Abel-Jacobi map, integral Hodge classes and decomposition of the diagonal

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Abstract

Given a smooth projective n-fold Y, with $H^{3,0}(Y) = 0$, the Abel-Jacobi map induces a morphism from each smooth variety parameterizing codimension 2-cycles in Y to the intermediate Jacobian $J(Y)$, which is an abelian variety. Assuming $n = 3$, we study in this paper the existence of families of 1-cycles in Y for which this induced morphism is surjective with rationally connected general fiber, and various applications of this property. When Y itself is rationally connected with trivial Brauer group, we relate this property to the existence of an integral cohomological decomposition of the diagonal of Y. We also study this property for cubic threefolds, completing the work of Iliev-Markushevich-Tikhomirov. We then conclude that the Hodge conjecture holds for degree 4 integral Hodge classes on fibrations into cubic threefolds over curves, with some restriction on singular fibers.

0 Introduction

The following result is proved by Bloch and Srinivas as a consequence of their decomposition of the diagonal:

Theorem 0.1. *([3]) Let Y be a smooth complex projective variety with $CH_0(Y)$ supported on a curve. Then, via the Abel-Jacobi map AJ_Y, the group of codimension 2-cycles homologous to 0 on Y modulo rational equivalence is up to torsion isomorphic to the intermediate Jacobian $J(Y)$ of Y,*

$$J(Y) := H^3(Y, \mathbb{C})/(F^2H^3(Y) \oplus H^3(Y, \mathbb{Z})). \tag{0.1}$$

This follows from the diagonal decomposition principle [3] (see also [30, II,10.2.1]). Indeed, this principle says that if Y is a smooth variety such that $CH_0(Y)$ is supported on some subvariety $W \subset Y$, there is an equality in $CH^d(Y \times Y)$, $d = dim\,Y$:

$$N\Delta_Y = Z_1 + Z_2, \tag{0.2}$$

where N is a nonzero integer and Z_1, Z_2 are codimension d cycles with

$$\operatorname{Supp} Z_1 \subset D \times Y,\ D \subsetneq Y, \quad \operatorname{Supp} Z_2 \subset Y \times W.$$

Assuming $\dim W \leq 1$, the integer N appearing in this decomposition is easily checked to annihilate the kernel and cokernel of AJ_Y.

Note that the integer N appearing above cannot in general be set equal to 1, and one purpose of this paper is to investigate the significance of this invariant, at least if we work on the level of cycles modulo homological equivalence. We will focus in this paper to the case of d-folds with CH_0 group supported on a curve. In this case, the diagonal decomposition (0.2) has a term Z_2 supported on $Y \times W$, for subvariety W of Y of dimension ≤ 1. We will say that Y admits a *cohomological decomposition of the diagonal* as in (0.2) if there is an equality of cycle classes in $H^{2d}(Y \times Y, \mathbb{Z})$ as in (0.2), with $\operatorname{Supp} Z_1 \subset D \times Y$ and $\operatorname{Supp} Z_2 \subset Y \times W$. We will say that Y admits an *integral cohomological decomposition of the diagonal* if one has such a decomposition with $N = 1$.

Recall that the existence of a cohomological decomposition of the diagonal as above has strong consequences (see [3] or [30, II,10.2.2,10.2.3]). For example, this implies the generalized Mumford theorem which says in this case that $H^i(Y, \mathcal{O}_Y) = 0$ for $i \geq 2$. Thus the Hodge structures on $H^2(Y, \mathbb{Q})$, hence on its Poincaré dual $H^{2d-2}(Y, \mathbb{Q})$ are trivial. Furthermore the intermediate Jacobian $J^3(Y)$ is an abelian variety (cf. [13]) and the Abel-Jacobi map $CH^2(Y)_{hom} \to J^3(Y)$ is surjective.

Going further and using the theory of Bloch-Ogus [2] together with Merkurjev-Suslin theorem, Bloch and Srinivas also prove the following:

Theorem 0.2. *[3] If Y is a smooth projective complex variety such that $CH_0(Y)$ is supported on a surface, the Griffiths group* $\operatorname{Griff}^2(Y) = CH^2(Y)_{hom}/\mathrm{alg}$ *is identically* 0.

The last result cannot be obtained as a consequence of the diagonal decomposition, which only shows that under the same assumption $\operatorname{Griff}^2(Y)$ is annihilated by the integer N introduced above.

We finally have the following improvement of Theorem 0.1 :

Theorem 0.3. *(cf. [24]) If Y is a smooth complex projective variety such that $CH_0(Y)$ is supported on a curve, then the Abel-Jacobi map induces an isomorphism:*

$$AJ_Y : CH^2(Y)_{hom} = CH^2(Y)_{alg} \cong J(Y).$$

This follows indeed from the fact that this map is surjective with kernel of torsion by Theorem 0.1, and that it can be shown by delicate arguments involving the Merkurjev-Suslin theorem, Gersten-Quillen resolution in K-theory, and Bloch-Ogus theory [2], that in general the Abel-Jacobi map is injective on torsion codimension 2 cycles.

The group on the left does not have a priori the structure of an algebraic variety, unlike the group on the right. However it makes sense to say that AJ_Y is algebraic, meaning that for any smooth algebraic variety B, and any codimension 2-cycle $Z \subset B \times Y$, with $Z_b \in CH^2(Y)_{hom}$ for any $b \in B$, the induced map

$$\phi_Z : B \to J(Y),\ b \mapsto AJ_Y(Z_b),$$

is a morphism of algebraic varieties.

Consider the case of a uniruled 3-fold Y with $CH_0(Y)$ supported on a curve. Then it is proved in [28] (cf. Theorem 1.3) that the integral degree 4 cohomology $H^4(Y, \mathbb{Z})$ is generated over $\mathbb{Z}$ by classes of curves, and thus the birational invariant

$$Z^4(Y) := \frac{Hdg^4(Y, \mathbb{Z})}{< [Z],\ Z \subset Y,\ \operatorname{codim} Z = 2 >} \tag{0.3}$$

studied in [20] and [7] (see also section 1), is trivial in this case.

One of the main results of this paper is the following theorem concerning the group Z^4 for certain fourfolds fibered over curves.

Theorem 0.4. *Let $f : X \to \Gamma$ be a fibration over a curve with general fiber a smooth cubic threefold or a complete intersection of two quadrics in $\mathbb{P}^5$. If the fibers of f have at worst ordinary quadratic singularities, then the Hodge conjecture holds for degree 4 integral Hodge classes on X. In other words, the group $Z^4(X)$ is trivial.*

Remark 0.5. The difficulty here is to prove the result for integral Hodge classes. Indeed, the fact that degree 4 rational Hodge classes are algebraic for X as above can be proved by using either the results of [8], since such an X is swept-out by rational curves, or those of Bloch-Srinivas [3], who prove this statement for any variety whose CH_0 group is supported on a subvariety of dimension ≤ 3, as a consequence of the decomposition of the diagonal (0.2), or by using the method of Zucker [31], who uses the theory of normal functions, which we will essentially follow here.

As we will recall from [7] in section 1, such a result can be obtained as a consequence of the study of the geometric properties of the Abel-Jacobi map for the fibers of f. In the case of the complete intersection of two quadrics in $\mathbb{P}^5$, this study was done by Reid [25] and Castravet [4], and Theorem 0.4 for this case is then an immediate consequence of Theorem 1.4 proved in section 1 (see Corollary 1.7). I thank the Referee for pointing out this application.

Another motivation for this study is the following: The conclusion of the above mentioned theorems 0.2, 0.3 and 1.3 is that for a uniruled threefold with CH_0 supported on a curve, all the interesting (and birationally invariant) phenomena concerning codimension 2 cycles, namely the kernel of the Abel-Jacobi map (Mumford [23]), the Griffiths group (Griffiths [13]) and the group $Z^4(X)$ versus degree 3 unramified cohomology with torsion coefficients (Soulé-Voisin [26], Colliot-Thélène-Voisin [7]) are trivial. In the rationally connected case, the only interesting cohomological invariant could be the Artin-Mumford invariant (or degree 2 unramified cohomology with torsion coefficients, cf. [6]), which is also equal to the Brauer group since $H^2(Y, \mathcal{O}_Y) = 0$. Still the geometric structure of the Abel-Jacobi map on families of 1-cycles on such threefolds is mysterious, in contrast to what happens in the curve case, where Abel's theorem shows that the Abel-Jacobi map on the family of effective 0-cycles of large degree has fibers isomorphic to projective spaces. Another goal of this paper is to underline substantial differences between 1-cycles on threefolds with small CH_0 on one side and 0-cycles on curves on the other side, coming from geometry of the fibers of the Abel-Jacobi map.

There are for example two natural questions (Questions 0.6 and 0.10) left open by Theorem 0.3 :

Question 0.6. *Let Y be a smooth projective threefold, such that $AJ_Y : CH_1(Y)_{alg} \to J(Y)$ is surjective. Is there a codimension 2 cycle $Z \subset J(Y) \times Y$ with $Z_b \in CH^2(Y)_{hom}$ for $b \in J(Y)$, such that the induced morphism*

$$\phi_Z : J(Y) \to J(Y),\ \phi_Z(b) := AJ_Y(Z_b)$$

is the identity?

Note that the surjectivity assumption is conjecturally implied by the vanishing $H^3(Y, \mathcal{O}_Y) = 0$, via the generalized Hodge conjecture (cf. [15]).

Remark 0.7. Although the *geometric* study of the Abel-Jacobi map will lead to consider flat families of curves on Y, we do not ask that the cycle Z above is a combination of codimension 2 algebraic subsets $Z_i \subset J(Y) \times Y$ which are flat over J. For a cycle $Z \in CH^2(B \times Y)$, this would be needed to define properly the restricted cycle $Z_b \in CH^2(b \times Y)$ if the base B of a family of cycles $Z \subset B \times Y$ was not smooth, but when it is smooth, we can use the restriction map $CH^2(B \times Y) \to CH^2(b \times Y)$ defined by Fulton [10].

Remark 0.8. One can more precisely introduce a birational invariant of Y defined as the gcd of the non zero integers N for which there exist a variety B and a cycle $Z \subset B \times Y$ as above, with $\deg \phi_Z = N$. Question 0.6 can then be reformulated by asking whether this invariant is equal to 1.

Remark 0.9. Question 0.6 has a positive answer if the Hodge conjecture for degree 4 integral Hodge classes on $Y \times J(Y)$ are algebraic. Indeed, the isomorphism $H_1(J(C), \mathbb{Z}) \cong H^3(Y, \mathbb{Z})$ is an isomorphism of Hodge structures which provides a degree 4 integral Hodge class α on $J(Y) \times Y$ (cf. [30, I,Lemma 11.41]). A codimension 2 algebraic cycle Z on $J(Y) \times Y$ with $[Z] = \alpha$ would provide a solution to Question 0.6.

The following question is an important variant of the previous one, which appears to be much more natural in specific geometric contexts (see section 2).

Question 0.10. *Is the following property (*) satisfied by Y?*

(*) There exists a mooth projective variety B and a codimension 2 cycle $Z \subset B \times Y$, with $Z_b \in CH^2(Y)_{hom}$ for any $b \in B$, such that the induced morphism $\phi_Z : B \to J(Y)$ is surjective with rationally connected general fiber.

This question has been solved by Iliev-Markushevich and Markushevich-Tikhomirov ([19], [22], see also [17] for similar results obtained independently) in the case where Y is a smooth cubic threefold in $\mathbb{P}^4$. The answer is also affirmative for the intersection X of two quadrics in $\mathbb{P}^5$ (cf. [25]): in this case the family of lines in X is a surface isomorphic via a choice of base point to the intermediate Jacobian $J(X)$.

Obviously a positive answer to Question 0.6 implies a positive answer to Question 0.10, as we can then just take $B = J(X)$. We will provide a more precise relation between these two questions in section 3.1. However, it seems that Question 0.10 is more natural, especially if we go to the following stronger version (0.11) that we wish to partially investigate in this paper:

Here we choose an integral cohomology class $\alpha \in H^4(Y, \mathbb{Z})$. Assuming $CH_0(Y)$ is supported on a curve, the Hodge structure on $H^4(Y, \mathbb{Q})$ is trivial and thus α is a Hodge class. Introduce the torsor $J(Y)_\alpha$ defined as follows: the Deligne cohomology group $H^4_D(Y, \mathbb{Z}(2))$ is an extension

$$0 \to J(Y) \to H^4_D(Y, \mathbb{Z}(2)) \stackrel{o}{\to} Hdg^4(Y, \mathbb{Z}) \to 0, \tag{0.4}$$

where o is the natural map from Deligne to Betti cohomology (cf. [30, I,Corollary 12.27]). Define

$$J(Y)_\alpha := o^{-1}(\alpha). \tag{0.5}$$

By definition, the Deligne cycle class map (cf. [9], [30, I,12.3.3]), restricted to codimension 2 cycles of class α, takes value in $J(Y)_\alpha$. Furthermore, for any family of 1-cycles $Z \subset B \times Y$ of class $[Z_b] = \alpha$, $b \in B$, parameterized by an algebraic variety B, the map ϕ_Z induced by the Abel-Jacobi map (or rather the Deligne cycle class map) of Y, that is

$$\phi_Z : B \to J(Y)_\alpha,\ \phi_Z(b) = AJ_Y(Z_b),$$

is a morphism of complex algebraic varieties. The following question makes sense for any smooth projective threefold Y satisfying the conditions $H^2(Y, \mathcal{O}_Y) = H^3(Y, \mathcal{O}_Y) = 0$:

Question 0.11. *Is the following property (**) satisfied by Y?*

(**) For any degree 4 integral cohomology class α on Y, there is a "naturally defined" *(up to birational transformations) smooth projective variety B_α, together with a codimension 2 cycle $Z_\alpha \subset B_\alpha \times Y$, with $[Z_{\alpha,b}] = \alpha$ in $H^4(Y, \mathbb{Z})$ for any $b \in B$, such that the morphism $\phi_{Z_\alpha} : B_\alpha \to J(Y)_\alpha$ is surjective with rationally connected general fiber.*

By "naturally defined", we have in mind that B_α should be determined by α by some natural geometric construction (eg, if α is sufficiently positive, a main component of the Hilbert scheme of curves of class α and given genus, or a moduli space of vector bundles with $c_2 = \alpha$), which would imply that B_α is defined over the same definition field as Y.

This question is solved by Castravet in [4] when Y is the complete intersection of two quadrics. Let us comment on the importance of Question 0.11 in relation with the Hodge conjecture with integral coefficients for degree 4 Hodge classes, (that is the study of the group Z^4 introduced in (0.3)): The important point here is that we want to consider fourfolds fibered over curves, or families of threefolds Y_t parameterized by a curve Γ. The generic fiber of this fibration is a threefold Y defined over $\mathbb{C}(\Gamma)$. Property (**) essentially says that property (*), being satisfied over the definition field, which is in this case $\mathbb{C}(\Gamma)$, holds in family. When we work in families, the necessity to look at all torsors $J(Y)_\alpha$, and not only at $J(Y)$, becomes obvious: for fixed Y the twisted Jacobians are all isomorphic (maybe not canonically) and if we can choose a cycle z_α in each given class α (for example if Y is uniruled so that $Z^4(Y) = 0$), we can use translations by the z_α to reduce the problem to the case where $\alpha = 0$; this is a priori not true in families, for example because non trivial torsors $\mathcal{J}_\alpha$ may appear. We will give more precise explanations in section 1 and explain one application of this property to the Hodge conjecture for degree 4 integral Hodge classes on fourfolds fibered over curves.

Our results in this paper are of two kinds. First of all, we extend the results of [19] and answer affirmatively Question 0.11 for cubic threefolds. As a consequence, we prove Theorem 0.4. Note that Castravet's work answers affirmatively Question 0.11 for $(2,2)$ complete intersections in $\mathbb{P}^5$, which implies Theorem 0.4 for a fourfold X fibered by complete intersections of two quadrics in $\mathbb{P}^5$ (see Corollary 1.7). However many such fourfolds X are rational over the base, (that is, birational to $\Gamma \times \mathbb{P}^3$) : this is the case for example if there is a section of the family of lines in the fibers of f. When X is rational over the base, the vanishing of $Z^4(X)$ is immediate because the group $Z^4(X)$ of (0.3) is a birational invariant of X.

By the results of [7], theses results can also be translated into statements concerning degree 3 unramified cohomology with $\mathbb{Q}/\mathbb{Z}$-coefficients of such fourfolds (see section 1).

Our second result relates Question 0.10 to the existence of a cohomological integral decomposition of the diagonal as in (0.2). Recall first (see [5]) that the intermediate Jacobian $J(Y)$ of a smooth projective threefold Y with $H^3(Y, \mathcal{O}_Y) = 0$ is naturally a principally polarized abelian variety, the polarization Θ being given by the intersection form on $H^3(Y, \mathbb{Z}) \cong H_1(J(Y), \mathbb{Z})$.

Theorem 0.12. *1) Let Y be a smooth projective 3-fold. If Y admits an integral cohomological decomposition of the diagonal as in (0.2), then :*

i) $H^4(Y, \mathbb{Z})$ is generated by classes of algebraic cycles,

ii) $H^p(Y, \mathbb{Z})$ has no torsion for any integer p.

iii) Y satisfies condition (*).

2) As a partial converse, assume i), ii) and iii).

If furthermore the intermediate Jacobian of Y admits a 1-cycle Γ of class $\frac{[\Theta]^{g-1}}{(g-1)!}$, $g = dim\, J(Y)$, then Y admits an integral cohomological decomposition of the diagonal as in (0.2).

Note that conditions i) and ii) are satisfied by a rationally connected threefold with no torsion in $H^3(Y, \mathbb{Z})$. Hence in this case, this theorem mainly relates question 0.10 to the existence of a cohomological decomposition of the diagonal.

We will also prove the following relation between Question 0.6 and 0.10 (cf. Theorem 3.1):

Theorem 0.13. *Assume that Question 0.10 has a positive answer for Y and that the intermediate Jacobian of Y admits a 1-cycle Γ of class $\frac{[\Theta]^{g-1}}{(g-1)!}$, $g = dim\, J(Y)$. Then Question 0.6 also has an affirmative answer for Y.*

The paper is organized as follows: in section 2, we give a positive answer to Question 0.10 for general cubic threefolds. We deduce from this Theorem 0.4. Section 3 is devoted to various relations between Question 0.6 and 0.10 and the relation between these questions and the cohomological decomposition of the diagonal with integral coefficients, in the spirit of Theorem 0.12.

Thanks. I wish to thank Jean-Louis Colliot-Thélène for many discussions leading to the present form of the paper and for numerous questions and suggestions. In particular, Theorem 0.4 answers one of his questions. I also thank Dimitri Markushevich and Jason Starr for helpful communications. I am also very grateful to the Referee for his careful reading and his numerous suggestions.

1 Preliminaries on integral Hodge classes and unramified cohomology

We give in this section a description of results and notations from the earlier papers [28], [7] which will be used later on in the paper. Let X be a smooth projective complex variety. We denote $Z^{2i}(X)$ the quotient $Hdg^{2i}(X,\mathbb{Z})/H^{2i}(X,\mathbb{Z})_{alg}$ of the group of degree $2i$ integral Hodge classes on X by the subgroup consisting of cycle classes. The group $Z^{2i}(X)$ is a birational invariant of X for $i=2,\,d-1$, where $d=dim\,X$ (cf. [26]). It is of course trivial in degrees $2i=0,\,2d,\,2$, where the Hodge conjecture holds for integral Hodge classes.

The paper [7] focuses on $Z^4(X)$ and relates its torsion to degree 3 unramified cohomology of X with torsion coefficients. Recall that for any abelian group A, unramified cohomology $H^i_{nr}(X,A)$ with coefficients in A was introduced in [6]. In the Betti context, the setup is as follows: Denote by X_{Zar} the variety X (or rather, the set $X(\mathbb{C})$) endowed with the Zariski topology, while X will be considered as endowed with the classical topology. The identity map

$$\pi : X \to X_{Zar}$$

is continuous and allows Bloch and Ogus [2] to introduce sheaves $\mathcal{H}^l(A)$ on X_{Zar} defined by

$$\mathcal{H}^l(A) := R^l\pi_*A.$$

Definition 1.1. (Ojanguren-Colliot-Thélène [6]) *Unramified cohomology $H^i_{nr}(X,A)$ of X with coefficients in A is the group $H^0(X_{Zar},\mathcal{H}^i(A))$.*

This is a birational invariant of X, as one can see easily using the Gersten-Quillen resolution for the sheaves $\mathcal{H}^i$ proved by Bloch-Ogus [2]. We refer to [7] for the description of other birational invariants constructed from the cohomology of the sheaves $\mathcal{H}^i$.

The following result is proved in [7], using the Bloch-Kato conjecture recently proved by Voevodsky [27]:

Theorem 1.2. *There is an exact sequence for any n:*

$$0 \to H^3_{nr}(X,\mathbb{Z})\otimes\mathbb{Z}/n\mathbb{Z} \to H^3_{nr}(X,\mathbb{Z}/n\mathbb{Z}) \to Z^4(X)[n] \to 0,$$

where the notation $[n]$ means that we take the n-torsion. There is an exact sequence:

$$0 \to H^3_{nr}(X,\mathbb{Z})\otimes\mathbb{Q}/\mathbb{Z} \to H^3_{nr}(X,\mathbb{Q}/\mathbb{Z}) \to Z^4(X)_{tors} \to 0,$$

where the subscript "tors" means that we consider the torsion part. Furthermore, the first term is 0 if $CH_0(X)$ is supported on a surface.

Concerning the vanishing of the group $Z^4(X)$, the following is proved in [28]. It will be used in section 3.2.

Theorem 1.3. *Let Y be a smooth projective threefold which is either uniruled or Calabi-Yau. Then $Z^4(Y) = 0$, that is, any integral degree 4 Hodge class on Y is algebraic. In particular, if Y is uniruled with $H^2(Y, \mathcal{O}_Y) = 0$, any integral degree 4 cohomology class on Y is algebraic.*

One of the results of [7] is the existence of smooth projective rationally connected varieties X of dimension ≥ 6 for which $Z^4(X) \neq 0$.

We will study in this paper the group $Z^4(X)$, where X is a smooth projective 4-fold, $f : X \to \Gamma$ is a surjective morphism to a smooth curve Γ, whose general fiber X_t satisfies $H^3(X_t, \mathcal{O}_{X_t}) = H^2(X_t, \mathcal{O}_{X_t}) = 0$. For X_t, $t \in \Gamma$, as above, the intermediate Jacobian $J(X_t)$ is an abelian variety, as a consequence of the vanishing $H^3(X_t, \mathcal{O}_{X_t}) = 0$ (cf. [30, I,12.2.2]). For any class

$$\alpha \in H^4(X_t, \mathbb{Z}) = Hdg^4(X_t, \mathbb{Z}),$$

we introduced above a torsor $J(X_t)_\alpha$ under $J(X_t)$, which is an algebraic variety non canonically isomorphic to $J(X_t)$.

Using the obvious extension of the formulas (0.1), (0.5) in the relative setting, the construction of $J(X_t)$, $J(X_t)_\alpha$ can be done in family on the Zariski open set $\Gamma_0 \subset \Gamma$, over which f is smooth. There is thus a family of abelian varieties $\mathcal{J} \to \Gamma_0$, and for any global section α of the locally constant system $R^4 f_* \mathbb{Z}$ on Γ_0, we get the twisted family $\mathcal{J}_\alpha \to \Gamma_0$. The construction of these families in the analytic setting (that is, as (twisted) families of complex tori) follows from Hodge theory (cf. [30, II,7.1.1]) and from their explicit set theoretic description given by formulas (0.1), (0.5). The fact that the resulting families are algebraic can be proved using the results of [24], when one knows that the Abel-Jacobi map is surjective. Indeed, it is shown under this assumption that the intermediate Jacobian is the universal abelian quotient of CH^2, and thus can be constructed algebraically in the same way as the Albanese variety.

Given a smooth algebraic variety B, a morphism $g : B \to \Gamma$ and a codimension 2 cycle $\mathcal{Z} \subset B \times_\Gamma X$ of relative class $[\mathcal{Z}_b] = \alpha_{g(b)} \in H^4(X_t, \mathbb{Z})$, the relative Abel-Jacobi map (or rather Deligne cycle class map) gives a morphism

$$\phi_{\mathcal{Z}} : B_0 \to \mathcal{J}_\alpha,\ b \mapsto AJ_Y(\mathcal{Z}_b)$$

over Γ_0, where $B_0 := g^{-1}(\Gamma_0)$. Again, the proof that $\phi_{\mathcal{Z}}$ is holomorphic is quite easy (cf. [30, II,7.2.1], while the algebraicity is more delicate.

The following result, which illustrates the importance of condition (**) as opposed to condition (*), appears in [7]. We recall the proof here, as we will need it to prove a slight improvement of the criterion, which will be used in section 2.1. As before, we assume that X is a smooth projective 4-fold, and that $f : X \to \Gamma$ is a surjective morphism to a smooth curve whose general fiber X_t satisfies $H^3(X_t, \mathcal{O}_{X_t}) = H^2(X_t, \mathcal{O}_{X_t}) = 0$.

Theorem 1.4. *Assume $f : X \to \Gamma$ satisfies the following assumptions:*

1. *The smooth fibers X_t have no torsion in $H^3(X_t(\mathbb{C}), \mathbb{Z})$.*
2. *The singular fibers of f are reduced with at worst ordinary quadratic singularities.*
3. *For any section α of $R^4 f_* \mathbb{Z}$ on Γ_0, there exists a variety $g_\alpha : B_\alpha \to \Gamma$ and a codimension 2 cycle $\mathcal{Z}_\alpha \subset B_\alpha \times_\Gamma X$ of relative class $g_\alpha^* \alpha$, such that the morphism $\phi_{\mathcal{Z}_\alpha} : B \to \mathcal{J}_\alpha$ is surjective with rationally connected general fiber.*

Then the Hodge conjecture is true for integral Hodge classes of degree 4 on X.

Proof. An integral Hodge class $\tilde{\alpha} \in Hdg^4(X,\mathbb{Z}) \subset H^4(X,\mathbb{Z})$ induces a section α of the constant system $R^4 f_*\mathbb{Z}$ which admits a lift to a section of the family of twisted Jacobians $\mathcal{J}_\alpha$. This lift is obtained as follows: The class $\tilde{\alpha}$ being a Hodge class on X admits a lift β in the Deligne cohomology group $H^4_D(X,\mathbb{Z}(2))$ by the exact sequence (0.4) for X. Then our section σ is obtained by restricting β to the fibers of f: $\sigma(t) := \beta_{|X_t}$. This lift is an *algebraic* section $\Gamma \to \mathcal{J}_\alpha$ of the structural map $\mathcal{J}_\alpha \to \Gamma$.

Recall that we have by hypothesis the morphism

$$\phi_{\mathcal{Z}_\alpha} : B_\alpha \to \mathcal{J}_\alpha$$

which is algebraic, surjective with rationally connected general fiber. We can now replace $\sigma(\Gamma)$ by a 1-cycle $\Sigma = \sum_i n_i \Sigma_i$ rationally equivalent to it in $\mathcal{J}_\alpha$, in such a way that the fibers of $\phi_{\mathcal{Z}_\alpha}$ are rationally connected over the general points of each component Σ_i of $\operatorname{Supp}\Sigma$.

According to [12], the morphism $\phi_{\mathcal{Z}_\alpha}$ admits a lifting over each Σ_i, which provides curves $\Sigma'_i \subset B_\alpha$.

Recall next that there is a codimension 2 cycle $\mathcal{Z}_\alpha \subset B_\alpha \times_\Gamma X$ of relative class α parameterized by a smooth projective variety B_α. We can restrict this cycle to each Σ'_i, getting codimension 2 cycles $\mathcal{Z}_{\alpha,i} \in CH^2(\Sigma'_i \times_\Gamma X)$. Consider the 1-cycle

$$Z := \sum_i n_i p_{i*}\mathcal{Z}_{\alpha,i} \in CH^2(\Gamma \times_\Gamma X) = CH^2(X),$$

where p_i is the restriction to Σ'_i of $p : B_\alpha \to \Gamma$. Recalling that Σ is rationally equivalent to $\sigma(\Gamma)$ in $\mathcal{J}_\alpha$, we find that the "normal function ν_Z associated to Z" (cf. [30, II,7.2.1]), defined by

$$\nu_Z(t) = AJ_{X_t}(Z_{|X_t})$$

is equal to σ. We then deduce from [13] (see also [30, II,8.2.2]), using the Leray spectral sequence of $f_U : X_U \to U$ and hypothesis 1, that the cohomology classes $[Z] \in H^4(X,\mathbb{Z})$ of Z and $\tilde{\alpha}$ coincide on any open set of the form X_U, where $U \subset \Gamma_0$ is an affine open set over which f is smooth.

On the other hand, the kernel of the restriction map $H^4(X,\mathbb{Z}) \to H^4(U,\mathbb{Z})$ is generated by the groups $i_{t*}H_4(X_t,\mathbb{Z})$ where $t \in \Gamma \setminus U$, and $i_t : X_t \to X$ is the inclusion map. We conclude using assumption 2 and the fact that the general fiber of f has $H^2(X_t,\mathcal{O}_{X_t}) = 0$, which imply that all fibers X_t (singular or not) have their degree 4 integral homology generated by homology classes of algebraic cycles; indeed, it follows from this and the previous conclusion that $[Z] - \tilde{\alpha}$ is algebraic, so that $\tilde{\alpha}$ is also algebraic.

∎

Remark 1.5. It is also possible in this proof, instead of moving the curve $\sigma(\Gamma)$ to a 1-cycle in general position, to use Theorem 1.9 below, which also guarantees that in fact σ itself lifts to B_α.

Remark 1.6. By Theorem 1.3, if X_t is uniruled and $H^2(X_t,\mathcal{O}_{X_t}) = 0$ (a geometric strengthening of our assumptions that $H^2(X_t,\mathcal{O}_{X_t}) = 0 = H^3(X_t,\mathcal{O}_{X_t}) = 0$), then any degree 4 integral cohomology class α_t on X_t is algebraic on X_t. Together with Bloch-Srinivas results [3] on the surjectivity of the Abel-Jacobi map for codimension 2-cycles under these assumptions, this shows that pairs (B_α, Z_α) with surjective $\phi_{\mathcal{Z}_\alpha} : B_\alpha \to \mathcal{J}_\alpha$ exist. In this case, the strong statement in assumption 3 is thus the rational connectedness of the fibers.

Corollary 1.7. *Let $f : X \to \Gamma$ be a fibration over a curve with general fiber a complete intersection of two quadrics in $\mathbb{P}^5$. If the fibers of f have at worst ordinary quadratic singularities, then the Hodge conjecture holds for degree 4 integral Hodge classes on X. In other words, the group $Z^4(X)$ is trivial.*

Proof. Indeed, the assumptions $H^i(X_t, \mathcal{O}_{X_t}) = 0$, $i > 0$ are a consequence of the fact that X_t is Fano in this case. Condition 1 in Theorem 1.4 is satisfied for complete intersections in projective space by Lefschetz hyperplane restriction theorem, and condition 3 is proved by Castravet [4].

■

In order to study cubic threefolds fibrations, we will need a technical strengthening of Theorem 1.4. We start with a smooth projective morphism $f : \mathcal{X} \to T$ of relative dimension 3 with T smooth and quasi-projective. We assume as before that smooth fibers $\mathcal{X}_t$ have no torsion in $H^3(\mathcal{X}_t, \mathbb{Z})$ and have $H^3(\mathcal{X}_t, \mathcal{O}_{\mathcal{X}_t}) = H^2(\mathcal{X}_t, \mathcal{O}_{\mathcal{X}_t}) = 0$. As before, for any global section α of $R^4f_*\mathbb{Z}$, we have the twisted family of intermediate Jacobians $\mathcal{J}_\alpha \to T$.

Theorem 1.8. *Assume that the following hold.*

(i) The local system $R^4f_*\mathbb{Z}$ is trivial.

(ii) For any section α of $R^4f_*\mathbb{Z}$, there exists a variety $g_\alpha : B_\alpha \to T$ and a family of relative 1-cycles $\mathcal{Z}_\alpha \subset B_\alpha \times_T \mathcal{X}$ of class α, such that the morphism $\phi_{\mathcal{Z}_\alpha} : B_\alpha \to \mathcal{J}_\alpha$ is surjective with rationally connected general fibers.

Then for any smooth quasi-projective curve Γ_0, any morphism $\phi : \Gamma_0 \to T$, and any smooth projective model $\psi : X \to \Gamma$ of $\mathcal{X} \times_T \Gamma_0 \to \Gamma_0$, assuming ψ has at worst nodal fibers, one has $Z^4(X) = 0$.

Proof. We mimick the proof of the previous theorem. We thus only have to prove that for any Hodge class $\tilde{\alpha}$ on X, inducing by restriction to the fibers of ψ a section α of $R^4\psi_*\mathbb{Z}$, the section $\sigma_{\tilde{\alpha}}$ of $\mathcal{J}_{\Gamma,\alpha} \to \Gamma_0$ induced by $\tilde{\alpha}$ lifts to a family of 1-cycles of class α in the fibers of ψ. Observe that by triviality of the local system $R^4f_*\mathbb{Z}$ on T, the section α extends to a section of $R^4f_*\mathbb{Z}$ on T. We then have by assumption the family of 1-cycles $\mathcal{Z}_\alpha \subset B_\alpha \times_T \mathcal{X}$ parameterized by B_α, with the property that the general fiber of the induced Abel-Jacobi map $\phi_{\mathcal{Z}_\alpha} : B_\alpha \to \mathcal{J}_\alpha$ is rationally connected. As $\phi^*\mathcal{J}_\alpha = \mathcal{J}_{\Gamma,\alpha}$, we can see the pair $(\phi, \sigma_{\tilde{\alpha}})$ as a general morphism from Γ_0 to $\mathcal{J}_\alpha$. The desired family of 1-cycles follows then from the existence of a lift of $\sigma_{\tilde{\alpha}}$ to B_α, given by the following result 1.9, due to Graber, Harris, Mazur and Starr (see also [18] for related results).

■

Theorem 1.9. *[11, Proposition 2.7]Let $f : Z \to W$ be a surjective projective morphism between smooth varieties over $\mathbb{C}$. Assume the general fiber of f is rationally connected. Then, for any rational map $g : C \dashrightarrow W$, where C is a smooth curve, there exists a rational lift $\tilde{g} : C \to Z$ of g in Z.*

■

2 On the fibers of the Abel-Jacobi map for the cubic threefold

The papers [22], [19], [17] are devoted to the study of the morphism induced by the Abel-Jacobi map, from the family of curves of small degree in a cubic threefold in $\mathbb{P}^4$ to its intermediate

Jacobian. In degree 4, genus 0, and degree 5, genus 0 or 1, it is proved that these morphisms are surjective with rationally connected fibers, but it is known that this is not true in degree 3 (and any genus), and, to our knowlege, the case of degree 6 has not been studied. As is clear from the proof of Theorem 1.4, we need for the proof of Theorem 2.11 to have such a statement for at least one naturally defined family of curves of degree divisible by 3. The following result provides such a statement.

Theorem 2.1. *Let $Y \subset \mathbb{P}^4_{\mathbb{C}}$ be a general cubic hypersurface. Then the map $\phi_{6,1}$ induced by the Abel-Jacobi map AJ_Y, from the family $M_{6,1}$ of elliptic curves of degree 6 contained in Y to $J(Y)$, is surjective with rationally connected general fiber.*

Remark 2.2. What we call here the family of elliptic curves is a desingularization of the closure, in the Hilbert scheme of Y, of the family parameterizing smooth degree 6 elliptic curves which are nondegenerate in $\mathbb{P}^4$.

Remark 2.3. As $J(Y)$ is not uniruled, an equivalent formulation of the result is the following (cf. [17]): the map $\phi_{6,1}$ is dominating and identifies birationally to the maximal rationally connected fibration (see [21]) of $M_{6,1}$.

Proof of Theorem 2.1. Notice that it suffices to prove the statement for very general Y, which we will assume occasionally. One can show that $M_{6,1}$ is for general Y irreducible of dimension 12. I suffices for this to argue as in [16]. One first shows that the universal variety $M_{6,1,univ}$ parameterizing pairs (E, Y) consisting of a nodal degree 6 nondegenerate elliptic curve in $\mathbb{P}^4$ and a smooth cubic 3-fold containing it, is smooth and irreducible. It remains to prove that the general fibers of the map $M_{6,1,univ} \to \mathbb{P}(H^0(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(3)))$ are irreducible. One uses for this the results of [22] to construct a subvariety $D_{5,1,univ}$ of $M_{6,1,univ}$ which dominates $\mathbb{P}(H^0(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(3)))$ with general irreducible fibers. One just takes for this the variety parameterizing elliptic curves obtained as the union of a degree 5 elliptic curve and a line meeting at one point. The results of [22] imply that the map $D_{5,1,univ} \to \mathbb{P}(H^0(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(3)))$ has irreducible general fibers (see below for an explicit description of the fiber $D_{5,1}$), and it follows easily that the same is true for the map $M_{6,1,univ} \to \mathbb{P}(H^0(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(3)))$.

Let $E \subset Y$ be a general nondegenerate smooth degree 6 elliptic curve. Then there exists a smooth $K3$ surface $S \subset Y$, which is a member of the linear system $|\mathcal{O}_Y(2)|$, containing E. The line bundle $\mathcal{O}_S(E)$ is then generated by two sections. Let $V := H^0(S, \mathcal{O}_S(E))$ and consider the rank 2 vector bundle $\mathcal{E}$ on Y defined by $\mathcal{E} := \mathcal{F}^*$, where $\mathcal{F}$ is the kernel of the surjective evaluation map $V \otimes \mathcal{O}_Y \to \mathcal{O}_S(E)$. We have the exact sequence

$$0 \to \mathcal{F} \to V \otimes \mathcal{O}_Y \to \mathcal{O}_S(E) \to 0. \tag{2.6}$$

One verifies by dualizing the exact sequence (2.6) that $H^0(Y, \mathcal{E})$ is of dimension 4, and that E is recovered as the zero locus of a transverse section of $\mathcal{E}$. Furthermore, the bundle $\mathcal{E}$ is stable when E is non degenerate. Indeed, $\mathcal{E}$ is stable if and only if $\mathcal{F}$ is stable. As $\det \mathcal{F} = \mathcal{O}_Y(-2)$, $\operatorname{rank} \mathcal{E} = 2$, the stability of $\mathcal{F}$ is equivalent to $H^0(Y, \mathcal{F}(1)) = 0$, which is equivalent by taking global sections in (2.6) to the fact that the product $V \otimes H^0(S, \mathcal{O}_S(1)) \to H^0(S, \mathcal{O}_S(E)(1))$ is injective. By the base-point free pencil trick, the kernel of this map is isomorphic to $H^0(S, \mathcal{O}_S(-E)(1))$.

The vector bundle $\mathcal{E}$ so constructed does not in fact depend on the choice of S, as it can also be obtained by the Serre construction starting from E (as is done in [22]), because by dualizing (2.6) and taking global sections, one sees that E is the zero set of a section of $\mathcal{E}$.

Using the fact that $H^i(Y, \mathcal{E}) = 0$ for $i > 0$, as follows from (2.6), one concludes that, denoting M_9 the moduli space of stable rank 2 bundles on Y with Chern classses

$$c_1(\mathcal{E}) = \mathcal{O}_Y(2),\ deg\, c_2(\mathcal{E}) = 6,$$

M_9 is of dimension 9 and one has a dominating rational map

$$\phi : M_{6,1} \dashrightarrow M_9$$

whose general fiber is isomorphic to $\mathbb{P}^3$ (more precisely, the fiber over $[\mathcal{E}]$ is the projective space $\mathbb{P}(H^0(Y,\mathcal{E}))$). It follows that the maximal rationally connected (MRC) fibration of $M_{6,1}$ (cf. [21]) factorizes through M_9.

Let us consider the subvariety $D_{3,3} \subset M_{6,1}$ parameterizing the singular elliptic curves of degree 6 consisting of two rational components of degree 3 meeting in two points. The family $M_{3,0}$ parameterizing rational curves of degree 3 is birationally a $\mathbb{P}^2$-fibration over the Theta divisor $\Theta \subset J(Y)$ (cf. [16]). More precisely, a general rational curve $C \subset Y$ of degree 3 is contained in a smooth hyperplane section $S \subset Y$, and deforms in a linear system of dimension 2 in S. The map $\phi_{3,0} : M_{3,0} \to J(Y)$ induced by the Abel-Jacobi map of Y has its image equal to $\Theta \subset J(Y)$ and its fiber passing through $[C]$ is the linear system $\mathbb{P}^2$ introduced above. This allows to describe $D_{3,3}$ in the following way: the Abel-Jacobi map of Y, applied to each component C_1, C_2 of a general curve $C_1 \cup C_2$ parameterized by $D_{3,3}$ takes value in the symmetric product $S^2\Theta$ and its fiber passing through $[C_1 \cup C_2]$ is described by the choice of two curves in the corresponding linear systems $\mathbb{P}^2_1$ and $\mathbb{P}^2_2$. These two curves have to meet in two points of the elliptic curve E_3 defined as the complete intersection of the two cubic surfaces S_1 and S_2. This fiber is thus birationally equivalent to S^2E_3.

Observe that the existence of this subvariety $D_{3,3} \subset M_{6,1}$ implies the surjectivity of $\phi_{6,1}$ because the restriction of $\phi_{6,1}$ to $D_{3,3}$ is the composition of the above-defined surjective map

$$\chi : D_{3,3} \to S^2\Theta$$

and of the sum map $S^2\Theta \to J(Y)$. Obviously the sum map is surjective.

We will show more precisely:

Lemma 2.4. *The map ϕ introduced above is generically defined along $D_{3,3}$ and $\phi_{|D_{3,3}} : D_{3,3} \dashrightarrow M_9$ is dominating. In particular $D_{3,3}$ dominates the base of the MRC fibration of $M_{6,1}$.*

Proof. Let $E = C_3 \cup C'_3$ be a general elliptic curve of Y parameterized by $D_{3,3}$. Then E is contained in a smooth $K3$ surface in the linear system $|\mathcal{O}_Y(2)|$. The linear system $|\mathcal{O}_S(E)|$ has no base point in S, and thus the construction of the vector bundle $\mathcal{E}$ can be done as in the smooth general case; furthermore it is verified in the same way that $\mathcal{E}$ is stable on Y. Hence ϕ is well-defined at the point $[E]$. One verifies that $H^0(Y,\mathcal{E})$ is of dimension 4 as in the general smooth case. As $\dim D_{3,3} = 10$ and $\dim M_9 = 9$, to show that $\phi_{|D_{3,3}}$ is dominating, it suffices to show that the general fiber of $\phi_{|D_{3,3}}$ is of dimension ≤ 1. Assume to the contrary that this fiber, which is contained in $\mathbb{P}(H^0(Y,\mathcal{E})) = \mathbb{P}^3$, is of dimension ≥ 2. Recalling that $D'_{3,3}$ is not swept-out by rational surfaces because it is fibered over $S^2\Theta$ into surfaces isomorphic to the second symmetric product of an elliptic curve, this fiber should be a surface of degree ≥ 3 in $\mathbb{P}(H^0(Y,\mathcal{E}))$. Any line in $\mathbb{P}(H^0(Y,\mathcal{E}))$ should then meet this surface in at least 3 points counted with multiplicities. For such a line, take the $\mathbb{P}^1 \subset \mathbb{P}(H^0(Y,\mathcal{E}))$ obtained by considering the base-point free pencil $|\mathcal{O}_S(E)|$. One verifies that this line is not tangent to $D_{3,3}$ at the point $[E]$ and thus should meet $D_{3,3}$ in another point. Choosing a component for each reducible fiber of the elliptic pencil $|\mathcal{O}_S(E)|$ provides two elements of $\operatorname{Pic} S$ of negative self-intersection, which are mutually orthogonal. Taking into account the class of E and the class of a hyperplane section, it follows that the surface S should thus have a Picard number $\rho(S) \geq 4$, which is easily excluded by a dimension count for a general pair (E, S), where E is an elliptic curve of type $(3,3)$ as above and S is the intersection of our general cubic Y and a quadric in $\mathbb{P}^4$ containing E. Namely, the conclusion is that the image of this space in the corresponding moduli space of $K3$ surfaces of type $(2,3)$ in $\mathbb{P}^5$ has codimension 2, and the study of the period map for $K3$ surfaces guarantees then that a general surface in this image has $\rho \leq 3$. ■

To construct other rational curves in $M_{6,1}$, we now make the following observation: if E is a nondegenerate degree 6 elliptic curve in $\mathbb{P}^4$, let us choose two distinct line bundles l_1, l_2 of degree 2 on E such that $\mathcal{O}_E(1) = 2l_1 + l_2$ in $\operatorname{Pic} E$. Then (l_1, l_2) provides an embedding of E in $\mathbb{P}^1 \times \mathbb{P}^1$, and denoting $h_i := pr_i^* \mathcal{O}_{\mathbb{P}^1}(1) \in \operatorname{Pic}(\mathbb{P}^1 \times \mathbb{P}^1)$, the line bundle $l = 2h_1 + h_2$ on $\mathbb{P}^1 \times \mathbb{P}^1$ has the property that $l_{|E} = 2l_1 + l_2 = \mathcal{O}_E(1)$ and the restriction map

$$H^0(\mathbb{P}^1 \times \mathbb{P}^1, l) \to H^0(E, \mathcal{O}_E(1))$$

is an isomorphism. The original morphism from E to $\mathbb{P}^4$ is given by a base-point free hyperplane in $H^0(E, \mathcal{O}_E(1))$. For a generic choice of l_2, this hyperplane also provides a base-point free hyperplane in $H^0(\mathbb{P}^1 \times \mathbb{P}^1, l)$, hence a morphism $\phi : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^4$, whose image is a surface Σ of degree 4. The residual curve of E in the intersection $\Sigma \cap Y$ is a curve in the linear system $|4h_1 + h_2|$ on $\mathbb{P}^1 \times \mathbb{P}^1$. This is thus a curve of degree 6 in $\mathbb{P}^4$ and of genus 0.

Let us now describe the construction in the reverse way: We start from a smooth general genus 0 and degree 6 curve C in $\mathbb{P}^4$, and want to describe morphisms $\phi : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^4$ as above, such that $C \subset \phi(\mathbb{P}^1 \times \mathbb{P}^1)$ is the image by ϕ of a curve in the linear system $|4h_1 + h_2|$ on $\mathbb{P}^1 \times \mathbb{P}^1$.

Lemma 2.5. *For a generic degree 6 rational curve C, the family of such morphisms ϕ is parameterized by a $\mathbb{P}^1$.*

Proof. Choosing an isomorphism $C \cong \mathbb{P}^1$, the inclusion $C \subset \mathbb{P}^4$ provides a base-point free linear subsystem $W \subset H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(6))$ of dimension 5. Let us choose a hyperplane $H \subset H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(6))$ containing W. The codimension 2 vector subspace W being given, such hyperplanes H are parameterized by a $\mathbb{P}^1$, which will be our parameter space. Indeed we claim that, when the hyperplane H is chosen generically, there exists a unique embedding $C \to \mathbb{P}^1 \times \mathbb{P}^1$ as a curve in the linear system $|4h_1 + h_2|$, such that H is recovered as the image of the restriction map

$$H^0(\mathbb{P}^1 \times \mathbb{P}^1, 2h_1 + h_2) \to H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(6)).$$

To prove the claim, notice that the embedding of $\mathbb{P}^1$ into $\mathbb{P}^1 \times \mathbb{P}^1$ as a curve in the linear system $|4h_1 + h_2|$ is determined up to the action of $\operatorname{Aut} \mathbb{P}^1$ by the choice of a degree 4 morphism from $\mathbb{P}^1$ to $\mathbb{P}^1$, which is equivalent to the data of a rank 2 base-point free linear system $W' \subset H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(4))$. The condition that H is equal to the image of the restriction map above is equivalent to the fact that

$$H = W' \cdot H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(2)) := Im\,(W' \otimes H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(2)) \to H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(6))),$$

where the map is given by multiplication of sections. Given H, let us set

$$W' := \cap_{t \in H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(2))} [H : t],$$

where $[H : t] := \{w \in H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(4)),\ wt \in H\}$. Generically, one then has $\dim W' = 2$ and $H = W' \cdot H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(2))$. This concludes the proof of the claim, and hence of the lemma. ■

If the curve C is contained in Y, the residual curve of C in the intersection $Y \cap \Sigma$ is an elliptic curve of degree 6 in Y. These two constructions are inverse of each other, which provides a birational map from the space of pairs $\{(E, \Sigma),\ E \in M_{6,1},\ E \subset \Sigma\}$ to the space of pairs $\{(C, \Sigma),\ C \in M_{6,0},\ C \subset \Sigma\}$, where $\Sigma \subset \mathbb{P}^4$ is a surface of the type considered above, and the inclusions of the curves C, E is Σ are in the linear systems described above.

This construction thus provides rational curves $R_C \subset M_{6,1}$ parameterized by $C \in M_{6,0}$ and sweeping-out $M_{6,1}$. To each curve C parameterized by $M_{6,0}$, one associates the family R_C of elliptic curves E_t which are residual to C in the intersection $\Sigma_t \cap Y$, where t runs over the $\mathbb{P}^1$ exhibited in Lemma 2.5.

Let $\Phi : M_{6,1} \dashrightarrow B$ be the maximal rationally connected fibration of $M_{6,1}$. We will use later on the curves R_C introduced above to prove the following Lemma 2.6. Consider the divisor $D_{5,1} \subset M_{6,1}$ parameterizing singular elliptic curves of degree 6 contained in Y, which are the union of an elliptic curve of degree 5 in Y and of a line $\Delta \subset Y$ meeting in one point. If E is such a generically chosen elliptic curve and $S \in |\mathcal{O}_Y(2)|$ is a smooth $K3$ surface containing E, the linear system $|\mathcal{O}_S(E)|$ contains the line Δ in its base locus, and thus the construction of the associated vector bundle $\mathcal{E}$ fails. Consider however a curve $E' \subset S$ in the linear system $|\mathcal{O}_S(2-E)|$. Then E' is again a degree 6 elliptic curve in Y and E' is smooth and nondegenerate for a generic choice of Y, E, S as above. The curves E' so obtained are parameterized by a divisor $D'_{5,1}$ of $M_{6,1}$, which also has the following description : the curves E' parameterized by $D'_{5,1}$ can also be characterized by the fact that they have a trisecant line contained in Y (precisely the line Δ introduced above).

Lemma 2.6. *The restriction of Φ to $D'_{5,1}$ is surjective.*

Proof. We will use the following elementary principle 2.7 : Let Z be a smooth projective variety, and $\Phi : Z \dashrightarrow B$ its maximal rationally connected fibration. Assume given a family of rational curves $(C_t)_{t\in M}$ sweeping-out Z.

Principle 2.7. *Let $D \subset Z$ be a divisor such that, through a general point $d \in D$, there passes a curve C_t, $t \in M$, meeting D properly at d. Then $\Phi_{|D} : D \dashrightarrow B$ is dominating.*

We apply now Lemma 2.7 to $Z = M_{6,1}$ and to the previously constructed family of rational curves R_C, $C \in M_{6,0}$. It thus suffices to show that for general Y, if $[E] \in D'_{5,1}$ is a general point, there exists an (irreducible) curve R_C passing through $[E]$ and not contained in $D'_{5,1}$. Let us recall that $D'_{5,1}$ is irreducible, and parameterizes elliptic curves of degree 6 in Y admitting a trisecant line contained in Y. Starting from a curve C of genus 0 and degree 6 which is nondegenerate in $\mathbb{P}^4$, we constructed a one parameter family $(\Sigma_t)_{t\in\mathbb{P}^1}$ of surfaces of degree 4 containing C and isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$. For general t, the trisecant lines of Σ_t in $\mathbb{P}^4$ form at most a 3-dimensional variety and thus, for a general cubic $Y \subset \mathbb{P}^4$, there exists no trisecant to Σ_t contained in Y. Let us choose a line Δ which is trisecant to Σ_0 but not trisecant to Σ_t for t close to 0, $t \neq 0$. A general cubic Y containing C and Δ contains then no trisecant to Σ_t for t close to 0, $t \neq 0$. Let E_0 be the residual curve of C in $\Sigma_0 \cap Y$. Then E_0 has the line $\Delta \subset Y$ as a trisecant, and thus $[E_0]$ is a point of the divisor $D'_{5,1}$ associated to Y. However the rational curve $R_C \subset M_{6,1}(Y)$ parameterizing the residual curves E_t of C in $\Sigma_t \cap Y$, $t \in \mathbb{P}^1$, is not contained in $D'_{5,1}$ because, by construction, no trisecant of E_t is contained in Y, for t close to 0, $t \neq 0$.

■

Remark 2.8. It is not true that the map $\phi_{|D'_{5,1}} : D'_{5,1} \dashrightarrow M_9$ is dominating. Indeed, the curves E' parameterized by $D'_{5,1}$ admit a trisecant line L in Y. Let $\mathcal{E}$ be the associated vector bundle. Then because $\det \mathcal{E} = \mathcal{O}_Y(2)$ and $\mathcal{E}_{|L}$ has a nonzero section vanishing in three points, $\mathcal{E}_{|L} \cong \mathcal{O}_L(3) \oplus \mathcal{O}_L(-1)$. Hence the corresponding vector bundles $\mathcal{E}$ are not globally generated and thus are not generic.

The proof of Theorem 2.1 is now concluded in the following way:

Lemma 2.4 says that the rational map $\Phi_{|D_{3,3}}$ is dominating. The dominating rational maps $\Phi_{|D_{3,3}}$ and $\Phi_{|D'_{5,1}}$, taking values in a non-uniruled variety, factorize through the base of the MRC fibrations of $D_{3,3}$ and $D'_{5,1}$ respectively. The maximal rationally connected fibrations of the divisors $D'_{5,1}$ and $D_{5,1}$ have the same base. Indeed, one way of seeing the construction of the correspondence between $D_{5,1}$ and $D'_{5,1}$ is by liaison : By definition, a general element E' of $D'_{5,1}$ is residual to a element E of $D_{5,1}$ in a complete intersection of two quadric hypersurfaces in Y. This implies that a $\mathbb{P}^2$-bundle over $D_{5,1}$ (parameterizing a reducible elliptic curve E of type $(5,1)$ and a rank 2 vector space of quadratic polynomials vanishing on E) is birationally equivalent to a $\mathbb{P}^2$-fibration over $D'_{5,1}$ defined in the same way.

The MRC fibration of the divisor $D_{5,1}$ is well understood by the works [19], [22]. Indeed, these papers show that the morphism $\phi_{5,1}$ induced by the Abel-Jacobi map of Y on the family $M_{5,1}$ of degree 5 elliptic curves in Y is surjective with general fiber isomorphic to $\mathbb{P}^5$: more precisely, an elliptic curve E of degree 5 in Y is the zero locus of a transverse section of a rank 2 vector bundle $\mathcal{E}$ on Y, and the fiber of $\phi_{5,1}$ passing through $[E]$ is for general E isomorphic to $\mathbb{P}(H^0(Y,\mathcal{E})) = \mathbb{P}^5$. The vector bundle $\mathcal{E}$, and the line $L \subset Y$ being given, the set of elliptic curves of type $(5,1)$ whose degree 5 component is the zero locus of a section of $\mathcal{E}$ and the degree 1 component is the line L identifies to

$$\bigcup_{x\in L} \mathbb{P}(H^0(Y,\mathcal{E}\otimes\mathcal{I}_x)),$$

which is rationally connected. Hence $D_{5,1}$, and thus also $D'_{5,1}$, is a fibration over $J(Y)\times F$ with rationally connected general fiber, where F is the Fano surface of lines in Y. As the variety $J(Y)\times F$ is not uniruled, it must be the base of the maximal rationally connected fibration of $D'_{5,1}$.

From the above considerations, we conclude that B is dominated by $J(Y)\times F$ and in particular $\dim B \leq 7$.

Let us now consider the dominating rational map $\Phi_{|D_{3,3}} : D_{3,3} \dashrightarrow B$, which will be denoted by Φ'. Recall that $D_{3,3}$ admits a surjective morphism χ to $S^2\Theta$, the general fiber being isomorphic to the second symmetric product S^2E_3 of an elliptic curve of degree 3. We postpone the proof of the following lemma:

Lemma 2.9. *The rational map Φ' factorizes through $S^2\Theta$.*

This lemma implies that B is rationally dominated by $S^2\Theta$, hence a fortiori by $\Theta\times\Theta$. We will denote $\Phi'' : \Theta\times\Theta \dashrightarrow B$ the rational map obtained by factorizing Φ' through $S^2\Theta$ and then by composing the resulting map with the quotient map $\Theta\times\Theta \to S^2\Theta$. We have a map $\phi_B : B \to J(Y)$ obtained as a factorization of the map $\phi_{6,1} : M_{6,1} \to J(Y)$, using the fact that the fibers of $\Phi : M_{6,1} \dashrightarrow B$ are rationally connected, hence contracted by $\phi_{6,1}$. Clearly, the composition of the map ϕ_B with the map Φ'' identifies to the sum map $\sigma : \Theta\times\Theta \to J(Y)$. Its fiber over an element $a \in J(Y)$ thus identifies in a canonical way to the complete intersection of the ample divisors Θ and $a-\Theta$ of $J(Y)$, where $a-\Theta$ is the divisor $\{a-x,\, x\in\Theta\}$ of $J(Y)$, which will be denoted Θ_a in the sequel. Consider the dominating rational map fiberwise induced by Φ'' over $J(Y)$:

$$\Phi''_a : \Theta\cap\Theta_a \dashrightarrow B_a,$$

where $B_a := \phi_B^{-1}(a)$ has dimension at most 2 and is not uniruled. The proof of Theorem 2.1 is thus concluded with the following lemma :

Lemma 2.10. *If Y is very general, and $a \in J(Y)$ is general, any dominating rational map Φ''_a from $\Theta\cap\Theta_a$ to a smooth non uniruled variety B_a of dimension ≤ 2 is constant.*

Indeed, this lemma tells that Φ''_a must be constant for very general Y and general a, hence also for general Y and a. This implies that B is in fact birationally isomorphic to $J(Y)$. ∎

To complete the proof of Theorem 2.1, it remains to prove Lemmas 2.9 and 2.10.

Proof of Lemma 2.9. Assume to the contrary that the dominating rational map $\Phi' : D_{3,3} \dashrightarrow B$ does not factorize through $S^2\Theta$. As the map $\chi : D_{3,3} \to S^2\Theta$ has for general fiber S^2E_3, where E_3 is an elliptic curve of degree 3 in $\mathbb{P}^4$, and B is not uniruled, Φ' factorizes through the corresponding fibration $\chi' : Z \dashrightarrow S^2\Theta$, with fiber Pic^2E_3. Let us denote $\Phi'' : Z \dashrightarrow B$ this second factorization. Recall that $\dim B \leq 7$ and that B maps surjectively onto $J(Y)$ via the morphism $\phi_B : B \dashrightarrow J(Y)$ induced by the Abel-Jacobi map $\phi_{6,1} : M_{6,1} \to J(Y)$. Clearly the elliptic curves Pic^2E_3 introduced above are contracted by the composite map $\phi_B \circ \Phi''$. If these curves are not contracted by Φ'', the fibers $B_a := \phi_B^{-1}(a)$, $a \in J(Y)$, are swept-out by elliptic curves. As the fibers B_a are either curves of genus > 0 or non uniruled surfaces (because $\dim B \leq 7$ and B is not uniruled), they have the property that there pass at most finitely many elliptic curves in a given deformation class through a general point $b \in B_a$. Under our assumption, there should thus exist :

- A variety J', and a dominating morphism $g : J' \to J(Y)$ such that the fiber J'_a, for a general point $a \in J(Y)$, has dimension at most 1 and parameterizes elliptic curves in a given deformation class in the fiber B_a.

- A family $B' \to J'$ of elliptic curves, and dominating rational maps :

$$\Psi : Z \dashrightarrow B',\ s : S^2\Theta \dashrightarrow J',\ r : B' \dashrightarrow B$$

such that Z is up to fiberwise isogeny birationally equivalent via (χ', Ψ) to the fibered product $S^2\Theta \times_{J'} B'$, r is generically finite, and $\Phi'' = r \circ \Psi$. We have just expressed above the fact that $\chi' : Z \to S^2\Theta$ is an elliptic fibration, and that, if the map Φ'' does not factors through χ', there is another elliptic fibration $B' \to B$ which sends onto B via the generically finite map r, and a commutative diagram of dominating rational maps

$$\begin{tikzcd}
Z \arrow[r, "\Psi"] \arrow[d, "\chi'"] & B' \arrow[r, "r"] \arrow[d] & B \arrow[d, "\phi_B"] \\
S^2\Theta \arrow[r, "s"] & J' \arrow[r, "g"] & J(Y)
\end{tikzcd}$$

where the first square is Cartesian up to fiberwise isogeny.

To get a contradiction out of this, observe that the family of elliptic curves $Z \to S^2\Theta$ is also birationally the inverse image of a family of elliptic curves parameterized by $G(2,4)$, via a natural rational map $f : S^2\Theta \dashrightarrow G(2,4)$. Indeed, an element of $S^2\Theta$ parameterizes two linear systems $|C_1|$, resp. $|C_2|$ of rational curves of degree 3 on two hyperplane sections $S_1 = H_1 \cap Y$, resp. $S_2 = H_2 \cap Y$ of Y. The map f associates to the unordered pair $\{|C_1|, |C_2|\}$ the plane $P := H_1 \cap H_2$. This plane $P \subset \mathbb{P}^4$ parameterizes the elliptic curve $E_3 := P \cap Y$, and there is a canonical isomorphism $E_3 \cong Pic^2E_3$ because $deg\, E_3 = 3$.

The contradiction then comes from the following fact: for $j \in \mathbb{P}^1$, let us consider the divisor D_j of Z which is swept-out by elliptic curves of fixed modulus determined by j, in the fibration $Z \dashrightarrow S^2\Theta$. This divisor is the inverse image by $f \circ \chi'$ of an ample divisor on $G(2,4)$ and furthermore it is also the inverse image by Ψ of the similarly defined divisor of B'. As r is generically finite and B is not uniruled, B' is not uniruled. Hence the rational map $\Psi : Z \dashrightarrow B'$ is "almost holomorphic", that is well-defined along its general fiber. One deduces from this that the divisor D_j has trivial restriction to a general fiber of Ψ. This fiber, which is of dimension

≥ 2 because $\dim Z = 9$ and $\dim B' \leq 7$, must then be sent to a point in $G(2,4)$ by $f \circ \chi'$ and its image in $S^2\Theta$ is thus contained in an union of components of fibers of f. Observing furthermore that the general fiber of f is irreducible of dimension 2, it follows that the rational dominating map $s : S^2\Theta \dashrightarrow J'$ factorizes through $f : S^2\Theta \dashrightarrow G(2,4)$. As $G(2,4)$ is rational, this contradicts the fact that J' dominates rationally $J(Y)$.

■

Proof of Lemma 2.10. Observe that, as $\dim \mathrm{Sing}\,\Theta = 0$ by [1], the general intersection $\Theta \cap \Theta_a$ is smooth. Assume first that $\dim B_a = 1$. Then either B_a is an elliptic curve or $H^0(B_a, \Omega_{B_a})$ has dimension ≥ 2. Notice that, by Lefschetz hyperplane restriction theorem, the Albanese variety of the complete intersection $\Theta \cap \Theta_a$ identifies to $J(Y)$. The curve B_a cannot be elliptic, otherwise $J(Y)$ would not be simple. But it is easy to prove by an infinitesimal variation of Hodge structure argument (cf. [30, II,6.2.1]) that $J(Y)$ is simple for very general Y. If $\dim H^0(B_a, \Omega_{B_a}) \geq 2$, this provides two independent holomorphic 1-forms on $\Theta \cap \Theta_a$ whose exterior product vanishes. These 1-forms come, by Lefschetz hyperplane restriction theorem, from independent holomorphic 1-forms on $J(Y)$, whose exterior product is a nonzero holomorphic 2-form on $J(Y)$. The restriction of this 2-form to $\Theta \cap \Theta_a$ vanishes, which contradicts Lefschetz hyperplane restriction theorem because $\dim \Theta \cap \Theta_a = 3$. This case is thus excluded. (Note that we could also argue using the fact that the fundamental group of $\Theta \cap \Theta_a$ is abelian, which prevents the existence of a surjective map to a curve of genus ≥ 2. In both cases, an argument of Lefschetz type is needed.)

Assume now $\dim B_a = 2$. Then the surface B_a has nonnegative Kodaira dimension, because it is not uniruled. Let L be the line bundle $\Phi_a''^*(K_{B_a})$ on $\Theta \cap \Theta_a$. The Iitaka dimension $\kappa(L)$ of L is nonnegative and there is a nonzero section of the bundle $\Omega^2_{\Theta\cap\Theta_a}(-L)$ given by the pullback via Φ_a'' of holomorphic 2-forms on B_a (as Φ_a'' is only a rational map, this pullback morphism $\Phi_a''^*(K_{B_a}) \to \Omega^2_{\Theta\cap\Theta_a}$ is first defined on the open set where Φ_a'' is well-defined, and then extended using the smoothness of $\Theta \cap \Theta_a$ and the fact that the indeterminacy locus has codimension ≥ 2. Notice that by Grothendieck-Lefschetz theorem [14], L is the restriction of a line bundle on $J(Y)$, hence must be numerically effective, because one can show, again by an infinitesimal variation of Hodge structure argument, that $\mathrm{NS}(J(Y)) = \mathbb{Z}$, for very general Y.

We claim that any such section is the restriction of a section of $\Omega^2_{J(Y)|\Theta\cap\Theta_a}(-L)$. Indeed, the conormal exact sequence of $\Theta \cap \Theta_a$ in $J(Y)$ writes

$$0 \to \mathcal{O}_{\Theta\cap\Theta_a}(-\Theta) \oplus \mathcal{O}_{\Theta\cap\Theta_a}(-\Theta_a) \to \Omega_{J|\Theta\cap\Theta_a} \to \Omega_{\Theta\cap\Theta_a} \to 0. \tag{2.7}$$

The exact Koszul complex associated to (2.7) twisted by $-L$ gives a long exact sequence:

$$0 \to Sym^2(\mathcal{O}_{\Theta\cap\Theta_a}(-\Theta) \oplus \mathcal{O}_{\Theta\cap\Theta_a}(-\Theta_a))(-L) \to \Omega_{J|\Theta\cap\Theta_a} \otimes (\mathcal{O}_J(-\Theta - L) \oplus \mathcal{O}_J(-\Theta_a - L)) \tag{2.8}$$
$$\to \Omega^2_{J|\Theta\cap\Theta_a}(-L) \to \Omega^2_{\Theta\cap\Theta_a}(-L) \to 0.$$

Using the fact that Ω_J is trivial, L is numerically effective and $\dim \Theta \cap \Theta_a \geq 3$, Kodaira's vanishing theorem and the splitting of (2.8) into short exact sequences imply the surjectivity of the restriction map

$$H^0(\Theta \cap \Theta_a, \Omega^2_{J|\Theta\cap\Theta_a}(-L)) \to H^0(\Theta \cap \Theta_a, \Omega^2_{\Theta\cap\Theta_a}(-L)),$$

which proves the claim.

As $\kappa(L) \geq 0$ and the fact that $\Omega_{J|\Theta\cap\Theta_a}$ is trivial, a nonzero section of $\Omega_{J|\Theta\cap\Theta_a}(-L)$ can exist only if L is trivial. Notice that, up to replacing $(\Theta \cap \Theta_a, \Phi_a'', B_a)$ by its Stein factorization (or rather, the Stein factorization of a desingularized model of Φ''_a), one may assume that Φ''_a has connected fibers. We claim that $h^0(B_a, K_{B_a}) = 1$. Indeed, choose a desingularization:

$$\widetilde{\Phi''_a} : \widetilde{\Theta \cap \Theta_a} \to B_a$$

of the rational map Φ''_a. Then we know that the line bundle $\widetilde{\Phi''_a}^* K_{B_a}$ has nonnegative Iitaka dimension and that it is trivial on the complement of the exceptional divisor of the desingularization map

$$\widetilde{\Theta \cap \Theta_a} \to \Theta \cap \Theta_a,$$

since it is equal to L on this open set. This is possible only if $\widetilde{\Phi''_a}^* K_{B_a}$ has exactly one section with zero set supported on this exceptional divisor. (We use here the fact that if a divisor supported on the exceptional divisor of a contraction has a multiple which is effective, then it is effective.) But then this unique section comes from a section of K_{B_a}, because we assumed that fibers of $\widetilde{\Phi''_a}$ are connected. This proves the claim.

Having this, we conclude that the Hodge structure on $H^2(\Theta \cap \Theta_a, \mathbb{Z}) \cong H^2(J(Y), \mathbb{Z})$ has a Hodge substructure with $h^{2,0} = 1$. But this can be also excluded for very general Y by an infinitesimal variation of Hodge structure argument.

∎

This concludes the proof of Theorem 2.1.

2.1 Application: Integral Hodge classes on cubic threefolds fibrations over curves

The main concrete application of Theorem 2.1 concerns fibrations $f : X \to \Gamma$ over a curve Γ with general fiber a cubic threefold in $\mathbb{P}^4$ (or the smooth projective models of smooth cubic hypersurfaces $X \subset \mathbb{P}^4_{\mathbb{C}(\Gamma)}$).

Theorem 2.11. *Let $f : X \to \Gamma$ be a cubic threefold fibration over a smooth curve. If the fibers of f have at worst ordinary quadratic singularities, then $Z^4(X) = 0$ and $H^3_{nr}(X, \mathbb{Z}/n\mathbb{Z}) = 0$ for any n.*

Note that the second statement is a consequence of the first by Theorem 1.2 and the fact that for X as above, $CH_0(X)$ is supported on a curve.

Theorem 2.11 applies in particular to cubic fourfolds. Indeed, a smooth cubic hypersurface X in $\mathbb{P}^5$ admits a Lefschetz pencil of hyperplane sections. It thus becomes, after blowing-up the base-locus of this pencil, birationally equivalent to a model of a cubic in $\mathbb{P}^4_{\mathbb{C}(t)}$ which satisfies all our hypotheses. Theorem 2.11 thus provides $Z^4(X) = 0$ for cubic fourfolds, a result first proved in [29].

Proof of Theorem 2.11. We only need to show that the conditions of Theorem 1.8 are satisfied by the universal family $\mathcal{X} \to T$ of cubic threefolds. Here $T \subset \mathbb{P}(H^0(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(3)))$ is the open set parameterizing smooth cubic threefolds, and $\mathcal{X} \overset{j}{\hookrightarrow} T \times \mathbb{P}^4$ is the universal hypersurface.

Cubic hypersurfaces in $\mathbb{P}^4$ do not have torsion in their degree 3 cohomology by Lefschetz hyperplane restriction theorem. They furthermore satisfy the vanishing conditions $H^i(X_t, \mathcal{O}_{X_t}) = 0$, $i > 0$, since they are Fano.

It thus suffices to prove that condition (ii) of Theorem 1.8 is satisfied.

The fibers of f being smooth hypersurfaces in $\mathbb{P}^4$, $R^4 f_* \mathbb{Z}$ is the trivial local system isomorphic to $\mathbb{Z}$. A section α of $R^4 f_* \mathbb{Z}$ over T is thus characterized by a number, its degree on the fibers

X_t with respect to the polarization $c_1(\mathcal{O}_{X_t}(1))$. When the degree of α is 5, one gets the desired family B_5 and cycle $\mathcal{Z}_5$ using the results of [19], in the following way: Iliev and Markushevich prove that if Y is a smooth cubic threefold, denoting $M_{5,1}$ a desingularization of the Hilbert scheme of degree 5, genus 1 curves in Y, the map $\phi_{5,1} : M_{5,1} \to J(Y)$ induced by the Abel-Jacobi map of Y is surjective with rationally connected fibers. Taking for B_5 a desingularization $\mathcal{M}_{5,1}$ of the relative Hilbert scheme of curves of degree 5 and genus 1 in the fibers of f, and for cycle $\mathcal{Z}_5$ the universal subscheme, the hypothesis (ii) of Theorem 1.8 is thus satisfied for the degree 5 section of $R^4 f_*\mathbb{Z}$.

For the degree 6 section α of $R^4 f_*\mathbb{Z}$, property (ii) is similarly a consequence of Theorem 2.1. Indeed, it says that the general fiber of the morphism $\phi_{6,1} : M_{6,1} \to J(Y)_6$ induced by the Abel-Jacobi map of Y on the family $M_{6,1}$ of degree 6 elliptic curves in Y is rationally connected for general Y. We thus take as before for family B_6 a desingularization $\mathcal{M}_{6,1}$ of the relative Hilbert scheme of curves of degree 6 and genus 1 in the fibers of f, and for cycle $\mathcal{Z}_6$ the universal subscheme.

To conclude, let us show that property (ii) for the sections α_5 of degree 5 and α_6 of degree 6 imply property (ii) for any section α. To see this, let us introduce the codimension 2 cycle h^2 on $\mathcal{X}$, where $h = j^*(pr_2^*\mathcal{O}_{\mathbb{P}^4}(1)) \in \mathrm{Pic}\,\mathcal{X}$. The codimension 2 cycle $h^2 \in CH^2(\mathcal{X})$ is thus of degree 3 on the fibers of f. The degree of a section α is congruent to 5, -5 or 6 modulo 3, and we can thus write $\alpha = \pm\alpha_5 + \mu\alpha_3$, or $\alpha = \alpha_6 + \mu\alpha_3$ for some integer μ. In the first case, consider the variety $B_\alpha = B_5$ and the cycle

$$\mathcal{Z}_\alpha = \pm\mathcal{Z}_5 + \mu(B_5 \times_\Gamma h^2) \subset B_5 \times_\Gamma X.$$

In the second case, consider the variety $B_\alpha = B_6$ and the cycle

$$\mathcal{Z}_\alpha = \mathcal{Z}_6 + \mu(B_6 \times \Gamma h^2) \subset B_6 \times_\Gamma X.$$

It is clear that the pair $(B_\alpha, \mathcal{Z}_\alpha)$ satisfies the condition (ii) of Theorem 1.8.

■

3 Structure of the Abel-Jacobi map and decomposition of the diagonal

3.1 Relation between Questions 0.6 and 0.10

We establish in this subsection the following relation between Questions 0.6 and 0.10:

Theorem 3.1. *Assume that Question 0.10 has an affirmative answer for Y and that the intermediate Jacobian of Y admits a 1-cycle Γ such that $\Gamma^{*g} = g!\,J(Y)$, $g = dim\,J(Y)$. Then Question 0.6 also has an affirmative answer for Y.*

Here we use the Pontryagin product $*$ on cycles of $J(Y)$ defined by

$$z_1 * z_2 = \mu_*(z_1 \times z_2),$$

where $\mu : J(Y) \times J(Y) \to J(Y)$ is the sum map (cf. [30, II,11.3.1]). The condition $\Gamma^{*g} = g!\,J(Y)$ is satisfied if the class of Γ is equal to $\frac{[\Theta]^{g-1}}{(g-1)!}$, for some principal polarization Θ. This is the case if $J(Y)$ is a Jacobian.

Proof of Theorem 3.1. There exists by assumption a variety B, and a codimension 2 cycle $Z \subset B \times Y$ cohomologous to 0 on fibers $b \times Y$, such that the morphism

$$\phi_Z : B \to J(Y)$$

induced by the Abel-Jacobi map of Y is surjective with rationally connected general fibers. Consider the 1-cycle Γ of $J(Y)$. We may assume by a moving lemma, up to changing the representative of Γ modulo homological equivalence, that $\Gamma = \sum_i n_i \Gamma_i$ where, for each component Γ_i of the support of Γ, the general fiber of ϕ_Z over Γ_i is rationally connected. We may furthermore assume that the Γ_i's are smooth. According to [12], the inclusion $j_i : \Gamma_i \hookrightarrow J(Y)$ has then a lift $\sigma_i : \Gamma_i \to B$. Denote $Z_i \subset \Gamma_i \times Y$ the codimension 2-cycle $(\sigma_i, Id_Y)^* Z$. Then the morphism $\phi_i : \Gamma_i \to J(Y)$ induced by the Abel-Jacobi map is equal to j_i.

For each g-uple of components $(\Gamma_{i_1}, \ldots, \Gamma_{i_g})$ of $\mathrm{Supp}\,\Gamma$, consider $\Gamma_{i_1} \times \ldots \times \Gamma_{i_g}$, and the codimension 2-cycle

$$Z_{i_1,\ldots,i_g} := (pr_1, Id_Y)^* Z_{i_1} + \ldots + (pr_g, Id_Y)^* Z_{i_g} \subset \Gamma_1 \times \ldots \times \Gamma_g \times Y.$$

The codimension 2-cycle

$$\mathcal{Z} := \sum_{i_1,\ldots,i_g} n_{i_1} \ldots n_{i_g} Z_{i_1,\ldots,i_g} \subset (\sqcup \Gamma_i)^g \times Y, \tag{3.9}$$

where $\sqcup \Gamma_i$ is the disjoint union of the Γ_i's (hence, in particular, is smooth), is invariant under the symmetric group $\mathfrak{S}_g$ acting on the factor $(\sqcup \Gamma_i)^g$ in the product $(\sqcup \Gamma_i)^g \times Y$. The part of $\mathcal{Z}$ dominating over a component of $(\sqcup \Gamma_i)^g$, (which is the only one we are interested in) is then the pullback of a codimension 2 cycle $\mathcal{Z}_{sym}$ on $(\sqcup \Gamma_i)^{(g)} \times Y$. Consider now the sum map

$$\sigma : (\sqcup \Gamma_i)^{(g)} \to J(Y).$$

Let $\mathcal{Z}_J := (\sigma, Id)_* \mathcal{Z}_{sym} \subset J(Y) \times Y$. The proof concludes with the following:

Lemma 3.2. *The Abel-Jacobi map :*

$$\phi_{\mathcal{Z}_J} : J(Y) \to J(Y)$$

is equal to $Id_{J(Y)}$.

Proof. Instead of the symmetric product $(\sqcup \Gamma_i)^{(g)}$ and the descended cycle $\mathcal{Z}_{sym}$, consider the product $(\sqcup \Gamma_i)^g$, the cycle $\mathcal{Z}$ and the sum map

$$\sigma' : (\sqcup \Gamma_i)^g \to J(Y).$$

Then we have $(\sigma', Id)_* \mathcal{Z} = g!(\sigma, Id)_* \mathcal{Z}_{sym}$ in $CH^2(J(Y) \times Y)$, so that writing $\mathcal{Z}'_J := (\sigma', Id)_* \mathcal{Z}$, it suffices to prove that $\phi_{\mathcal{Z}'_J} : J(Y) \to J(Y)$ is equal to $g!\, Id_{J(Y)}$.

This is done as follows: let $j \in J(Y)$ be a general point, and let $\{x_1, \ldots, x_N\}$ be the fiber of σ' over j. Thus each x_l parameterizes a g-uple $(i_1^l, \ldots, i_g^l)$ of components of $\mathrm{Supp}\,\Gamma$, and points $\gamma_{i_1}^l, \ldots, \gamma_{i_g}^l$ of $\Gamma_{i_1}, \ldots, \Gamma_{i_g}$ respectively, such that:

$$\sum_{1 \le k \le g} \gamma_{i_k}^l = j. \tag{3.10}$$

On the other hand, recall that

$$\gamma_{i_k}^l = AJ_Y(Z_{i_k^l,\gamma_{i_k}^l}). \tag{3.11}$$

It follows from (3.10) and (3.11) that for each $l \in \{1, \dots, N\}$, we have:

$$AJ_Y(\sum_{1\leq k\leq g} Z^l_{i_k,\gamma^l_{i_k}}) = AJ_Y(Z_{i^l_1,\dots,i^l_g,(\gamma^l_{i_1},\dots,\gamma^l_{i_g})}) = j. \tag{3.12}$$

Recall now that $\Gamma = \sum_i n_i\Gamma_i$ and that $(\Gamma)^{*g} = g!J(Y)$, which is equivalent to the following equality :

$$\sigma'_*(\sum_{i_1,\dots,i_g} n_{i_1}\dots n_{i_g}\Gamma_{i_1}\times\dots\times\Gamma_{i_g}) = g!J(Y).$$

This exactly says that $\sum_{1\leq l\leq N}\sum_{i^l_1,\dots,i^l_g} n_{i^l_1}\dots n_{i^l_g} = g!$ which together with (3.9) and (3.12) proves the desired equality $\phi_{\mathcal{Z}'_J} = g!\,Id_{J(Y)}$. ■

The proof of Theorem 3.1 is now complete. ■

Remark 3.3. When $NS\,J(Y) = \mathbb{Z}\Theta$, the existence of a 1-cycle Γ in $J(Y)$ such that $\Gamma^{*g} = g!\,J(Y)$, $g = dim\,J(Y)$ is equivalent to the existence of a 1-cycle Γ of class $\frac{[\Theta]^{g-1}}{(g-1)!}$. The question whether the intermediate Jacobian of Y admits a 1-cycle Γ of class $\frac{[\Theta]^{g-1}}{(g-1)!}$ is unknown even for the cubic threefold. However it has a positive answer for $g\leq 3$ because any principally polarized abelian variety of dimension ≤ 3 is the Jacobian of a curve.

3.2 Decomposition of the diagonal modulo homological equivalence

This section is devoted to the study of Question 0.6 or condition (*) of Question 0.10.

Assume Y is a smooth projective threefold such that $CH_0(Y)$ is supported on a curve. The Bloch-Srinivas decomposition of the diagonal (0.2) says that there exists a nonzero integer N such that, denoting $\Delta_Y \subset Y\times Y$ the diagonal :

$$N\Delta_Y = Z + Z' \text{ in } CH^3(Y\times Y), \tag{3.13}$$

where the support of Z' is contained in $Y\times W$ for a curve $W\subset Y$ and the support of Z is contained in $D\times Y$, $D \subsetneqq Y$.

We wish to study the invariant of Y defined as the gcd of the non zero integers N appearing above. This is a birational invariant of Y. One can also consider the decomposition (3.13) modulo homological equivalence, and our results below relate the triviality of this invariant, that is the existence of an integral cohomological decomposition of the diagonal, to condition (*) (among other things).

Theorem 3.4. *Let Y be a smooth projective 3-fold. Assume Y admits a cohomological decomposition of the diagonal as in (3.13). Then we have:*

(i) The integer N annihilates the torsion of $H^p(Y,\mathbb{Z})$ for any p.

(ii) The integer N annihilates $Z^4(Y)$.

(iii) $H^i(Y,\mathcal{O}_Y) = 0$, $\forall i>1$ and there exists a codimension 2 cycle $Z\subset J(Y)\times Y$ such that ϕ_Z is equal to $N\,Id_{J(Y)}$.

Corollary 3.5. *If Y admits an integral cohomological decompoosition of the diagonal, then:*
i) $H^p(Y,\mathbb{Z})$ is without torsion for any p.
ii) $Z^4(Y)=0$.
iii) Question 0.6 has an affirmative answer for Y.

Remark 3.6. That the integral decomposition of the diagonal as in (3.13), with $N=1$, and in the Chow group $CH(Y\times Y)$ implies that $H^3(Y,\mathbb{Z})$ has no torsion was observed by Colliot-Thélène. Note that when $H^2(X,\mathcal{O}_X)=0$, the torsion of $H^3(Y,\mathbb{Z})$ is the Brauer group of Y.

Proof of Theorem 3.4. There exist by assumption a proper algebraic subset $D\subsetneqq Y$, which one may assume of pure dimension 2, and $Z\in CH^3(Y\times Y)$ with support contained in $D\times Y$, a curve $W\subset Y$ and a cycle $Z'\in CH^3(Y\times Y)$ with support contained in $Y\times W$, such that

$$N[\Delta_Y]=[Z']+[Z] \text{ in } H^6(Y\times Y,\mathbb{Z}). \tag{3.14}$$

Codimension 3 cycles z of $Y\times Y$ act on $H^p(Y,\mathbb{Z})$ for any p and on the intermediate Jacobian of Y, and this action, which we will denote

$$z^*: H^p(Y,\mathbb{Z})\to H^p(Y,\mathbb{Z}),\ z^*: J(Y)\to J(Y),$$

depends only on the cohomology class of z. As the diagonal of Y acts by the identity map on $H^p(Y,\mathbb{Z})$ for $p>0$ and on $J(Y)$, one concludes that

$$N\,Id_{H^p(Y,\mathbb{Z})}=Z'^*+Z^*: H^p(Y,\mathbb{Z})\to H^p(Y,\mathbb{Z}), \text{ for } p>0. \tag{3.15}$$

It is clear that Z'^* acts trivially on $J(Y)$ since Z' is supported over a curve in Y. We thus conclude that

$$N\,Id_{J(Y)}=Z^*: J(Y)\to J(Y). \tag{3.16}$$

Let $\tau:\widetilde{D}\to Y$ be a desingularization of D and $i_{\widetilde{D}}=i_D\circ\tau:\widetilde{D}\to Y$. Similarly, let $\widetilde{W}\to W$ be a desingularization of W, and $i_{\widetilde{W}}:\widetilde{W}\to Y$ be the natural morphism. The part of the cycle Z which dominates D can be lifted to a cycle $\widetilde{Z}$ in $\widetilde{D}\times Y$, and the remaining part acts trivially on $H^p(Y,\mathbb{Z})$ for $p\leq 3$ for codimension reasons. Thus the map Z^* acting on $H^p(Y,\mathbb{Z})$ for $p\leq 3$ can be written as

$$Z^*=i_{\widetilde{D}*}\circ\widetilde{Z}^*. \tag{3.17}$$

Similarly, we can lift Z' to a cycle $\widetilde{Z'}$ supported on $Y\times\widetilde{W}$. We note now that the action of $\widetilde{Z}^*$ on cohomology sends $H^p(Y,\mathbb{Z})$, $p\leq 3$, to $H^{p-2}(\widetilde{D},\mathbb{Z})$, $p\leq 3$. The last groups have no torsion. It follows that $\widetilde{Z}^*$ annihilates the torsion of $H^p(Y,\mathbb{Z})$, $p\leq 3$. On the other hand, the morphism Z'^* factors as $\widetilde{Z'}^*\circ i_{\widetilde{W}}^*$, and, as the integral cohomology of a smooth curve has no torsion, it follows that $i_{\widetilde{W}}^*$, hence Z'^*, annihilate the torsion of $H^p(Y,\mathbb{Z})$ for $p\leq 3$. Formula (3.15) implies then that the torsion of $H^p(Y,\mathbb{Z})$ is annihilated by $N\,Id$ for $p\leq 3$.

To deal with the torsion of $H^p(Y,\mathbb{Z})$ with $p\geq 4$, we rather use the actions Z_*, Z'_* of Z, Z' on $H^p(Y,\mathbb{Z})$, $p=4$, 5. This action again factors through $\widetilde{Z}_*$, $\widetilde{Z'}_*$. Now, $\widetilde{Z}_*$ factors through the restriction map:

$$H^p(Y,\mathbb{Z})\to H^p(\widetilde{D},\mathbb{Z}),$$

while $\widetilde{Z'}_*$ factors through the Gysin map

$$i_{\widetilde{W},*} : H^{p-4}(\widetilde{W},\mathbb{Z}) \to H^p(Y,\mathbb{Z}).$$

Again, the integral cohomology of the curve $\widetilde{W}$ having no torsion, we conclude that $\widetilde{Z'}_*$ annihilates the torsion of $H^p(Y,\mathbb{Z})$. On the other hand, as $\dim\widetilde{D} \leq 2$, $H^p(\widetilde{D},\mathbb{Z})$ has no torsion for $p = 4, 5$. It follows that we also have $Z_*(H^p(Y,\mathbb{Z})_{tors}) = 0$ for $p = 4, 5$, and as Z_* acts as $N\,Id = Z_*$ on these groups, we conclude that $N\,H^p(Y,\mathbb{Z})_{tors} = 0$ for $p = 4, 5$. This proves (i).

To prove (ii), let us consider again the action $Z_* = N\,Id - Z'_*$ on the cohomology $H^4(Y,\mathbb{Z})$. Observe again that the part Z' of Z not dominating Z has a trivial action on $H^4(Y,\mathbb{Z})$, while the dominating part lifts as above to a cycle $\widetilde{Z}$ in $\widetilde{D} \times Y$. Then we find that $N\,Id_{H^4(Y,\mathbb{Z})} - Z'_*$ factors as $\widetilde{Z}_* \circ i^*_{\widetilde{D}}$, hence through the restriction map $i^*_{\widetilde{D}} : H^4(Y,\mathbb{Z}) \to H^4(\widetilde{D},\mathbb{Z})$. As $\dim \widetilde{D} = 2$, the group on the right is generated by classes of algebraic cycles, and thus $(N\,Id - Z'_*)(H^4(Y,\mathbb{Z}))$ is generated by classes of algebraic cycles on Y. On the other hand, $\operatorname{Im} Z'_*$ is contained in

$$\operatorname{Im} i_{\widetilde{W},*} : H^0(\widetilde{W},\mathbb{Z}) \to H^4(Y,\mathbb{Z}),$$

hence consists of algebraic classes. It follows that $N\,H^4(Y,\mathbb{Z})$ is generated by classes of algebraic cycles on Y, that is $N\,Z^4(Y) = 0$.

(iii) The vanishing $H^i(Y,\mathcal{O}_Y) = 0$, $\forall i > 1$, is a well-known consequence of the decomposition (3.13) of the diagonal with $\dim W \leq 1$ (cf. [3], [30, II,10.2.2]). This is important to guarantee that $J(Y)$ is an abelian variety.

It is well-known, and this is a consequence of the Lefschetz theorem on $(1,1)$-classes applied to $\mathrm{Pic}^0(\widetilde{D}) \times \widetilde{D}$ (see Remark 0.9), that there exists a universal divisor $\mathcal{D} \in \mathrm{Pic}(\mathrm{Pic}^0(\widetilde{D}) \times \widetilde{D})$, such that the induced morphism : $\phi_{\mathcal{D}} : \mathrm{Pic}^0(\widetilde{D}) \to \mathrm{Pic}^0(\widetilde{D})$ is the identity. On the other hand, we have the morphism

$$\widetilde{Z}^* : J(Y) \to J^1(\widetilde{D}) = Pic^0(\widetilde{D}),$$

which is a morphism of abelian varieties.

Let us consider the cycle

$$\mathcal{Z} := (Id_{J(Y)}, i_{\widetilde{D}})_* \circ (\widetilde{Z}^*, Id_{\widetilde{D}})^*(\mathcal{D}) \in CH^2(J(Y) \times Y).$$

Then $\phi_{\mathcal{Z}} : J(Y) \to J(Y)$ is equal to

$$i_{\widetilde{D}*} \circ \phi_{\mathcal{D}} \circ \widetilde{Z}^* : J(Y) \to J^1(\widetilde{D}) \to J^1(\widetilde{D}) \to J(Y).$$

As $\phi_{\mathcal{D}}$ is the identity map acting on $\mathrm{Pic}^0(\widetilde{D})$ and $i_{\widetilde{D}*} \circ \widetilde{Z}^*$ is equal to $N\,Id$ acting on $J(Y)$ according to (3.16) and (3.17), one concludes that the endomorphism $\phi_{\mathcal{Z}}$ of $J(X)$ is equal to $N\,Id_{J(Y)}$. ∎

Recall from section 1 that for any smooth projective complex variety Y, $Z^4(Y)$ is a quotient of $H^3_{nr}(Y,\mathbb{Q}/\mathbb{Z})$, if $Z^4(Y)$ is of torsion (which is always the case in dimension 3). We can now get a result better than (ii) if instead of considering the decomposition of the diagonal modulo homological equivalence, we consider it modulo algebraic equivalence. For example, we have the following statement, which improves (iii) above:

Proposition 3.7. *Let Y be a smooth projective 3-fold. Assume Y admits a decomposition of the diagonal as in (3.13), modulo algebraic equivalence. Then the integer N annihilates $H^3_{nr}(Y,A) = 0$ for any abelian group A.*

Proof. We use the fact that correspondences modulo algebraic equivalence act on cohomology groups $H^p(X_{Zar}, \mathcal{H}^q)$. We refer to the appendix of [7] for this fact which is precisely stated as follows:

Proposition 3.8. *If X, Y are smooth projective and $Z \subset X \times Y$ is a cycle defined up to algebraic equivalence, satisfying $\dim Y - \dim Z = r$, then there is an induced morphism*

$$Z_* : H^p(X_{Zar}, \mathcal{H}^q(A)) \to H^{p+r}(Y_{Zar}, \mathcal{H}^{q+r}(A)).$$

These actions are compatible with the composition of correspondences.

Assume now that Y admits a decomposition of the diagonal of the form

$$N\Delta_Y = Z_1 + Z_2 \text{ in } CH(Y \times Y)/\text{alg},$$

where $Z_1 \subset D \times Y$ for some $D \subsetneqq Y$ and $Z_2 \subset Y \times W$, with $\dim W \leq 1$. The diagonal acts on $H^3_{nr}(Y, A)$ as the identity. Thus we get

$$N\, Id_{H^3_{nr}(Y,A)} = Z_{1*} + Z_{2*} : H^3_{nr}(Y, A) \to H^3_{nr}(Y, A).$$

We observe now (by introducing again a desingularization $\widetilde{D}$ of D) that $Z_{1*} = 0$ on $H^3_{nr}(Y, A)$, because Z_{1*} factors through the restriction map

$$H^3_{nr}(Y, A) \to H^3_{nr}(\widetilde{D}, A)$$

and the group on the right is 0, because $\dim \widetilde{D} \leq 2$. Furthermore Z_{2*} also vanishes on $H^3_{nr}(Y, A) = H^0(Y_{Zar}, \mathcal{H}^3(A))$ because $Im\, Z_{2*}$ factors through $i_{\widetilde{W}*}$ which shifts $H^p(\mathcal{H}^q)$, $p \geq 0$, $q \geq 0$, to $H^{p+2}(\mathcal{H}^{q+2})$. Thus $Z_{1*} + Z_{2*} = N\, Id_{H^3_{nr}(Y,A)} = 0$ and $H^3_{nr}(Y, A) = 0$. ∎

A partial converse to Theorem 3.4 is as follows.

Theorem 3.9. *Assume the smooth projective threefold Y satisfies the following conditions.*

i) $H^i(Y, \mathcal{O}_Y) = 0$ for $i > 0$.

ii) $Z^4(Y) = 0$.

iii) $H^p(Y, \mathbb{Z})$ has no torsion for any integer p and the intermediate Jacobian of Y admits a 1-cycle Γ of class $\frac{[\Theta]^{g-1}}{(g-1)!}$, $g = \dim J(Y)$.

Then condition (*) on Y implies the existence of an integral cohomological decomposition of the diagonal as in (3.13) with $\dim W = 0$.

Remark 3.10. The condition $Z^4(Y) = 0$ is satisfied if Y is uniruled. Condition i) is satisfied if Y is rationally connected. For a rationally connected threefold Y, $\oplus_p H^p(Y, \mathbb{Z})$ is without torsion if $H^3(Y, \mathbb{Z})$ is without torsion.

Proof of Theorem 3.9. When the integral cohomology of Y has no torsion, the class of the diagonal $[\Delta_Y] \in H^6(Y \times Y, \mathbb{Z})$ has an integral Künneth decomposition.

$$[\Delta_Y] = \delta_{6,0} + \delta_{5,1} + \delta_{4,2} + \delta_{3,3} + \delta_{2,4} + \delta_{1,5} + \delta_{0,6},$$

where $\delta_{i,j} \in H^i(Y, \mathbb{Z}) \otimes H^j(Y, \mathbb{Z})$. The class $\delta_{0,6}$ is the class of $Y \times y$ for any point y of Y. As we assumed $CH_0(Y) = 0$, we have

$$H^1(Y, \mathcal{O}_Y) = 0,\ H^2(Y, \mathcal{O}_Y) = 0. \tag{3.18}$$

The first condition implies that the groups $H^1(Y,\mathbb{Q})$ and $H^5(Y,\mathbb{Q})$ are trivial, hence the groups $H^1(Y,\mathbb{Z})$ and $H^5(Y,\mathbb{Z})$ must be trivial since they have no torsion by assumption. It follows that $\delta_{5,1}=\delta_{1,5}=0$.

Next, the condition i) implies that the Hodge structure on $H^2(Y,\mathbb{Q})$, hence also on $H^4(Y,\mathbb{Q})$ by duality, is trivial. Hence $H^4(Y,\mathbb{Z})$ and $H^2(Y,\mathbb{Z})$ are generated by Hodge classes, and because we assumed $Z^4(Y)=0$, it follows that $H^4(Y,\mathbb{Z})$ and $H^2(Y,\mathbb{Z})$ are generated by cycle classes. From this one concludes that $\delta_{4,2}$ et $\delta_{2,4}$ are represented by algebraic cycles whose support does not dominate Y by the first projection. The same is true for $\delta_{6,0}$ which is the class of $y\times Y$. The existence of a decomposition as in (3.14) is thus equivalent to the fact that there exists a cycle $Z\subset Y\times Y$, such that the support of Z is contained in $D\times Y$, with $D\subsetneqq Y$, and $Z^*:H^3(Y,\mathbb{Z})\to H^3(Y,\mathbb{Z})$ is the identity map. This last condition is indeed equivalent to the fact that the component of type $(3,3)$ of $[Z]$ is equal to $\delta_{3,3}$.

Let now $\Gamma=\sum_i n_i\Gamma_i$ be a 1-cycle of $J(Y)$ of class $\frac{[\Theta]^{g-1}}{(g-1)!}$, where $\Gamma_i\subset J(Y)$ are smooth curves in general position. If (*) is satisfied, there exist a variety B and a codimension 2 cycle $\mathcal{Z}\subset B\times Y$ homologous to 0 on the fibers $b\times Y$, such that $\phi_{\mathcal{Z}}:B\to J(Y)$ is surjective with rationally connected general fiber. By [12], $\phi_{\mathcal{Z}}$ admits a section σ_i over each Γ_i, and $(\sigma_i,Id)^*\mathcal{Z}$ provides a family $Z_i\subset\Gamma_i\times Y$ of 1-cycles homologous to 0 in Y, parameterized by Γ_i, such that $\phi_{Z_i}:\Gamma_i\to J(Y)$ identifies to the inclusion of Γ_i in $J(Y)$.

Let us consider the cycle $Z\in CH^3(Y\times Y)$ defined by

$$Z=\sum_i n_iZ_i\circ{}^tZ_i.$$

The proof that the cycle Z satisfies the desired property is then given in the following Lemma 3.11.

■

Lemma 3.11. *The map $Z^*:H^3(Y,\mathbb{Z})\to H^3(Y,\mathbb{Z})$ is the identity map.*

Proof. We have

$$Z^*=\sum_i n_i{}^tZ_i^*\circ Z_i^*.$$

Let us study the composite map

$${}^tZ_i^*\circ Z_i^*:H^3(Y,\mathbb{Z})\to H^1(\Gamma_i,\mathbb{Z})\to H^3(Y,\mathbb{Z}).$$

Recalling that $Z_i\in CH^2(\Gamma_i\times Y)$ is the restriction to $\sigma(\Gamma_i)$ of $\mathcal{Z}\in CH^2(\Gamma_i\times Y)$, one finds that this composite map can also be written as:

$${}^tZ_i^*\circ Z_i^*={}^t\mathcal{Z}^*\circ([\sigma(\Gamma_i)]\cup)\circ\mathcal{Z}^*,$$

where $[\sigma(\Gamma_i)]\cup$ is the morphism of cup-product with the class $[\sigma(\Gamma_i)]$ One uses for this the fact that, denoting $j_i:\sigma(\Gamma_i)\to B$ the inclusion, the composition

$$j_{i*}\circ j_i^*:H^1(B,\mathbb{Z})\to H^{2n-1}(B,\mathbb{Z}),\ n=dim\,B$$

is equal to $[\sigma(\Gamma_i)]\cup$.

We thus obtain:

$$Z^*={}^t\mathcal{Z}^*\circ(\sum_i n_i[\sigma(\Gamma_i)]\cup)\circ\mathcal{Z}^*.$$

But we know that the map $\phi_{\mathcal{Z}} : B \to J(Y)$ has rationally connected fibers, and thus induces isomorphisms:

$$\phi_{\mathcal{Z}}^* : H^1(J(Y), \mathbb{Z}) \cong H^1(B, \mathbb{Z}),\ \phi_{\mathcal{Z}*} : H^{2n-1}(B, \mathbb{Z}) \cong H^{2g-1}(J(Y), \mathbb{Z}). \tag{3.19}$$

Via these isomorphisms, $\mathcal{Z}^*$ becomes the canonical isomorphism

$$H^3(Y, \mathbb{Z}) \cong H^1(J(Y), \mathbb{Z})$$

(which uses the fact that $H^3(Y, \mathbb{Z})$ is torsion free) and ${}^t\mathcal{Z}^*$ becomes the dual canonical isomorphism

$$H^{2g-1}(J(Y), \mathbb{Z}) \cong H^3(Y, \mathbb{Z}).$$

Finally, the map $\sum_i n_i[\sigma(\Gamma_i)]\cup : H^1(B, \mathbb{Z}) \to H^{2n-1}(B, \mathbb{Z})$ identifies via (3.19) to the map $\sum_i n_i[\Gamma_i]\cup : H^1(J(Y), \mathbb{Z}) \to H^{2g-1}(J(Y), \mathbb{Z})$. Recalling that $\sum_i n_i[\Gamma_i] = \frac{[\Theta]^{g-1}}{(g-1)!}$, we identified $Z^* : H^3(Y, \mathbb{Z}) \to H^3(Y, \mathbb{Z})$ to the composite map

$$H^3(Y, \mathbb{Z}) \cong H^1(J(Y), \mathbb{Z}) \stackrel{\frac{[\Theta]^{g-1}}{(g-1)!}\cup}{\to} H^{2g-1}(J(Y), \mathbb{Z}) \cong H^3(Y, \mathbb{Z}),$$

where the last isomorphism is Poincaré dual of the first, and using the definition of the polarization Θ on $J(Y)$ (as being given by Poincaré duality on Y: $H^3(Y, \mathbb{Z}) \cong H^3(Y, \mathbb{Z})^*$) we find that this composite map is the identity. ■

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