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Yann Demichel

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YANN DEMICHEL

ABSTRACT. This paper considers random balls in a D-dimensional Euclidean space whose centers are prescribed by a homogeneous Poisson point process and whose radii are forced to be in a finite interval with a power-law distribution. Random fields are constructed by counting the number of covering balls at each point. We are mainly interested in the simulation of these fields and in the empirical estimation of its index H. Finally simulations are given.

INTRODUCTION

Various random fields are obtained by summing elementary patterns properly rescaled and normalized. A pioneer work in this area is due to Cioczek-Georges and Mandelbrot [5] where a sum of random micropulses in dimension one, or generalizations in higher dimensions, are rescaled and normalized in order to get a fractional Brownian field of index $H < 1/2$ (antipersistent fBf). In that paper, it is emphasized that the power law distribution prescribed for the length of the micropulses makes it impossible to get $H > 1/2$. Using similar models in dimension one, recent works ([6, 7]) have examined the internet traffic modeling. The resulting signal is proved to exhibit a long range dependence ($H > 1/2$), in accordance with observations. Such a range for index H is made possible either by prescribing the connection lengths with heavy tails, or by forcing the number of long connections.

In the present paper, we build elementary fields by counting the number of balls whose centers and radii are distributed according a Poisson point process. If the centers are uniformly distributed in the space, we force the radii to be in each given slice $(\alpha^{j+1}, \alpha^j]$ ($\alpha \in (0,1)$ is fixed and j ranges in $\mathbb{Z}$). Moreover their distribution has a power-law density of the type $r^{-D-1+2H}$, $H \in \mathbb{R}$. Next we build a piling field $F_{j_{\min}, j_{\max}}$ summing all the slices from $j_{\min}$ to $j_{\max}$.

We are mainly interested in the simulation of fields obtained by piling elementary slices and in the estimation of its index H. Let us note that simulating such fields appears as very tractable since the basic objects are balls and the basic operation consists in counting. We first simulate each slice and then proceed to the piling of the slices. This procedure is similar to the construction of the "general multitype Boolean model" in [10].

Concerning the index estimation, we use the structure functions as introduced in [17]. Roughly speaking, the q-structure function of a given function f is equal to the L^q-norm of the ε-increments of f. We use these tools to provide two different empirical estimators of the H index: The first one takes into account the small scales and the second one the intermediate scales.

OUTLINE OF THE PAPER

The random fields we deal with are introduced in Section 1. First the piling field is obtained by summing the elementary slices between a lower and an upper scale. In Section 2,

we introduce a notion of D-dimensional q-structure functions. They are used to estimate the H index of the piling field. Two cases are presented: working at small scales (i.e. working with small balls) in dimension $D = 1$ and working at intermediate scales for any D. Section 3 is devoted to the simulation procedure of the fields on a cube as well as the numerical computation of the structure functions. We conclude this section by numerical examples.

1. Some multiscale random models

1.1. Elementary slices.

Let $H \in \mathbb{R}$. For $\alpha \in (0,1)$ and $j \in \mathbb{Z}$ we consider $\Phi_j = \left\{ (X_j^n, R_j^n)_n \right\}$ a Poisson point process in $\mathbb{R}^D \times \mathbb{R}^+$ of intensity

$$\nu_j(dx, dr) = dx \otimes r^{-D-1+2H} \mathbb{1}_{(\alpha^{j+1}, \alpha^j]}(r) dr \,. \tag{1}$$

Note that Φ_j is well-defined since ν_j is a nonnegative measure on $\mathbb{R}^D \times \mathbb{R}^+$. We write $B(x, r)$ for the closed ball of center x and radius $r > 0$ with respect to the Euclidean norm $\|\cdot\|$. We consider the associated so called "random balls field" T_j as defined in [3] that provides, at each point $y \in \mathbb{R}^D$, the number of balls $B(X_j^n, R_j^n)$ that contain the point y, namely

$$T_j(y) = \sum_{(X_j^n, R_j^n) \in \Phi_j} \mathbb{1}_{B(X_j^n, R_j^n)}(y) \,. \tag{2}$$

Equivalently, we can represent the field T_j through a stochastic integral

$$T_j(y) = \int_{\mathbb{R}^D \times \mathbb{R}^+} \mathbb{1}_{B(x,r)}(y) N_j(dx, dr), \tag{3}$$

where N_j is a Poisson random measure on $\mathbb{R}^D \times \mathbb{R}^+$ of intensity ν_j.

In order to ensure that the right hand side of (3) is well defined, it is sufficient to remark that, by Fubini's theorem,

$$\int_{\mathbb{R}^D \times \mathbb{R}^+} \mathbb{1}_{B(x,r)}(y) \, \nu_j(dx, dr) = V_D \int_{\alpha^{j+1}}^{\alpha^j} r^{-1+2H} dr = V_D \left(\frac{1 - \alpha^{2H}}{2H} \right) \alpha^{2Hj} < +\infty \,,$$

where V_D is the Lebesgue measure of $B(0,1)$.

Let us remark that due to the translation invariance of the Lebesgue measure, the random field T_j is stationary. Moreover T_j admits moments of any order and according to [1], for $n \geqslant 1$, the n-th moment of $T_j(y)$ is given by

$$\mathbb{E}(T_j(y)^n) = \sum_{(r_1, \ldots, r_n) \in I(n)} K_n(r_1, \ldots, r_n) \prod_{k=1}^{n} \left(\int_{\mathbb{R}^D \times \mathbb{R}^+} \mathbb{1}_{B(x,r)}(y)^k \, \nu_j(dx, dr) \right)^{r_k}$$

$$= \sum_{(r_1, \ldots, r_n) \in I(n)} K_n(r_1, \ldots, r_n) \left(V_D \left(\frac{1 - \alpha^{2H}}{2H} \right) \alpha^{2Hj} \right)^{\sum_{k=1}^{n} r_k} ,$$

where $I(n) = \left\{ (r_1, \ldots, r_n) \in \mathbb{N}^n \,;\, \sum_{k=1}^{n} k r_k = n \right\}$ and $K_n(r_1, \ldots, r_n) = n! \left(\prod_{k=1}^{n} r_k! (k!)^{r_k} \right)^{-1}$.

Finally let us mention that the covariance of T_j is simply given by

$$\mathrm{Cov}(T_j(y), T_j(y')) = \int_{\alpha^{j+1}}^{\alpha^j} \left| B(y, r) \cap B(y', r) \right| r^{-D-1+2H} dr \,,$$

where $|A|$ denotes the Lebesgue measure of a set A.

Notice that the covariance function of the field $\left(T_j(\alpha^l y)\right)_{y \in \mathbb{R}^D}$ is obtained by considering the field $(T_{j-l}(y))_{y \in \mathbb{R}^D}$ according to the following scale invariance

$$\mathrm{Cov}(T_j(\alpha^l y), T_j(\alpha^l y')) = \alpha^{2Hl}\, \mathrm{Cov}(T_{j-l}(y), T_{j-l}(y')).$$

It yields a kind of aggregate similarity property as defined in [2]:

Proposition 1.1. *Let $\alpha \in (0,1)$ and $j \in \mathbb{Z}$. Then, for any $l \in \mathbb{Z}$ such that $m = \alpha^{2Hl} \in \mathbb{N}$,*

$$\left\{ T_{j+l}(\alpha^l y)\,;\, y \in \mathbb{R}^D \right\} \overset{fdd}{=} \left\{ \sum_{k=1}^m T_j^{(k)}(y)\,;\, y \in \mathbb{R}^D \right\},$$

where $\left(T_j^{(k)}\right)_{k \geqslant 1}$ are iid copies of T_j.

Proof. Let us assume that there exists $l \in \mathbb{Z}$ such that $m = \alpha^{2Hl} \in \mathbb{N}$. Let $p \geqslant 1$ and $y_1, \ldots, y_p \in \mathbb{R}^D$, $u_1, \ldots, u_p \in \mathbb{R}$. Then

$$\log \mathbb{E} \exp\left(i \sum_{n=1}^p u_n T_{j+l}(\alpha^l y_n) \right) = \int_{\mathbb{R}^D \times \mathbb{R}^+} \left(e^{i \sum_{n=1}^p \mathbb{1}_{B(x,r)}(\alpha^l y_n)} - 1 \right) \nu_{j+l}(dx, dr)$$

$$= \alpha^{2Hl} \int_{\mathbb{R}^D \times \mathbb{R}^+} \left(e^{i \sum_{n=1}^p u_n \mathbb{1}_{B(x,r)}(y_n)} - 1 \right) \nu_j(dx, dr) = \sum_{k=1}^m \log \mathbb{E} \exp\left(i \sum_{n=1}^p u_n T_j(y_n) \right),$$

where the second line is obtained by a change of variables. Hence the result follows. $\qquad\square$

1.2. **Piling of the elementary slices.**

Now let us consider the associated piling random field. Let $(\Phi_j)_{j \in \mathbb{Z}}$ be independent Poisson point processes in $\mathbb{R}^D \times \mathbb{R}^+$, with each Φ_j of intensity $\nu_j(dx, dr)$ given by (1). For $j_{\min}, j_{\max} \in \mathbb{Z}$ with $j_{\min} \leqslant j_{\max}$ the piling random field

$$F_{j_{\min}, j_{\max}}(y) = \sum_{j=j_{\min}}^{j_{\max}} T_j(y)\,,\ y \in \mathbb{R}^D \tag{4}$$

can be considered as the random balls field associated with a Poisson random measure $N_{j_{\min}, j_{\max}}$ of intensity

$$\nu_{j_{\min}, j_{\max}}(dx, dr) = dx \otimes r^{-D-1+2H} \mathbb{1}_{(\alpha^{j_{\max}+1},\, \alpha^{j_{\min}}]}(r) dr\,. \tag{5}$$

Increments.

Let us mention some facts about the regularity of $F_{j_{\min}, j_{\max}}$. Since $F_{j_{\min}, j_{\max}}$ is discrete valued, it is not continuous. However one can look at the second order regularity. The increments of $F_{j_{\min}, j_{\max}}$ are clearly centered. Let us compute their variance.

For all $y, y' \in \mathbb{R}^D$ one has, using (4) and (3) :

$$F_{j_{\min}, j_{\max}}(y) - F_{j_{\min}, j_{\max}}(y') = \int_{\mathbb{R}^D \times \mathbb{R}^+} (\mathbb{1}_{B(x,r)}(y) - \mathbb{1}_{B(x,r)}(y')) N_{j_{\min}, j_{\max}}(dx, dr)$$

$$= \int_{\mathbb{R}^D \times \mathbb{R}^+} (\mathbb{1}_{B(y-y',r)}(x - y') - \mathbb{1}_{B(0,r)}(x - y')) N_{j_{\min}, j_{\max}}(dx, dr)\,.$$

Therefore, using the translation invariance of the Lebesgue measure :

$$\mathrm{Var}(F_{j_{\min},j_{\max}}(y) - F_{j_{\min},j_{\max}}(y')) = \int_{\mathbb{R}^D} \int_{\alpha^{j_{\max}+1}}^{\alpha^{j_{\min}}} (\mathbb{1}_{B(y-y',r)}(u) - \mathbb{1}_{B(0,r)}(u))^2 r^{-D-1+2H} du\,dr\,.$$

Since the increments are centered, using Fubini's theorem, we deduce the exact expression :

$$\mathbb{E}(F_{j_{\min},j_{\max}}(y) - F_{j_{\min},j_{\max}}(y'))^2 = \int_{\alpha^{j_{\max}+1}}^{\alpha^{j_{\min}}} \left| B(y-y',r) \triangle B(0,r) \right| r^{-D-1+2H} dr\,, \qquad (6)$$

where $A \triangle B$ is the symmetrical difference between two subsets of $\mathbb{R}^D$.

Moreover one has the bound

$$\mathbb{E}\left(F_{j_{\min},j_{\max}}(y) - F_{j_{\min},j_{\max}}(y') \right)^2 \leqslant \int_{\mathbb{R}^D \times \mathbb{R}^+} \left| B(y-y',r) \triangle B(0,r) \right| r^{-D-1+2H} du\,dr\,.$$

Therefore, using now the rotation invariance of the Lebesgue measure :

$$\mathbb{E}(F_{j_{\min},j_{\max}}(y) - F_{j_{\min},j_{\max}}(y'))^2 \leqslant C_{D,H} \left\| y - y' \right\|^{2H} \qquad (7)$$

with

$$C_{D,H} = \int_{\mathbb{R}^+} |B(e_1,r) \triangle B(0,r)| \, r^{-D-1+2H} \, dr \quad \text{and} \quad e_1 = (1,0,\dots,0) \in \mathbb{R}^D. \qquad (8)$$

Finally, notice that for any $u \in \mathbb{R}^D$ and $r \in \mathbb{R}^+$, one can find a constant $C(u) \in (0,+\infty)$ such that

$$|B(u,r) \triangle B(0,r)| \leqslant C(u) \min\left(r^D, r^{D-1} \right).$$

Thus $C_{D,H}$ is a finite constant for all $H \in (0,1/2)$. In these cases Inequality (7) appears as an Holder condition as the second order.

Convergence of the piling.
When $j_{\min} \to -\infty$ and $j_{\max} \to +\infty$, Equality (5) invites us to consider a Poisson random measure N of intensity

$$\nu(dx,dr) = dx \otimes r^{-D-1+2H} \mathbb{1}_{(0,+\infty)}(r) dr\,.$$

Note that ν satisfies

$$\forall\, y \in \mathbb{R}^D \quad \int_{\mathbb{R}^D \times \mathbb{R}^+} \mathbb{1}_{B(x,r)}(y)\, \nu(dx,dr) = +\infty$$

so that the stochastic integral (3) is not well-defined. However, in view of (7), when $H \in (0,1/2)$, one can consider the random field F_H defined as

$$F_H(y) = \int_{\mathbb{R}^D \times \mathbb{R}^+} (\mathbb{1}_{B(x,r)}(y) - \mathbb{1}_{B(x,r)}(0)) N(dx,dr)\,,\ y \in \mathbb{R}^D. \qquad (9)$$

We obtain a random balls field known as *fractional Poisson field* (fPf) (see [2, 14]). The fPf is a centered process with stationary increments. Even though the fPf is not Gaussian, it shares the same covariance function as the fractional Brownian field B_H of index H (see [15]):

$$\mathrm{Cov}(F_H(y), F_H(y')) = \frac{C_{D,H}}{2} \left(\|y\|^{2H} + \|y'\|^{2H} - \|y-y'\|^{2H} \right) \qquad (10)$$

where $C_{D,H}$ is given by (8). Similar random fields are obtained in [9, 8] where Poisson random measures are also used but yield a different model.

We can easily link the limit behaviour of the piling field $F_{j_{\min}, j_{\max}}$ when $j_{\min} \to -\infty$ and $j_{\max} \to +\infty$ to F_H.

Proposition 1.2. *Let $H \in (0, 1/2)$. Then, for all $p \geqslant 1$, $y_1, \ldots, y_p \in \mathbb{R}^D$, the sequence*

$$(F_{j_{\min}, j_{\max}}(y_1) - F_{j_{\min}, j_{\max}}(0), \ldots, F_{j_{\min}, j_{\max}}(y_p) - F_{j_{\min}, j_{\max}}(0))$$

converges almost surely as $j_{\min} \to -\infty$ and $j_{\max} \to +\infty$ to $(F_H(y_1), \ldots, F_H(y_p))$ where F_H is a fPf.

Proof. Let us fix $y \in \mathbb{R}^D$ and write $M_n = F_{0,n}(y) - F_{0,n}(0)$, $n \geqslant 0$. The sequence $(M_n)_{n \geqslant 0}$ is a martingale with respect to its natural filtration $\sigma(M_{n'}, 0 \leqslant n' \leqslant n)$. Thus in order to prove that $(M_n)_{n \geqslant 0}$ converges almost surely it is enough to prove that it is bounded in $L^2(\Omega)$. Since $H \in (0, 1/2)$, (7) yields :

$$\mathbb{E}(M_n^2) \leqslant C_{D,H} \, \|y\|^{2H}, \ \ C_{D,H} < \infty \, ,$$

which proves the required condition. Hence $(M_n)_{n \geqslant 0}$ converges almost surely to a limit M_∞^+ as $n \to +\infty$.

Similarly, we prove that the sequence $\left(F_{-n,0}(y) - F_{-n,0}(0)\right)_{n \geqslant 0}$ converges almost surely to a limit M_∞^- as $n \to \infty$.

Thus, the sequence $\left(F_{j_{\min}, j_{\max}}(y) - F_{j_{\min}, j_{\max}}(0)\right)_{j_{\min}, j_{\max}}$ converges almost surely to $M_\infty = M_\infty^- + M_\infty^+$ as $j_{\min} \to -\infty$ and $j_{\max} \to +\infty$. Finally it remains to check that the limit M_∞ is a fPf. This may be simply done computing the limit of the characteristic function of

$$(F_{j_{\min}, j_{\max}}(y_1) - F_{j_{\min}, j_{\max}}(0), \ldots, F_{j_{\min}, j_{\max}}(y_p) - F_{j_{\min}, j_{\max}}(0))$$

as $j_{\min} \to -\infty$ and $j_{\max} \to +\infty$. $\qquad\square$

Normal convergence.

A classical normal convergence is obtained for shot noise fields when the number of shots tends to $+\infty$ (see [11] for instance). Here we can obtain such a convergence. Let $\left(F_{j_{\min}, j_{\max}}^{(k)}\right)_{k \geqslant 1}$ be iid copies of $F_{j_{\min}, j_{\max}}$. According to the Central Limit Theorem :

$$\left\{ \frac{1}{\sqrt{K}} \sum_{k=1}^{K} \left(F_{j_{\min}, j_{\max}}^{(k)}(y) - F_{j_{\min}, j_{\max}}^{(k)}(0)\right) ; y \in \mathbb{R}^D \right\} \xrightarrow[K \to +\infty]{fdd} \left\{ W_{j_{\min}, j_{\max}}(y) ; y \in \mathbb{R}^D \right\} \quad (11)$$

where $W_{j_{\min}, j_{\max}}$ is a centered Gaussian process with stationary increments, $W_{j_{\min}, j_{\max}}(0) = 0$ and a variance given by

$$\mathrm{Var}(W_{j_{\min}, j_{\max}}(y)) = \int_{\alpha^{j_{\max}+1}}^{\alpha^{j_{\min}}} |B(y, r) \triangle B(0, r)| \, r^{-D-1+2H} \, dr. \quad (12)$$

Looking at Figure 2 one can observe that the limit field seems more regular than the original piling $F_{j_{\min}, j_{\max}}$. This is due to the fact that second order regularity implies almost surely regularity for a Gaussian field.

2. Empirical estimation of the H index

According to (4) and (5), H appears as the important parameter of the field $F_{j_{\min}, j_{\max}}$. This section is devoted to its estimation. Our method is inspired by the inequality (7). One can think of the quadratic variation method (see [12]) but here, in practice, it is very difficult to get arbitrary small increments. Thus we present an alternative method using simulated samples of the field $F_{j_{\min}, j_{\max}}$.

2.1. **General tools.**
If $f : \mathbb{R}^D \to \mathbb{R}$ is a continuous function then one can consider, for $q > 0$, the functional

$$S_q(f, \varepsilon) = \frac{1}{D} \sum_{i=1}^{D} \int_{[0,1]^D} |f(t + \varepsilon e_i) - f(t - \varepsilon e_i)|^q \, dt \,, \ \varepsilon \geqslant 0 \,, \tag{13}$$

where $\{e_1, \ldots, e_D\}$ denoted the canonical basis of $\mathbb{R}^D$. Such quantity, called the q-structure function of f, has been used to study the fractal behaviour of various functions f (see [17, 13]).

Here $F_{j_{\min}, j_{\max}}$ is not continuous but $\mathbb{E}\, |F_{j_{\min}, j_{\max}}(\varepsilon e_i) - F_{j_{\min}, j_{\max}}(-\varepsilon e_i)|^q < \infty$. Thus, by Fubini's theorem and stationarity, we can consider

$$\mathbb{E}\left(S_q(F_{j_{\min}, j_{\max}}, \varepsilon)\right) = \frac{1}{D} \sum_{i=1}^{D} \int_{[0,1]^D} \mathbb{E}\, |F_{j_{\min}, j_{\max}}(t + \varepsilon e_i) - F_{j_{\min}, j_{\max}}(t - \varepsilon e_i)|^q \, dt. \tag{14}$$

The value $q = 2$ allows us to give exact and explicit formulas especially in dimension $D = 1$.

The $D = 1$ case.

Proposition 2.1. *Assume that $D = 1$. Then one has* $\mathbb{E}\left(S_2(F_{j_{\min}, j_{\max}}, \varepsilon)\right) =$

$$\begin{cases} \dfrac{4}{1 - 2H} \left(\alpha^{(2H-1)(j_{\max}+1)} - \alpha^{(2H-1)j_{\min}} \right) \varepsilon & \text{if } \varepsilon \in [0, \alpha^{j_{\max}+1}) \,, \\[2ex] \dfrac{4}{1 - 2H} \varepsilon \left(\varepsilon^{2H-1} - \alpha^{(2H-1)j_{\min}} \right) + \dfrac{2}{H} \left(\varepsilon^{2H} - \alpha^{2H(j_{\max}+1)} \right) & \text{if } \varepsilon \in [\alpha^{j_{\max}+1}, \alpha^{j_{\min}}] \,, \\[2ex] \dfrac{2}{H} \left(\alpha^{2Hj_{\min}} - \alpha^{2H(j_{\max}+1)} \right) & \text{if } \varepsilon > \alpha^{j_{\min}} \,. \end{cases} \tag{15}$$

Proof. When $D = 1$, Equation (6) gives

$$\mathbb{E}\left(F_{j_{\min}, j_{\max}}(t + \varepsilon) - F_{j_{\min}, j_{\max}}(t - \varepsilon)\right)^2 = \int_{\alpha^{j_{\max}+1}}^{\alpha^{j_{\min}}} \left| [2\varepsilon - r, 2\varepsilon + r] \triangle [-r, r] \right| r^{-2+2H} dr$$

One checks that

$$[2\varepsilon - r, 2\varepsilon + r] \triangle [-r, r] = \begin{cases} [2\varepsilon - r, 2\varepsilon + r] \cup [-r, r] & \text{if } r \leqslant \varepsilon \,, \\ [-r, 2\varepsilon - r] \cup [r, 2\varepsilon + r] & \text{if } r > \varepsilon \,, \end{cases}$$

where the two unions are disjoined. Therefore

$$\mathbb{E}\left(F_{j_{\min}, j_{\max}}(t + \varepsilon) - F_{j_{\min}, j_{\max}}(t - \varepsilon)\right)^2 = 4 \int_{\alpha^{j_{\max}+1}}^{\alpha^{j_{\min}}} \min\{r, \varepsilon\} r^{-2+2H} dr \,.$$

To conclude we just have to consider the position of ε and the fact that the latter quantity does not depend on t. $\qquad\qquad\square$

The $D \geqslant 1$ case. For general D the volume of the set $B(2\varepsilon e_i, r) \Delta B(0, r)$ is not given by tractable formulas. Thus it seems quite difficult to expand the previous calculus to higher dimensions. However it is possible to obtain an exact expression for the expectation of $S_2(F_{j_{\min}, j_{\max}}, \varepsilon)$ by looking at the large ε.

Proposition 2.2. *For all $\varepsilon > \alpha^{j_{\min}}$, one has :*

$$\mathbb{E}\left(S_2(F_{j_{\min}, j_{\max}}, \varepsilon)\right) = \frac{\pi^{D/2}}{\Gamma(1 + D/2)\,H}\left(\alpha^{2Hj_{\min}} - \alpha^{2H(j_{\max}+1)}\right).$$

Proof. If $\varepsilon > r$ then

$$\forall\, i \in \{1, \ldots, D\} \quad \left|B(2\varepsilon e_i, r)\Delta B(0, r)\right| = 2V_D\, r^D$$

since the two balls are disjoined.
Thus (6) gives

$$\mathbb{E}(F_{j_{\min}, j_{\max}}(t + \varepsilon e_i) - F_{j_{\min}, j_{\max}}(t - \varepsilon e_i))^2 = \frac{V_D}{H}\left(\alpha^{2Hj_{\min}} - \alpha^{2H(j_{\max}+1)}\right).$$

Hence the result follows, since the latter quantity does not depend on t nor on i, and V_D is expressed using the Gamma Euler's function. $\square$

2.2. Empirical estimators.
Now we show how to derive several estimators of the H index from Propositions 2.1 and 2.2. We assume that we have independent realizations of $F_{j_{\min}, j_{\max}}$. Moreover we fix an integer $N \geqslant 1$ such that $\alpha^{j_{\max}+1} = N^{-1}$. In practice it will be linked to the resolution δ of the grid by $\delta = N^{-1}$.

The $D = 1$ case. First we assume that $D = 1$ and $j_{\min} \geqslant 0$. We focus on the small balls or equivalently on the small scales and use $S_2(F_{j_{\min}, j_{\max}}, \varepsilon)$ for small ε.

Proposition 2.3. *Assume that $D = 1$. Let $0 \leqslant j_{\min} < j_{\max}$ and let $\left(F_{j_{\min}, j_{\max}}^{(k)}\right)_{k \geqslant 1}$ be iid copies of $F_{j_{\min}, j_{\max}}$ of index H. Finally, let us define for all $K \geqslant 1$:*

$$\widehat{\gamma_K} = \frac{1}{2\log 2}\,\log\frac{\displaystyle\sum_{k=1}^{K} S_2\left(F_{j_{\min}, j_{\max}}^{(k)}, 2N^{-1}\right)}{\displaystyle\sum_{k=1}^{K} S_2\left(F_{j_{\min}, j_{\max}}^{(k)}, N^{-1}\right)}. \tag{16}$$

Then, when K goes to $+\infty$, $\widehat{\gamma_K}$ converges almost surely to $h_1(H)$ where

$$h_1(H) = H + \frac{1}{2\log 2}\log\left(\frac{1 - (2\alpha^{-j_{\min}}N^{-1})^{1-2H} + (1-2H)(1-2^{-2H})/(2H)}{1 - (\alpha^{-j_{\min}}N^{-1})^{1-2H}}\right). \tag{17}$$

Proof. Since $\alpha^{j_{\max}+1} = N^{-1}$, Proposition 2.1 implies that for all $\varepsilon \in [N^{-1}, \alpha^{j_{\min}}]$:

$$\mathbb{E}\left(S_2\left(F_{j_{\min}, j_{\max}}, \varepsilon\right)\right) = \frac{4}{1-2H}\varepsilon^{2H} - \frac{4}{1-2H}\varepsilon\,\alpha^{(1-2H)j_{\min}} + \frac{2}{H}\left(\varepsilon^{2H} - N^{-2H}\right).$$

Therefore we obtain :

$$\frac{\mathbb{E}\left(S_2\left(F_{j_{\min}, j_{\max}}, 2N^{-1}\right)\right)}{\mathbb{E}\left(S_2\left(F_{j_{\min}, j_{\max}}, N^{-1}\right)\right)} = 2^{2H}\left(\frac{1 - (2\alpha^{-j_{\min}}N^{-1})^{1-2H} + (1-2H)(1-2^{-2H})/(2H)}{1 - (\alpha^{-j_{\min}}N^{-1})^{1-2H}}\right).$$

Thus :

$$\frac{1}{2\log 2}\log\frac{\mathbb{E}\left(S_2\left(F_{j_{\min}, j_{\max}}, 2N^{-1}\right)\right)}{\mathbb{E}\left(S_2\left(F_{j_{\min}, j_{\max}}, N^{-1}\right)\right)}$$
$$= H + \frac{1}{2\log 2}\log\left(\frac{1 - (2\alpha^{-j_{\min}}N^{-1})^{1-2H} + (1-2H)(1-2^{-2H})/(2H)}{1 - (\alpha^{-j_{\min}}N^{-1})^{1-2H}}\right). \tag{18}$$

But one has, by the law of large numbers :

$$\mathbb{E}\left(S_2\left(F_{j_{\min},j_{\max}},\cdot\right)\right) = \lim_{K\to+\infty} \frac{1}{K}\sum_{k=1}^{K} S_2\big(F^{(k)}_{j_{\min},j_{\max}},\cdot\big).$$

So $\widehat{\gamma_K} \underset{K\to+\infty}{\longrightarrow} h_1(H)$ almost surely and the result follows from (18). $\qquad\square$

From this result, we deduce a first estimator $\widehat{H_K}$ for index H setting $\widehat{H_K} = h_1^{-1}\left(\widehat{\gamma_K}\right)$.

The $D \geqslant 1$ and $j_{\min} > 0$ case. Since $j_{\min} > 0$ it is possible to see all balls whatever their radii are. This allows us to propose an estimator for H in the general multidimensional case $D \geqslant 1$. We will focus on the large scales *ie* use $S_2(F_{j_{\min},j_{\max}},\varepsilon)$ for ε near 1.

Under the assumption $j_{\min} > 0$ the interval $(\alpha^{j_{\min}}, 1]$ is not empty. Let M^* be the smallest integer such that $\varepsilon_{M^*} = n_{M^*}N^{-1} > \alpha^{j_{\min}}$. Then for all $m \in \{M^*, \dots, M\}$, $\varepsilon_m \in (\alpha^{j_{\min}}, 1]$ (note that the larger $j_{\min}$ is the more points we have).

Proposition 2.4. *Let $0 < j_{\min} < j_{\max}$ and let $\left(F^{(k)}_{j_{\min},j_{\max}}\right)_{k\geqslant 1}$ be iid copies of $F_{j_{\min},j_{\max}}$ of index H. Finally, let us define for all $K \geqslant 1$:*

$$\widehat{\pi_K} = \frac{1}{K}\sum_{k=1}^{K}\left(\frac{1}{M-M^*+1}\sum_{m=M^*}^{M} S_2\left(F^{(k)}_{j_{\min},j_{\max}},\varepsilon_m\right)\right). \tag{19}$$

Then, when K goes to $+\infty$, $\widehat{\pi_K}$ converges almost surely to $h_2(H)$ where

$$h_2(H) = \frac{\pi^{D/2}}{\Gamma(1+D/2)H}\left(\alpha^{2Hj_{\min}} - N^{-2H}\right). \tag{20}$$

Proof. Proposition 2.2 implies :

$$\forall\, m \in \{M^*, \dots, M\}\quad \mathbb{E}\left(S_2\left(F_{j_{\min},j_{\max}},\varepsilon_m\right)\right) = \frac{\pi^{D/2}}{\Gamma(1+D/2)H}\left(\alpha^{2Hj_{\min}} - N^{-2H}\right).$$

Thus:

$$\frac{1}{M-M^*+1}\sum_{m=M^*}^{M} \mathbb{E}\left(S_2\left(F_{j_{\min},j_{\max}},\varepsilon_m\right)\right) = \frac{\pi^{D/2}}{\Gamma(1+D/2)H}\left(\alpha^{2Hj_{\min}} - N^{-2H}\right). \tag{21}$$

Now, by the law of large numbers, one has:

$$\mathbb{E}\left(S_2\left(F_{j_{\min},j_{\max}},\cdot\right)\right) = \lim_{K\to+\infty}\frac{1}{K}\sum_{k=1}^{K} S_2\left(F^{(k)}_{j_{\min},j_{\max}},\cdot\right).$$

So $\widehat{\pi_K} \underset{K\to+\infty}{\longrightarrow} h_2(H)$ almost surely and the result follows from (21). $\qquad\square$

As previously, we deduce a second estimator $\widehat{H'_K}$ for index H considering $\widehat{H'_K} = h_2^{-1}\left(\widehat{\pi_K}\right)$.

3. Simulations and numerical examples

All the Matlab codes are available at http://www.math-info.univ-paris5.fr/~demichel

3.1. **Simulation of the piling field on a cube.**

We focus here on the simulation of the piling field $F_{j_{\min}, j_{\max}}$ for $H \in (0, 1/2)$. We generate exact simulations of the fields T_j and $F_{j_{\min}, j_{\max}}$ (see (2) and (4)) on the cube $[c, c + d]^D$ with $c \in \mathbb{R}$ and $d \geqslant 1$. Let us recall that T_j may be written as the sum

$$T_j = \sum_{(X_j^n, R_j^n) \in \Phi_j} \mathbb{1}_{B(X_j^n, R_j^n)}.$$

If X_j^n is at a distance of the cube larger than R_j^n then no point of the cube is covered by $B(X_j^n, R_j^n)$. Since $R_j^n \leqslant \alpha^j$, when simulating the slice number j, it is enough to pick up centers of balls randomly in the enlarged cube $[c - \alpha^j, c + d + \alpha^j]^D$.

Let us denote $c_{\alpha, H} = (\alpha^{-D+2H} - 1)/(D - 2H)$ and consider the measure on $\mathbb{R}^D \times \mathbb{R}^+$

$$\widetilde{\nu}_j(dx, dr) = c_{\alpha, H}(d + 2\alpha^j)^D \alpha^{-j(D-2H)} \mu_j(dx) \otimes \rho_j(dr) \tag{22}$$

where

$$\begin{cases} \mu_j(dx) = \dfrac{1}{(d + 2\alpha^j)^D} \, \mathbb{1}_{[c-\alpha^j, \, c+d+\alpha^j]^D}(x) \, dx \\[2mm] \rho_j(dr) = c_{\alpha, H}^{-1} \, \alpha^{j(D-2H)} r^{-D-1+2H} \mathbb{1}_{(\alpha^{j+1}, \, \alpha^j]}(r) \, dr \end{cases} \tag{23}$$

are respectively two probability measures for centers and radii of random balls.

We simulate T_j considering

$$T_j(y) = \sum_{n=1}^{\Lambda_j} \mathbb{1}_{B(X_j^n, R_j^n)}(y), \quad y \in [c, c + d]^D, \tag{24}$$

where

- $(X_j^n)_n$ is a family of iid random variables with law $\mu_j(dx)$
- $(R_j^n)_n$ is a family of iid random variables with law $\rho_j(dr)$
- Λ_j is a Poisson random variable with parameter $c_{\alpha, H}(d + 2\alpha^j)^D \alpha^{-j(D-2H)}$.

Let us recall that a simple way to generate the sequence $(R_j^n)_n$ is to use the pseudo-inverse method : we get $R_j^n = \alpha^j(\alpha^{-D+2H} - (\alpha^{-D+2H} - 1)V_j^n)^{-1/(D-2H)}$ with V_j^n a uniform random variable on $[0, 1]$.

Consequently, considering independent realizations of $(T_j)_{j_{\min} \leqslant j \leqslant j_{\max}}$ one simulate $F_{j_{\min}, j_{\max}}$ using (4) and

$$\sum_{j=j_{\min}}^{j_{\max}} T_j(y) = \sum_{j=j_{\min}}^{j_{\max}} \left(\sum_{n=1}^{\Lambda_j} \mathbb{1}_{B(X_j^n, R_j^n)}(y) \right), \quad y \in [c, c + d]^D. \tag{25}$$

In practice $F_{j_{\min}, j_{\max}}$ is only obtained on a discrete subset of $[c, c + d]^D$, say a grid with a step of size $\delta \in (0, 1)$. Thus it no longer makes sense to consider balls with radii smaller than δ. Since the smallest radius is greater than $\alpha^{j_{\max}+1}$, one will assume that $\alpha^{j_{\max}+1} \geqslant \delta$. On the other hand, to get the most precise details, it seems natural to assume exactly that $\alpha^{j_{\max}+1} = \delta$. Consequently, given the resolution δ, the slice factor α will be fixed by $\alpha^{j_{\max}+1} = \delta$. Finally, one is often interested in the small scales behaviour of the field and not

in the general "look-like" geometry. Thus, in order to not see the large balls, one may take $j_{\min} \geqslant 0$.

We have simulated, in dimension $D = 2$, the piling field $F_{0,15}$ on the cube $[0,1]^2$ with $\delta = 0.005$. Figure 1 shows the construction at different steps, precisely the three slices T_5, T_{10}, T_{15}, and the corresponding pilings $F_{0,5}$, $F_{0,10}$ and $F_{0,15}$. Figure 2 illustrates the Central Limit Theorem of Section 1.2 for the same field $F_{0,15}$.
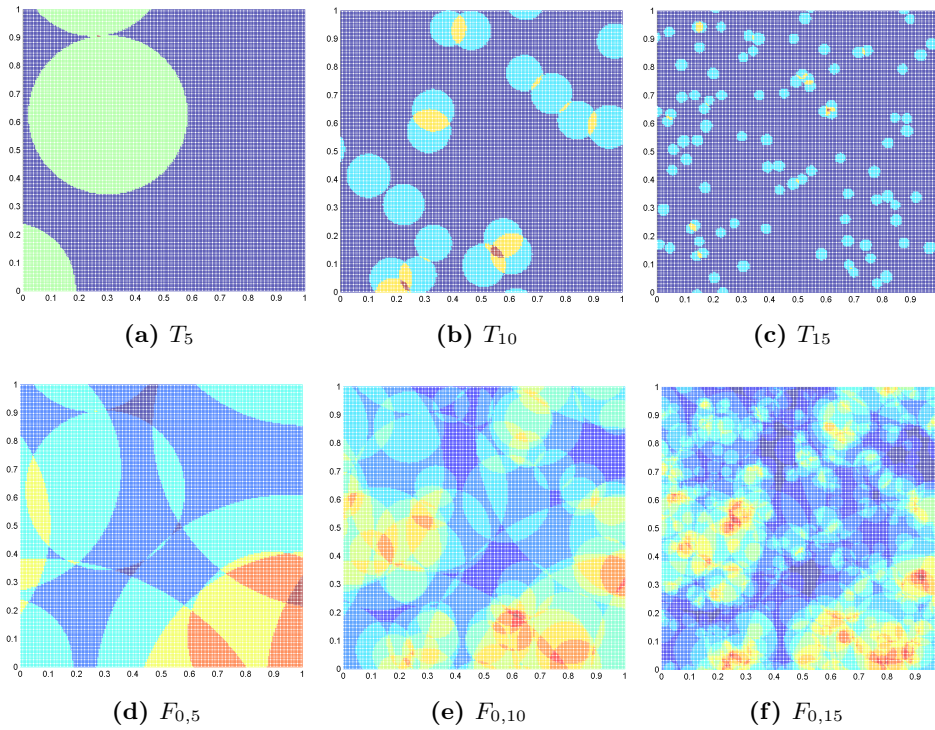

(a) T_5 (b) T_{10} (c) T_{15}

(d) $F_{0,5}$ (e) $F_{0,10}$ (f) $F_{0,15}$

FIGURE 1. Step by step piling.

(a) $K = 1$ (b) $K = 5$ (c) $K = 10$

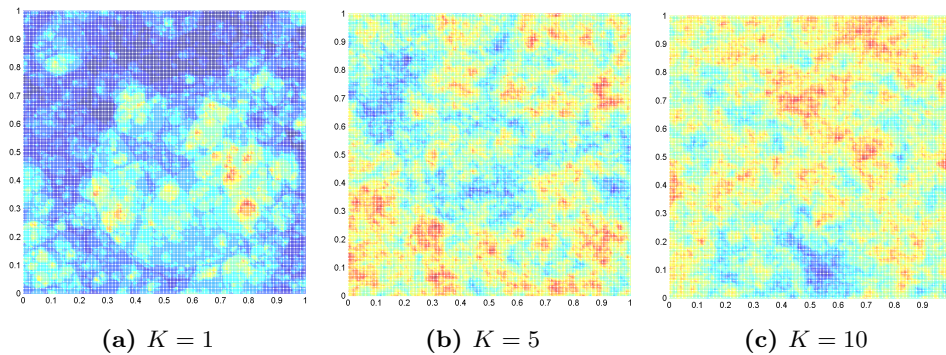FIGURE 2. Normal convergence when $K \to +\infty$.

3.2. Simulation of $S_q(f, \varepsilon)$.

We explain now how to use the $q-$structure functions in a practical way. To simplify we only deal with the case $D = 1$ and $[c, c+d]^D = [0,1]$. Let us consider a function $f : \mathbb{R} \to \mathbb{R}$. We suppose that we can simulate f on a regular subdivision $\tau = \{\tau_i = iN^{-1}\}$ of $[-1, 2]$ with step $\delta = N^{-1}$ (where $N \geqslant 2$). We consider $S_q(f, \varepsilon)$ given by (13) for $\varepsilon \in [N^{-1}, 1] \cap \tau$. More precisely, we choose a finite sequence of $M \geqslant 2$ integers n_m (with $n_1 = 1$ and $n_M = N$) and put

$$\varepsilon_m = \tau_{n_m} = n_m N^{-1}. \tag{26}$$

Let us note that the smallest ε considered corresponds to the resolution of the grid N^{-1}. Then, for $1 \leqslant m \leqslant M$, we approximate the integral defining $S_q(f, \varepsilon_m)$ by its Riemann sum :

$$S_q(f, \varepsilon_m) = \int_0^1 |f(t + \varepsilon_m) - f(t - \varepsilon_m)|^q \, dt \approx \frac{1}{N} \sum_{i=1}^N |f(\tau_{i+n_m}) - f(\tau_{i-n_m})|^q. \tag{27}$$

Let us emphasize that Propositions 2.1 and 2.2 still hold when replacing $S_2(f, \varepsilon_m)$ by the discrete sum (27).

Now we focus on the behaviour of $S_q(f, \varepsilon_m)$ with respect to ε_m. In general, the $\varepsilon-$increments of f behave like a power of ε as ε goes to 0. Thus this invites us to use log-log plots. Notice that at the resolution of the grid, it is what happens for the variance of the increments of $F_{j_{\min}, j_{\max}}$ (see formulas (15)).

For $1 \leqslant m \leqslant M$, let us write $\eta_m = \log n_m = \log(N\varepsilon_m)$ and

$$L_q(f, \eta_m) = \frac{1}{q} \log S_q(f, \varepsilon_m) = \frac{1}{q} \log S_q\left(f, N^{-1}10^{\eta_m}\right)$$

and consider the log-log plot $\{(\eta_m, L_q(f, \eta_m)) \, ; \, 1 \leqslant m \leqslant M\}$, which is called the $q-structure$ curve of f. In order to obtain the η_m approximately equally spaced, one usually assumes that the $\varepsilon_m s$ have an arithmetic progression (but n_m should be an integer). However since $n_1 = 1$ and $n_2 \geqslant 2$, we always have $\eta_2 - \eta_1 \geqslant \log 2$. In order to get the minimal lag $\eta_2 - \eta_1$ we set $n_2 = 2$.

3.3. Numerical examples of estimation in dimension $D = 1$.

We focus here on the empirical estimation of the H index. In practice the $F_{j_{\min}, j_{\max}}^{(k)}$ are simulated on a regular grid with step $\delta = N^{-1}$ using the results of Section 3.1. We compute $S_2^2\left(F_{j_{\min}, j_{\max}}^{(k)}, N^{-1}\right)$ and $S_2^2\left(F_{j_{\min}, j_{\max}}^{(k)}, 2N^{-1}\right)$ for each $F_{j_{\min}, j_{\max}}^{(k)}$ using the discrete sums (27). This gives $\widehat{\gamma_K}$ and we find $\widehat{H_K}$ by solving the equation $h_1(h) = \widehat{\gamma_K}$ with a numerical approximation procedure (e.g. the standard Newton method).

Let us give a first example. We consider $F_{j_{\min}, j_{\max}}$ for different values of H. The processes are simulated on a regular grid of $[-1, 2]$ with step $\delta = 5.10^{-4}$ (so $N = 2000$). We chose $j_{\min} = 0$ and $j_{\max} = 15$. Table 1 shows the results for $K = 500$.

We see that H is well approximated by $\widehat{H_K}$ whatever H is.

Let us give a second example. We look again at $F_{j_{\min}, j_{\max}}$, for different values of H. We assume $D = 1$. The processes are again simulated on a regular grid of $[-1, 2]$ with step

H	$h_1(H)$	$\widehat{\gamma_K}$	$\widehat{H_K}$
0.45	0.47399	0.47425	0.45159
0.40	0.46519	0.46553	0.40182
0.35	0.45531	0.45589	0.35282
0.30	0.44473	0.44452	0.29906
0.25	0.43375	0.43492	0.25528
0.20	0.42265	0.42196	0.19688
0.15	0.41161	0.41051	0.14496
0.10	0.40075	0.40111	0.10169
0.05	0.39013	0.39028	0.05074

TABLE 1.   Estimation of H with $(j_{\min}, j_{\max}) = (0, 15)$ and $K = 500$.

$\delta = 5.10^{-4}$ (so $N = 2000$). We chose $j_{\min} = 5$ and $j_{\max} = 15$. We use the two estimators $\widehat{H_K}$ and $\widehat{H'_K}$ with $K = 500$. Table 2 reports the different results.

(a) Using $\widehat{H_K}$				(b) Using $\widehat{H'_K}$			
H	$h_1(H)$	$\widehat{\gamma_K}$	$\widehat{H_K}$	H	$h_2(H)$	$\widehat{\pi_K}$	$\widehat{H'_K}$
0.45	0.46579	0.46579	0.45001	0.45	0.51932	0.52374	0.44877
0.40	0.45784	0.45853	0.40415	0.40	0.73624	0.74679	0.39800
0.35	0.44910	0.44847	0.34656	0.35	1.05565	1.05701	0.34982
0.30	0.43974	0.43863	0.29427	0.30	1.53341	1.52188	0.30099
0.25	0.42993	0.42895	0.24513	0.25	2.26068	2.27908	0.24897
0.20	0.41983	0.41942	0.19800	0.20	3.38875	3.41444	0.19909
0.15	0.40959	0.41122	0.15795	0.15	5.17488	5.20930	0.14924
0.10	0.39934	0.39921	0.09937	0.10	8.06343	8.05267	0.10015
0.05	0.38917	0.38904	0.04938	0.05	12.83808	12.87241	0.04971

TABLE 2. Estimation of H with $(j_{\min}, j_{\max}) = (5, 15)$ and $K = 500$.

We see that in both cases $j_{\min} = 0$ and $j_{\min} = 5$, the estimates given by $\widehat{H_K}$ are with the same precision. When $j_{\min} = 5$, $\widehat{\pi_K}$ is far from $h_2(H)$ whereas $\widehat{\gamma_K}$ is close to $h_1(H)$. However the estimator $\widehat{H'_K}$ is better than $\widehat{H_K}$ as H decreases. Actually the function h_2 is more convenient for a numerical inversion since its derivative is larger than the h_1 one (see Figure 3).

Finally, Figure 4 shows the 2-structure curves of $F_{j_{\min}, j_{\max}}$ for $H = 0.25$ (see (3.2)) where $S_2(F_{j_{\min}, j_{\max}}, \varepsilon_m)$ is replaced by its empirical mean $\frac{1}{K} \sum_{k=1}^{K} S_2(F_{j_{\min}, j_{\max}}^{(k)}, \varepsilon_m)$ with $K = 500$.

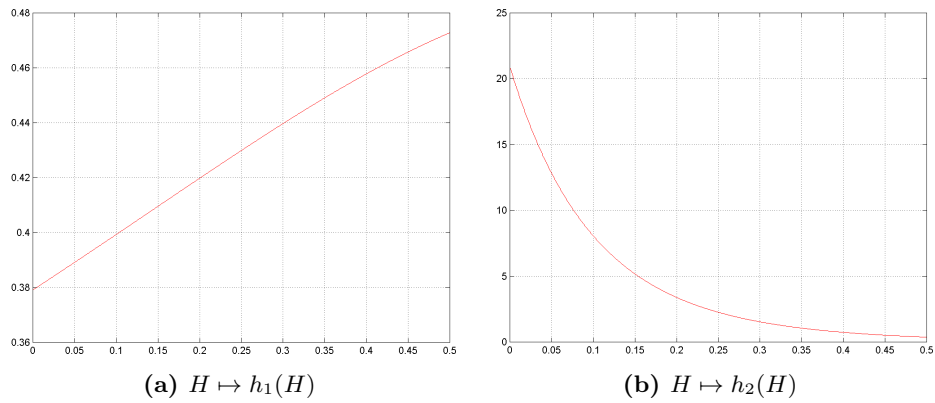**(a)** $H \mapsto h_1(H)$ **(b)** $H \mapsto h_2(H)$

FIGURE 3. Functions h_1 et h_2 with $(j_{\min}, j_{\max}) = (5, 15)$ and $N = 2000$.

Near 0 one cannot distinguish the curve from its tangent: it is the principe of the estimation based on $\widehat{\gamma_K}$. For $\eta \geqslant \eta_{M^*} = 2.2695$ the curve is near a constant: it is the principe of the estimation based on $\widehat{\pi_K}$.
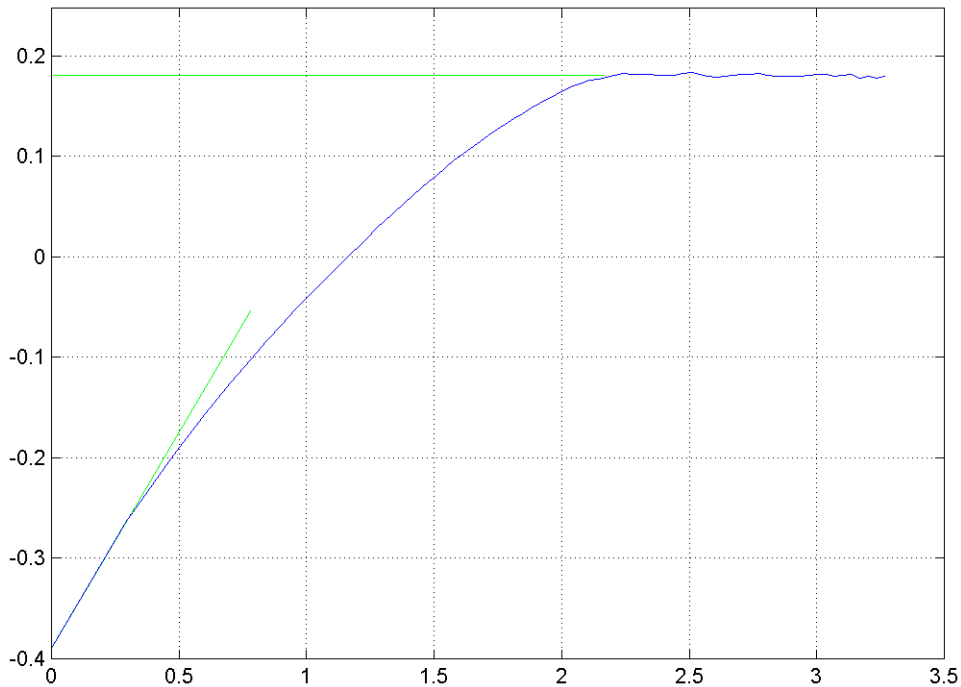

FIGURE 4. The 2-structure curve of $F_{j_{\min}, j_{\max}}$ for $(j_{\min}, j_{\max}) = (5, 15)$, $H = 0.25$ and $K = 500$.

CONCLUSION

The several random models presented in this paper are easy to simulate and it is possible to estimate their main parameter H. Thus they will be very useful in practice to modelize $D-$dimensional signals involving a parameter which may be related to the H index.

Moreover, when $H \in (0, 1/2)$ and $(j_{\min}, j_{\max}) \to (-\infty, +\infty)$, we note that $\mathrm{Var}(W_{j_{\min}, j_{\max}}) \to C_{D,H} \|y\|^{2H}$ (see (12) and (8)), so that $W_{j_{\min}, j_{\max}}$ converges to B_H when $(j_{\min}, j_{\max}) \to (-\infty, +\infty)$. Thus it seems quite natural to use the piling fields $F_{j_{\min}, j_{\max}}$ as a microscopic description of a fractional brownian field B_H.

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E-mail address: yann.demichel@mi.parisdescartes.fr