

Maximum (cardinality) matchings in bipartite graphs

So far we have seen simple matching algorithms. In particular a simple algorithm to find a maximum matching in a tree.

Trees are a particular class of bipartite graphs. So the natural question is: what is the complexity of the maximum matching problem in bipartite graphs?

Historically some of the most important theorems for bipartite graphs were proved directly and only later were these bipartite results obtained as corollaries of more general non-bipartite theorems.

Moreover bipartite graphs deserve special treatment because the majority of real world applications of matching theory deal with bipartite graph models.

Exercise. $\alpha(G) + \tau(G) = |V|$, for any graph G .

Lemma 1. $\nu(G) + \rho(G) = |V|$, for any graph G with no isolated vertex.

Proof. Let C be a minimum edge cover. $G[V(C)]$, the graph induced by the set $V(C)$, consist of a number of “stars” (because any other “larger” graph would not be minimal).

Therefore the number of connected components in $G[V(C)]$ is $n - \rho(G)$ (because each component on k vertices has $k - 1$ edges).

Picking one edge from each of them we get a matching M of size $n - \rho(G)$. Hence $\nu(G) \geq n - \rho(G)$.

Conversely let M be a maximum matching in G .

For every vertex v NOT incident to an edge of M there must be an edge connecting v with some edge in M .

We can thus define a set of lines covering every vertex of G of size $\nu(G) + (n - 2\nu(G))$. Hence $\rho(G) \leq n - \nu(G)$.

Useful functions

Let G be any graph. Then

$\nu(G)$ denote the matching number of G (i.e. the cardinality of the largest matchings in G)

$\tau(G)$ the vertex covering number (i.e. the cardinality of the smallest sets of vertices incident to all edges in G)

$\alpha(G)$ the independence number (i.e. the cardinality of the largest sets U of vertices such that $G[U]$ is empty) and

$\rho(G)$ the edge covering number (i.e. the smallest number of edges that are incident to any vertex in G).

Next we will describe and prove a number of results relating these quantities (valid for any graph).

In proving the first of these two lemma you will notice the nice relationship between minimal vertex covers and independent sets of points, if S is one such cover then $V \setminus S$ is an independent set. There is no such relationship between edge covers and matchings.

So what? Are questions about $\tau(G)$ or $\alpha(G)$ going to be “easier” than questions about $\rho(G)$ and $\nu(G)$? Quite the contrary!

Exercises. Prove the following results.

1. A minimal edge cover is minimum if and only if it contains a maximum matching.
2. A maximal matching is maximum if and only if it is contained in a minimum line cover.

König's theorem

Let's now turn to bipartite graphs with three very beautiful theorems.

Theorem 1. $\nu(G) = \tau(G)$ if G is bipartite.

Proof It is enough to prove $\nu(G) \geq \tau(G)$ (the other inequality is fairly simple and left as an exercise).

Define G' to be the graph with the minimum number of edges which is a subgraph of G and has $\tau(G') = \tau(G)$ (hence $\tau(G' \setminus e) < \tau(G')$ for any $e \in E(G')$).

Claim. $E(G')$ is a matching in G .

Notice that König's theorem follows from the claim as $\tau(G) = \tau(G') = |E(G')| \leq \nu(G)$.

7. Furthermore, since G is bipartite, there is a set T (the smaller of the two colour classes of G'') which covers G'' and has size at most $\frac{|V(G'')|}{2} - 1 = \tau(G') - 1 - t$.
8. But $T \cup (S_x \cap S_y)$ covers G' ... and $|T \cup (S_x \cap S_y)| \leq \tau(G') - 1 - t + t = \tau(G') - 1$. Contradiction.

Proof of the main claim

We will prove the claim by contradiction.

1. Assume there were two edges x and y incident to a vertex a in G' , then consider $G' - x$ and $G' - y$.
2. There is a cover S_x in $G' - x$ with $|S_x| = \tau(G') - 1$ and neither end-points of x belong to S_x .
3. Similarly there is a cover S_y in $G' - y$ with $|S_y| = \tau(G') - 1$ and neither end-point of y are in S_y .
4. Hence $|S_x| = |S_y|$.
5. Consider the subgraph G'' of G' induced by a (the common endpoint of x and y) and $S_x \oplus S_y = (S_x \setminus S_y) \cup (S_y \setminus S_x)$.
6. Let $t = |S_x \cap S_y|$. Then, trivially, $|V(G'')| = 2(\tau(G') - 1 - t) + 1$.

Hall's theorem

König's theorem will be needed to prove the correctness of our matching algorithm. But the result also has a number of important consequences.

Hall's theorem is an important corollary of König's theorem.

We prefer to give an explicit inductive proof of this, as this is of interest in its own right. We first need one bit of notation.

If X is a set of vertices in a graph G , let denote by $\Gamma(X)$ the set of vertices that are adjacent to at least one vertex in X .

Theorem 2. Let $G = (V_1, V_2, E)$ be a bipartite graph. Then G has a matching of V_1 into V_2 if and only if $|\Gamma(X)| \geq |X|$ for all $X \subseteq V_1$.

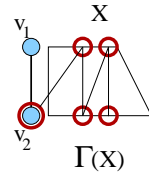
Proof. First notice that if G does have a matching of V_1 into V_2 then $|\Gamma(X)| \geq |X|$ for all $X \subseteq V_1$, because for every $X \subseteq V_1$ we can find a distinct element in V_2 , i.e. $\Gamma(X)$ is at least as large as X .

Proof of the “difficult” implication

Let's suppose $|\Gamma(X)| \geq |X|$ for all $X \subseteq V_1$. We prove that there is a matching of V_1 into V_2 by induction on $|V_1|$. If V_1 has size zero or one the result is trivial.

(Case 1.) $|\Gamma(X)| > |X|$ for all X such that $\emptyset \subset X \subset V_1$.

Let v_1 and v_2 be adjacent with $v_i \in V_i$ for $i = 1, 2$. Let $G' = G - v_1 - v_2$ and X be any subset of $V_1 - v_1$. We have that $|\Gamma(X)| > |X|$ by our assumption and therefore, counting w.r.t. G' , $|\Gamma_{G'}(X)| \geq |\Gamma(X)| - 1 \geq |X|$. Thus we can apply the induction hypothesis, find a matching of $V_1 - v_1$ into $V_2 - v_2$, and add $\{v_1, v_2\}$ to it, end of story.



Exercises

1. Can you design a recursive algorithm based on the proof of Hall's theorem given above?
2. What is the complexity of the algorithm under point 1.?

A final corollary

A *perfect matching* (or *1-factor*) is a matching which covers all points of G . The following is an obvious corollary of Hall's theorem.

Theorem 3. (Frobenius' Marriage theorem) A bipartite graph G has a perfect matching if and only if $|V_1| = |V_2|$ and Hall's condition holds for any $X \subseteq V_1$.

(Case 2.) There is at least one proper subset of V_1 , X' such that $|\Gamma(X')| = |X'|$. Let G_1 be the subgraph of G induced by X' and $\Gamma(X')$ and G_2 the graph induced by $V \setminus X' \setminus \Gamma(X')$.

Suppose $X \subseteq X'$, then $\Gamma(X)$ must be a subset of $\Gamma(X')$ therefore, as for every subset of V_1 , $|\Gamma_{G_1}(X)| \geq |X|$.

Secondly if $X \subseteq V_1 \setminus X'$, then we look at $\Gamma(X \cup X')$. This is equal to $\Gamma_{G_2}(X) \cup \Gamma(X')$ (and the two sets are disjoint). Therefore $|\Gamma_{G_2}(X)| = |\Gamma_{G_2}(X) \cup \Gamma(X')| - |\Gamma(X')| \geq |X \cup X'| - |\Gamma(X')| = |X \cup X'| - |X'| = |X|$.

Therefore we can apply induction to G_1 and G_2 and get two matchings M_1 and M_2 whose union is the solution to the given problem.

