



**HAL**  
open science

# A remark on ill-posedness issues for KdV and mKdV equations

Luc Molinet

► **To cite this version:**

| Luc Molinet. A remark on ill-posedness issues for KdV and mKdV equations. 2010. hal-00473966v1

**HAL Id: hal-00473966**

**<https://hal.science/hal-00473966v1>**

Preprint submitted on 17 Apr 2010 (v1), last revised 15 Sep 2010 (v2)

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# A remark on ill-posedness issues for KdV and mKdV equations

**Luc Molinet**

*L.M.P.T., Université François Rabelais Tours, Fédération Denis Poisson-CNRS,  
Parc Grandmont, 37200 Tours, France*

April 17, 2010

## **Abstract**

In this brief article we give an elementary proof of the following alternative for the Cauchy problem associated with the KdV and the mKdV equations: Either there exists no  $T > 0$  such that the solution map  $u_0 \mapsto u$  associated with mKdV is continuous at the origin from  $H^s(\mathbb{R})$ ,  $s < 0$ , into  $\mathcal{D}'(]0, T[ \times \mathbb{R})$  or there exists no  $T > 0$  and no  $R > 0$  such that the solution map associated with KdV is continuous from the ball  $B(0, R)$  of  $H^s(\mathbb{R})$ ,  $s < -1$ , into  $\mathcal{D}'(]0, T[ \times \mathbb{R})$ .

## **1 Statement of the result**

The KdV and mKdV equations are physical relevant equations that respectively read

$$v_t + v_{xxx} + 6vv_x = 0 \tag{1.1}$$

and

$$u_t + u_{xxx} \mp 6u^2u_x = 0, \tag{1.2}$$

where  $u(t, x)$  and  $v(t, x)$  are real valued functions.

One distinguishes between two different mKdV equations depending on the sign in front of the nonlinear term. The sign "+" corresponds to the focussing mKdV equation whereas the sign "-" corresponds to the defocussing mKdV. Equations (1.1) and (1.2) are known to be completely integrable. In the end of the sixties, Miura discovered that the KdV equation is related to the mKdV equations by the now so-called Miura transforms. Indeed, the Miura transforms  $\Phi_d : v \mapsto v_x + v^2$  and  $\Phi_f : v \mapsto v_x + iv^2$  map respectively a smooth solution of the defocussing mKdV to a solution of the real-valued

KdV equation and a smooth solution of the focussing mKdV equation to a solution of the complex valued KdV equation.

The Cauchy problem, in Sobolev spaces, associated with these equations has been extensively studied since the end of the eighties. The lowest Sobolev index reached for the well-posedness theory is  $s_u(KdV) = -3/4$  for the KdV equation (cf. [8] or [3]) and  $s_u(mKdV) = 1/4$  for the mKdV equations (cf. [7]). Note that the difference of one derivative between these two results agrees with the lost of one derivative in the Miura transforms. These indexes have been proved to be optimal if one requires moreover the solution-map to be uniformly continuous on bounded set (cf. [7], [1]). However, one may expect that the solution-map associated with these equations can be continuously extended below these indexes especially since they are far above the critical Sobolev indexes for the dilatation symetrie that are respectively  $s_d(KdV) = -3/2$  and  $s_d(mKdV) = -1/2$ . It is worth emphasizing that such results have been proved in the periodic case by Kappeler and Topalov (cf. [5], [4]). More precisely, they proved the existence of such continuous extensions in  $H^{-1}(\mathbb{T})$  for KdV and  $L^2(\mathbb{T})$  for mKdV. Note that, in the periodic setting, the associated flow-maps are known to be non uniformly continuous on bounded sets for respectively  $s < -1/2$  and  $s < 1/2$ .

In this note we combine the Miura transforms and the so-called local Kato smoothing effect to prove that at least one of the solution-map associated with theses equations cannot be continuously extended till its associated critical index  $s_d$ . More precisely we prove the following alternative on the Cauchy problem associated with these equations.

**Theorem 1.1.** *One at least of the following assertions is true :*

- i) Let  $s < 0$  be given. There exists no  $T > 0$  such that the solution map  $u_0 \mapsto u$  associated with the defocussing mKdV is continuous at the origin from  $H^\infty(\mathbb{R})$ , equipped with the  $H^s(\mathbb{R})$ -topology, into  $\mathcal{D}'(]0, T[ \times \mathbb{R})$ .*
- ii) Let  $\gamma \geq 0$  and  $s < -1$  be given. There exists no  $T > 0$  and no  $R > 0$  such that the solution map associated with the (real-valued) KdV equation is continuous from the ball  $B(0, R)$  of  $H^\gamma(\mathbb{R})$ , equipped with the  $H^s(R)$  topology, into  $\mathcal{D}'(]0, T[ \times \mathbb{R})$ .*

*In the same way, one at least of the following assertions is true :*

- iii) Let  $s < 0$  be given. There exists no  $T > 0$  such that the solution map  $u_0 \mapsto u$  associated with the focussing mKdV is continuous at the origin from  $H^s(\mathbb{R})$  into  $\mathcal{D}'(]0, T[ \times \mathbb{R})$ .*

*iv) Let  $\gamma \geq 0$  and  $s < -1$  be given . There exists no  $T > 0$  and no  $R > 0$  such that the solution map associated with the complex-valued KdV equation is continuous from the ball  $B(0, R)$  of  $H^\gamma(\mathbb{R})$ , equipped with the  $H^s(\mathbb{R})$  topology, into  $\mathcal{D}'(]0, T[ \times \mathbb{R})$ .*

## 2 Proof.

The idea of the proof is the following: The Miura transformation being discontinuous from  $H^s(\mathbb{R})$  into  $H^{s-1}(\mathbb{R})$  as soon as  $s < 0$ , we can find a sequence of initial data, bounded in  $L^2(\mathbb{R})$ , that converges to 0 in  $H^s(\mathbb{R})$  but which Miura transform converges in  $H^{s-1}(\mathbb{R})$  to some non identically vanishing element of  $H^\infty(\mathbb{R})$ . On the other hand, thanks to the Kato smoothing effect, the Miura transformation is continuous for the  $\mathcal{D}'(]0, T[ \times \mathbb{R})$ -topology on sequences of solutions to (1.2) that emanate from sequences of initial data that are bounded in  $L^2(\mathbb{R})$ . Assuming that *i)* (resp. *iii)* ) does not hold, this will ensure that *ii)* (resp. *iv)*) holds.

Let  $\{h_N\} \subset H^\infty(\mathbb{R})$  be the sequence defined by

$$\hat{h}_N(\xi) = \chi_{[N, N+1]}(|\xi|).$$

Clearly  $\{h_N\}$  is bounded in  $L^2(\mathbb{R})$  and tends to 0 in  $H^s(\mathbb{R})$  for any  $s < 0$ . Let us fix  $0 < \varepsilon < 1$ .

Straightforward calculations lead to

$$\Phi_d(\varepsilon h_N) = \varepsilon^2 \theta + \varepsilon h'_N + \varepsilon^2 \alpha_N \quad \text{and} \quad \Phi_f(\varepsilon h_N) = i\varepsilon^2 \theta + \varepsilon h'_N + i\varepsilon^2 \alpha_N,$$

where

$$\begin{aligned} \hat{\theta}(\xi) &= (1 - |\xi|) \chi_{[0,1]}(|\xi|) \\ \text{and } \hat{\alpha}_N(\xi) &= \left(1 - \left| |\xi| - 2N - 1 \right| \right) \chi_{[0,1]}(|\xi| - 2N - 1). \end{aligned}$$

Clearly  $\theta \in H^\infty(\mathbb{R})$  with  $\|\theta\|_{H^\gamma} \lesssim 1$  for any  $\gamma \in \mathbb{R}$ .  $\alpha_N \rightarrow 0$  in  $H^{-1}(\mathbb{R})$  and  $h'_N \rightarrow 0$  in  $H^s(\mathbb{R})$  for  $s < -1$ . In particular, for any  $s < -1$ ,

$$\Phi_d(\varepsilon h_N) \rightarrow \varepsilon^2 \theta \quad \text{and} \quad \Phi_f(\varepsilon h_N) \rightarrow \varepsilon^2 \theta \quad \text{in } H^s(\mathbb{R}) \quad \text{as } N \rightarrow +\infty. \quad (2.1)$$

Since  $h_N \in H^\infty(\mathbb{R})$  it follows from classical well-posedness results for mKdV (see for instance [2]) that the solution  $u_N := u(\cdot, \varepsilon h_N)$  of (1.2) emanating from  $\varepsilon h_N$  exists for all times and belongs to  $C(\mathbb{R}; H^\infty(\mathbb{R}))$ . Moreover, it is well known that the  $L^2$ -norm of the solution is a constant of the motion and that the so-called Kato smoothing effect holds. For sake of completeness we give hereafter a version of this smoothing effect that is suitable for our purpose (see for instance [2] for a general setting).

**Proposition 2.1.** *Let  $u_0 \in H^\infty(\mathbb{R})$ . Then for any  $T > 0$  and  $R > 0$  there exists  $C(T, R) > 0$  such that the emanating solution of (1.2) satisfies*

$$\int_{-T}^T \int_{\mathbb{R}} |\partial_x u(t, x)|^2 dx dt \leq C(R, T) (\|u_0\|_{L^2}^2 + \|u_0\|_{L^2}^4)$$

*Proof.* Let  $\phi \in C_0^\infty([-2, 2])$  with  $\phi \equiv 1$  on  $[-1, 1]$  and  $0 \leq \phi \leq 1$  on  $[-2, 2]$ . For  $R > 1$  we set  $h'(x) := \phi(x/R)$  and  $h(x) := \int_{-\infty}^x h'(t) dt$ . Integrating (1.2) against  $hu$  one obtains

$$\frac{d}{dt} \int_{\mathbb{R}} hu^2 + 3 \int_{\mathbb{R}} h'(u_x)^2 - \int_{\mathbb{R}} h''' u^2 \mp \int_{\mathbb{R}} h' u^4 = 0$$

Using that  $h'$  has compact support and Schwarz inequality, we get

$$\begin{aligned} \|h'u^2\|_{L^\infty} &\leq \int_{\mathbb{R}} |(h'u^2)'| \leq \int_{\mathbb{R}} |h''u^2| + 2 \int_{\mathbb{R}} |h'uu_x| \\ &\leq C \|u\|_{L^2}^2 + \|u\|_{L^2} \left( \int_{\mathbb{R}} h'(u_x)^2 \right)^{1/2} \end{aligned}$$

Hence, by Schwarz and Young inequalities,

$$\left| \int_{\mathbb{R}} h'u^4 \right| \leq \|u\|_{L^2}^2 \|h'u^2\|_{L^\infty} \leq C \|u\|_{L^2}^4 + 2 \int_{\mathbb{R}} h'(u_x)^2$$

and, since the  $L^2$ -norm is conserved by the flow of (1.2), we get

$$\frac{d}{dt} \int_{\mathbb{R}} hu^2 + \int_{\mathbb{R}} h'(u_x)^2 \leq C (\|u_0\|_{L^2}^2 + \|u_0\|_{L^2}^4).$$

Integrating this inequality in  $(-T, T)$  and using the properties of  $h'$ , we thus infer that

$$\left| \int_{-T}^T \int_{-R}^R (u_x)^2 \right| \leq \left| \int_{-T}^T \int_{\mathbb{R}} h'(u_x)^2 \right| \leq C(T+R) (\|u_0\|_{L^2}^2 + \|u_0\|_{L^2}^4).$$

□

Let us now assume that the assertion *i*) (respectively *iii*) of Theorem 1.1 is false, i.e. there exists  $s < 0$  and  $T > 0$  such that the solution map  $u_0 \mapsto u$  associated with the defocussing mKdV is continuous at the origin from  $H^\infty(\mathbb{R})$ , equipped with the  $H^s(\mathbb{R})$ -topology, into  $\mathcal{D}'(]0, T[ \times \mathbb{R})$ . We claim that  $\Phi_d(u_N) \rightharpoonup 0$  (respectively  $\Phi_f(u_N) \rightharpoonup 0$ ) in  $L^\infty(0, T; H^{-1}(\mathbb{R}))$ . Indeed, first by the conservation of the  $L^2$ -norm,  $\{u_N\}$

is bounded in  $L^\infty(0, T; L^2(\mathbb{R}))$ . Second, by the equation,  $\{\partial_t u_N\}$  is bounded in  $L^\infty(0, T; H^{-3}(\mathbb{R}))$ . Third, from Proposition 2.1,  $\{\partial_x u_N\}$  is bounded in  $L^2_{\text{loc}}(] - T, T[ \times \mathbb{R})$ . It thus follows from Aubin-Lions compactness theorem that there exists  $u \in L^\infty(0, T; H^{-1}) \cap L^2_{\text{loc}}(]0, T[ \times \mathbb{R})$  and an increasing sequence of integers  $\{N_k\}$  such that

$$\begin{aligned} u_{N_k} &\rightarrow u && \text{a.e. in } ]0, T[ \times \mathbb{R}, \\ \partial_x u_{N_k} &\rightharpoonup \partial_x u && \text{in } L^\infty(0, T; H^{-1}(\mathbb{R})), \\ u_{N_k}^2 &\rightharpoonup u^2 && \text{in } L^\infty(0, T; H^{-1}(\mathbb{R})). \end{aligned}$$

On the other hand, since we assumed that  $u_0 \mapsto u$  is continuous at the origin from  $H^\infty(\mathbb{R})$  equipped with the  $H^s(\mathbb{R})$ -topology into  $\mathcal{D}'(]0, T[ \times \mathbb{R})$ , we must have that  $u_N \rightarrow 0$  in  $\mathcal{D}'(]0, T[ \times \mathbb{R})$ . This ensures that  $u = 0$  and that actually the whole sequence  $\{u_N\}$  (and not only a subsequence) converges to 0 in all the senses above. This shows that

$$\Phi_d(u_N) \rightharpoonup 0 \text{ (respectively } \Phi_f(u_N) \rightharpoonup 0) \text{ in } L^\infty(0, T; H^{-1}(\mathbb{R})) \quad (2.2)$$

as claimed.

*Conclusion :* Let us fixed  $s < -1$  and  $\varepsilon > 0$ . From (2.1),  $\Phi_d(\varepsilon h_N) \rightarrow \varepsilon^2 \theta \neq 0$  in  $H^s(\mathbb{R})$  and the solution  $v(\cdot, \varepsilon^2 \theta)$  to (1.1) emanating from  $\varepsilon^2 \theta$  satisfies  $\|v(t, \varepsilon^2 \theta)\|_{L^2} = \varepsilon^2 \|\theta\|_{L^2} = \varepsilon^2 \sqrt{2/3}$  for all  $t \geq 0$ . On the other hand, assuming that assertion *i*) is false, (2.2) shows that the sequence of solutions  $\{v(\cdot, \Phi_d(\varepsilon h_N))\} = \{\Phi_d(u(\cdot, \varepsilon h_N))\}$  of KdV emanating from  $\{\Phi_d(\varepsilon h_N)\}$  tends weakly to 0 in  $L^\infty(0, T; H^{-1}(\mathbb{R}))$  for some  $T > 0$ . Hence, the flow-map associated with KdV is not continuous at  $\varepsilon^2 \theta$  from  $H^\infty(\mathbb{R})$  equipped with the  $H^s(\mathbb{R})$ -topology into  $\mathcal{D}'(]0, T[ \times \mathbb{R})$  and thus assertion *ii*) holds.

Finally, the fact that one at least of the assertions *iii*) and *iv*) holds, follows exactly in the same way.

**Acknowledgements:** L.M. was partially supported by the ANR project "Equa-Disp". He would like to thank Didier Pilod for his careful reading of a preliminary version of this work.

## References

- [1] M. Christ, J. Colliander, and T. Tao, *Asymptotics, frequency modulation and low regularity ill-posedness for canonical defocusing equations*, Amer. J. Math., **125** (2003), no. 6, 1235-1293.

- [2] J. Ginibre, Y. Tsutsumi and G. Velo, *Existence and uniqueness of solutions for the generalized Korteweg de Vries equation*, Math. Z., **203** (1990), no. 1, 9–36
- [3] Z. Guo, *Global well-posedness of Korteweg-de Vries equation in  $H^{-3/4}(\mathbb{R})$* , J. Math. Pures Appl., **91** (2009), 583–597.
- [4] T. Kappeler and P. Topalov, *Global well-posedness of mKdV in  $L^2(\mathbb{T}, \mathbb{R})$* , Comm. Partial Differential Equations **30** (2005), no. 1-3, 435–449.
- [5] T. Kappeler and P. Topalov, *Global wellposedness of KdV in  $H^{-1}(\mathbb{T}, \mathbb{R})$* , Duke Math. J. **135** (2006), no. 2, 327–360.
- [6] C. E. Kenig, G. Ponce, and L. Vega, *Well-posedness and scattering results for the generalized Korteweg-de Vries equation via the contraction principle*, Comm. Pure Appl. Math. **46** (1993), no. 4, 527–620.
- [7] C. E. Kenig, G. Ponce, and L. Vega, *On the ill-posedness of some canonical dispersive equations*, Duke Math. J., **106**, no. 3 (2001), 617–633.
- [8] N. Kishimoto, *Well-posedness of the Cauchy problem for the Korteweg-de Vries equation at the critical regularity*, Differential Integral Equations **22** (2009), no. 5-6, 447–464.