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À Eckart Viehweg

Abstract

Given a smooth projective 3-fold Y, with $H^{3,0}(Y) = 0$, the Abel-Jacobi map induces a morphism from each smooth variety parameterizing 1-cycles in Y to the intermediate Jacobian $J(Y)$. We study in this paper the existence of families of 1-cycles in Y for which this induced morphism is surjective with rationally connected general fiber, and various applications of this property. When Y itself is rationally connected, we relate this property to the existence of an integral homological decomposition of the diagonal. We also study this property for cubic threefolds, completing the work of Iliev-Markoushevich. We then conclude that the Hodge conjecture holds for degree 4 integral Hodge classes on fibrations into cubic threefolds over curves, with restriction on singular fibers.

0 Introduction

The following result is proved by Bloch and Srinivas [2] as a consequence of their decomposition of the diagonal:

Theorem 0.1. *Let Y be a smooth projective 3-fold with $CH_0(Y)$ supported on a curve. Then, via the Abel-Jacobi map AJ_Y, the group of 1-cycles homologous to 0 on Y modulo rational equivalence is up to torsion isomorphic to the intermediate Jacobian $J(Y)$ of Y,*

$$J(Y) := H^3(Y, \mathbb{C})/(F^2H^3(Y) \oplus H^3(Y, \mathbb{Z})).$$

This follows from the diagonal decomposition principle [2] (see also [21, II, 10.2.1]). Indeed, this principle says that if Y is a smooth variety such that $CH_0(Y)$ is supported on some subvariety $W \subset Y$, there is an equality in $CH^d(Y \times Y)$, $d = dim\, Y$:

$$N\Delta_Y = \Gamma_1 + \Gamma_2, \tag{0.1}$$

where N is a nonzero integer and Z_1, Z_2 are codimension d cycles with

$$|Z_1| \subset D \times Y,\ D \subsetneq Y, \quad |Z_2| \subset Y \times W.$$

Assuming $\dim W \leq 1$, the integer N appearing in this decomposition is easily checked to annihilate the kernel and cokernel of AJ_Y.

Note that the integer N appearing above cannot be in general set to be 1, and one purpose of this paper is to investigate the significance of this invariant, at least if we work on the level of cycles modulo homological equivalence. We will say that Y admits an integral homological decomposition of the diagonal as in (0.1) if there is an equality as in (0.1) (with the same conditions on $\mathrm{Supp}\, Z_1$ and $\mathrm{Supp}\, Z_2$), with $N = 1$, and in $CH^d(Y \times Y)/\mathrm{hom}$.

Going further and using the theory of Bloch-Ogus [1] together with Merkurjev-Suslin theorem, Bloch and Srinivas also prove the following:

Theorem 0.2. *If Y is a smooth projective complex variety such that $CH_0(Y)$ is supported on a surface, the Griffiths group* $\mathrm{Griff}^2(Y) = CH^2(Y)_{hom}/\mathrm{alg}$ *is identically* 0.

The last result cannot be obtained as a consequence of the diagonal decomposition, which only shows that under the same assumption $\mathrm{Griff}^2(Y)$ is annihilated by the integer N introduced above.

We finally have the following improvement of Theorem 0.1 :

Theorem 0.3. *(cf. [17]) If Y is a smooth projective 3-fold such that $CH_0(Y)$ is supported on a curve, then the Abel-Jacobi map induces an isomorphism:*

$$AJ_Y : CH_1(Y)_{hom} = CH_1(Y)_{alg} \to J(Y).$$

This follows indeed from the fact that this map is surjective with kernel of torsion by Theorem 0.1, and that it can be shown by delicate arguments involving the Merkurjev-Suslin theorem, Gersten-Quillen resolution in K-theory, and Bloch-Ogus theory [1], that this map induces an isomorphism on torsion.

The group on the left does not have a priori the structure of an algebraic variety, unlike the group on the right. However it makes sense to say that AJ_Y is algebraic, meaning that for any smooth algebraic variety B, and any codimension 2-cycle $Z \subset B \times Y$, with $Z_b \in CH^2(Y)_{hom}$ for any $b \in B$, the induced map

$$\phi_Z : B \to J(Y),\ b \mapsto AJ_Y(Z_b),$$

is a morphism of algebraic varieties.

Theorem 0.3 may give the feeling that for such threefolds, the study of 1-cycles parallels that of 0-cycles on curves. Still the structure of the Abel-Jacobi map on families of 1-cycles on such threefolds is mysterious, in contrast to what happens in the curve case, where Abel's theorem shows that the Abel-Jacobi map on the family of effective 0-cycles of large degree has fibers isomorphic to projective spaces. Another goal of this paper is to underline substantial differences between 1-cycles on threefolds with small CH_0 on one side and 0-cycles on curves on the other side.

There are for example two natural questions left open by the above results :

Question 0.4. *Let Y be a smooth projective threefold, such that $AJ_Y : CH_1(Y)_{alg} \to J(Y)$ is surjective. Is there a family of codimension 2-cycles homologous to 0 on Y, parameterized by $J(Y)$, say $Z \subset J(Y) \times Y$, such that the induced morphism $\phi_Z : J(Y) \to J(Y)$ is the identity?*

Note that the surjectivity assumption is conjecturally implied by the vanishing $H^3(Y, \mathcal{O}_Y) = 0$, via the generalized Hodge conjecture (cf. [8]).

There are two other related questions that we wish to partially investigate in this paper:

Question 0.5. *Is the following property (*) satisfied by Y?*

(*) There exists a family of codimension 2-cycles homologous to 0 on Y, parameterized by a smooth projective variety B, say $Z \subset B \times Y$, with $Z_b \in CH^2(Y)_{hom}$ for any $b \in B$, such that the induced morphism $\phi_Z : B \to J(Y)$ is surjective with rationally connected general fiber.

This question has been solved by Iliev-Markoushevich ([12], see also [10] for similar results obtained independently) in the case where Y is a smooth cubic threefold in $\mathbb{P}^4$. Obviously a positive answer to Question 0.4 implies a positive answer to Question 0.5, as we can then just take $B = J(X)$. However, it seems that Question 0.5 is more natural, especially if we go to the following stronger version : Here we choose an integral cohomology class $\alpha \in H^4(Y,\mathbb{Z})$. Assuming $CH_0(Y)$ is supported on a curve, one has $H^2(Y,\mathcal{O}_Y)=0$ by Mumford's theorem [16] (see also [21, II,10.2.2]) and thus α is a Hodge class. Introduce the torsor $J(Y)_\alpha$ defined as follows: the Deligne cohomology group $H^4_D(Y,\mathbb{Z}(2))$ is an extension

$$0 \to J(Y) \to H^4_D(Y,\mathbb{Z}(2)) \stackrel{o}{\to} Hdg^4(Y,\mathbb{Z}) \to 0,$$

where o is the natural map from Deligne to Betti cohomology. Define

$$J(Y)_\alpha := o^{-1}(\alpha). \tag{0.2}$$

By definition, the Deligne cycle class (cf. [5], [21, I,12.3.3]), restricted to codimension 2 cycles of class α, takes value in $J(Y)_\alpha$. For any family of 1-cycles $Z \subset B \times Y$ of class $[Z_t]=\alpha$, $t \in B$ on Y parameterized by an algebraic variety B, the Abel-Jacobi map (or rather the Deligne cycle class map) of Y induces a morphism complex algebraic varieties:

$$\phi_Z : B \to J(Y)_\alpha.$$

Question 0.6. *Is the following property (**) satisfied by Y?*

(**) For any degree 4 integral cohomology class α on Y, there is a canonically defined variety B_α, together with a family of 1-cycles of class α on Y, parameterized by B_α, say $Z_\alpha \subset B_\alpha \times Y$, with $[Z_{\alpha,b}]=\alpha$ in $H^4(Y,\mathbb{Z})$ for any $b \in B$, such that the morphism $\phi_{Z_\alpha} : B_\alpha \to J(Y)_\alpha$ induced by Deligne cycle class is surjective with rationally connected general fiber.

Let us comment on the importance of Question 0.6 in relation with the Hodge conjecture with integral coefficients for degree 4 Hodge classes, (which has been recently related in [4] to degree 3 unramified cohomology with finite coefficients): The important point here is that we want to consider fourfolds fibered over curves, or families of threefolds parameterized by a curve. Property (**) essentially says that property (*), being satisfied in a canonical way, holds in family. When we work in families, the necessity to look at all twistors $J(Y)_\alpha$, and not only at $J(Y)$ becomes obvious: for fixed Y the twisted Jacobians are all isomorphic (maybe not canonically); this is not true in families: non trivial twistors $\mathcal{J}_\alpha$ can appear. We will give more precise explanations in subsection 1 of this introduction and explain one application of this property to the Hodge conjecture for degree 4 integral Hodge classes on fourfolds fibered over curves.

Our results in this paper are of two kinds. First of all, we extend the results of [12] and answer affirmatively Question 0.6 for cubic threefolds. As a consequence, we get the following result (Theorem 2.10):

Theorem 0.7. *Let $f : X \to \Gamma$ be a fibration over a curve with general fiber a smooth cubic threefold. If the fibers of f have at worst ordinary quadratic singularities, then the Hodge conjecture holds for degree 4 integral Hodge classes on X.*

By the results of [4], this result can also be translated into a statement concerning degree 3 unramified cohomology with $\mathbb{Q}/\mathbb{Z}$-coefficients of such fourfolds (see section 1).

Our second result relates Question 0.5 to the existence of an homological integral decomposition of the diagonal as in (0.1).

Theorem 0.8. *i) Let Y be a smooth projective 3-fold. If Y admits an integral homological decomposition of the diagonal as in (0.1), then $H^4(Y,\mathbb{Z})$ is generated by classes of algebraic cycles, $H^*(Y,\mathbb{Z})$ has no torsion and Y satisfies condition (*).*

ii) Conversely, assume $CH_0(Y) = \mathbb{Z}$, $H^4(Y, \mathbb{Z})$ is generated by classes of algebraic cycles and $H^*(Y, \mathbb{Z})$ is without torsion (these conditions are satisfied by a rationally connected threefold with no torsion in $H^3(Y, \mathbb{Z})$).

If furthermore the intermediate Jacobian of Y admits a 1-cycle Γ of class $\frac{[\Theta]^{g-1}}{(g-1)!}$, $g = dim\, J(Y)$, then condition (*) implies that Y admis an integral homological decomposition of the diagonal as in (0.1).

Here we recall that the intermediate Jacobian $J(Y)$ is naturally a principally polarized abelian variety, the polarization Θ being given by the intersection form on $H^3(Y, \mathbb{Z}) \cong H_1(J(Y), \mathbb{Z})$.

We will also prove the following relation between Question 0.4 and 0.5 (cf. Theorem 3.1):

Theorem 0.9. *Assume that Question 0.5 has a positive answer for Y and that the intermediate Jacobian of Y admits a 1-cycle Γ of class $\frac{[\Theta]^{g-1}}{(g-1)!}$, $g = dim\, J(Y)$. Then Question 0.4 also has an affirmative answer for Y.*

The paper is organized as follows: in section 2, we give a positive answer to Question 0.5 for very general cubic threefolds. We deduce from this Theorem 0.7. Section 3 is devoted to various relations between Question 0.4 and 0.5 and the relation between these questions and the homological decomposition of the diagonal with integral coefficients, in the spirit of Theorem 0.8.

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1 Preliminaries on integral Hodge classes and unramified cohomology

We give in this section a description of results and notations from the earlier papers [19], [4] which will be used later on in the paper. Let X be a smooth projective complex variety. We denote $Z^{2i}(X)$ the quotient $Hdg^{2i}(X, \mathbb{Z})/H^{2i}(X, \mathbb{Z})_{alg}$ of the group of degree $2i$ integral Hodge classes on X by the subgroup consisting of cycle classes. The group $Z^{2i}(X)$ is a birational invariant of X for $i = 2$, $d-1$, where $d = dim\, X$ (cf. [18]). It is of course trivial in degrees $2i = 0$, $2d$, 2, where the Hodge conjecture holds for integral Hodge classes.

The paper [4] focuses on $Z^4(X)$ and relates its torsion to degree 3 unramified cohomology of X with finite coefficients. Recall that for any constant coefficients sheaf A on X, unramified cohomology $H^i_{nr}(X, A)$ with coefficients in A was introduced in [3] : denote by X_{Zar} the variety X endowed with the Zariski topology, while X will be considered as endowed with the classical topology. The identity map

$$\pi : X \to X_{Zar}$$

is continuous and allows Bloch and Ogus [1] to introduce sheaves $\mathcal{H}^l(A)$ on X_{Zar} defined by

$$\mathcal{H}^l(A) := R^l\pi_*A.$$

Definition 1.1. *(Ojanguren-Colliot-Thélène) Unramified cohomology $H^i_{nr}(X, A)$ of X with coefficients in A is the group $H^0(X_{Zar}, \mathcal{H}^i(A))$.*

This is a birational invariant of X, as shown by the Gersten-Quillen resolution for the sheaves $\mathcal{H}^i$ proved by Bloch-Ogus [1]. We refer to [4] for the description of other birational invariants constructed from the cohomology of the sheaves $\mathcal{H}^i$.

The following result is proved in [4] (assuming the Bloch-Kato conjecture announced by Voevodsky, or restricting to $n = 2$ for an unconditional result):

Theorem 1.2. *There is an exact sequence for any n:*

$$0 \to H^3_{nr}(X, \mathbb{Z}) \otimes \mathbb{Z}/n\mathbb{Z} \to H^3_{nr}(X, \mathbb{Z}/n\mathbb{Z}) \to Z^4(X)[n] \to 0,$$

where the notation $[n]$ means that we take the n-torsion. There is an exact sequence:

$$0 \to H^3_{nr}(X, \mathbb{Z}) \otimes \mathbb{Q}/\mathbb{Z} \to H^3_{nr}(X, \mathbb{Q}/\mathbb{Z}) \to Z^4(X)_{tors} \to 0,$$

where the subscript "tors" means that we consider the torsion part. Furthermore, the first term is 0 if $CH_0(X)$ is supported on a surface.

Concerning the vanishing of the group $Z^4(X)$, the following is proved in [19]. It will be used in section 3.2

Theorem 1.3. *Let Y be a smooth projective threefold which is either uniruled or Calabi-Yau. Then $Z^4(Y) = 0$, that is, any integral degree 4 Hodge class on Y is algebraic. In particular, if Y is uniruled with $H^2(Y, \mathcal{O}_Y) = 0$, any integral degree 4 cohomology class on Y is algebraic.*

We will now be interested in the group $Z^4(X)$, where X is a smooth projective 4-fold, $f : X \to \Gamma$ is a surjective morphism to a smooth curve Γ, whose general fiber X_t satisfies $H^3(X_t, \mathcal{O}_{X_t}) = H^2(X_t, \mathcal{O}_{X_t}) = 0$. For X_t, $t \in \Gamma$, as above, the intermediate Jacobian $J(X_t)$ is an abelian variety, as a consequence of the vanishing $H^3(X_t, \mathcal{O}_{X_t}) = 0$ (cf. [21, I,12.2.2]). For any class

$$\alpha \in H^4(X_t(\mathbb{C}), \mathbb{Z}) = Hdg^4(X_t, \mathbb{Z}),$$

we introduced above a torsor $J(X_t)_\alpha$ under $J^2(X_t)$ which is an algebraic variety non canonically isomorphic to $J(X_t)$.

This construction works in family on the Zariski open set $\Gamma_0 \subset \Gamma$, over which f is smooth. There is thus a family of abelian varieties $\mathcal{J} \to \Gamma_0$, and for any locally constant section α of the locally constant system $R^4 f_* \mathbb{Z}$, we get the twisted family $\mathcal{J}_\alpha \to \Gamma_0$, defined by the obvious extension of formula (0.2) in the relative setting. The construction of these families in the analytic setting is easy. The fact that the resulting families are algebraic is much harder.

Given a smooth algebraic variety B, a morphism $g : B \to \Gamma$ and a family of relative 1-cycles $\mathcal{Z} \subset B \times_\Gamma X$ of class $[Z_t] = \alpha_t \in H^4(X_t, \mathbb{Z})$, the relative Abel-Jacobi map (or rather Deligne cycle class) gives a morphism $\phi_\mathcal{Z} : B_0 \to \mathcal{J}_\alpha$ over Γ_0, where $B_0 := g^{-1}(\Gamma_0)$.

The following result, which illustrates the importance of condition (**) as opposed to condition (*), appears in [4]. We recall the proof here, as we will need it to prove a slight improvement of the criterion, which will be used in section 2.1. Here we assume that X is a smooth projective 4-fold, and that $f : X \to \Gamma$ is a morphism to a smooth curve.

Theorem 1.4. *Assume $f : X \to \Gamma$ satisfies the following assumptions:*

1. *The smooth fibers X_t of f satisfy the vanishing conditions $H^3(X_t, \mathcal{O}_{X_t}) = H^2(X_t, \mathcal{O}_{X_t}) = 0$.*
2. *The smooth fibers X_t have no torsion in $H^3(X_t(\mathbb{C}), \mathbb{Z})$.*

3. *The singular fibers of f are reduced with at worst ordinary quadratic singularities.*
4. *For any section α of $R^4 f_*\mathbb{Z}$, there exists a variety $g_\alpha : B_\alpha \to \Gamma$ and a family of relative 1-cycles $\mathcal{Z}_\alpha \subset B_\alpha \times_\Gamma X$ of class α, such that the morphism $\phi_{\mathcal{Z}_\alpha} : B \to \mathcal{J}_\alpha$ is surjective with rationally connected general fiber.*

Then the Hodge conjecture is true for integral Hodge classes of degree 4 on $\mathcal{X}$.

Proof. Following Zucker [22], an integral Hodge class $\tilde{\alpha} \in Hdg^4(X, \mathbb{Z}) \subset H^4(X, \mathbb{Z})$ induces a section α of the constant system $R^4 f_*\mathbb{Z}$ which admits a canonical lift in the family of twisted Jacobians $\mathcal{J}_\alpha$. This lift is an *algebraic* section $\sigma : \Gamma \to \mathcal{J}_\alpha$ of the structural map $\mathcal{J}_\alpha \to \Gamma$.

Recall that we have by hypothesis the morphism

$$\phi_{\mathcal{Z}_\alpha} : B_\alpha \to \mathcal{J}_\alpha$$

which is algebraic, surjective with rationally connected general fiber. We can now replace σ by a 1-cycle $\Sigma = \sum_i n_i \Sigma_i$ rationally equivalent to it in $\mathcal{J}_\alpha$, in such a way that the fibers of $\phi_{\mathcal{Z}_\alpha}$ are rationally connected over the general points of each component Σ_i of $\mathrm{Supp}\,\Sigma$.

According to [6], the morphism $\phi_{\mathcal{Z}_\alpha}$ admits a lifting over each Σ_i, which provides curves $\Sigma_i' \subset B_\alpha$.

Recall next that B_α parameterizes a family $\mathcal{Z}_\alpha \subset B_\alpha \times_\Gamma X$ of relative 1-cycles of class α in the fibers X_t. We can restrict this family to each Σ_i', getting $\mathcal{Z}_{\alpha,i} \subset \Sigma_i' \times_\Gamma X$. Consider the 1-cycle

$$Z := \sum_i n_i p_{i*} \mathcal{Z}_{\alpha,i} \subset \Gamma \times_\Gamma X = X,$$

where p_i is the restriction to Σ_i' of $p : B_\alpha \to \Gamma$. Recalling that Σ is rationally equivalent to $\sigma(\Gamma)$ in $\mathcal{J}_\alpha$, we find that the normal function ν_Z associated to Z, defined by

$$\nu_Z(t) = AJ_{X_t}(Z_{|X_t})$$

is equal to σ. We then deduce from [7] (see also [21, II,8.2.2]), using the Leray spectral sequence of $f_U : X_U \to U$ and hypothesis 2, that the cohomology classes $[Z] \in H^4(X, \mathbb{Z})$ of Z and $\tilde{\alpha}$ coincide on any open set of the form X_U, where $U \subset \Gamma_0$ is an affine open set over which f is smooth.

On the other hand, the kernel of the restriction map $H^4(X, \mathbb{Z}) \to H^4(U, \mathbb{Z})$ is generated by the groups $i_{t*}H_4(X_t, \mathbb{Z})$ where $t \in \Gamma \setminus U$, and $i_t : X_t \to X$ is the inclusion map. We conclude using assumption 3 and the fact that the general fiber of f has $H^2(X_t, \mathcal{O}_{X_t}) = 0$, which imply that all fibers X_t (singular or not) have their degree 4 integral homology generated by homology classes of algebraic cycles; indeed, it follows from this and the previous conclusion that $[Z] - \tilde{\alpha}$ is algebraic, so that $\tilde{\alpha}$ is also algebraic.

■

Remark 1.5. It is also possible in this proof, instead of moving the section σ to a 1-cycle in general position, to use Theorem 1.8 below, which also guarantees that in fact σ itself lifts to B_α.

Remark 1.6. By Theorem 1.3, if X_t is uniruled and $H^2(X_t, \mathcal{O}_{X_t}) = 0$ (a strengthening of assumption 1), then any degree 4 integral cohomology class α_t on X_t is algebraic on X_t. Together with Bloch-Srinivas results [2] on the surjectivity of the Abel-Jacobi map for codimension 2-cycles under these assumptions, this shows that pairs (B_α, Z_α) with surjective $\phi_{\mathcal{Z}_\alpha} : B_\alpha \to \mathcal{J}_\alpha$ exist. In this case, the strong statement in assumption 4 is thus the rational connectedness of the fibers.

In order to study cubic threefolds fibrations, we will need a technical strenghtening of Theorem 1.4. We start with a smooth projective morphism $f : \mathcal{X} \to T$ of relative dimension 3. We assume as before that smooth fibers $\mathcal{X}_t$ have no torsion in $H^3(\mathcal{X}_t, \mathbb{Z})$ and have $H^3(\mathcal{X}_t, \mathcal{O}_{\mathcal{X}_t}) = H^2(\mathcal{X}_t, \mathcal{O}_{\mathcal{X}_t}) = 0$. As before, for any global section α of $R^4 f_*\mathbb{Z}$, we have the twisted family of intermediate Jacobians $\mathcal{J}_\alpha \to T$, where T is smooth quasi-projective.

Theorem 1.7. *Assume that:*

(i) The local system $R^4 f_*\mathbb{Z}$ is trivial.

(ii) For any section α of $R^4 f_*\mathbb{Z}$, there exists a variety $g_\alpha : B_\alpha \to T$ and a family of relative 1-cycles $\mathcal{Z}_\alpha \subset B_\alpha \times_T \mathcal{X}$ of class α, such that the morphism $\phi_{\mathcal{Z}_\alpha} : B_\alpha \to \mathcal{J}_\alpha$ is surjective with rationally connected general fibers.

Then for any smooth projective curve Γ, any morphism $\phi : \Gamma_0 \to T$, where Γ_0 is a Zariski open set of Γ, and any smooth projective model $\psi : X \to \Gamma$ of $\mathcal{X} \times_T \Gamma_0$, if furthermore g has only nodal fibers, one has $Z^4(X) = 0$.

Proof. We mimick the proof of the previous theorem. We thus only have to prove that for any Hodge class $\tilde{\alpha}$ on X, inducing by restriction to the fibers of ψ a section α of $R^4\psi_*\mathbb{Z}$, the section $\sigma_{\tilde{\alpha}}$ of $\mathcal{J}_{\Gamma,\alpha} \to \Gamma_0$ induced by $\tilde{\alpha}$ lifts to a family of 1-cycles of class α in the fibers of ψ. Observe that by triviality of the local system $R^4 f_*\mathbb{Z}$ on T, the section α extends to a section of $R^4 f_*\mathbb{Z}$ on T. We then have by assumption the family of 1-cycles $\mathcal{Z}_\alpha \subset B_\alpha \times_T \mathcal{X}$ parameterized by B_α, with the property that the general fiber of the induced Abel-Jacobi map $\phi_{\mathcal{Z}_\alpha} : B_\alpha \to \mathcal{J}_\alpha$ is rationally connected. As $\phi^*\mathcal{J}_\alpha = \mathcal{J}_{\Gamma,\alpha}$, we can see the pair $(\phi, \sigma_{\tilde{\alpha}})$ as a general morphism from Γ_0 to $\mathcal{J}_\alpha$. The desired family of 1-cycles follows then from the existence of a lift of $\sigma_{\tilde{\alpha}}$ to B_α, given by the following Theorem 1.8, due to Hogadi and Xu.

■

Theorem 1.8. *Let $f : Z \to W$ be a surjective projective morphism between smooth varieties over $\mathbb{C}$. Assume the general fiber of f is rationally connected. Then, for any rational map $g : C \dashrightarrow W$, where C is a smooth curve, there exists a rational lift $\tilde{g} : C \to Z$ of g in Z.*

When the image curve $g(C)$ is in general position, so that the fiber of f over the general point of $Im\, g$ is actually rationally connected, this result is a consequence of [6]. The degenerate case is a consequence of the work of Hogadi and Xu [11] combined with [6]. Indeed, they prove that the variety $C_0 \times_W Z$ contains a subvariety fibered over C_0 with rationally connected general fibers, where C_0 is the definition locus of g. ■

2 On the fibres of the Abel-Jacobi map for the cubic threefold

The papers [15], [12], [10] are devoted to the study of the morphism induced by the Abel-Jacobi map, from the family of curves of small degree in a cubic threefold in $\mathbb{P}^4$ to its intermediate Jacobian. In degree 4, genus 0, and degree 5, genus 0 or 1, it is proved that these morphisms are surjective with rationally connected fibers, but it is known that this is not true in degree 3 (and any genus), and, to our knowlege, the case of degree 6 has not been studied. As is clear from the proof of Theorem 1.4, we need for the proof of Theorem 2.10 to have such a statement for at least one canonically defined family of curves of degree divisible by 3. The following result provides such a statement.

Theorem 2.1. *Let $Y \subset \mathbb{P}^4_{\mathbb{C}}$ be a general cubic hypersurface. Then the map $\phi_{6,1}$ induced by the Abel-Jacobi map AJ_Y, from the family $M_{6,1}$ of elliptic curves of degree 6 contained in Y to $J(Y)$, is surjective with rationally connected general fiber.*

Remark 2.2. What we call here the family of elliptic curves is a desingularization of the closure, in the Hilbert scheme of Y, of the family parameterizing smooth degree 6 elliptic curves which are nondegenerate in $\mathbb{P}^4$.

Remark 2.3. As $J(Y)$ is not uniruled, an equivalent formulation of the result is the following (cf. [10]): the map $\phi_{6,1}$ is dominating and identifies birationally to the maximal rationally connected fibration (see [14]) of $M_{6,1}$.

Proof of Theorem 2.1. Notice that it suffices to prove the statement for very general Y, which we will assume occasionally. One easily checks that $M_{6,1}$ is for general Y irreducible of dimension 12. Let $E \subset Y$ be a general nondegenerate smooth degree 6 elliptic curve. Then there exists a smooth $K3$ surface $S \subset Y$, which is a member of the linear system $|\mathcal{O}_Y(2)|$, containing E. The line bundle $\mathcal{O}_S(E)$ is then generated by two sections. Let $V := H^0(S, \mathcal{O}_S(E))$ and consider the rank 2 vector bundle $\mathcal{E}$ on Y defined by $\mathcal{E} = \mathcal{F}^*$, where $\mathcal{F}$ is the kernel of the surjective evaluation map

$$V \otimes \mathcal{O}_Y \to \mathcal{O}_S(E). \tag{2.3}$$

One verifies by dualizing the exact sequence (2.3) that $H^0(Y, \mathcal{E})$ is of dimension 4, and that E is recovered as the zero locus of a transverse section of $\mathcal{E}$. Furthermore, the bundle $\mathcal{E}$ is stable because E is nondegenerate. One concludes that, denoting M_9 the moduli space (of dimension 9) of stable rank 2 bundles on Y with Chern classses

$$c_1(\mathcal{E}) = \mathcal{O}_Y(2), \ deg\, c_2(\mathcal{E}) = 6,$$

one has a dominating rational map

$$\phi : M_{6,1} \dashrightarrow M_9$$

whose general fiber is isomorphic to $\mathbb{P}^3$ (more precisely, the fiber over $[\mathcal{E}]$ is the projective space $\mathbb{P}(H^0(Y, \mathcal{E}))$). It follows that the maximal rationally connected (MRC) fibration of $M_{6,1}$ (cf. [14]) factorizes through M_9.

Let us consider the subvariety $D_{3,3} \subset M_{6,1}$ parameterizing the singular elliptic curves of degree 6 consisting of two rational components of degree 3 meeting in two points. The family $M_{3,0}$ parameterizing rational curves of degree 3 is birationally a $\mathbb{P}^2$-fibration over the Theta divisor $\Theta \subset J(Y)$ (cf. [9]). More precisely, a general rational curve $C \subset Y$ of degree 3 is contained in a smooth hyperplane section $S \subset Y$, and deforms in a linear system of dimension 2 in S. The map $\phi_{3,0} : M_{3,0} \to J(Y)$ induced by the Abel-Jacobi map of Y has its image equal to $\Theta \subset J^2(Y)$ and its fiber passing through $[C]$ is the linear system $\mathbb{P}^2$ introduced above. This allows to describe $D_{3,3}$ in the following way: the Abel-Jacobi map of Y, applied to each component C_1, C_2 of a general curve $C_1 \cup C_2$ parameterized by $D_{3,3}$ takes value in the symmetric product $S^2\Theta$ and its fiber passing through $[C_1 \cup C_2]$ is described by the choice of two curves in the corresponding linear systems $\mathbb{P}^2_1$ and $\mathbb{P}^2_2$. These two curves have to meet in two points of the elliptic curve E_3 defined as the complete intersection of the two cubic surfaces S_1 and S_2. This fiber is thus birationally equivalent to S^2E_3.

Observe that the existence of this subvariety $D_{3,3} \subset M_{6,1}$ implies the surjectivity of $\phi_{6,1}$ because the restriction of $\phi_{6,1}$ to $D_{3,3}$ is the composition of the above-defined surjective map

$$\chi : D_{3,3} \to S^2\Theta$$

and of the sum map $S^2\Theta \to J(Y)$. Obviously the sum map is surjective.

We will show more precisely:

Lemma 2.4. *The map ϕ introduced above is generically defined along $D_{3,3}$ and $\phi_{|D_{3,3}} : D_{3,3} \dashrightarrow M_9$ is dominating. In particular $D_{3,3}$ dominates the base of the MRC fibration of $M_{6,1}$.*

To construct other rational curves in $M_{6,1}$, we now make the following observation: if E is a nondegenerate degree 6 elliptic curve in $\mathbb{P}^4$, let us choose two distinct line bundles l_1, l_2 of degree 2 on E such that $\mathcal{O}_E(1) = 2l_1 + l_2$ in $\operatorname{Pic} E$. Then (l_1, l_2) provides an embedding of E in $\mathbb{P}^1 \times \mathbb{P}^1$, and denoting $h_i := pr_i^* \mathcal{O}_{\mathbb{P}^1}(1) \in \operatorname{Pic}(\mathbb{P}^1 \times \mathbb{P}^1)$, the line bundle $l = 2h_1 + h_2$ on $\mathbb{P}^1 \times \mathbb{P}^1$ has the property that $l_{|E} = 2l_1 + l_2 = \mathcal{O}_E(1)$ and the restriction map

$$H^0(\mathbb{P}^1 \times \mathbb{P}^1, l) \to H^0(E, \mathcal{O}_E(1))$$

is an isomorphism. The original morphism from E to $\mathbb{P}^4$ is given by a base-point free hyperplane in $H^0(E, \mathcal{O}_E(1))$. For a generic choice of l_2, this hyperplane also provides a base-point free hyperplane in $H^0(\mathbb{P}^1 \times \mathbb{P}^1, l)$, hence a morphism $\phi : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^4$, whose image is a surface Σ of degree 4. The residual curve of E in the intersection $\Sigma \cap Y$ is a curve in the linear system $|4h_1 + h_2|$ on $\mathbb{P}^1 \times \mathbb{P}^1$. This is thus a curve of degree 6 in $\mathbb{P}^4$ and of genus 0.

Conversely, if we start from a smooth general genus 0 and degree 6 curve C in $\mathbb{P}^4$, we claim that there exists a $\mathbb{P}^1$ parameterizing morphisms $\phi : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^4$ as above, such that $C \subset \phi(\mathbb{P}^1 \times \mathbb{P}^1)$ is the image by ϕ of a curve in the linear system $|4h_1 + h_2|$ on $\mathbb{P}^1 \times \mathbb{P}^1$. To see this, we note that the morphism from $\mathbb{P}^1$ to $\mathbb{P}^4$ given up to the action of $Aut\, \mathbb{P}^1$ by $C \subset \mathbb{P}^4$ provides a base-point free linear subsystem $W \subset |\mathcal{O}_{\mathbb{P}^1}(6)|$ of rank 5. Let us choose a hyperplane $H \subset |\mathcal{O}_{\mathbb{P}^1}(6)|$ containing W. W being given, such hyperplanes H are parameterized by a $\mathbb{P}^1$. H being chosen generically, there exists a unique embeding $C \to \mathbb{P}^1 \times \mathbb{P}^1$ as a curve in the linear system $|4h_1 + h_2|$, such that H is recovered as the image of the restriction map

$$H^0(\mathbb{P}^1 \times \mathbb{P}^1, 2h_1 + h_2) \to H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(6)).$$

To see this last fact, notice that the embedding of $\mathbb{P}^1$ into $\mathbb{P}^1 \times \mathbb{P}^1$ as a curve in the linear system $|4h_1 + h_2|$ is determined up to the action of $Aut\, \mathbb{P}^1$ by the choice of a degree 4 morphism from $\mathbb{P}^1$ to $\mathbb{P}^1$, which is equivalent to the data of a rank 2 base-point free linear system $W' \subset H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(4))$. The condition that H is equal to the image of the restriction map above is equivalent to the fact that

$$H = W' \cdot H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(2)) := Im\,(W' \otimes H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(2)) \to H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(6)),$$

where the map is given by multiplication of sections. Given H, let us set

$$W' := \cap_{t \in H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(2))} [H : t],$$

where $[H : t] := \{w \in H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(4)),\ wt \in H\}$. Generically, one then has $\dim W' = 2$ et $H = W' \cdot H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(2))$, and this concludes the proof of the claim.

If the curve C is contained in Y, the residual curve of C in the intersection $Y \cap \Sigma$ is an elliptic curve of degree 6 in Y. These two constructions are inverse of each other, which provides a birational map from the space of pairs $\{(E, \Sigma),\ E \in M_{6,1},\ E \subset \Sigma\}$ to the space of pairs $\{(C, \Sigma),\ C \in M_{6,0},\ C \subset \Sigma\}$, where $\Sigma \subset \mathbb{P}^4$ is a surface of the type considered above, and the inclusions of the curves C, E is Σ are in the linear systems described above.

This construction thus provides rational curves $R_C \subset M_{6,1}$ parameterized by $C \in M_{6,0}$ and sweeping-out $M_{6,1}$. To each curve C parameterized by $M_{6,0}$, one associates the family R_C of elliptic curves E_t which are residual to C in the intersection $\Sigma_t \cap Y$, where t runs over the $\mathbb{P}^1$ exhibited in the above claim.

Let $\Phi : M_{6,1} \dashrightarrow B$ be the maximal rationally connected fibration of $M_{6,1}$. We will use later on the curves R_C introduced above to prove the following Lemma 2.5. Notice first that the divisor $D_{5,1} \subset M_{6,1}$ parameterizing singular elliptic curves of degree 6 contained in Y, which are the union of an elliptic curve of degree 5 in Y and of a line $\Delta \subset Y$ meeting in one point. If E is such a generically chosen elliptic curve and $S \in |\mathcal{O}_Y(2)|$ is a smooth $K3$ surface containing E, the linear system $|\mathcal{O}_S(E)|$ contains the line Δ in its base locus, and thus the construction of the associated vector bundle $\mathcal{E}$ fails. Consider however a curve $E' \subset S$ in the linear system $|\mathcal{O}_S(2-E)|$. Then E' is again a degree 6 elliptic curve in Y and E' is smooth and nondegenerate for a generic choice of Y, E, S as above. The curves E' so obtained are parameterized by a divisor $D'_{5,1}$ of $M_{6,1}$, which also has the following description : the curves E' parameterized by $D'_{5,1}$ can also be characterized by the fact that they have a trisecant line contained in Y (precisely the line Δ introduced above).

Lemma 2.5. *The restriction of Φ to $D'_{5,1}$ is surjective.*

Remark 2.6. It is not true that the map $\phi_{|D'_{5,1}} : D'_{5,1} \dashrightarrow M_9$ is dominating. Indeed, curves E parameterized by $D'_{5,1}$ admit a trisecant line L in Y. Let $\mathcal{E}$ be the associated vector bundle. Then because $\det \mathcal{E} = \mathcal{O}_Y(2)$ and $\mathcal{E}_{|L}$ has a nonzero section vanishing in three points, $\mathcal{E}_{|L} \cong \mathcal{O}_L(3) \oplus \mathcal{O}_L(-1)$. Hence the corresponding vector bundles $\mathcal{E}$ are not globally generated and thus are not generic.

Admitting Lemmas 2.4 and 2.5, the proof of Theorem 2.1 is concluded in the following way: Lemma 2.4 says that the rational map $\Phi_{|D_{3,3}}$ is dominating. The dominating rational maps $\Phi_{|D_{3,3}}$ and $\Phi_{|D'_{5,1}}$, taking values in a non uniruled variety, factorize through the base of the MRC fibrations of $D_{3,3}$ and $D'_{5,1}$ respectively. The maximal rationally connected fibrations of the divisors $D'_{5,1}$ and $D_{5,1}$ have the same base. Indeed, the construction of the correspondence between $D_{5,1}$ and $D'_{5,1}$ is obtained by liaison, and it follows that a $\mathbb{P}^2$-bundle over $D_{5,1}$ (parameterizing a reducible elliptic curve E of type $(5,1)$ and a rank 2 vector space of quadratic polynomials vanishing on E) is birationally equivalent to a $\mathbb{P}^2$-fibration over $D'_{5,1}$ defined in the same way. The MRC fibration of the divisor $D_{5,1}$ is well understood by the works [12], [15]. Indeed, these papers show that the morphism $\phi_{5,1}$ induced by the Abel-Jacobi map of Y on the family $M_{5,1}$ of degree 5 elliptic curves in Y is surjective with general fiber isomorphic to $\mathbb{P}^5$: more precisely, an elliptic curve E of degree 5 in Y is the zero locus of a transverse section of a rank 2 vector bundle $\mathcal{E}$ on Y, and the fiber of $\phi_{5,1}$ passing through $[E]$ is for general E isomorphic to $\mathbb{P}(H^0(Y,\mathcal{E})) = \mathbb{P}^5$. The vector bundle $\mathcal{E}$, and the line $L \subset Y$ being given, the set of elliptic curves of type $(5,1)$ whose degree 5 component is the zero locus of a section of $\mathcal{E}$ and the degree 1 component is the line L identifies to

$$\bigcup_{x \in L} \mathbb{P}(H^0(Y, \mathcal{E} \otimes \mathcal{I}_x)),$$

which is rationally connected. Hence $D_{5,1}$, and thus also $D'_{5,1}$, is a fibration over $J(Y) \times F$ with rationally connected general fiber, where F is the Fano surface of lines in Y. As the variety $J(Y) \times F$ is not uniruled, it must be the base of the maximal rationally connected fibration of $D'_{5,1}$.

From the above considerations, we conclude that B is dominated by $J(Y) \times F$ and in particular $\dim B \leq 7$.

Let us now consider the dominating rational map $\Phi' := \Phi_{|D_{3,3}} : D_{3,3} \dashrightarrow B$. Recall that $D_{3,3}$ admits a surjective morphism χ to $S^2\Theta$, the general fiber being isomorphic to the second symmetric product S^2E_3 of an elliptic curve of degree 3. We postpone the proof of the following lemma:

Lemma 2.7. *The rational map Φ' factorizes through $S^2\Theta$.*

This lemma implies that B is rationally dominated by $S^2\Theta$, hence a fortiori by $\Theta \times \Theta$. We will denote $\Phi'' : \Theta \times \Theta \dashrightarrow B$ the rational map obtained by factorizing Φ' through $S^2\Theta$ and then by composing the resulting map with the quotient map $\Theta \times \Theta \to S^2\Theta$. We have a map $\phi_B : B \to J(Y)$ obtained as a factorization of the map $\phi_{6,1} : M_{6,1} \to J(Y)$, using the fact that the fibers of $\Phi : M_{6,1} \dashrightarrow B$ are rationally connected, hence contracted by $\phi_{6,1}$. Clearly, the composition of the map ϕ_B with the map Φ'' identifies to the sum map $\sigma : \Theta \times \Theta \to J(Y)$. Its fiber over an element $a \in J^2(Y)$ thus identifies in a canonical way to the complete intersection of the ample divisors Θ and $a - \Theta$ of $J(Y)$. Consider the dominating rational map fiberwise induced by Φ'' over $J(Y)$:

$$\Phi''_a : \Theta \cap (a - \Theta) \dashrightarrow B_a,$$

where $B_a := \phi_B^{-1}(a)$ has dimension at most 2 and is not uniruled. The proof is thus concluded with Lemma 2.8 below, which implies that B is in fact birationally isomorphic to $J(Y)$. Indeed, this lemma tells that Φ''_a must be constant for very general Y and general a, hence also for general Y and a. ∎

Lemma 2.8. *If Y is very general, and $a \in J(Y)$ is general, any dominating rational map Φ''_a from $\Theta \cap (a - \Theta)$ to a smooth non uniruled variety B_a of dimension ≤ 2 is constant.*

To complete the proof of Theorem 2.1, it remains to prove Lemmas 2.4, 2.5, 2.7 and 2.8.

Proof of Lemma 2.4. Let $E = C_3 \cup C'_3$ be a general elliptic curve $(3,3)$ contained in Y. Then E is contained in a smooth $K3$ surface in the linear system $|\mathcal{O}_Y(2)|$. The linear system $|\mathcal{O}_S(E)|$ has no base point in S, and thus the construction of the vector bundle $\mathcal{E}$ can be done as in the smooth general case, and furthermore it is easily verified that $\mathcal{E}$ is stable on Y. Hence ϕ is well-defined at the point $[E]$. One verifies that $H^0(Y, \mathcal{E})$ is of dimension 4 as in the general smooth case. As $\dim D_{3,3} = 10$ and $\dim M_9 = 9$, to show that $\phi_{|D_{3,3}}$ is dominating, it suffices to show that the general fiber of $\phi_{|D_{3,3}}$ is of dimension ≤ 1. Assume to the contrary that this fiber, which is contained in $\mathbb{P}(H^0(Y, \mathcal{E})) = \mathbb{P}^3$, is of dimension ≥ 2. Recalling that $D'_{3,3}$ is not swept-out by rational surfaces because it is fibered over $S^2\Theta$ into surfaces isomorphic to the second symmetric product of an elliptic curve, this fiber should be a surface of degree ≥ 3 in $\mathbb{P}(H^0(Y, \mathcal{E}))$. A general line in $\mathbb{P}(H^0(Y, \mathcal{E}))$ should then meet this surface in at least 3 points counted with multiplicities. For such a line, take the $\mathbb{P}^1 \subset \mathbb{P}(H^0(Y, \mathcal{E}))$ obtained by considering the base-point free pencil $|\mathcal{O}_S(E)|$. One verifies that this line is not tangent to $D_{3,3}$ at the point $[E]$ and thus should meet $D_{3,3}$ in another point. The surface S should thus have a Picard number $\rho(S) \geq 4$, which is easily excluded by a dimension count for a general pair (E, S), where E is an elliptic curve of type $(3,3)$ as above and S is the intersection of our general cubic Y and a quadric in $\mathbb{P}^4$ containing E. ∎

For the proof of Lemma 2.5 we will use the following elementary Lemma 2.9 : Let Z be a smooth projective variety, and $\Phi : Z \dashrightarrow B$ its maximal rationally connected fibration. Assume given a family of rational curves $(C_t)_{t \in M}$ sweeping-out Z.

Lemma 2.9. *Let $D \subset Z$ be a divisor such that, through a general point $d \in D$, there passes a curve C_t, $t \in M$, meeting D properly at d. Then $\Phi_{|D} : D \dashrightarrow B$ is dominating.*

Proof of Lemma 2.5. The proof is obtained by applying Lemma 2.9 to $Z = M_{6,1}$ and to the previously constructed family of rational curves R_C, $C \in M_{6,0}$. It thus suffices to show that for general Y, if $[E] \in D'_{5,1}$ is a general point, there exists a (irreducible) curve R_C passing through $[E]$ and not contained in $D'_{5,1}$. Let us recall that $D'_{5,1}$ is irreducible, and parameterizes elliptic curves of degree 6 in Y admitting a trisecant line contained in Y. Starting from a curve C of genus 0 and degree 6 which is nondegenerate in $\mathbb{P}^4$, we constructed a one parameter family $(\Sigma_t)_{t\in\mathbb{P}^1}$ of surfaces of degree 4 containing C and isomorphic to $\mathbb{P}^1\times\mathbb{P}^1$. For general t, the trisecant lines of Σ_t in $\mathbb{P}^4$ form at most a 3-dimensional variety and thus, for a general cubic $Y\subset\mathbb{P}^4$, there exists not trisecant to Σ_t contained in Y. Let us choose a line Δ which is trisecant to Σ_0 but not trisecant to Σ_t for t close to 0, $t\neq 0$. A general cubic Y containing C and Δ contains then no trisecant to Σ_t for t close to 0, $t\neq 0$. Let E_0 be the residual curve of C in $\Sigma_0\cap Y$. Then E_0 has the line $\Delta\subset Y$ as a trisecant, and thus $[E_0]$ is a point of the divisor $D'_{5,1}$ associated to Y. However the rational curve $R_C\subset M_{6,1}(Y)$ parameterizing the residual curves E_t of C in $\Sigma_t\cap Y$, $t\in\mathbb{P}^1$, is not contained in $D'_{5,1}$ because, by construction, no trisecant of E_t is contained in Y, for t close to 0, $t\neq 0$. ∎

Proof of Lemma 2.7. Assume to the contrary that the dominating rational map $\Phi' : D_{3,3} \dashrightarrow B$ does not factorize through $S^2\Theta$. As the map $\chi : D_{3,3}\to S^2\Theta$ has for general fiber S^2E_3, where E_3 is an elliptic curve of degree 3 in $\mathbb{P}^4$, and B is not uniruled, Φ' factorizes through the corresponding fibration $\chi' : Z\dashrightarrow S^2\Theta$, with fiber Pic^2E_3. Let us denote $\Phi'' : Z\dashrightarrow B$ this second factorization. Recall that $\dim B\leq 7$ and that B maps surjectively onto $J(Y)$ via the rational map $\phi_B : B\dashrightarrow J(Y)$ induced by the Abel-Jacobi map $\phi_{6,1} : M_{6,1}\to J(Y)$. Clearly the elliptic curves Pic^2E_3 introduced above are contracted by the composite map $\phi_B\circ\Phi''$. If these curves are not contracted by Φ'', the fibers $B_a := \phi_B^{-1}(a)$, $a\in J^2(Y)$, are swept-out by elliptic curves. As the fibers B_a are either curves of genus >0 or non uniruled surfaces (because $\dim B\leq 7$ and B is not uniruled), they have the property that there pass at most finitely many elliptic curves in a given deformation class through a general point $b\in B_a$. Under our assumption, there should thus exist :

- A variety J', and a dominating morphism $g : J'\to J(Y)$ such that the fiber J'_a, for a general point $a\in J^2(Y)$, has dimension at most 1 and parameterizes elliptic curves in a given deformation class in the fiber B_a.

- A family $B'\to J'$ of elliptic curves, and dominating rational maps :

$$\Psi : Z\dashrightarrow B',\ s : S^2\Theta\dashrightarrow J',\ r : Z'\dashrightarrow B$$

such that Z is birationally equivalent to the fibered product $S^2\Theta\times_{J'}B'$, r is generically finite, and $\Phi'' = r\circ\Psi$.

To get a contradiction out of this, observe that the family of elliptic curves $Z\to S^2\Theta$ is also birationally the inverse image of a family of elliptic curves parameterized by $G(2,4)$, via a natural rational map $f : S^2\Theta\dashrightarrow G(2,4)$. Indeed, an element of $S^2\Theta$ parameterizes two linear systems $|C_1|$, resp. $|C_2|$ of rational curves of degree 3 on two hyperplane sections $S_1 = H_1\cap Y$, resp. $S_2 = H_2\cap Y$ of Y. The map f associates to the unordered pair $\{|C_1|,|C_2|\}$ the plane $P := H_1\cap H_2$. This plane $P\subset\mathbb{P}^4$ parameterizes the elliptic curve $E_3 := P\cap Y$, and there is a canonical isomorphism $E_3\cong Pic^2E_3$ because $deg\,E_3 = 3$.

The contradiction then comes from the following fact: for $j\in\mathbb{P}^1$, let us consider the divisor D_j of Z which is swept-out by elliptic curves of fixed modulus determined by j, in the fibration $Z\dashrightarrow S^2\Theta$. This divisor is the inverse image by $f\circ\chi'$ of an ample divisor on $G(2,4)$ and furthermore it is also the inverse image by Ψ of the similarly defined divisor of B'. As r is generically finite and B is not uniruled, B' is not uniruled. Hence the rational map $\Psi : Z\dashrightarrow B'$ is "almost holomorphic", that is well-defined along its general fiber. One deduces from this that the divisor D_j has trivial restriction to a general fiber of Ψ. This fiber, which is of dimension ≥ 2 because $\dim Z = 9$ and $\dim B'\leq 7$, must then be sent to a point in $G(2,4)$ by $f\circ\chi'$

and its image in $S^2\Theta$ is thus contained in an union of components of fibers of f. Observing furthermore that the general fiber of f is irreducible of dimension 2, it follows that the rational dominating map $s : S^2\Theta \dashrightarrow J'$ factorizes through $f : S^2\Theta \dashrightarrow G(2,5)$. As $G(2,5)$ is rational, this contradicts the fact that J' dominates rationally $J(Y)$.

■

Proof of Lemma 2.8. Observe first that, as $\dim \mathrm{Sing}\,\Theta = 0$, the general intersection $\Theta \cap (a-\Theta)$ is smooth. Assume first that $\dim B_a = 1$. Then either B_a is an elliptic curve or $H^0(B_a, \Omega_{B_a})$ has dimension ≥ 2. Notice that, by Lefschetz hyperplane restriction theorem, the Albanese variety of the complete intersection $\Theta \cap (a-\Theta)$ identifies to $J(Y)$. B_a cannot be elliptic, otherwise $J(Y)$ would not be simple. But it is easy to prove by an infinitesimal variation of Hodge structure argument (cf. [21, II, 6.2.1]) that $J(Y)$ is simple for very general Y. If $\dim H^0(B_a, \Omega_{B_a}) \geq 2$, this provides two independent holomorphic 1-forms on $\Theta \cap (a-\Theta)$ whose exterior product vanishes. These 1-forms come, by Lefschetz hyperplane restriction theorem, from independent holomorphic 1-forms on $J(Y)$, whose exterior product is a nonzero holomorphic 2-form on $J(Y)$. The restriction of this 2-form to $\Theta \cap (a-\Theta)$ vanishes, which contradicts Lefschetz hyperplane restriction theorem because $\dim \Theta \cap (a-\Theta) = 3$. This case is thus excluded.

Assume now $\dim B_a = 2$. Then the surface B_a has nonnegative Kodaira dimension, because it is not uniruled. Let L be the line bundle $\Phi_a''^*(K_{B_a})$ on $\Theta \cap (a-\Theta)$. The Iitaka dimension $\kappa(L)$ of L is nonnegative and there is a nonzero section of the bundle $\Omega^2_{\Theta\cap(a-\Theta)}(-L)$. Notice that by Grothendieck-Lefschetz theorem, L is the restriction of a line bundle on $J(Y)$, hence must be numerically effective, because one can show, again by an infinitesimal variation of Hodge structure argument, that $\mathrm{NS}(J(Y)) = \mathbb{Z}$, for very general Y. An easy argument using the conormal exact sequence of $\Theta \cap (a-\Theta)$ in $J^2(Y)$, vanishing theorems, and the fact that $\dim \Theta \cap (a-\Theta) \geq 3$, allows to show that any such section is the restriction of a section of $(\Omega^2_{J(Y)})_{|\Theta\cap(a-\Theta)}(-L)$. As $\kappa(L) \geq 0$, such a nonzero section can exist only if L is trivial. Notice that, up to replacing $(\Theta \cap (a-\Theta), \Phi_a'', B_a)$ by its Stein factorization (or rather, the Stein factorization of a desingularized model of Φ_a''), one may assume that Φ_a'' has connected fibers. In this case, the triviality of L implies the existence of a nonzero section of K_{B_a} and the fact that $h^0(B_a, K_{B_a}) = 1$. This is clear if Φ_a'' is a morphism. Otherwise, choose a desingularization:

$$\widetilde{\Phi_a''} : \Theta \widetilde{\cap (a-\Theta)} \to B_a$$

of the rational map Φ_a''. Then we know that $\widetilde{\Phi_a''}^* K_{B_a}$ has nonnegative Iitaka dimension and that it is trivial away from the exceptional divisor of the desingularization map

$$\Theta \widetilde{\cap (a-\Theta)} \to \Theta \cap (a-\Theta).$$

This is possible only if $\widetilde{\Phi_a''}^* K_{B_a}$ is effective supported on this exceptional divisor. But then it has only one section, which provides a unique section of K_{B_a}, because we assumed that fibers of $\widetilde{\Phi_a''}$ are connected.

Having this, we conclude that the Hodge structure on $H^2(\Theta \cap (a-\Theta), \mathbb{Z}) \cong H^2(J^2(Y), \mathbb{Z})$ has a Hodge substructure with $h^{2,0} = 1$. But this can be also excluded for very general Y by an infinitesimal variation of Hodge structure argument.

■

This concludes the proof of Theorem 2.1.

2.1 Application: Integral Hodge classes on cubic threefolds fibrations over curves

The main concrete application of Theorem 2.1 concerns fibrations $f : X \to \Gamma$ over a curve Γ with general fiber a cubic threefold in $\mathbb{P}^4$ (or the smooth projective models of smooth cubic hypersurfaces $X \subset \mathbb{P}^4_{\mathbb{C}(\Gamma)}$).

Theorem 2.10. *Let $f : X \to \Gamma$ be a cubic threefold fibration over a smooth curve. If the fibers of f have at worst ordinary quadratic singularities, $Z^4(X) = 0$ and $H^3_{nr}(X, \mathbb{Z}/n\mathbb{Z}) = 0$ for any n.*

Note that the second statement is a consequence of the first by Theorem 1.2 and the fact that for X as above, $CH_0(X)$ is supported on a curve.

Theorem 2.10 applies in particular to cubic fourfolds. Indeed, a smooth cubic hypersurface X in $\mathbb{P}^5$ admits a Lefschetz pencil of hyperplane sections. It thus becomes, after blowing-up the base-locus of this pencil, birationally equivalent to a model of a cubic in $\mathbb{P}^4_{\mathbb{C}(t)}$ which satisfies all our hypotheses. Theorem 2.10 thus provides $Z^4(X) = 0$ for cubic fourfolds, a result first proved in [20].

Proof of Theorem 2.10. We first prove the theorem for a fibration $X \to \Gamma$ whose general fiber satisfies the conclusion of Theorem 2.1.

Cubic hypersurfaces in $\mathbb{P}^4$ do not have torsion in their degree 3 cohomology by Lefschetz hyperplane restriction theorem. They furthermore satisfy the vanishing conditions $H^i(X_t, \mathcal{O}_{X_t}) = 0$, $i > 0$.

It thus suffices to prove that condition 4 of Theorem 1.4 is satisfied.

The fibers of f being smooth hypersurfaces in $\mathbb{P}^4$, $R^4 f_*\mathbb{Z}$ is, over the regularity open set Γ_0, the trivial local system isomorphic to $\mathbb{Z}$. A section α of $R^4 f_*\mathbb{Z}$ over Γ_0 is thus characterized by a number, its degree on the fibers X_t with respect to the polarization $c_1(\mathcal{O}_{X_t}(1))$. When the degree of α is 5, one gets the desired family B_5 and cycle $\mathcal{Z}_5$ using the results of [12], in the following way: Iliev and Markoushevich prove that if Y is a smooth cubic threefold, denoting $M_{5,1}$ a desingularization of the Hilbert scheme of degree 5, genus 1 curves in Y, the map $\phi_{5,1} : M_{5,1} \to J(Y)$ induced by the Abel-Jacobi map of Y is surjective with rationally connected fibers. Taking for B_5 a desingularization $\mathcal{M}_{5,1}$ of the relative Hilbert scheme of curves of degree 5 and genus 1 in the fibers of f, and for cycle $\mathcal{Z}_5$ the universal subscheme, the hypothesis 4 of Theorem 1.4 is thus satisfied for the degree 5 section of $R^4 f_*\mathbb{Z}$.

For the degree 6 section α of $R^4 f_*\mathbb{Z}$, property 4 is similarly a consequence of Theorem 2.1. Indeed, it says that the general fiber of the morphism $\phi_{6,1} : M_{6,1} \to J(Y)_6$ induced by the Abel-Jacobi map of Y on the family $M_{6,1}$ of degree 6 elliptic curves in Y is rationally connected for general Y, which by assumption remains true for the general fiber of f. We thus take as before for family B_6 a desingularization $\mathcal{M}_{6,1}$ of the relative Hilbert scheme of curves of degree 6 and genus 1 in the fibers of f, and for cycle $\mathcal{Z}_6$ the universal subscheme.

To conclude, let us show that property 4 for the sections α_5 of degree 5 and α_6 of degree 6 imply property 4 for any section α. To see this, let us introduce the cycle h^2 of codimension 2 over X, where $h \in Pic\,X$ induces by restriction to a general fibre X_t the positive generator of $Pic\,X_t$. The cycle h^2 is thus of degree 3 on the fibers of f. The degree of a section α is congruent to 5, -5 or 6 modulo 3, and we can thus write $\alpha = \pm\alpha_5 + \mu\alpha_3$, or $\alpha = \alpha_6 + \mu\alpha_3$ for an adequate number μ. In the first case, consider the variety $B_\alpha = B_5$ and the cycle

$$\mathcal{Z}_\alpha = \pm\mathcal{Z}_5 + \mu(B_5 \times_\Gamma h^2) \subset B_5 \times_\Gamma X.$$

In the second case, consider the variety $B_\alpha = B_6$ and the cycle

$$\mathcal{Z}_\alpha = \mathcal{Z}_6 + \mu(B_6 \times \Gamma h^2) \subset B_6 \times_\Gamma X.$$

It is clear that the pair $(B_\alpha, \mathcal{Z}_\alpha)$ satisfies the condition 4 of Theorem 1.4.

The general case, where the general fiber of $f : X \to \Gamma$ is not assumed to satisfy the conclusion of Theorem 2.1, is treated in the same way, replacing the reference to Theorem 1.4 by a reference to its variant (Theorem 1.7) and applying the arguments above, reducing the verification of the assumption (ii) in Theorem 1.7 to the case of the sections α of degree 5 or 6. ■

3 Structure of the Abel-Jacobi map and decomposition of the diagonal

3.1 Relation between Questions 0.4 and 0.5

We establish in this subsection the following relation between Questions 0.4 and 0.5:

Theorem 3.1. *Assume that Question 0.5 has a positive answer for Y and that the intermediate Jacobian of Y admits a 1-cycle Γ of class $\frac{[\Theta]^{g-1}}{(g-1)!}$, $g = dim\, J(Y)$. Then Question 0.4 also has an affirmative answer for Y.*

Proof. There exists by assumption a variety B, and a codimension 2 cycle $Z \subset B \times Y$ cohomologous to 0 on fibers $b \times Y$, such that the morphism

$$\phi_Z : B \to J(Y)$$

induced by the Abel-Jacobi map of Y is surjective with rationally connected general fibers. Consider the 1-cycle Γ of $J(Y)$. We may assume by moving lemma, up to changing the representative of Γ modulo homological equivalence, that $\Gamma = \sum_i n_i \Gamma_i$ where, for each component Γ_i of the support of Γ, the general fiber of ϕ_Z over Γ_i is rationally connected. According to [6], the inclusion $j_i : \Gamma_i \hookrightarrow J(Y)$ has then a lift $\sigma_i : \Gamma_i \to B$. Denote $Z_i \subset \Gamma_i \times Y$ the codimension 2-cycle $(\sigma_i, Id_Y)^* Z$. Then the morphism $\phi_i : \Gamma_i \to J(Y)$ induced by the Abel-Jacobi map is equal to j_i.

Observe now the following: there is the Pontryagin product $*$ on cycles of $J(Y)$. Another way of rephrasing the condition that the class of Γ is equal to $\frac{[\Theta]^{g-1}}{(g-1)!}$, $g = dim\, J(Y)$, for some principal polarization Θ is to say that

$$(\Gamma)^{*g} = J(Y)$$

as dimension g cycles of $J(Y)$.

For each g-uple of components $(\Gamma_{i_1}, \ldots, \Gamma_{i_g})$ of $\mathrm{Supp}\,\Gamma$, consider $\Gamma_{i_1} \times \ldots \times \Gamma_{i_g}$, and the codimension 2-cycle

$$Z_{i_1,\ldots,i_g} := (pr_1, Id_Y)^* Z_{i_1} + \ldots + (pr_g, Id_Y)^* Z_{i_g} \subset \Gamma_1 \times \ldots \times \Gamma_g \times Y.$$

The codimension 2-cycle

$$\mathcal{Z} := \sum_{i_1,\ldots,i_g} n_{i_1} \ldots n_{i_g} Z_{i_1,\ldots,i_g} \subset (\sqcup \Gamma_i)^g \times Y$$

is invariant under the symmetric group $\mathfrak{S}_g$ acting on the factor $(\sqcup\Gamma_i)^g$ in the product $(\sqcup\Gamma_i)^g \times Y$. The part of $\mathcal{Z}$ dominating over a component of $(\sqcup\Gamma_i)^g$, (which is the only one we are interested in) is then the pull-back of a codimension 2-cycle $\mathcal{Z}_{sym}$ on $(\sqcup\Gamma_i)^{(g)} \times Y$. Consider now the sum map

$$\sigma : (\sqcup\Gamma_i)^{(g)} \to J(Y).$$

Let $\mathcal{Z}_J := (\sigma, Id)_*\mathcal{Z}_{sym} \subset J(Y) \times Y$. The proof concludes with the following:

Lemma 3.2. *The Abel-Jacobi map :*

$$\phi_{\mathcal{Z}_J} : J(Y) \to J(Y)$$

is equal to $Id_{J(Y)}$.

Proof. Instead of the symmetric product $(\sqcup\Gamma_i)^{(g)}$ and the descended cycle $\mathcal{Z}_{sym}$, consider the product $(\sqcup\Gamma_i)^g$, the cycle $\mathcal{Z}$ and the sum map

$$\sigma' : (\sqcup\Gamma_i)^g \to J(Y).$$

Then we have $(\sigma', Id)_*\mathcal{Z} = g!(\sigma, Id)_*\mathcal{Z}_{sym}$ in $CH^2(J(Y) \times Y)$, so that writing $\mathcal{Z}'_J := (\sigma', Id)_*\mathcal{Z}$, it suffices to prove that:

$$g!\, Id_{J(Y)} = \phi_{\mathcal{Z}'_J} : J(Y) \to J(Y). \tag{3.4}$$

This is done as follows: let $j \in J(Y)$ be a general point, and let $\{x_1, \ldots, x_N\}$ be the fiber of σ' over j. Thus each x_l parameterizes a g-uple $(i_1^l, \ldots, i_g^l)$ of components of $\mathrm{Supp}\,\Gamma$, and points $\gamma_{i_1}^l, \ldots, \gamma_{i_g}^l$ of $\Gamma_{i_1}, \ldots, \Gamma_{i_g}$ respectively, such that:

$$\sum_{1 \le k \le g} \gamma_{i_k}^l = j. \tag{3.5}$$

On the other hand, recall that

$$\gamma_{i_k}^l = AJ_Y(Z_{i_k^l, \gamma_{i_k}^l}). \tag{3.6}$$

It follows from (3.5) and (3.6) that for each l, we have:

$$AJ_Y(\sum_{1 \le k \le g} Z_{i_k}^l, \gamma_{i_k}^l) = AJ_Y(Z_{i_1^l, \ldots, i_g^l, (\gamma_{i_1}^l, \ldots, \gamma_{i_g}^l)}) = j. \tag{3.7}$$

Recall now that $\Gamma = \sum_i n_i \Gamma_i$ and recall that $(\Gamma)^{*g} = J(Y)$, which can be translated by saying that

$$\sigma'_*(\sum_{i_1, \ldots, i_g} n_{i_1} \ldots n_{i_g} \Gamma_{i_1} \times \ldots \times \Gamma_{i_g}) = g! J(Y).$$

This exactly says that

$$\sum_{1 \le l \le N} \sum_{i_1^l, \ldots, i_g^l} n_{i_1^l} \ldots n_{i_g^l} = g!$$

which together with (3.7) proves (3.4). ∎

The proof of Theorem 3.1 is now complete. ∎

Remark 3.3. The question whether the intermediate Jacobian of Y admits a 1-cycle Γ of class $\frac{[\Theta]^{g-1}}{(g-1)!}$, $g = dim\, J(Y)$ is unknown even for the cubic threefold. However it has a positive answer for $g \le 3$ because any principally polarized abelian variety of dimension ≤ 3 is the Jacobian of a curve.

3.2 Decomposition of the diagonal modulo homological equivalence

This section is devoted to the study of assumption 4 in Theorem 1.4, (see also Question 0.5). We study here this condition when the family $f : X \to \Gamma$ is a product $Y \times \Gamma$, where Y is a smooth projective threefold staisfying $Z^4(Y) = 0$ (for example, by theorem 1.3, if Y is a uniruled threefold). First of all, we observe that assumption 4 in theorem 1.4 is equivalent to the following condition on the 3-fold Y with $H^2(Y, \mathcal{O}_Y) = 0$ and $Z^4(Y) = 0$.

(*) *There exists a smooth variety B and a family of 1-cycles $\mathcal{Z} \subset B \times Y$ homologous to 0 in Y, such that the Abel-Jacobi map AJ_Y induces a surjective morphism $\phi_{\mathcal{Z}} : B \to J^2(Y)$ with rationally connected general fiber.*

Indeed, as Y satisfies $H^2(Y, \mathcal{O}_Y) = 0$ and $Z^4(Y) = 0$, any class $\alpha \in H^4(Y, \mathbb{Z})$ is algebraic, that is, there exists $z_\alpha \in CH^2(Y)$, such that $[z] = \alpha$. If B and $\mathcal{Z}$ exist as above, the family $B \times \Gamma$ and the cycle $\mathcal{Z} \times \Gamma + B \times z_\alpha \times \Gamma \subset B \times Y \times \Gamma$ induce a surjective map with rationally connected general fiber

$$\phi_\alpha : B \times \Gamma \to J(Y)_\alpha \times \Gamma = \mathcal{J}_\alpha$$

and thus property 4 is satisfied by $Y \times \Gamma$.

Assume furthermore Y satisfies $CH_0(Y) = \mathbb{Z}$. Bloch-Srinivas decomposition of the diagonal [2] says that there exists a nonzero integer N such that, denoting $\Delta_Y \subset Y \times Y$ the diagonal :

$$N\Delta_Y = Y \times y + Z \text{ in } CH^3(Y \times Y), \tag{3.8}$$

where y is any point of Y and the support of Z is contained in $D \times Y$, $D \subsetneqq Y$.

We wish to study the invariant of Y defined as the gcd of the integers N appearing above. This is a birational invariant of Y. One can also consider the decomposition (3.8) modulo homological equivalence, and our results relate the triviality of this invariant, that is the existence of an integral homological decomposition of the diagonal as in (3.8), to condition (*) (among other things). We first recall that the intermediate Jacobian

$$J(Y) = H^3(Y, \mathbb{C})/(F^2H^3(Y, \mathbb{C}) \oplus H^3(Y, \mathbb{Z})/\text{torsion}$$

of a smooth projective threefold Y with $H^3(Y, \mathcal{O}_Y) = 0$ is an abelian variety, which is principally polarized by the unimodular intersection form on the lattice $H^3(Y, \mathbb{Z})/\text{torsion}$. We denote by Θ a divisor in the numerical divisor class defined by this principal polarization.

Theorem 3.4. *Let Y be a smooth projective 3-fold. Assume Y admits an integral homological decomposition of the diagonal as in (3.8); then we have:*
(i) $H^*(Y, \mathbb{Z})$ is without torsion.
(ii) $Z^4(Y) = 0$.
(iii) $H^i(Y, \mathcal{O}_Y) = 0$, $\forall i > 0$ and Y satisfies condition (*).

Proof. There exists by assumption $D \subsetneqq Y$, which one may assume of pure dimension 2, and $Z \in CH^3(Y \times Y)$ such that the support of Z is contained in $D \times Y$, and

$$[\Delta_Y] = [Y \times y] + [Z] \text{ in } H^6(Y \times Y, \mathbb{Z}). \tag{3.9}$$

Codimension 3 cycles z of $Y \times Y$ act on $H^*(Y, \mathbb{Z})$ and on the intermediate Jacobian of Y, and this action, which we will denote

$$z^* : H^*(Y, \mathbb{Z}) \to H^*(Y, \mathbb{Z}),\ z^* : J(Y) \to J(Y)$$

depends only on the cohomology class of z. As the diagonal of Y acts by the identity map on $H^*(Y,\mathbb{Z})$ for $*>0$ and on $J(Y)$, and the cycle $Y\times y$ acts trivially on both of these groups, one concludes that

$$Id_{H^*(Y,\mathbb{Z})}=Z^*:H^*(Y,\mathbb{Z})\to H^*(Y,\mathbb{Z}),\text{ for }*>0, \tag{3.10}$$

$$Id_{J(Y)}=Z^*:J(Y)\to J(Y). \tag{3.11}$$

Let $\tau:\widetilde{D}\to Y$ be a desingularization of D and $i_{\widetilde{D}}=i_D\circ\tau:\widetilde{D}\to Y$. The part of the cycle Z which dominates D can be lifted to a cycle $\widetilde{Z}$ in $\widetilde{D}\times Y$, and the remaining part acts trivially on $H^*(Y,\mathbb{Z})$ for $*\leq 3$ for codimension reasons. Thus the map Z^* acting on $H^*(Y,\mathbb{Z})$ for $*\leq 3$ can be written as

$$Z^*=i_{\widetilde{D}*}\circ\widetilde{Z}^*. \tag{3.12}$$

We note that the action of $\widetilde{Z}^*$ on cohomology sends $H^*(Y,\mathbb{Z})$, $*\leq 3$, to $H^{*-2}(\widetilde{D},\mathbb{Z})$, $*\leq 3$. The last groups have no torsion. It follows that $\widetilde{Z}^*$ annihilates the torsion of $H^*(Y,\mathbb{Z})$, $*\leq 3$. Formula (3.10) implies then that $H^*(Y,\mathbb{Z})$ has no torsion for $*\leq 3$.

To deal with the torsion of $H^*(Y,\mathbb{Z})$ with $*\geq 4$, we rather use the action z_* of z on $H^*(Y,\mathbb{Z})$, $*=4,\,5$. This action again factors through $\widetilde{Z}_*$, (indeed, the part of Z which does not dominate D acts trivially on these groups for dimension reasons,) hence through the restriction map:

$$H^*(Y,\mathbb{Z})\to H^*(\widetilde{D},\mathbb{Z}).$$

As $\dim\widetilde{D}\leq 2$, $H^*(\widetilde{D},\mathbb{Z})$ has no torsion for $*=4,\,5$. It follows that we also have $H^*(Y,\mathbb{Z})_{tors}=0$ for $*=4,\,5$. This proves (i).

To prove (ii), let us consider again the action $Z_*=Id$ on the cohomology $H^4(Y,\mathbb{Z})$. Observe again that the part Z' of Z not dominating Z has a trivial action on $H^4(Y,\mathbb{Z})$, while the dominating part lifts as above to a cycle $\widetilde{Z}$ in $\widetilde{D}\times Y$. Then we find that $Id_{H^4(Y,\mathbb{Z})}$ factors as $\widetilde{Z}_*\circ i^*_{\widetilde{D}}$, hence through the restriction map $i^*_{\widetilde{D}}:H^4(Y,\mathbb{Z})\to H^4(\widetilde{D},\mathbb{Z})$. As $\dim\widetilde{D}=2$, the group on the right is generated by classes of algebraic cycles, and thus $H^4(Y,\mathbb{Z})$ is generated by classes of algebraic cycles on Y. In particular $Z^4(Y)=0$.

(iii) The vanishing $H^i(Y,\mathcal{O}_Y)=0$, $\forall i>0$ is a well-known consequence of the decomposition of the diagonal (3.8) (cf. [2], [21, II,10.2.2]). This is important to guarantee that $J(Y)$ is an abelian variety.

It is well-known, and this is a consequence of Lefschetz theorem on $(1,1)$-classes, that there exists a universal divisor $\mathcal{D}\subset\mathrm{Pic}^0(\widetilde{D})\times\widetilde{D}$, such that the induced morphism : $\phi_{\mathcal{D}}:\mathrm{Pic}^0(\widetilde{D})\to\mathrm{Pic}^0(\widetilde{D})$ is the identity. On the other hand, we have the morphism

$$\widetilde{Z}^*:J(Y)\to J^1(\widetilde{D})=Pic^0(\widetilde{D}),$$

which is a morphism of abelian varieties.

Let us consider the cycle

$$\mathcal{Z}:=(Id_{J(Y)},i_{\widetilde{D}})_*\circ(\widetilde{Z}^*,Id_{\widetilde{D}})^*(\mathcal{D})\in CH^2(J(Y)\times Y).$$

Then $\phi_{\mathcal{Z}}:J(Y)\to J(Y)$ is equal to

$$i_{\widetilde{D}*}\circ\phi_{\mathcal{D}}\circ\widetilde{Z}^*:J(Y)\to J^1(\widetilde{D})\to J^1(\widetilde{D})\to J(Y).$$

As $\phi_{\mathcal{D}}$ is the identity map acting on $\mathrm{Pic}^0(\widetilde{D})$ and $i_{\widetilde{D}*}\circ\widetilde{Z}^*$ is the identity map acting on $J(Y)$ according to (3.11) and (3.12), one concludes that $\phi_{\mathcal{Z}}=Id:J(Y)\to J(Y)$, which of course implies (*). ∎

Remark 3.5. The proof even shows that under our assumptions, Question 0.4 has an affirmative answer for Y.

Recall from section 1 that for any smooth projective complex variety Y, $Z^4(Y)$ is a quotient of $H^3_{nr}(Y,\mathbb{Q}/\mathbb{Z})$, if $Z^4(Y)$ is of torsion (which is always the case in dimension 3). We can now get a result better than (ii) if instead of considering the decomposition of the diagonal modulo homological equivalence, we consider it modulo algebraic equivalence. For example, we have the following statement, which improves (iii) above:

Proposition 3.6. *Let Y be a smooth projective 3-fold. Assume Y admits an integral decomposition of the diagonal as in (3.8), modulo algebraic equivalence. Then $H^3_{nr}(Y,A)=0$ for any abelian group A.*

Proof. We use the fact that correspondences modulo algebraic equivalence act on cohomology groups $H^p(X_{Zar},\mathcal{H}^q)$. We refer to the appendix of [4] for this fact which is precisely stated as follows:

Proposition 3.7. *If X, Y are smooth projective and $Z\subset X\times Y$ is a cycle defined up to algebraic equivalence, satisfying $\dim Y - dim\, Z = r$, then there is an induced morphism*

$$Z_*: H^p(X_{Zar},\mathcal{H}^q(A))\to H^{p+r}(Y_{Zar},\mathcal{H}^{q+r}(A)).$$

These actions are compatible with the composition of correspondences.

Assume now that Y admits a decomposition of the diagonal of the form

$$\Delta_Y = Z_1 + Z_2 \text{ in } CH(Y\times Y)/\mathrm{alg},$$

where $Z_1\subset D\times Y$ for some $D\subsetneqq Y$ and $Z_2\subset Y\times y_0$. The diagonal acts on $H^3_{nr}(Y,A)$ as the identity. Thus we get

$$Id_{H^3_{nr}(Y,A)} = Z_{1*}+Z_{2*} : H^3_{nr}(Y,A)\to H^3_{nr}(Y,A).$$

We observe now (by introducing again a desingularization $\widetilde{D}$ of D) that $Z_{1*}=0$ on $H^3_{nr}(Y,A)$, because Z_{1*} factors through the restriction map

$$H^3_{nr}(Y,A)\to H^3_{nr}(\widetilde{D},A)$$

and the group on the right is 0, because $\dim\widetilde{D}\leq 2$. Furthermore Z_{2*} also vanishes on $H^3_{nr}(Y,A)=H^0(Y_{Zar},\mathcal{H}^3(A))$ because $Im\, Z_{2*}$ factors through j_{y_0*} which shifts $H^p(\mathcal{H}^q)$, $p\geq 0$, $q\geq 0$, to $H^{p+3}(\mathcal{H}^{q+3})$.

Thus $Id_{H^3_{nr}(Y,A)}=0$ and $H^3_{nr}(Y,A)=0$. ∎

A partial converse to Theorem 3.4 is as follows.

Theorem 3.8. *Assume that Y satisfies*

i) $CH_0(Y)=\mathbb{Z}$.

ii) $Z^4(Y)=0$.

iii) $H^*(Y,\mathbb{Z})$ has no torsion and the intermediate Jacobian of Y admits a 1-cycle Γ of class $\frac{[\Theta]^{g-1}}{(g-1)!}$, $g=dim\, J(Y)$.

Then condition (*) implies the existence of an integral homological decomposition of the diagonal as in (3.8).

Remark 3.9. The condition $Z^4(Y) = 0$ is satisfied if Y is uniruled. Condition ii) is satisfied if Y is rationally connected. For a rationally connected threefold Y, $H^*(Y, \mathbb{Z})$ is without torsion if $H^3(Y, \mathbb{Z})$ is without torsion.

Remark 3.10. That the integral decomposition of the diagonal as in (3.8) and in the Chow group $CH(Y \times Y)$ implies that $H^3(Y, \mathbb{Z})$ has no torsion was observed by Colliot-Thélène.

Proof of Theorem 3.8. When the integral cohomology of Y has no torsion, the class of the diagonal $[\Delta_Y] \in H^6(Y \times Y, \mathbb{Z})$ has an integral Künneth decomposition.

$$[\Delta_Y] = \delta_{6,0} + \delta_{5,1} + \delta_{4,2} + \delta_{3,3} + \delta_{2,4} + \delta_{1,5} + \delta_{0,6},$$

where $\delta_{i,j} \in H^i(Y, \mathbb{Z}) \otimes H^j(Y, \mathbb{Z})$. The class $\delta_{0,6}$ is the class of $Y \times y$ for any point y of Y. As we assumed $CH_0(Y) = 0$, we have

$$H^1(Y, \mathcal{O}_Y) = 0,\ H^2(Y, \mathcal{O}_Y) = 0. \tag{3.13}$$

The first condition implies that the groups $H^1(Y, \mathbb{Q})$ and $H^5(Y, \mathbb{Q})$ are trivial, hence the groups $H^1(Y, \mathbb{Z})$ and $H^5(Y, \mathbb{Z})$ must be trivial since they have no torsion by assumption. It follows that $\delta_{5,1} = \delta_{1,5} = 0$.

Next, the second vanishing condition in (3.13) implies that the Hodge structure on $H^2(Y, \mathbb{Q})$, hence also on $H^4(Y, \mathbb{Q})$ by duality, is trivial. Hence $H^4(Y, \mathbb{Z})$ and $H^2(Y, \mathbb{Z})$ are generated by Hodge classes, and because we assumed $Z^4(Y) = 0$, it follows that $H^4(Y, \mathbb{Z})$ and $H^2(Y, \mathbb{Z})$ are generated by cycle classes. From this one concludes that $\delta_{4,2}$ et $\delta_{2,4}$ are represented by algebraic cycles whose support does not dominate Y by the first projection. The same is true for $\delta_{6,0}$ which is the class of $y \times Y$. The existence of a decomposition as in (3.9) is thus equivalent to the fact that there exists a cycle $Z \subset Y \times Y$, such that the support of Z is contained in $D \times Y$, with $D \subsetneqq Y$, and $Z^* : H^3(Y, \mathbb{Z}) \to H^3(Y, \mathbb{Z})$ is the identity map. This last condition is indeed equivalent to the fact that the component of type $(3, 3)$ of $[Z]$ is equal to $\delta_{3,3}$.

Let now $\Gamma = \sum_i n_i \Gamma_i$ be a 1-cycle of $J(Y)$ of class $\frac{[\Theta]^{g-1}}{(g-1)!}$, where $\Gamma_i \subset J(Y)$ are smooth curves in general position. If (*) is satisfied, there exist a variety B and a family $\mathcal{Z} \subset B \times Y$ of 1-cycles homologous to 0 in Y, such that $\phi_{\mathcal{Z}} : B \to J(Y)$ is surjective with rationally connected general fiber. By [6], $\phi_{\mathcal{Z}}$ admits a section σ_i over each Γ_i, and $(\sigma_i, Id)^* \mathcal{Z}$ provides a family $Z_i \subset \Gamma_i \times Y$ of 1-cycles homologous to 0 in Y, parameterized by Γ_i, such that $\phi_{Z_i} : \Gamma_i \to J(Y)$ identifies to the inclusion of Γ_i in $J(Y)$.

Let us consider the cycle $Z \in CH^3(Y \times Y)$ defined by

$$Z = \sum_i n_i Z_i \circ {}^t Z_i.$$

The proof that the cycle Z satisfies the desired property is then given in the following Lemma 3.11.

■

Lemma 3.11. *The map $Z^* : H^3(Y, \mathbb{Z}) \to H^3(Y, \mathbb{Z})$ is the identity map.*

Proof. We have

$$Z^* = \sum_i n_i {}^t Z_i^* \circ Z_i^*.$$

Let us study the composite map

$${}^t Z_i^* \circ Z_i^* : H^3(Y, \mathbb{Z}) \to H^1(\Gamma_i, \mathbb{Z}) \to H^3(Y, \mathbb{Z}).$$

Recalling that $Z_i \in CH^2(\Gamma_i \times Y)$ is the restriction to $\sigma(\Gamma_i)$ of $\mathcal{Z} \in CH^2(\Gamma_i \times Y)$, one finds that this composite map can also be written as:

$$^tZ_i^* \circ Z_i^* = {}^t\mathcal{Z}^* \circ \cup[\sigma(\Gamma_i)] \circ \mathcal{Z}^*.$$

One uses for this the fact that, denoting $j_i : \sigma(\Gamma_i) \to B$ the inclusion, the composition

$$j_{i*} \circ j_i^* : H^1(B, \mathbb{Z}) \to H^{2n-1}(B, \mathbb{Z}),\ n = dim\, B$$

is equal to $\cup[\sigma(\Gamma_i)]$.

We thus obtain:

$$Z^* = {}^t\mathcal{Z}^* \circ \cup(\sum_i n_i[\sigma(\Gamma_i)]) \circ \mathcal{Z}^*.$$

But we know that the map $\phi_{\mathcal{Z}} : B \to J(Y)$ has rationally connected fibers, and thus induces isomorphisms:

$$\phi_{\mathcal{Z}}^* : H^1(J(Y), \mathbb{Z}) \cong H^1(B, \mathbb{Z}),\ \phi_{\mathcal{Z}*} : H^{2n-1}(B, \mathbb{Z}) \cong H^{2g-1}(J(Y), \mathbb{Z}). \tag{3.14}$$

Via these isomorphisms, $\mathcal{Z}^*$ becomes the canonical isomorphism

$$H^3(Y, \mathbb{Z}) \cong H^1(J(Y), \mathbb{Z})$$

(which uses the fact that $H^3(Y, \mathbb{Z})$ is torsion free) and ${}^t\mathcal{Z}^*$ becomes the dual canonical isomorphism

$$H^{2g-1}(J(Y), \mathbb{Z}) \cong H^3(Y, \mathbb{Z}).$$

Finally, the map $\cup(\sum_i n_i[\sigma(\Gamma_i)]) : H^1(B, \mathbb{Z}) \to H^{2n-1}(B, \mathbb{Z})$ identifies via (3.14) to the map $\cup(\sum_i n_i[(\Gamma_i)]) : H^1(J(Y), \mathbb{Z}) \to H^{2g-1}(J(Y), \mathbb{Z})$. Recalling that $\sum_i n_i[(\Gamma_i)] = \frac{[\Theta]^{g-1}}{(g-1)!}$, we identified $Z^* : H^3(Y, \mathbb{Z}) \to H^3(Y, \mathbb{Z})$ to the composite map

$$H^3(Y, \mathbb{Z}) \cong H^1(J(Y), \mathbb{Z}) \stackrel{\cup\frac{[\Theta]^{g-1}}{(g-1)!}}{\to} H^{2g-1}(J(Y), \mathbb{Z}) \cong H^3(Y, \mathbb{Z}),$$

where the last isomorphism is Poincaré dual of the first, and using the definition of the polarization Θ on $J(Y)$ (as being given by Poincaré duality on Y: $H^3(Y, \mathbb{Z}) \cong H^3(Y, \mathbb{Z})^*$) we find that this composite map is the identity. ■

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