite dimensional* subspace of the Hilbert space $H_{g_j}(\mathbb{R}^{n-1})$, such that the microlocal spaces $H_{g_j}(V^j_\pm)$ are both approximated by it modulo $\mathcal{O}(h^\infty)$.

For each $j = 1, \dots, N$, let S'_j, S_j be two families of open sets with smooth boundaries in $T^*\mathbb{R}^{n-1}$, satisfying

$$\widetilde\kappa_j^{-1}(\widehat{\mathcal{T}}_j) \Subset S'_j \Subset S_j \subset \widetilde D_j\,, \quad j = 1, \dots, N\,. \tag{5.14}$$

In particular, each S_j, S'_j splits into disjoint components $S'_{kj} \Subset S_{kj} \subset \widetilde D_{kj}$.

Once these sets are chosen, we need to adjust the set V in Lemma 5.3, making it thinner if necessary:

Lemma 5.5. *For $\delta > 0$ small enough, there exists $V = \mathrm{neigh}(\widehat K_\delta, V_1)$ and $t_0 > 0$ such that the following property holds.*

For any indices $j = 1, \dots, N$, $k \in J_+(j)$, any $z \in [-\delta, \delta]$ and any point $\rho \in \widetilde D_{kj} \cap S_j$ such that its successor $\widetilde F_{kj,z}(\rho)$ does not belong to S'_k, then the trajectory between $\kappa_{j,z}(\rho)$ and $F_{kj,z}(\kappa_{j,z}(\rho))$ spends a time $t \ge t_0$ outside of $W = V \cup \bigcup_{j=1}^N \Omega''_j$.

The time t_0 is necessarily smaller than the maximal return time $t_{\max}$ of (4.27); on the other hand, t_0 increases if we decrease the width $\sim \epsilon$ of the sets Ω''_j. See figure 8 for a sketch. Now, let

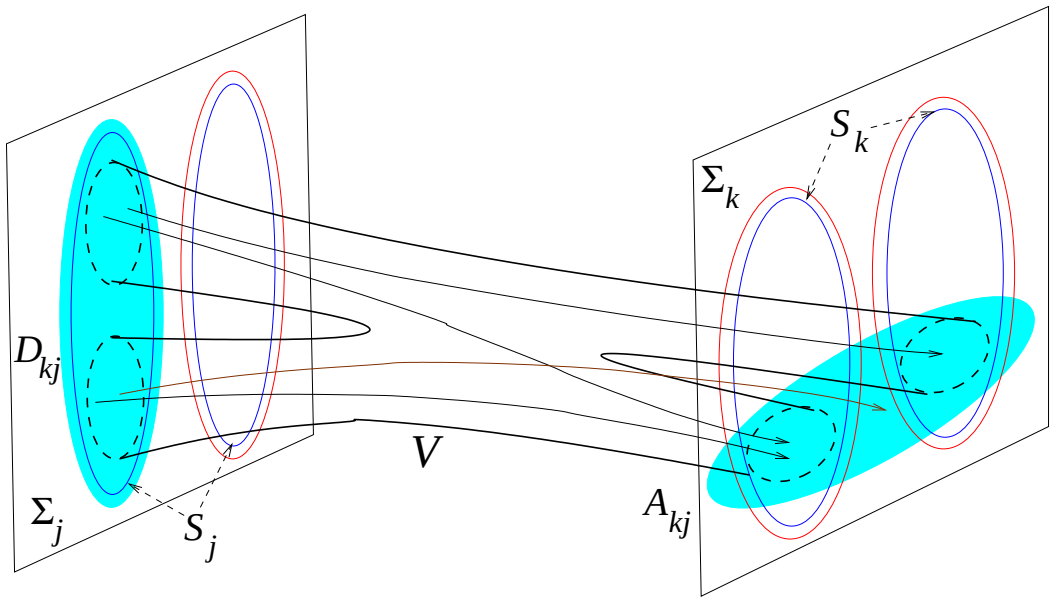

FIGURE 8. Schematic representation (inside some energy layer $p^{-1}(z)$) of the neighbourhood V and the sets S_k, S_j. The departure/arrival sets D_{kj}, A_{kj} are similar to the ones appearing in figure 4. The sets S_k, S_j are represented through their images in Σ_k, Σ_j through $\kappa_{k,z}$, $\kappa_{j,z}$. We showe 3 trajectories staying inside V all the time, and one ending outside S_k.

$$Q_j = Q_j(x', \xi'; h) \in S(T^*\mathbb{R}^{n-1}),$$

with leading symbol q_j independent of h (the leading symbol is meant in the sense of (3.23)). We choose that leading symbol to be real and have the following properties:

$$\begin{aligned} &q_j(\rho) < 0\,, \quad \rho \in S_j\,, \\ &q_j(\rho) > 0\,, \quad \rho \in T^*\mathbb{R}^{n-1} \setminus \overline{S}_j\,, \quad \liminf_{\rho\to\infty} q_j(\rho) > 0. \end{aligned} \tag{5.15}$$

Lemma 3.1 shows that one can choose Q_j so that

$$Q_j^w(x', hD_{x'}) \;:\; H_{g_j}(\mathbb{R}^{n-1}) \longrightarrow H_{g_j}(\mathbb{R}^{n-1}) \quad \text{is self-adjoint.}$$

Under the assumptions (5.15), we know that Q_j has discrete spectrum in a fixed neighbourhood of $\mathbb{R}_-$ when $h > 0$ is small enough. Let

$$\mathcal{H}_j \overset{\text{def}}{=} \Pi_j\big(H_{g_j}(\mathbb{R}^{n-1})\big)\,, \quad \text{where } \Pi_j \overset{\text{def}}{=} \mathbb{1}_{\mathbb{R}_-}\big(Q_j^w(x', hD_{x'})\big)\,, \tag{5.16}$$

that is, Π_j is the spectral projection corresponding to the negative spectrum of Q_j^w. In particular,

$$\|\Pi_j\|_{H_{g_j}\to H_{g_j}} = 1, \qquad \dim(\mathcal{H}_j) \sim c_j\, h^{1-n}\,, \quad c_j > 0\,. \tag{5.17}$$

We group together these projectors in a diagonal matrix $\Pi_h \stackrel{\rm def}{=} {\rm diag}(\Pi_1,\dots,\Pi_N)$ projecting $H_{g_1}(\mathbb{R}^{n-1})\times\cdots H_{g_N}(\mathbb{R}^{n-1})$ onto $\mathcal{H}\stackrel{\rm def}{=}\mathcal{H}_1\times\cdots\mathcal{H}_N$.

The space $\mathcal{H}_j$ will be equipped with the norm $\|\bullet\|_{H_{g_j}}$. For future reference we record the following lemma based on functional calculus of pseudodifferential operators (see for instance [12, Chapter 7]):

Lemma 5.6. *For any uniformly bounded family of states $u=(u(h)\in L^2(\mathbb{R}^{n-1}))_{h\to 0}$,*

$$ {\rm WF}_h(u)\Subset S_j \implies \|u-\Pi_j u\|_{H_{g_j}}=\mathcal{O}(h^\infty)\|u\|_{H_{g_j}}\,. $$

In §5.1 we used the microlocally defined operators

$$ R^j_+(z): H_G(\Omega_j)\to H_{g_j}(\widetilde{\Sigma}_j)\,. $$

Renaming them $R^j_{+,m}(z)$ (where m stands for *microlocal*) we now define

$$ R^j_+(z)\stackrel{\rm def}{=}\Pi_j\, R^j_{+,m}: H_G(X)\to\mathcal{H}_j\,. \tag{5.18} $$

The estimate (5.8) together with the above Lemma shows that

$$ \|R^j_+(z)\|_{H_G(X)\to\mathcal{H}_j}=\mathcal{O}(h^{-M_0\epsilon})\,,\qquad z\in\mathcal{R}(\delta,M_0,h)\,. \tag{5.19} $$

The operators $R^j_+(z)$ are globally well-defined once we choose a specific realization of $R^j_{+,m}(z)$, which gives a unique definition mod $\mathcal{O}(h^\infty)$. We have thus obtained a family of operators

$$ R_+(z)\stackrel{\rm def}{=}(R^1_+,\dots,R^N_+)\ :\ H_G(X)\longrightarrow\mathcal{H}_1\times\cdots\mathcal{H}_N\,. $$

In turn, the operators $R^j_-(z)$ are obtained by selecting a realization of the microlocally defined operator $R^j_{-,m}(z)$ on $H_{g_j}(\widetilde{\Sigma}_j)$, and restricting that realization to $\mathcal{H}_j$:

$$ R^j_-(z)=R^j_{-,m}(z)\,\Pi_j\ :\ \mathcal{H}_j\longrightarrow H_G(X)\,. \tag{5.20} $$

Again, these operators are well defined mod $\mathcal{O}(h^\infty)$. Putting together (5.8) with (5.17) ensures that

$$ \|R^j_-(z)\|_{\mathcal{H}_j\to H_G}=\mathcal{O}(h^{-M_0\epsilon})\,. $$

We group these operators into

$$ \begin{aligned} &R_-(z)\ :\ \mathcal{H}_1\times\cdots\mathcal{H}_N\longrightarrow H_G(X)\\ &R_-(z)u_-=\sum_{j=1}^N R^j_-(z)\,u^j_-\,,\qquad u_-=(u^1_-,\dots,u^N_-)\,. \end{aligned} \tag{5.21} $$

5.3. **A well posed Grushin problem.** With these definitions we consider the following Grushin problem:

$$\begin{gathered}\mathcal{P}(z)\ :\ H^2_G\times\mathcal{H}\to H_G\times\mathcal{H},\qquad \mathcal{H}\overset{\text{def}}{=}\mathcal{H}_1\times\cdots\mathcal{H}_N\,,\\ \mathcal{P}(z)\overset{\text{def}}{=}\begin{pmatrix}(i/h)(P_{\theta,R}(h)-z) & R_-(z)\\ R_+(z) & 0\end{pmatrix},\qquad z\in\mathcal{R}(\delta,M_0,h)\,.\end{gathered}\tag{5.22}$$

Since $P_{\theta,R}(h)-z$ (which we will denote by $P-z$ for short) is a Fredholm operator, so is $\mathcal{P}(z)$, as we have only added finite dimensional spaces. For $\operatorname{Im} z>0$ the operator $(P-z)$ is invertible, so Lemma 3.3 shows that the index of $\mathcal{P}(z)$ is 0. Hence, in order to prove that $\mathcal{P}(z)$ is bijective it suffices to to construct an approximate right inverse and then use a Neumann series. The rest of this section will be devoted to the proof of this (approximate) right invertibility of $\mathcal{P}(z)$.

5.3.1. *A well-posed homogeneous problem.* As before we first consider the homogeneous problem

$$(i/h)(P-z)u+R_-(z)u_-=0\,,\qquad R_+(z)u=v_+\,,\tag{5.23}$$

where only one component v_+^k is nonzero (we may assume that $\|v_+^k\|_{\mathcal{H}_1}=1$). For that we adapt the methods of §4.2.2. We construct an approximate solution using the extended Poisson operator $K_k^+(z)$ (that operator acts on the microlocal space $H_{g_k}(\widetilde{\Sigma}_k)$, so its action on $\mathcal{H}_k$ is well-defined modulo $\mathcal{O}(h^\infty)$), and take

$$u=\chi_b^k\,K_k^+(z)\,v_+^k\,,$$

where χ_b^k is the backwards cutoff function with properties given in (4.12),(4.23) and (4.25). The fact that G increases along the trajectories implies that u satisfies the same norm bound as with the "old norms" (see (4.28)):

$$\|u\|_{H_G(X)}\lesssim h^{-M_0(t_{\max}+\epsilon)}\,\|v_+^k\|_{\mathcal{H}_k}\,.$$

The microlocally defined operator satisfies

$$R^k_{+,m}(z)\,u\equiv v_+^k+\mathcal{O}_{H_{g_k}}(h^\infty)\,,\qquad R^j_{+,m}(z)\,u=\mathcal{O}_{H_{g_j}}(h^\infty),\quad j\neq k\,.$$

As a result, projecting the left hand side onto $\mathcal{H}^k$ has a negligible effect:

$$R^k_+(z)\,u=\Pi_k(v_+^k+\mathcal{O}(h^\infty))=v_+^k+\mathcal{O}_{\mathcal{H}_k}(h^\infty)\,.$$

Following (4.29) we write

$$(i/h)(P-z)u\equiv[(i/h)P,\chi_b^k]\,K_k^+(z)v_+\in H_G(X)\,.\tag{5.24}$$

As noticed in §4.2.2, the transport properties of $K_k^+(z)$ show that u is microlocalized inside the union of tubes $\cup_{j\in J_+(k)}T_{jk}^{++}(z)$, so the right hand side in (5.24) splits into a component

concentrated near $\widetilde{D}_k$, and other components concentrated near the arrival sets $A_{jk}(z)$, $j \in J_+(k)$. We rewrite (4.34) for the present data:

$$(i/h)(P-z)u \equiv R^k_{-,m}(z)v^k_+ - \sum_{j\in J_+(k)} R^j_{-,m}(z)\mathcal{M}_{jk}(z)v^k_+ \,. \tag{5.25}$$

Each state $\mathcal{M}_{jk}(z)v^k_+$ is microlocalized inside the arrival set $\widetilde{A}_{jk}(z) \subset \widetilde{\Sigma}_j$, which is not contained in S_j in general – see the remark at the end of §4 and Fig. 8.

Consequently one could fear that replacing the operators $R^j_{-,m}(z)$ by the truncated operators $R^j_-(z)$ would drastically modify the above right hand side. The microlocally weighted spaces H_G, H_{g_j} have been constructed precisely to avoid this problem. The mechanism is a direct consequence of the relative properties of the sets S_j and V explained in Lemma 5.5. Namely, a point $\rho_k \in S_{jk}$ is either "good", if its image $\rho_j = F_{jk,z}(\rho_k) \in S'_j$, or "bad", in which case

$$G_0(\rho_j) - G_0(\rho_k) \geq t_0 \,. \tag{5.26}$$

Let us choose a cutoff

$$\chi_j \in \mathcal{C}^\infty_{\rm c}(S_j), \quad \chi_j = 1 \ \text{ on } S'_j \,, \quad \chi_j = 0 \ \text{ outside } \ \mathrm{neigh}(S'_j, S_j) \,. \tag{5.27}$$

Since the Fourier integral operator $\mathcal{M}_{jk}(z) : H(\widetilde{D}_k) \to H(\widetilde{A}_{jk}(z))$ is uniformly bounded, (5.26) implies the norm estimate (see Lemma 5.4)

$$\forall v^k_+ \in \mathcal{H}_k \,, \qquad \|(1-\chi^w_j)\,\mathcal{M}_{jk}(z)\,v^k_+\|_{H_{g_j}} \lesssim h^{Mt_0 - M_0 t_{\max}}\,\|v^k_+\|_{\mathcal{H}_k}, \qquad z \in \mathcal{R}(\delta, M_0, h) \,.$$

For this estimate to be small when $h \to 0$, we require the ratio M_0/M to be small enough to ensure the condition

$$t_0 - \frac{M_0}{M} t_{\max} \geq t_0/2 > 0 \,.$$

(The bounds (3.31) and $M_0 \leq M_1$ show that the ratio M_0/M can indeed be chosen arbitrary small.)

On the other hand, $\chi^w_j\,\mathcal{M}_{jk}(z)\,v^k_+$ is microlocalized inside $\mathrm{neigh}(S'_j, S_j)$, so Lemma 5.6 implies that $(\Pi_j - 1)\chi^w_j\,\mathcal{M}_{jk}(z)\,v^k_+ = \mathcal{O}_{H_{g_j}}(h^\infty)$. Putting these estimates altogether, we find that

$$\forall v^k_+ \in \mathcal{H}_k, \qquad \mathcal{M}_{jk}(z)\,v^k_+ = \Pi_j\,\mathcal{M}_{jk}(z)\,v^k_+ + \mathcal{O}(h^{Mt_0/2})\,\|v^k_+\| \,. \tag{5.28}$$

This crucial estimate shows that the projection of $\mathcal{M}_{jk}(z)\,v^k_+$ on $\mathcal{H}_j$ has a negligible effect. We now define the finite rank operators

$$\widetilde{M}_{jk}(z) \stackrel{\rm def}{=} \begin{cases} \Pi_j\,\mathcal{M}_{jk}(z)\,\Pi_k : \mathcal{H}_k \to \mathcal{H}_j \,, \ j \in J_+(k) \,, \\ 0 \quad \text{otherwise} \,, \end{cases} \quad \text{in short } \widetilde{M}(z) = \Pi_h \mathcal{M}(z)\,\Pi_h \,. \tag{5.29}$$

This operators satisfy the same norm bounds (4.36) as their infinite rank counterparts. Using these operators, and remembering that the operators $R_-^j : \mathcal{H}_j \to H_G(X)$ are bounded by $\mathcal{O}(h^{-M_0\epsilon})$, we rewrite (5.25) as

$$(i/h)(P-z)u \equiv R_-^k(z)v_+^k - \sum_{j\in J_+(k)} R_-^j(z)\widetilde{M}_{jk}(z)\, v_+^k + \mathcal{O}(h^{Mt_0/3})\,\|v_+^k\|\,.$$

Generalizing the initial data to arbitrary $v_+ \in \mathcal{H}_1 \times \cdots \times \mathcal{H}_N$, we obtain the

Proposition 5.7. *Assume $z \in \mathcal{R}(\delta, M_0, h)$. Let $v_+ \in \mathcal{H}$. Then there exists $(u, u_-) \in H^2_G(X) \times \mathcal{H}$ such that*

$$(i/h)(P-z)u + R_-(z)\,u_- = \mathcal{O}(h^{Mt_0/3})\|v_+\|_{\mathcal{H}} \qquad \text{in } H_G(X), \tag{5.30}$$

$$R_+(z)u = v_+ + \mathcal{O}(h^\infty)\,\|v_+\|_{\mathcal{H}} \qquad \text{in } \mathcal{H}, \tag{5.31}$$

$$\|u\|_{H_G(X)} \lesssim h^{-M_0(t_{\max}+\epsilon)}\,\|v_+\|_{\mathcal{H}}, \qquad \|u_-\|_{\mathcal{H}} \lesssim h^{-M_0 t_{\max}}\|v_+\|_{\mathcal{H}}. \tag{5.32}$$

The second part of the solution, u_-, is of the form

$$u_- = (\widetilde{M}(z) - Id)v_+\,, \quad \|\widetilde{M}(z)\|_{\mathcal{H}\to\mathcal{H}} \lesssim h^{-M_0 t_{\max}}\,,$$

where $\widetilde{M}(z) = (\widetilde{M}_{jk}(z))_{j,k=1,..,N}$ is the matrix of operators defined in (5.29).

We collect some properties of the operators $\widetilde{M}_{jk}(z)$, $j \in J_+(k)$, for $z \in [-\delta, \delta]$:

- $\widetilde{M}_{jk}(z)$ is uniformly bounded, and $\mathrm{WF}'_h(\widetilde{M}_{jk}(z)) \subset \overline{S}_j \times \overline{S}_k$.
- take $\rho_k \in \overline{S}_k$, $\rho_j = \tilde{F}_{jk,z}(\rho_k) \in \overline{S}_j$:
  (1) if the trajectory segment connecting the points $\kappa_{k,z}(\rho_k)$, $\kappa_{j,z}(\rho_j)$ is contained in W, then microlocally near (ρ_j, ρ_k), $\widetilde{M}_{jk}(z)$ is an h-Fourier integral operator of order zero with associated canonical transformation $\tilde{F}_{jk,z} = \kappa_{j,z}^{-1} \circ F_{jk,z} \circ \kappa_{k,z}$
  (2) if furthermore the above segment is disjoint from the support of G, then $\widetilde{M}_{jk}(z)$ is microlocally unitary near (ρ_j, ρ_k).
  (3) if, on the opposite, this segment contains a part outside W, then there exist $\chi_j \in \mathcal{C}^\infty_{\mathrm{c}}(\mathrm{neigh}(\rho_j))$, $\chi_k \in \mathcal{C}^\infty_{\mathrm{c}}(\mathrm{neigh}(\rho_k))$, equal to 1 near ρ_j and ρ_k respectively, and a time $t(\rho_k) > 0$ independent of the exponent M, such that

$$\chi_j^w\, \widetilde{M}_{jk}(z)\chi_k^w = \mathcal{O}(h^{M\,t(\rho_k)}) : H_{g_k} \to H_{g_j}\,.$$

For $z \in \mathcal{R}(\delta, M_0, h)$ similar statements hold, modulo the fact that the symbol of the Fourier integral operator is multiplied by $\exp(-izt_+/h)$, which modifies the order of the operator.

5.3.2. *A well-posed inhomogeneous problem.* Let us now consider the inhomogeneous problem

$$(i/h)(P_{\theta,R} - z)u + R_-(z)u_- = v \qquad v \in H_G(X)\,. \tag{5.33}$$

We will use a partition of unity to decompose v into several component.

Take $\psi_\delta \in S(T^*X)$, $\psi_\delta = 1$ near $p^{-1}([-\delta/2,\delta/2])$, and $\psi_\delta = 0$ outside $p^{-1}([-\delta,\delta])$. The operator $(P_{\theta,R} - z)$ is elliptic outside $p^{-1}[-\delta/2,\delta/2]$. Taking $\widetilde{\psi}_\delta$ similar with ψ_δ but with $\operatorname{supp}\widetilde{\psi}_\delta \subset p^{-1}([-\delta/2,\delta/2])$, the operator

$$L \stackrel{\text{def}}{=} (P_{\theta,R} - z - i\widetilde{\psi}_\delta^w) : H^2_G \to H_G$$

is invertible, with uniformly bounded inverse $L^{-1} \in \Psi^0_h$. Hence, by taking

$$u = (h/i)L^{-1}\,(1-\psi_\delta^w)\,v\,,$$

we find

$$(i/h)(P_{\theta,R}-z)u = (i/h)(P_{\theta,R}-z-i\widetilde{\psi}_\delta^w)u + \mathcal{O}(h^\infty)\,\|u\| = (1-\psi_\delta^w)\,v + \mathcal{O}(h^\infty)\|v\|\,,$$

which solves our problem for the data $(1-\psi_\delta^w)\,v$. The first equality uses pseudodifferential calculus and the fact that $\psi_\delta \equiv 1$ on the support of $\widetilde{\psi}_\delta$:

$$\widetilde{\psi}_\delta^w L^{-1}(1-\psi_\delta^w) = \mathcal{O}_{\mathscr{S}'\to\mathscr{S}}(h^\infty)\,.$$

Let us now consider the data $(\psi_\delta^w v)$ microlocalized in $p^{-1}([-\delta,\delta])$. We split this state using a spatial cutoff $\psi_R \in \mathcal{C}^\infty_{\rm c}(X)$, such that $\psi_R = 1$ in $B(0,R)$, $\psi_R = 0$ outside $B(0,2R)$. To solve the equation

$$(i/h)(P_{\theta,R}-z)u = \widetilde{v}, \qquad \widetilde{v} = (1-\psi_R)\,\psi_\delta^w\,v\,, \tag{5.34}$$

we take the Ansatz

$$u = E(z)\,\widetilde{v}\,, \tag{5.35}$$

with $E(z)$ the parametrix of (4.15) (with P replaced by $P_{\theta,R}$), for the same time $T = t_{\max}+\epsilon$ as in (4.39). It satisfies

$$(i/h)(P_{\theta,R}-z)u = \widetilde{v} - e^{-iT(P_{\theta,R}-z)/h}\,\widetilde{v}\,. \tag{5.36}$$

The time T is small enough, so that

$$\Phi^t\left(p^{-1}([-\delta,\delta]) \setminus T^*_{B(0,R)}X\right) \cap T^*_{B(0,R/2)}X = \emptyset\,, \quad 0 \le t \le T\,.$$

Hence, the states

$$\widetilde{v}(t) \stackrel{\text{def}}{=} e^{-it(P_{\theta,R}-z)/h}\widetilde{v}$$

are all microlocalized outside $T^*_{B(0,R/2)}X$ for $t \in [0,T]$. The estimate (3.30) (adapted to the weight G_0) then implies that [31, Lemma 6.4]

$$\partial_t\|\widetilde{v}(t)\|^2_{H_G} = \frac{2}{h}\,\mathrm{Im}\langle(P_{\theta,R}-z)\widetilde{v}(t),\widetilde{v}(t)\rangle_{H_G} \le (-M_1/C_1 + 2M_0)\,\log(1/h)\,, \quad \forall t\in[0,T]\,,$$

where $C_1 > 0$ is independent of the choice of M_1. Once more, we assume M_0/M_1 is small enough so that $-M_1/C_1 + 2M_0 \le -M_1/2C_1$, and hence

$$\|e^{-iT(P_{\theta,R}-z)/h}\,\widetilde{v}\|_{H_G} \le C\,h^{M_1T/2C_1}\|\widetilde{v}\|_{H_G}\,,$$

so the problem (5.34) is solved modulo a remainder $\mathcal{O}(h^{M_1T/2C_1})$.

We now consider the component $(\psi_R\psi_\delta^w v)$ microlocalized in $T^*_{B(0,2R)}\cap p^{-1}([-\delta,\delta])$. We split it again using a cutoff $\psi_{V_1}\in\mathcal{C}^\infty_{\rm c}(V_1)$, $\psi_{V_1}=1$ in the set $V_1'\Subset V_1$ (see the discussion after Lemma 5.3). To solve the problem for the inhomogeneous data

$$\widetilde v=(1-\psi_{V_1}^w)\psi_R\psi_\delta^w v\,,$$

we use the Ansatz (5.35), resulting in the estimate (5.36). The microlocalization of $\widetilde v$ outside of V_1', together with the assumption (5.11), implies the norm estimate (see Lemma 5.4)

$$\|e^{-iT(P_{\theta,R}-z)/h}\,\widetilde v\|_{H_G}\le C\,h^{Mt_1/2-M_0T}\|\widetilde v\|_{H_G}\,.$$

Again, we assume M_0/M small enough, so that $Mt_1/2-M_0T\ge Mt_1/3$. We have solved the problem for $\widetilde v$ up to a remainder $\mathcal{O}(h^{Mt_1/3})\|\widetilde v\|_{H_G}$.

We finally consider the data $\widetilde v=\psi_{V_1}^w\psi_R\psi_\delta^w v$ microlocalized inside V_1. For this data, we can use the microlocal analysis of §4.2.3. If $\mathrm{WF}_h(\widetilde v)$ is contained inside $V_1\cap\widehat T_{jk}^{--}$, then $\mathrm{WF}_h(\chi_b^k\,E(z)\,\widetilde v)$ (see the Ansatz (4.40)) will intersect Σ_j inside the arrival set $\widetilde A_{jk}(z)$, but not necessarily inside S_j. However, the same phenomenon as in Lemma 5.5 occurs: there exists a time $t_3>0$ such that, for any $z\in[-\delta,\delta]$ and any $\rho\in V_1\cap T_{jk}^{--}(z)$,

$$\rho_+(\rho)\in\Sigma_j(z)\setminus\kappa_{j,z}(S_j')\Longrightarrow G_0(\rho_+(\rho))-G_0(\rho)\ge t_3\,. \tag{5.37}$$

If we decompose $R^j_{+,m}(z)E(z)\widetilde v$ using the cutoff χ_j of (5.27), the property (5.37) implies that

$$\|(1-\chi_j^w)\,R^j_{+,m}(z)E(z)\widetilde v\|_{H_{g_j}}=\mathcal{O}(h^{Mt_3/2-M_0T})\|\widetilde v\|_{H_G}\,.$$

Again we assume M_0/M small enough, so that $Mt_3/2-M_0T\ge Mt_3/3$. Hence, if we set

$$\begin{aligned}u_-^j&=R_+^j(z)\chi_j^w\,E(z)\widetilde v\\&=R^j_{+,m}(z)\chi_j^wE(z)\widetilde v+\mathcal{O}(h^\infty)\|\widetilde v\|_{H_G}\\&=R^j_{+,m}(z)E(z)\widetilde v+\mathcal{O}(h^{Mt_3/3})\|\widetilde v\|_{H_G}\,,\end{aligned}$$

we end up with a solution of (5.33) modulo a remainder $\mathcal{O}(h^{Mt_3/3})\|\widetilde v\|_{H_G}$.

We recall that M_1/M is bounded by (3.31), so all the above error estimates can be put in the form $\mathcal{O}(h^{cM})\|\widetilde v\|_{H_G}$, with $c>0$ independent of M: we have thus shown that the problem (5.33) admits a solution for any $v\in H_G$, up to this remainder. We may then apply Proposition 5.7 to solve the resulting homogeneous problem, and get an approximate solution for the full problem (5.22). We summarize this solution in the following

Proposition 5.8. *Assume $z\in\mathcal{R}(\delta,M_0,h)$. Let $(v,v_+)\in H_G\times\mathcal{H}$. Then there exists $(u,u_-)\in H^2_G\times\mathcal{H}$ such that*

$$\begin{cases}(i/h)(P-z)u+R_-(z)u_- &= v+\mathcal{O}(h^{cM})(\|v\|_{H_G}+\|v_+\|_{\mathcal{H}}) \qquad \text{in } H_G(X)\,,\\ R_+(z)u &= v_+ +\mathcal{O}(h^\infty)\,(\|v\|_{H_G}+\|v_+\|_{\mathcal{H}}) \qquad \text{in } \mathcal{H}\,,\end{cases} \tag{5.38}$$

$$\|u\|_{H^2_G}+\|u_-\|_{\mathcal{H}}\lesssim h^{-M_0(2t_{\max}+2\epsilon)}\left(\|v\|_{H_G}+\|v_+\|_{\mathcal{H}}\right). \tag{5.39}$$

5.4. **Invertibility of the Grushin problem.** We can transform this approximate solution into an exact one. The system (5.38) can be expressed as an approximate inverse of $\mathcal{P}(z)$:

$$\begin{pmatrix} u \\ u_- \end{pmatrix} = \widetilde{\mathcal{E}}(z) \begin{pmatrix} v \\ v_+ \end{pmatrix}, \tag{5.40}$$
$$\mathcal{P}(z)\,\widetilde{\mathcal{E}}(z) = I + \mathcal{R}(h) : H_G \times \mathcal{H} \longrightarrow H_G \times \mathcal{H}\,, \quad \|\mathcal{R}(h)\| = \mathcal{O}(h^{cM})\,.$$

For h small enough the operator $I + \mathcal{R}(h)$ can be inverted by a Neumann series, so we obtain an exact right inverse of $\mathcal{P}(z)$,

$$\mathcal{E}(z) = \widetilde{\mathcal{E}}(z)\,(I + \mathcal{R}(z))^{-1}\,.$$

Since $\mathcal{P}(z)$ is of index zero, $\mathcal{E}(z)$ is also a left inverse, which proves the well-posedness of our Grushin problem (5.22).

Theorem 2. *We consider $h > 0$ small enough, and $z \in \mathcal{R}(\delta, M_0, h)$. For every $(v, v_+) \in H_G \times \mathcal{H}$, there exists a unique $(u, u_-) \in H^2_G \times \mathcal{H}$ such that*

$$\begin{cases} (i/h)(P_{\theta,R} - z)u + R_-(z)u_- & = v \\ R_+(z)u & = v_+\,, \end{cases} \tag{5.41}$$

where $R_\pm(z)$ are defined by (5.18) *and* (5.20)*. The estimates (5.39) hold, so if we write*

$$\begin{pmatrix} u \\ u_- \end{pmatrix} = \mathcal{E}(z) \begin{pmatrix} v \\ v_- \end{pmatrix}, \qquad \mathcal{E}(z) = \begin{pmatrix} E & E_+ \\ E_- & E_{-+} \end{pmatrix}\,,$$

then the following operator norms (between the appropriate Hilbert spaces) are bounded by:

$$\|E\|\,, \quad \|E_+\|, \quad \|E_-\|\,, \quad \|E_{-+}\| = \mathcal{O}(h^{-M_0(2t_{\max}+2\epsilon)}). \tag{5.42}$$

Moreover, we have a precise expression for the effective Hamiltonian:

$$E_{-+}(z) = -I + \widetilde{M}(z) + \mathcal{O}_{\mathcal{H}\to\mathcal{H}}(h^{c'M}) \stackrel{\text{def}}{=} -I + M(z,h)\,, \tag{5.43}$$

where $\widetilde{M}(z)$ is the matrix of "open quantum maps" defined in (5.29) *and described after Proposition 5.7.*

Remark. If we restrict the parameter z to a rectangle of height $|\operatorname{Im} z| \leq Ch$ instead of $|\operatorname{Im} z| \leq M_0 h \log(1/h)$, the bounds (5.43) become $\|E_*(z)\| = \mathcal{O}(1)$.

Theorem 1 and the formula (1.8) follow from this more precise result. In fact, the equality (3.13) shows that

$$\operatorname{rank} \oint_z \chi R(w)\chi\, dw = \operatorname{rank} \oint_z \chi R_{\theta,R}(w)\chi\, dw = -\frac{1}{2\pi i} \operatorname{tr} \oint_z R_{\theta,R}(w)\, dw\,, \tag{5.44}$$

see [41, Proposition 3.6] for the proof of the last identity in the simpler case of compactly supported perturbations, and [39, Section 5] for the general case.

The well-posedness of our Grushin problem means that we can apply formula (3.33) recalled in §3.7. It shows that the right hand side in (5.44) is equal to

$$\frac{1}{2\pi i}\,\mathrm{tr}\oint_z E_{-+}(w)^{-1}E'_{-+}(w)\,dw\,,$$

which in view of (5.43) gives (1.8). The exponent $L \overset{\text{def}}{=} c'M$ in the remainder of (5.43) depends on the integer $M > 0$ used in the scaling of the weight function G, which can be chosen arbitrary large, independently of $c' > 0$.

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