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Jacques Fejoz

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# A SIMPLE PROOF OF THE INVARIANT TORI THEOREM

JACQUES FÉJOZ

ABSTRACT. We first reduce the proof of Kolmogorov's theorem on the persistence of quasiperiodic invariant tori in Hamiltonian systems, to Herman's normal form, a well-posed inversion problem. The latter problem is then addressed with a simple inverse function theorem in the analytic category, whose own proof relies on the Newton algorithm and on interpolation inequalities.

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## 1. KOLMOGOROV'S THEOREM

Let  $\mathcal{H}$  be the space of germs along  $T_0^n := \mathbb{T}^n \times \{0\}$  of real analytic Hamiltonians in  $\mathbb{T}^n \times \mathbb{R}^n = \{(\theta, r)\}$  ( $\mathbb{T}^n = \mathbb{R}^n/2\pi\mathbb{Z}^n$ ). The vector field associated with  $H \in \mathcal{H}$  is

$$\vec{H} : \quad \dot{\theta} = \partial_r H, \quad \dot{r} = -\partial_\theta H.$$

For  $\alpha \in \mathbb{R}^n$ , let  $\mathcal{K}$  be the affine subspace of Hamiltonians  $K \in \mathcal{H}$  such that  $K|_{T_0^n}$  is constant (i.e.  $T_0^n$  is invariant) and  $\vec{K}|_{T_0^n} = \alpha$ . Those Hamiltonians are characterized by their first order expansion along  $T_0^n$ , of the form  $c + \alpha \cdot r$  for some  $c \in \mathbb{R}$ , that is, their expansion is constant with respect to  $\theta$  and the coefficient of  $r$  is  $\alpha$ .

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Let  $\mathbb{Z}_*^n := \mathbb{Z}^n \setminus \{0\}$  and

$$D_{\gamma,\tau} = \{\alpha \in \mathbb{R}^n, \forall k \in \mathbb{Z}_*^n |k \cdot \alpha| \geq \gamma |k|^{-\tau}\}, \quad |k| := |k|_1 = |k_1| + \cdots + |k_n|;$$

see Rüssmann [2001] or Féjóz [2010] for (less restrictive) Brjuno arithmetic conditions.

**Theorem 1** (Kolmogorov [1954], Chierchia [2008]). *Let  $\alpha \in D_{\gamma,\tau}$  and  $K^o \in \mathcal{K}$  such that the averaged Hessian*

$$\int_{\mathbb{T}^n} \frac{\partial^2 K^o}{\partial r^2}(\theta, 0) d\theta$$

*is non degenerate. Every  $H \in \mathcal{H}$  close to  $K^o$  possesses an  $\alpha$ -quasiperiodic invariant torus.*

This theorem has far reaching consequences. In particular it has led to a partial answer to the long standing question of the stability of the Solar system (Arnold [1964], Féjóz [2004], Celletti and Chierchia [2007]). See Bost [1986], Sevryuk [2003], de la Llave [2001] for references and background.

Kolmogorov's theorem is a consequence of the following normal form. Let  $\mathcal{G}$  be the space of germs along  $\mathbb{T}_0^n$  of real analytic exact symplectomorphisms  $G$  in  $\mathbb{T}^n \times \mathbb{R}^n$  of the following form:

$$G(\theta, r) = (\varphi(\theta), {}^t\varphi'(\theta)^{-1}(r + \rho(\theta))),$$

where  $\varphi$  is a real analytic isomorphism of  $\mathbb{T}^n$  fixing the origin, and  $\rho$  is an exact 1-form on  $\mathbb{T}^n$ .

**Theorem 2** (Herman). *Let  $\alpha \in D_{\gamma,\tau}$  and  $K^o \in \mathcal{K}$ . For every  $H \in \mathcal{H}$  close enough to  $K^o$ , there exists a unique  $(K, G, \beta) \in \mathcal{K} \times \mathcal{G} \times \mathbb{R}^n$  close to  $(K^o, \text{id}, 0)$  such that*

$$H = K \circ G + \beta \cdot r$$

*in some neighborhood of  $G^{-1}(\mathbb{T}_0^n)$ . Moreover,  $\beta$  is  $C^1$ -smooth with respect to  $H$ .*

In other words, the orbits of Hamiltonians  $K \in \mathcal{K}$  under the action of symplectomorphisms of  $\mathcal{G}$  locally form a subspace of finite codimension  $n$ . The offset  $\beta \cdot r$  usually breaks the dynamical conjugacy between  $K$  and  $H$ ; hence Herman's normal form is of geometrical nature and can be called a *twisted conjugacy*. The strategy for deducing the existence of an  $H$ -invariant torus (namely,  $G^{-1}(\mathbb{T}_0^n)$ ) from that of a  $K$ -invariant torus (namely,  $\mathbb{T}_0^n$ ) is to show that  $\beta$  vanishes on some subset of large measure in some parameter space (in some cases, the frequency  $\alpha$  cannot be fixed and needs to be varied).

In the paper,  $O(r^n)$  will denote the ideal of functions of  $(\theta, r)$  of the  $n$ -th order with respect to  $r$ .

*Proof of theorem 1 assuming theorem 2.* Let  $K_2^o(\theta) := \frac{1}{2} \frac{\partial^2 K^o}{\partial r^2}(\theta, 0)$ . Let  $F$  be the quadratic form valued, analytic function which solves the cohomological equation  $\mathcal{L}_\alpha F = K_2^o - \int_{\mathbb{T}_0^n} K_2^o d\theta$  (see lemma 5), and  $\varphi$  be the germ along  $\mathbb{T}_0^n$  of the (well defined) time-one map of the flow of the Hamiltonian  $F(\theta) \cdot r^2$ . The map  $\varphi$  is symplectic and restricts

to the identity on  $\mathbb{T}_0^n$ . At the expense of substituting  $K^o \circ \varphi$  and  $H \circ \varphi$  for  $K^o$  and  $H$  respectively, one can thus assume that

$$K^o = c + \alpha \cdot r + Q \cdot r^2 + O(r^3), \quad Q := \int_{\mathbb{T}_0^n} K_2^o(\theta) d\theta.$$

The germs so obtained from the initial  $K^o$  and  $H$  are close to one another.

Consider the family of trivial perturbations obtained by translating  $K^o$  in the direction of actions:

$$K_R^o(\theta, r) := K^o(\theta, R + r), \quad R \in \mathbb{R}^n, \quad R \text{ small,}$$

and its approximation obtained by truncating the first order jet of  $K_R^o$  along  $\mathbb{T}_0^n$  from its terms  $O(R^2)$  which possibly depend on  $\theta$ :

$$\hat{K}_R^o(\theta, r) := (c + \alpha \cdot R) + (\alpha + 2Q \cdot R) \cdot r + O(r^2) = K_R^o + O(R^2).$$

For the Hamiltonian  $\hat{K}_R^o$ ,  $\mathbb{T}_0^n$  is invariant and quasiperiodic of frequency  $\alpha + 2Q \cdot R$ . Hence the Herman normal form of  $\hat{K}_R^o$  with respect to the frequency  $\alpha$  is

$$\hat{K}_R^o = \left( \hat{K}_R^o - \hat{\beta}_R^o \cdot r \right) \circ \text{id} + \hat{\beta}_R^o \cdot r, \quad \hat{\beta}_R^o := 2Q \cdot R.$$

By assumption the matrix  $\left. \frac{\partial \hat{\beta}^o}{\partial R} \right|_{R=0} = 2Q$  is invertible and the map  $R \mapsto \hat{\beta}^o(R)$  is a local diffeomorphism.

Now, theorem 2 asserts the existence of an analogous map  $R \rightarrow \beta(R)$  for  $H_R$ , which is a small  $C^1$ -perturbation of  $R \mapsto \hat{\beta}^o(R)$ , and thus a local diffeomorphism, with a domain having a lower bound locally uniform with respect to  $H$ . Hence if  $H$  is close enough to  $K^o$  there is a unique small  $R$  such that  $\beta = 0$ . For this  $R$  the equality  $H_R = K \circ G$  holds, hence the torus obtained by translating  $G^{-1}(\mathbb{T}_0^n)$  by  $R$  in the direction of actions is invariant and  $\alpha$ -quasiperiodic for  $H$ .  $\square$

**Exercise 3** Simplify this proof when starting from an integrable Hamiltonian  $K^o(r)$ .

It is the aim of the rest of the paper to prove theorem 2, by locally inverting some operator

$$\phi : (K, G, \beta) \mapsto H = K \circ G + \beta \cdot r$$

when  $\alpha$  is diophantine.

## 2. COMPLEXIFICATION AND THE FUNCTIONAL SETTING

For various sets  $U$  and  $V$ ,  $\mathcal{A}(U, V)$  will denote the set of continuous maps  $U \rightarrow V$  which are real analytic on the interior  $\overset{\circ}{U}$ , and  $\mathcal{A}(U) := \mathcal{A}(U, \mathbb{C})$ .

Recall notations for the abstract torus and its embedding in the phase space:

$$\mathbb{T}^n = \mathbb{R}^n / 2\pi\mathbb{Z}^n \quad \text{and} \quad \mathbb{T}_0^n = \mathbb{T}^n \times \{0\} \subset \mathbb{T}^n \times \mathbb{R}^n.$$

Define complex extensions

$$\mathbb{T}_{\mathbb{C}}^n = \mathbb{C}^n / 2\pi\mathbb{Z}^n \quad \text{and} \quad \mathbb{T}_{\mathbb{C}}^n = \mathbb{T}_{\mathbb{C}}^n \times \mathbb{C}^n$$

as well as bases of neighborhoods

$$\mathbb{T}_s^n = \{\theta \in \mathbb{T}_{\mathbb{C}}^n, \max_{1 \leq j \leq n} |\operatorname{Im} \theta_j| \leq s\} \quad \text{and} \quad \mathbb{T}_s^n = \{(\theta, r) \in \mathbb{T}_{\mathbb{C}}^n, |(\theta, r)| \leq s\},$$

with  $|(\theta, r)| := \max_{1 \leq j \leq n} \max(|\operatorname{Im} \theta_j|, |r_j|)$ .

**2.1. Spaces of Hamiltonians.** – Let  $\mathcal{H}_s = \mathcal{A}(\mathbb{T}_s^n)$ , endowed with the Banach norm

$$|H|_s := \sup_{(\theta, r) \in \mathbb{T}_s^n} |H(\theta, r)|,$$

so that  $\mathcal{H}$  be the inductive limit of the spaces  $\mathcal{H}_s$ .

– For  $\alpha \in \mathbb{R}^n$ , let  $\mathcal{K}_s$  be the affine subspace consisting of those  $K \in \mathcal{H}_s$  such that  $K(\theta, r) = c + \alpha \cdot r + O(r^2)$  for some  $c \in \mathbb{R}$ .

– If  $G$  is a real analytic isomorphism on some open set of  $\mathbb{T}_{\mathbb{C}}^n$  and if  $G$  is transverse to  $\mathbb{T}_s^n$ , let  $G^* \mathcal{A}(\mathbb{T}_s^n) := \mathcal{A}(G^{-1}(\mathbb{T}_s^n))$  be endowed with the Banach norm

$$|H|_{G,s} := |H \circ G^{-1}|_s.$$

**2.2. Spaces of conjugacies.**

**2.2.1. Diffeomorphisms of the torus.** Let  $\mathcal{D}_s$  be the space of maps  $\varphi \in \mathcal{A}(\mathbb{T}_s^n, \mathbb{T}_{\mathbb{C}}^n)$  which are analytic isomorphisms from  $\mathring{\mathbb{T}}_s^n$  to their image and which fix the origin.

Let also

$$\chi_s := \{v \in \mathcal{A}(\mathbb{T}_s^n)^n, v(0) = 0\}$$

be the space of vector fields on  $\mathbb{T}_s^n$  which vanish at 0, endowed with the Banach norm

$$|v|_s := \max_{\theta \in \mathbb{T}_s^n} \max_{1 \leq j \leq n} |v_j(\theta)|.$$

According to corollary 14, the map

$$\sigma \mathring{B}_{s+\sigma}^\chi := \{v \in \chi_{s+\sigma}, |v|_s < \sigma\} \rightarrow \mathcal{D}_s, \quad v \mapsto \operatorname{id} + v$$

is defined and locally bijective. It endows  $\mathcal{D}_s$  with a local structure of Banach manifold in the neighborhood of the identity.

We will consider the contragredient action of  $\mathcal{D}_s$  on  $\mathbb{T}_s^n$  (with values in  $\mathbb{T}_{\mathbb{C}}^n$ ) :

$$\varphi(\theta, r) := (\varphi(\theta), {}^t \varphi'(\theta)^{-1} \cdot r),$$

in order to linearize the dynamics on the alleged invariant tori.

**2.2.2. Straightening tori.** Let  $\mathcal{B}_s$  be the space of exact one-forms over  $\mathbb{T}_s^n$ , with

$$|\rho|_s = \max_{\theta \in \mathbb{T}_s^n} \max_{1 \leq j \leq n} |\rho_j(\theta)|, \quad \rho = (\rho_1, \dots, \rho_n).$$

We will consider its action on  $\mathbb{T}_s^n$  by translation of the actions:

$$\rho(\theta, r) := (\theta, r + \rho(\theta)),$$

in order to straighten the perturbed invariant tori.

2.2.3. *Our space of conjugacies.* Let  $\mathcal{G}_s = \mathcal{D}_s \times \mathcal{B}_s$ , identified with a space of Hamiltonian symplectomorphisms by

$$(\varphi, \rho)(\theta, r) := \varphi \circ \rho(\theta, r) = (\varphi(\theta), {}^t\varphi'(\theta)^{-1}(r + \rho(\theta))).$$

Endow its tangent space at the identity  $T_{\text{id}}\mathcal{G}_s = \mathfrak{g}_s := \chi_s \times \mathcal{B}_s$  with the norm

$$|\dot{G}|_s = |(v, \rho)|_s := \max(|v|_s, |\rho|_s),$$

and its tangent space at  $G = (\varphi, \rho)$  with the norm

$$|\delta G|_s := |\delta G \circ G^{-1}|_s, \quad \delta G \in T_G\mathcal{G}.$$

Here and elsewhere, the notation  $\delta G$ , as well as similar ones, should be taken as a whole; there is no separate  $\delta \in \mathbb{R}$  in the present paper.

Also consider the following neighborhoods of the identity:

$$\mathcal{G}_s^\sigma = \left\{ G \in \mathcal{G}_s, \max_{(\theta, r) \in T_s^n} |(\Theta - \theta, R - r)| \leq \sigma, (\Theta, R) = G(\theta, r) \right\}, \quad \sigma > 0.$$

The operators (commuting with inclusions of source and target spaces)

$$\phi_s : E_s := \mathcal{K}_{s+\sigma} \times \mathcal{G}_s^\sigma \times \mathbb{R}^n \rightarrow \mathcal{H}_s, \quad (K, G, \beta) \mapsto K \circ G + \beta \cdot r$$

are now defined.

### 3. HERMAN'S NORMAL FORM

**Theorem 4.** *Let  $\alpha \in D_{\gamma, \tau}$ . For all  $0 < s < s + \sigma < 1$ ,  $\phi_{s+\sigma}$  has a local inverse: if  $|H - K^o|_{s+\sigma}$  is small, there is a unique  $(K, G, \beta) \in E_s$ ,  $|\cdot|_s$ -close to  $(K^o, \text{id}, 0)$  (in a manner worsening when  $\sigma$  is small), such that  $H = K \circ G + \beta \cdot r$ . Moreover  $\beta \circ \phi^{-1}$  is a  $C^1$ -function locally in the neighborhood of  $K^o$  in  $\mathcal{H}_{s+\sigma}$ .*

This follows from the inverse function theorem of appendix A, whose two hypotheses (one on  $\phi'^{-1}$  and one on  $\phi''$ ) will now be checked.

Let  $\mathcal{L}_\alpha$  be the Lie derivative operator in the direction of the constant vector field  $\alpha$ :

$$\mathcal{L}_\alpha : \mathcal{A}(\mathbb{T}_s^n) \rightarrow \mathcal{A}(\mathbb{T}_s^n), \quad f \mapsto f' \cdot \alpha = \sum_{1 \leq j \leq n} \alpha_j \frac{\partial f}{\partial \theta_j}.$$

We will need the following lemma in two instances in the proof of lemma 6.

**Lemma 5** (Cohomological equation). *If  $g \in \mathcal{A}(\mathbb{T}_{s+\sigma}^n)$  has 0-average ( $\int_{\mathbb{T}} g \, d\theta = 0$ ), there exists a unique function  $f \in \mathcal{A}(\mathbb{T}_s^n)$  of 0-average such that  $\mathcal{L}_\alpha f = g$ , and there exists a  $C_0 = C_0(n, \tau)$  such that, for any  $\sigma$ :*

$$|f|_s \leq C_0 \gamma^{-1} \sigma^{-\tau-n} |g|_{s+\sigma}.$$

*Proof.* Let  $g(\theta) = \sum_{k \in \mathbb{Z}_*^n} g_k e^{ik \cdot \theta}$  be the Fourier expansion of  $g$ . The unique formal solution to the equation  $\mathcal{L}_\alpha f = g$  is given by  $f(\theta) = \sum_{k \in \mathbb{Z}_*^n} \frac{g_k}{ik \cdot \alpha} e^{ik \cdot \theta}$ .

Since  $g$  is analytic, its Fourier coefficients decay exponentially: we find

$$|g_k| = \left| \int_{\mathbb{T}^n} g(\theta) e^{-ik \cdot \theta} \frac{d\theta}{2\pi} \right| \leq |g|_{s+\sigma} e^{-|k|(s+\sigma)}$$

by shifting the torus of integration to a torus  $\text{Im } \theta_j = \pm(s + \sigma)$ .

Using this estimate and replacing the small denominators  $k \cdot \alpha$  by the estimate defining the diophantine property of  $\alpha$ , we get

$$\begin{aligned} |f|_s &\leq \frac{|g|_{s+\sigma}}{\gamma} \sum_k |k|^\tau e^{-|k|\sigma} \\ &\leq \frac{2^n |g|_{s+\sigma}}{\gamma} \sum_{\ell \geq 1} \binom{\ell+n-1}{\ell} \ell^\tau e^{-\ell\sigma} \leq \frac{4^n |g|_{s+\sigma}}{\gamma (n-1)!} \sum_{\ell} (\ell+n-1)^{\tau+n-1} e^{-\ell\sigma}, \end{aligned}$$

where the latter sum is bounded by

$$\begin{aligned} \int_1^\infty (\ell+n-1)^{\tau+n-1} e^{-(\ell-1)\sigma} d\ell &= \sigma^{-\tau-n} e^{n\sigma} \int_{n\sigma}^\infty \ell^{\tau+n-1} e^{-\ell} d\ell \\ &< \sigma^{-\tau-n} e^{n\sigma} \int_0^\infty \ell^{\tau+n-1} e^{-\ell} d\ell = \sigma^{-\tau-n} e^{n\sigma} \Gamma(\tau+n). \end{aligned}$$

Hence  $f$  belongs to  $\mathcal{A}(\mathbb{T}_s^n)$  and satisfies the wanted estimate.  $\square$

We will write  $x = (K, G, \beta, c)$ ,  $\delta x = (\delta K, \delta G, \delta \beta, \delta c)$  and  $\delta \hat{x} = (\delta \hat{K}, \delta \hat{G}, \delta \hat{\beta}, \delta \hat{c})$ .

Fix  $0 < s < s + \sigma < 1$ .

**Lemma 6.** *There exists  $C' > 0$  which is locally uniform with respect to  $x \in E_s$  in the neighborhood of  $G = \text{id}$  such that the linear map  $\phi'(x)$  has an inverse  $\phi'(x)^{-1}$  satisfying*

$$|\phi'(x)^{-1} \cdot \delta H|_s \leq \sigma^{-\tau-n-1} C' |\delta H|_{G, s+\sigma}.$$

*Proof.* A function  $\delta H \in G^* \mathcal{A}(T_{s+\sigma})$  being given, we want to solve the equation

$$\delta \phi(x) \cdot \delta x = \delta K \circ G + K' \circ G \cdot \delta G + \delta \beta \cdot r + \delta c = \delta H,$$

for the unknowns  $\delta K \in T_K \mathcal{K}_s \subset \mathcal{A}(\mathbb{T}_s^n)$ ,  $\delta G \in T_G \mathcal{G}_s$ ,  $\delta \beta \in \mathbb{R}^n$  and  $\delta c \in \mathbb{R}$ , or, equivalently, after composing with  $G^{-1}$  to the right,

$$\delta K + K' \cdot \dot{G} + \delta \beta \cdot r \circ G^{-1} + \delta c = \dot{H},$$

where we have set  $\dot{G} := \delta G \circ G^{-1} \in \mathfrak{g}_s$  and  $\dot{H} := \delta H \circ G^{-1} \in \mathcal{A}(\mathbb{T}_s^n)$ .

More specifically,  $G^{-1}$  and  $\dot{G}$  are of the form

$$G^{-1}(\theta, r) = (\varphi^{-1}(\theta), {}^t \varphi' \circ \varphi^{-1}(\theta) \cdot (r - \rho \circ \varphi^{-1}(\theta))), \quad \dot{G} = (\dot{\varphi}, \dot{\rho} - r \cdot \dot{\varphi}'),$$

where  $\dot{\varphi} \in \chi_{s+\sigma}$  and  $\dot{\rho} \in \mathcal{B}_{s+\sigma}$ , and we can expand

$$K = \alpha \cdot r + K_2(\theta) \cdot r^2 + O(r^3) \quad \text{and} \quad \dot{H} = \dot{H}_0(\theta) + \dot{H}_1(\theta) \cdot r + O(r^2).$$

The equation becomes

$$(1) \quad [\dot{\rho} \cdot \alpha + \delta c - (\rho \cdot \varphi') \circ \varphi^{-1} \cdot \delta \beta] + r \cdot [-\dot{\varphi}' \cdot \alpha + \varphi' \circ \varphi^{-1} \cdot \delta \beta + 2K_2 \cdot \dot{\rho}] + \dot{K} = \dot{H} + O(r^2),$$

where the term  $O(r^2)$  in the right hand side depends only on  $K$  and  $\dot{G}$ , and not on  $\dot{K}$ . The equation turns out to be triangular in the five unknowns. The existence of a unique solution with the wanted estimate follows from repeated applications of lemma 5 and Cauchy's inequality:

– The average over  $T_0^n$  of the first order terms with respect to  $r$  in equation (1) yields

$$\delta\beta = \left( \int_{T_0^n} \varphi' \circ \varphi^{-1} d\theta \right)^{-1} \cdot \int_{T_0^n} \dot{H}_1 d\theta,$$

which does exist if  $\varphi$  is close to the identity (proposition 14).

– Similarly, the average of the restriction to  $T_0^n$  of (1) yields:

$$\delta c = \int_{T_0^n} \dot{H}_0 d\theta + \int_{T_0^n} \rho \circ \varphi^{-1} d\theta \cdot \int_{T_0^n} \dot{H}_1 d\theta.$$

– Next, the restriction to  $T_0^n$  of (1) can be solved uniquely with respect to  $\delta\rho$  according to lemma 5 (applied with  $\rho = f'$ ).

– The part of degree one can then be solved for  $\dot{\varphi}$  similarly.

– Terms of order  $\geq 2$  in  $r$  determine  $\dot{K}$ . □

**Lemma 7.** *There exists a constant  $C'' > 0$  which is locally uniform with respect to  $x \in E_{s+\sigma}$  in the neighborhood of  $G = \text{id}$  such that the bilinear map  $\phi''(x)$  satisfies*

$$|\phi''(x) \cdot \delta x \otimes \delta \hat{x}|_{G,s} \leq \sigma^{-1} C'' |\delta x|_{s+\sigma} |\delta \hat{x}|_{s+\sigma}.$$

*Proof.* Differentiating  $\phi$  twice yields

$$\phi''(x) \cdot \delta x \otimes \delta \hat{x} = \delta K' \circ G \cdot \delta G + \delta \hat{K}' \circ G \cdot \delta G + K'' \circ G \cdot \delta G \otimes \delta \hat{G},$$

whence the estimate. □

## A. AN INVERSE FUNCTION THEOREM

Let  $E = (E_s)_{0 < s < 1}$  be a decreasing family of Banach spaces with increasing norms  $|\cdot|_s$ , and  $\epsilon B_s^E = \{x \in E_s, |x|_s < \epsilon\}$ ,  $\epsilon > 0$ , be its balls centered at 0.

Let  $(F_s)$  be an analogous family. Endow  $F$  with additional norms  $|\cdot|_{x,s}$ ,  $x \in E_s$ ,  $0 < s < 1$ , satisfying

$$|y|_{0,s} = |y|_s \quad \text{and} \quad |y|_{x',s} \leq |y|_{x,s+|x'-x|_s}.$$

These norms allow for dealing with composition operators without artificially loosing some fixed “width of analyticity”  $\sigma$  at each step of the Newton algorithm.

Let  $\phi : \sigma B_{s+\sigma}^E \rightarrow F_s$ ,  $s < s + \sigma$ ,  $\phi(0) = 0$ , be maps commuting with inclusions, twice differentiable, such that the differential  $\phi'(x) : E_{s+\sigma} \rightarrow F_s$  has a right inverse  $\phi'(x)^{-1} : F_{s+\sigma} \rightarrow E_s$ , and

$$\begin{cases} |\phi'(x)^{-1}\eta|_s \leq C' \sigma^{-\tau'} |\eta|_{x,s+\sigma} \\ |\phi''(x)\xi^{\otimes 2}|_{x,s} \leq C'' \sigma^{-\tau''} |\xi|_{x,s+\sigma}^2 \end{cases} \quad (\forall s, \sigma, x, \xi, \eta)$$



with  $C', C'', \tau', \tau'' \geq 1$ . Let  $C := C' C''$  and  $\tau := \tau' + \tau''$ .

**Theorem 8.**  $\phi$  is locally surjective and, more precisely, for any  $s, \eta$  and  $\sigma$  with  $\eta < s$ ,

$$\epsilon B_{s+\sigma}^F \subset \phi(\eta B_s^E), \quad \epsilon := 2^{-8\tau} C^{-2} \sigma^{2\tau} \eta.$$

In other words,  $\phi$  has a right-inverse  $\psi : \epsilon B_{s+\sigma}^F \rightarrow \eta B_s^E$ .

*Proof.* Some numbers  $s, \eta$  and  $\sigma$  and  $y \in B_{s+\eta}^F$  being given, let

$$f : \sigma B_{s+\eta+\sigma}^E \rightarrow E_s, \quad x \mapsto x + \phi'(x)^{-1}(y - \phi(x))$$

and

$$Q : \sigma B_{s+\sigma}^E \times \sigma B_{s+\sigma}^E \rightarrow F_s, \quad (x, \hat{x}) \mapsto \phi(\hat{x}) - \phi(x) - \phi'(x)(\hat{x} - x).$$

**Lemma 9.** The function  $Q$  satisfies:  $|Q(x, \hat{x})|_{x,s} \leq 2^{-1} C'' \sigma^{-\tau''} |\hat{x} - x|_{s+\sigma+|\hat{x}-x|_s}^2$ .

*Proof of the lemma.* Let  $\hat{x}_t := (1-t)x + t\hat{x}$ . Taylor's formula yields

$$Q(x, \hat{x}) = \int_0^1 (1-t) \phi''(\hat{x}_t) (\hat{x} - x)^2 dt,$$

hence

$$|Q(x, \hat{x})|_{x,s} \leq \int_0^1 (1-t) |\phi''(\hat{x}_t) (\hat{x} - x)^2|_{x,s} dt \leq \int_0^1 (1-t) |\phi''(\hat{x}_t) (\hat{x} - x)^2|_{\hat{x}_t, s+|\hat{x}_t-x|_s} dt,$$

whence the estimate.  $\square$

Now, let  $s, \eta$  and  $\sigma$  be fixed, with  $\eta < s$  and  $y \in \epsilon B_{s+\sigma}^F$  for some  $\epsilon$ . We will see that if  $\epsilon$  is small enough, the sequence  $x_0 = 0, x_n := f^n(0)$  is defined for all  $n \geq 0$  and converges towards some preimage  $x \in \eta B_s^E$  of  $y$  by  $\phi$ .

Let  $(\sigma_n)_{n \geq 0}$  be a sequence of positive real numbers such that  $3 \sum \sigma_n = \sigma$ , and  $(s_n)_{n \geq 0}$  be the sequence decreasing from  $s_0 := s + \sigma$  to  $s$  defined by induction by the formula  $s_{n+1} = s_n - 3\sigma_n$ .

Assuming the existence of  $x_0, \dots, x_{n+1}$ , we see that  $\phi(x_k) = y + Q(x_{k-1}, x_k)$ , hence

$$x_{k+1} - x_k = \phi'(x_k)^{-1}(y - \phi(x_k)) = -\phi'(x_k)^{-1}Q(x_{k-1}, x_k) \quad (1 \leq k \leq n).$$

Further assuming that  $|x_{k+1} - x_k|_{s_k} \leq \sigma_k$ , the estimate of the right inverse and lemma 9 entail that

$$|x_{n+1} - x_n|_{s_{n+1}} \leq c_n |x_n - x_{n-1}|_{s_n}^2 \leq \dots \leq c_n c_{n-1}^2 \dots c_1^{2^{n-1}} |x_1|_{s_1}^{2^{n-1}}, \quad c_k := 2^{-1} C \sigma_k^{-\tau}.$$

The estimate

$$|x_1|_{s_1} \leq C'(3\sigma_0)^{-\tau'} |y|_{s_0} \leq 2^{-1} C \sigma_0^{-\tau} \epsilon = c_0 \epsilon$$

and the fact, to be checked later, that  $c_k \geq 1$  for all  $k \geq 0$ , show :

$$|x_{n+1} - x_n|_{s_{n+1}} \leq \left( \epsilon \prod_{k \geq 0} c_k^{2^{-k}} \right)^{2^n}.$$

Since  $\sum_{n \geq 0} \rho^{2^n} \leq 2\rho$  if  $2\rho \leq 1$ , in order to have all  $x_n$ 's defined and to have  $\sum |x_{n+1} - x_n|_s \leq \eta$ , it suffices that

$$\epsilon \leq \frac{\eta}{2} \prod_{k \geq 0} c_k^{-2^{-k}}.$$

Maximizing the upper bound of  $\epsilon$  under the constraint  $3 \sum_{n \geq 0} \sigma_n = \sigma$  yields  $\sigma_k := \frac{\sigma}{6} 2^{-k}$ . A posteriori it is straightforward that  $|x_{n+1} - x_n|_{s_n} \leq \sigma_n$  (as earlier assumed to apply lemma 9) and  $c_n \geq 1$  for all  $n \geq 0$ . Besides, using that  $\sum k 2^{-k} = \sum 2^{-k} = 2$  we get

$$\frac{\eta}{2} \prod_{k \geq 0} c_k^{-2^{-k}} = \frac{\eta}{2} \prod_{k \geq 0} \frac{1}{2^{\tau k 2^{-k}}} \left( \frac{2}{C} \left( \frac{\sigma}{6} \right)^\tau \right)^{2^{-k}} = \frac{2\eta}{C^2} \left( \frac{\sigma}{12} \right)^{2\tau} > \frac{\sigma^{2\tau} \eta}{2^{8\tau} C^2},$$

whence the theorem.  $\square$

**Exercise 10** The domain of  $\psi$  contains  $\epsilon B_S^F$ ,  $\epsilon = 2^{-12\tau} \tau^{-1} C^{-2} S^{3\tau}$ , for any  $S$ .

*Proof.* The above function  $\epsilon(\eta, \sigma) = 2^{-8\tau} C^{-2} \sigma^{2\tau} \eta$  attains its maximum with respect to  $\eta < s$  for  $\eta = s$ . Besides, under the constraint  $s + \sigma = S$  the function  $\epsilon(s, \sigma)$  attains its maximum when  $\sigma = 2\tau s$  and  $s = \frac{S}{1+2\tau}$ . Hence,  $S$  being fixed, the domain of  $\psi$  contains  $\epsilon B_S^F$  if

$$\epsilon < 2^{-8\tau} C^{-2} \frac{S}{1+2\tau} \left( \frac{2\tau S}{12(1+2\tau)} \right)^{2\tau}.$$

Given that  $S < 1 < \tau$  by hypothesis, it suffices that  $\epsilon$  be equal to the stated value.  $\square$

**A.1. Regularity of the right-inverse.** In the proof of theorem 8 we have built right inverses  $\psi : \epsilon B_{s+\eta+\sigma}^F \rightarrow \eta B_{s+\eta}^E$ , of  $\phi$ , commuting with inclusions. The estimate given in the statement shows that  $\psi$  is continuous at 0; due to the invariance of the hypotheses of the theorem by small translations,  $\psi$  is locally continuous.

We further make the following two assumptions:

- The maps  $\phi'(x)^{-1} : F_{s+\sigma} \rightarrow E_s$  are left (as well as right) inverses (in theorem 4 we have restricted to an adequate class of symplectomorphisms);
- The scale  $(|\cdot|_s)$  of norms of  $(E_s)$  satisfies some interpolation inequality:

$$|x|_{s+\sigma}^2 \leq |x|_s |x|_{s+\tilde{\sigma}} \quad \text{for all } s, \sigma, \tilde{\sigma} = \sigma \left( 1 + \frac{1}{s} \right)$$

(according to the remark after corollary 16, this estimate is satisfied in the case of interest to us, since  $\sigma + \log(1 + \sigma/s) \leq \tilde{\sigma}$ ).

**Lemma 11** (Lipschitz regularity). *If  $\sigma < s$  and  $y, \hat{y} \in \epsilon B_{s+\sigma}^F$  with  $\epsilon = 2^{-14\tau} C^{-3} \sigma^{3\tau}$ ,*

$$|\psi(\hat{y}) - \psi(y)|_s \leq C_L |\hat{y} - y|_{s+\sigma}, \quad C_L = 2C' \sigma^{-\tau'}.$$

*Proof.* Fix  $\eta < \zeta < \sigma < s$ ; the impatient reader can readily look at the end of the proof how to choose the auxiliary parameters  $\eta$  and  $\zeta$  more precisely.

Let  $\epsilon = 2^{-8\tau}C^{-2}\zeta^{2\tau}\eta$ , and  $y, \hat{y} \in \epsilon B_{s+\sigma}^F$ . According to theorem 8,  $x := \psi(y)$  and  $\hat{x} := \psi(\hat{y})$  are in  $\eta B_{s+\sigma-\zeta}^E$ , provided the condition, to be checked later, that  $\eta < s + \sigma - \zeta$ . In particular, we will use a priori that

$$|\hat{x} - x|_{s+\sigma-\zeta} \leq |\hat{x}|_{s+\sigma-\zeta} + |x|_{s+\sigma-\zeta} \leq 2\eta.$$

We have

$$\begin{aligned} \hat{x} - x &= \phi'(x)^{-1}\phi'(x)(\hat{x} - x) \\ &= \phi'(x)^{-1}(\hat{y} - y - Q(x, \hat{x})) \end{aligned}$$

and, according to the assumed estimate on  $\phi'(x)^{-1}$  and to lemma 9,

$$|\hat{x} - x|_s \leq C'\sigma^{-\tau'}|\hat{y} - y|_{s+\sigma} + 2^{-1}C\zeta^{-\tau}|\hat{x} - x|_{s+2\eta+|\hat{x}-x|_s}^2.$$

In the norm index of the last term, we will coarsely bound  $|\hat{x} - x|_s$  by  $2\eta$ . Additionally using the interpolation inequality:

$$|\hat{x} - x|_{s+4\eta}^2 \leq |\hat{x} - x|_s |\hat{x} - x|_{s+\tilde{\sigma}}, \quad \tilde{\sigma} = 4\eta \left(1 + \frac{1}{s}\right),$$

yields

$$(1 - 2^{-1}C\zeta^{-\tau}|\hat{x} - x|_{s+\tilde{\sigma}}) |\hat{x} - x|_s \leq C'\sigma^{-\tau'}|\hat{y} - y|_{s+\sigma}.$$

Now, we want to choose  $\eta$  small enough so that

– first,  $\tilde{\sigma} \leq \sigma - \zeta$ , which implies  $|\hat{x} - x|_{s+\tilde{\sigma}} \leq 2\eta$ . By definition of  $\tilde{\sigma}$ , it suffices to have  $\eta \leq \frac{\sigma-\zeta}{4(1+1/s)}$ .

– second,  $2^{-1}C\zeta^{-\tau}2\eta \leq 1/2$ , or  $\eta \leq \frac{\zeta^\tau}{2C}$ , which implies that  $2^{-1}C\zeta^{-\tau}|\hat{x} - x|_{s+\tilde{\sigma}} \leq 1/2$ , and hence  $|\hat{x} - x|_s \leq 2C'\sigma^{-\tau'}|\hat{y} - y|_{s+\sigma}$ .

A choice is  $\zeta = \frac{\sigma}{2}$  and  $\eta = \frac{\sigma^\tau}{16C} < s$ , whence the value of  $\epsilon$  in the statement.  $\square$

**Proposition 12** (Smoothness). *For every  $s, \sigma$ , there exists  $\eta, \epsilon, C_1$  such that  $\eta \geq \sigma$  and, for every  $y, \hat{y} \in \epsilon B_{s+\eta}^F$ ,*

$$|\psi(\hat{y}) - \psi(y) - \phi'(\psi(y))^{-1}(\hat{y} - y)|_s \leq C_1|\hat{y} - y|_{s+\sigma}^2.$$

Moreover, the map  $\psi' : \epsilon B_{s+\eta}^F \rightarrow L(F_{s+\eta}, E_s)$  defined locally by  $\psi'(y) = \phi'(\psi(y))^{-1}$  is continuous.

*Proof.* Fix  $y, \hat{y} \in \epsilon B_{s+\sigma}^F$  as in the previous proof. Let  $x = \psi(y)$ ,  $\eta = \hat{y} - y$ ,  $\xi = \psi(y + \eta) - \psi(y)$  (thus  $\eta = \phi(x + \xi) - \phi(x)$ ), and  $\Delta := \psi(y + \eta) - \psi(y) - \phi'(x)^{-1}\eta$ . Definitions yield

$$\Delta = \phi'(x)^{-1}(\phi'(x)\xi - \eta) = -\phi'(x)^{-1}Q(x, x + \xi).$$

Using the estimates on  $\phi'(x)^{-1}$  and  $Q$  and the latter lemma,

$$|\Delta|_s \leq C_1|\eta|_{s+\sigma}^2$$

for some  $C_1 > 0$  depending on  $\sigma$ . The number  $\sigma$  can be chose arbitrarily small, provided  $\eta$  itself is small enough.

The inversion of linear operators between Banach spaces being analytic,  $y \mapsto \phi(\psi(y))^{-1}$  is continuous in the stated sense.  $\square$

**Corollary 13.** *If  $\pi \in L(E_s, V)$  is a family of linear maps, commuting with inclusions, into a fixed Banach space  $V$ , then  $\pi \circ \psi$  is  $C^1$  and  $(\pi \circ \psi)' = \pi \cdot \phi' \circ \psi$ .*

This corollary will be used with  $\pi : (K, G, \beta) \mapsto \beta$  in the proof of theorem 4.

## B. SOME ESTIMATES ON ANALYTIC ISOMORPHISMS

In this appendix, we give a quantitative inverse function theorem for real analytic isomorphisms on  $\mathbb{T}_s^n$ . This is used in section 2, to parametrize locally  $\mathcal{D}_s$  by vector fields.

Recall that we have set  $\mathbb{T}_s^n := \{\theta \in \mathbb{C}^n / 2\pi\mathbb{Z}^n, \max_{1 \leq j \leq n} |\operatorname{Im} \theta_j| \leq s\}$ . We will denote by  $p : \mathbb{R}_s^n := \mathbb{R}^n \times i[-s, s]^n \rightarrow \mathbb{T}_s^n$  its universal covering.

**Proposition 14.** *Let  $v \in \mathcal{A}(\mathbb{T}_{s+2\sigma}^n, \mathbb{C}^n)$ ,  $|v|_{s+2\sigma} < \sigma$ . The map  $\operatorname{id} + v : \mathbb{T}_{s+2\sigma}^n \rightarrow \mathbb{R}_{s+3\sigma}^n$  induces a map  $\varphi : \mathbb{T}_{s+2\sigma}^n \rightarrow \mathbb{T}_{s+3\sigma}^n$  whose restriction  $\varphi : \mathbb{T}_{s+\sigma}^n \rightarrow \mathbb{T}_{s+2\sigma}^n$  has a unique right inverse  $\psi : \mathbb{T}_s^n \rightarrow \mathbb{T}_{s+\sigma}^n$ :*

$$\begin{array}{ccc} \mathbb{T}_{s+\sigma}^n & \xrightarrow{\varphi} & \mathbb{T}_{s+2\sigma}^n \\ & \swarrow \psi & \uparrow \\ & & \mathbb{T}_s^n \end{array}$$

Furthermore,

$$|\psi - \operatorname{id}|_s \leq |v|_{s+\sigma}$$

and, provided  $2\sigma^{-1}|v|_{s+2\sigma} \leq 1$ ,

$$|\psi' - \operatorname{id}| \leq 2\sigma^{-1}|v|_{s+2\sigma}.$$

*Proof.* Let  $\Phi : \mathbb{R}_{s+2\sigma}^n \rightarrow \mathbb{R}_{s+3\sigma}^n$  be a continuous lift of  $\operatorname{id} + v$  and  $k \in M_n(\mathbb{Z})$ ,  $k(l) := \Phi(x + l) - \Phi(x)$ .

- (1) *Injectivity of  $\Phi : \mathbb{R}_{s+\sigma}^n \rightarrow \mathbb{R}_{s+2\sigma}^n$ .* Suppose that  $x, \hat{x} \in \mathbb{R}_{s+\sigma}^n$  and  $\Phi(x) = \Phi(\hat{x})$ . By the mean value theorem,

$$|x - \hat{x}| = |v(p\hat{x}) - v(px)| \leq |v'|_{s+\sigma}|x - \hat{x}|,$$

and, by Cauchy's inequality,

$$|x - \hat{x}| \leq \frac{|v|_{s+2\sigma}}{\sigma}|x - \hat{x}| < |\hat{x} - x|,$$

hence  $x = \hat{x}$ .

- (2) *Surjectivity of  $\Phi : \mathbb{R}_s^n \subset \Phi(\mathbb{R}_{s+\sigma}^n)$ .* For any given  $y \in \mathbb{R}_s^n$ , the contraction

$$f : \mathbb{R}_{s+\sigma}^n \rightarrow \mathbb{R}_{s+\sigma}^n, \quad x \mapsto y - v(x)$$

has a unique fixed point, which is a pre-image of  $y$  by  $\Phi$ .

- (3) *Injectivity of  $\varphi : \mathbb{T}_{s+\sigma}^n \rightarrow \mathbb{T}_{s+2\sigma}^n$ .* Suppose that  $px, p\hat{x} \in \mathbb{R}_{s+\sigma}^n$  and  $\varphi(px) = \varphi(p\hat{x})$ , i.e.  $\Phi(x) = \Phi(\hat{x}) + \kappa$  for some  $\kappa \in \mathbb{Z}^n$ . That  $k$  be in  $GL(n, \mathbb{Z})$ , follows from the invertibility of  $\Phi$ . Hence,  $\Phi(x - k^{-1}(\kappa)) = \Phi(\hat{x})$ , and, due to the injectivity of  $\Phi$ ,  $px = p\hat{x}$ .
- (4) *Surjectivity of  $\varphi : \mathbb{T}_s^n \subset \varphi(\mathbb{T}_{s+\sigma}^n)$ .* This is a trivial consequence of that of  $\Phi$ .
- (5) *Estimate on  $\psi := \varphi^{-1} : \mathbb{T}_s^n \rightarrow \mathbb{T}_{s+\sigma}^n$ .* Note that the wanted estimate on  $\psi$  is in the sense of  $\Psi := \Phi^{-1} : \mathbb{R}_s^n \rightarrow \mathbb{R}_{s+\sigma}^n$ . If  $y \in \mathbb{R}_s^n$ ,

$$\Psi(y) - y = -v(p\Psi(y)),$$

hence  $|\Psi - \text{id}|_s \leq |v|_{s+\sigma}$ .

- (6) *Estimate on  $\psi'$ .* We have  $\psi' = \varphi'^{-1} \circ \varphi$ , where  $\varphi'^{-1}(x)$  stands for the inverse of the map  $\xi \mapsto \varphi'(x) \cdot \xi$ . Hence

$$\psi' - \text{id} = \varphi'^{-1} \circ \varphi - \text{id},$$

and, under the assumption that  $2\sigma^{-1}|v|_{s+2\sigma} \leq 1$ ,

$$|\psi' - \text{id}|_s \leq |\varphi'^{-1} - \text{id}|_{s+\sigma} \leq \frac{|v'|_{s+\sigma}}{1 - |v'|_{s+\sigma}} \leq \frac{\sigma^{-1}|v|_{s+2\sigma}}{1 - \sigma^{-1}|v|_{s+2\sigma}} \leq 2\sigma^{-1}|v|_{s+2\sigma}.$$

□

### C. INTERPOLATION OF SPACES OF ANALYTIC FUNCTIONS

In this section we prove some Hadamard interpolation inequalities, which are used in A.1.

Recall that we denote by  $\mathbb{T}_{\mathbb{C}}^n$  the infinite annulus  $\mathbb{C}^n/2\pi\mathbb{Z}^n$ , by  $\mathbb{T}_s^n$ ,  $s > 0$ , the bounded sub-annulus  $\{\theta \in \mathbb{T}_{\mathbb{C}}^n, |\text{Im}\theta_j| \leq s, j = 1\dots n\}$  and by  $\mathbb{D}_t^n$ ,  $t > 0$ , the polydisc  $\{r \in \mathbb{C}^n, |r_j| \leq t, j = 1\dots n\}$ . The supremum norm of a function  $f \in \mathcal{A}(\mathbb{T}_s^n \times \mathbb{D}_t^n)$  will be denoted by  $|f|_{s,t}$ .

Let  $0 < s_0 \leq s_1$  and  $0 < t_0 \leq t_1$  be such that

$$\log \frac{t_1}{t_0} = s_1 - s_0.$$

Let also  $0 \leq \rho \leq 1$  and

$$s = (1 - \rho)s_0 + \rho s_1 \quad \text{and} \quad t = t_0^{1-\rho} t_1^\rho.$$

**Proposition 15.** *If  $f \in \mathcal{A}(\mathbb{T}_{s_1}^n \times \mathbb{D}_{t_1}^n)$ ,*

$$|f|_{s,t} \leq |f|_{s_0,t_0}^{1-\rho} |f|_{s_1,t_1}^\rho.$$

*Proof.* Let  $\tilde{f}$  be the function on  $\mathbb{T}_{s_1}^n \times \mathbb{D}_{t_1}^n$ , constant on  $2n$ -tori of equations  $(\text{Im}\theta, r) = cst$ , defined by

$$\tilde{f}(\theta, r) = \max_{\mu, \nu \in \mathbb{T}^n} |f((\pm\theta_1 + \mu_1, \dots, \pm\theta_n + \mu_n), (r_1 e^{i\nu_1}, \dots, r_n e^{i\nu_n}))|$$

(with all possible combinations of signs). Since  $\log |f|$  is subharmonic and  $\mathbb{T}^{2n}$  is compact,  $\log \tilde{f}$  too is upper semi-continuous. Besides,  $\log \tilde{f}$  satisfies the mean inequality, hence is plurisubharmonic.

By the maximum principle, the restriction of  $|f|$  to  $\mathbb{T}_s^n \times \mathbb{D}_t^n$  attains its maximum on the distinguished boundary of  $\mathbb{T}_s^n \times \mathbb{D}_t^n$ . Due to the symmetry of  $\tilde{f}$ :

$$|f|_{s,t} = \tilde{f}(is\epsilon, t\epsilon), \quad \epsilon = (1, \dots, 1).$$

Now, the function

$$\varphi(z) := \tilde{f}(z\epsilon, e^{-(iz+s)}t\epsilon)$$

is well defined on  $\mathbb{T}_{s_1}$ , for it is constant with respect to  $\operatorname{Re} z$  and, due to the relations imposed on the norm indices, if  $|\operatorname{Im} z| \leq s_1$  then  $|e^{-(iz+s)}t| \leq e^{s_1-s}t = t_1$ .

The estimate

$$\log \varphi(z) \leq \frac{s_1 - \operatorname{Im} z}{s_1 - s_0} \varphi(s_0 i) + \frac{\operatorname{Im} z - s_0}{s_1 - s_0} \varphi(s_1 i)$$

trivially holds if  $\operatorname{Im} z = s_0$  or  $s_1$ , for, as noted above for  $j = 1$ ,  $e^{s_j - s}t = t_j$ ,  $j = 0, 1$ . But note that the left and right hand sides respectively are subharmonic and harmonic. Hence the estimate holds whenever  $s_0 \leq \operatorname{Im} z \leq s_1$ , whence the claim for  $z = is$ .  $\square$

Recall that we have let  $\mathbb{T}_s^n := \mathbb{T}_s^n \times \mathbb{D}_s^n$ ,  $s > 0$ , and, for a function  $f \in \mathcal{A}(\mathbb{T}_s^n)$ , let  $|f|_s = |f|_{s,s}$  denote its supremum norm on  $\mathbb{T}_s^n$ . As in the rest of the paper, we now restrict the discussion to widths of analyticity  $\leq 1$ .

**Corollary 16.** *If  $\sigma_1 = -\log(1 - \frac{\sigma_0}{s})$  and  $f \in \mathcal{A}(\mathbb{T}_{s+\sigma_1}^n)$ ,*

$$|f|_s^2 \leq |f|_{s-\sigma_0} |f|_{s+\sigma_1}.$$

In A.1, we will use the equivalent fact that, if  $\tilde{\sigma} = s + \log(1 + \frac{\sigma}{s})$  and  $f \in \mathcal{A}(\mathbb{T}_{s+\tilde{\sigma}}^n)$ ,

$$|f|_{s+\sigma}^2 \leq |f|_s |f|_{s+\tilde{\sigma}}.$$

*Proof.* In proposition 15, consider the following particular case :

- $\rho = 1/2$ . Hence

$$s = \frac{s_0 + s_1}{2} \quad \text{and} \quad t = \sqrt{t_0 t_1}.$$

- $s = t$ . Hence in particular  $t_0 = s e^{s_0 - s}$  and  $t_1 = s e^{s_1 - s}$ .

Then

$$|f|_s^2 = |f|_{s,s}^2 \leq |f|_{s_0, t_0} |f|_{s_1, t_1}.$$

We want to determine  $\max(s_0, t_0)$  and  $\max(s_1, t_1)$ . Let  $\sigma_1 := s - s_0 = s_1 - s$ . Then  $t_0 = s e^{-\sigma_1}$  and  $t_1 = s e^{\sigma_1}$ . The expression  $s + \sigma - s e^\sigma$  has the sign of  $\sigma$  (in the relevant region  $0 \leq s + \sigma \leq 1$ ,  $0 \leq s \leq 1$ ); by evaluating it at  $\sigma = \pm\sigma_1$ , we see that  $s_0 \leq t_0$  and  $s_1 \geq t_1$ .

Therefore, since the norm  $|\cdot|_{s,t}$  is non-decreasing with respect to both  $s$  and  $t$ ,

$$|f|_s^2 \leq |f|_{t_0, t_0} |f|_{s_1, s_1} = |f|_{t_0} |f|_{s_1}$$

(thus giving up estimates uniform with respect to small values of  $s$ ). By further setting  $\sigma_0 = s - t_0 = s(1 - e^{-\sigma_1})$ , we get the wanted estimate, and the asserted relation between  $\sigma_0$  and  $\sigma_1$  is readily verified.  $\square$

## D. COMMENTS

*Section 1.* The proof of Kolmogorov's theorem presented here differs from others chiefly for the following reasons:

- The seeming detour through Herman's normal form reduces Kolmogorov's theorem to a functionally well posed inversion problem (compare with Zehnder [1975, 1976]). The remaining, finite dimensional problem is then to show that the frequency offset  $\beta \in \mathbb{R}^n$  may vanish; in general, it is met using a non-degeneracy hypothesis of one kind or another.

The formulation chosen here adapts to more degenerate cases, including lower dimensional, normally hyperbolic or elliptic, tori (compare the two proofs of Herman described in Bost [1986] and Féjóz [2004]).

- The analytic (or Gevrey) category is simpler, in KAM theory, than Hölder or Sobolev categories because the Newton algorithm can be carried out without intercalating smoothing operators (compare with Sergeraert [1972], Bost [1986]).
- Incidentally, Hadamard interpolation inequalities are simple to infer for analytic norms because, again, they do not depend on regularizing operators, as it is done in appendix C (see [Hörmander, 1976, Theorem A.5]).
- The use of auxiliary norms ( $|\cdot|_{G,s}$  in lemmas 5 and 7,  $|\cdot|_{x,s}$  in appendix A) prevents from artificially loosing, due to compositions, a fixed width of analyticity at each step of the Newton algorithm –the domains of analyticity being deformed rather than shrunk (compare with the inverse function theorem of [Jacobowitz, 1972, Theorem 1]).

*Section 1. Theorem 4.* Herman's normal form is the Hamiltonian analogue of the normal form of vector fields on the torus in the neighborhood of Diophantine constant vector fields (Arnold [1961], Moser [1966a]). The normal form for Hamiltonians implies the normal form for vector fields on the torus [Féjóz, 2004, Théorème 40] and is rather simpler to prove from the algebraic point of view.

*Section 3. Lemma 5.* . The estimate is obtained by bounding the terms of Fourier series one by one; for sharper estimates which much improve the exponent of  $\sigma$  in the right hand side, making it independent of the dimension), see Moser [1966b] and Rüssmann [1975].

*Appendix A. Theorem 8.* – The two competing small parameters  $\eta$  and  $\sigma$  being fixed, our choice of the sequence  $(\sigma_n)$  maximizes  $\epsilon$  for the Newton algorithm. It does not modify the sequence  $(x_k)$  but only the information we retain from  $(x_k)$ .

- In the expression of  $\epsilon$ , the square exponent of  $C$  is inherent in the quadratic convergence of Newton's algorithm. From this follows the dependance, in KAM theory, of the size  $\epsilon$  of the allowed perturbation with respect to the small diophantine constant  $\gamma$ :  $\epsilon = O(\gamma^2)$ .
- The method of Jacobowitz [1972] (see Moser [1966b] also) in order to deduce an inverse function theorem in the smooth category from its analogue in the analytic category does

not work here. The idea would be to use Jackson's theorem in approximation theory to characterize the Hölder spaces by their approximation properties in terms of analytic functions and, then, to find a smooth preimage  $x$  by  $\phi$  of a smooth function  $y$  as the limit of analytic preimages  $x_j$  of analytic approximations  $y_j$  of  $y$ . However, in our inversion function theorem we require the operator  $\phi$  to be defined only on balls  $\sigma B_{s+\sigma}$  with shrinking radii when  $s + \sigma$  tends to 0. This domain is too small in general to include the analytic approximations  $y_j$  of a smooth  $y$ . Such a restriction is inherent in the presence of composition operators, which Jacobowitz [1972] did not have to deal with. Yet we *could* generalize Jacobowitz's proof at the expense of making additional hypotheses on the form of our operator  $\phi$ , which would take into account the specificity of directions  $K$  and  $G$ , as well as of the real phase space and of its complex extension.

*Appendix A.1.* It is possible to prove that  $\psi$  is  $C^1$  without additional assumptions, just by patterning [Sergeraert, 1972, p. 626]). Yet the proof simplifies and the estimates improve under the combined two additional assumptions.

*Appendix B.* We include this section for the sake of completeness, although the quantitative estimates are needed only if one wants a quantitative version of Kolmogorov's theorem, with an explicit value of  $\epsilon$ .

A similar proposition is proved in Pöschel [2001] using an argument from degree theory.

*Appendix C.* In this paragraph, the obtained interpolation inequalities generalize the standard Hadamard convexity inequalities. Yet they cannot be interpreted as convexity inequalities, due to the geometry of the complex phase space. See [Narasimhan, 1995, Chap. 8] for more general but less precise inequalities.

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