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Stella Krell

FINITE VOLUME METHOD FOR GENERAL MULTIFLUID FLOWS GOVERNED BY THE INTERFACE STOKES PROBLEM

STELLA KRELL*

Abstract. We study the approximation of solutions to the stationary Stokes problem with a piecewise constant viscosity coefficient (interface Stokes problem) in the discrete duality finite volume (DDFV) framework. In order to take into account the discontinuities and to prevent consistency defect in the scheme, we propose to modify the definition of the numerical fluxes on the edges of the mesh where the discontinuity occurs. We first show how to design our new scheme, called m-DDFV, and we analyze the well-posedness of the scheme and its convergence properties. Finally, we provide numerical results which confirm that the m-DDFV scheme significantly improves the convergence rate of the usual DDFV method for Stokes problems with discontinuous viscosity.

Key words. Interface Stokes problem, discontinuous coefficients, DDFV methods.

1. Introduction.

In many numerical simulations the two phases are modeled by a single set of conservation laws for the whole computational domain. Such an approach leads to Navier-Stokes equations with discontinuous density and viscosity coefficients. For highly viscous flows then the Stokes equations with discontinuous viscosity are a reasonable model problem.

1.1. Stokes model.

In this paper, we are concerned with the finite volume approximation of solutions to the steady interface Stokes problem with homogeneous Dirichlet boundary conditions: Find a velocity $\mathbf{u} : \Omega \to \mathbb{R}^2$ and a pressure $p : \Omega \to \mathbb{R}$ such that:

$$\begin{aligned} \operatorname{div}(-2\eta \mathrm{D}\mathbf{u} + p\mathrm{Id}) = \mathbf{f}, \quad \operatorname{div}(\mathbf{u}) = 0, \text{ in } \Omega_i, \text{ for } i = 1,2, \quad \textstyle\int_\Omega p(x)\mathrm{d}x = 0, \\ \mathbf{u} = 0, \text{ on } \partial\Omega, \quad [\mathbf{u}] = 0, \quad [2\eta \mathrm{D}\mathbf{u}\vec{\mathbf{n}} - p\vec{\mathbf{n}}] = 0, \text{ on } \Gamma, \end{aligned} \tag{1.1}$$

where Ω is a polygonal connected open bounded subset of $\mathbb{R}^2$, $\mathrm{D}\mathbf{u} = \frac{1}{2}(\nabla\mathbf{u} + {}^t\nabla\mathbf{u})$, $\mathbf{f}$ is a function in $(L^2(\Omega))^2$, a piecewise constant viscosity η, equal to $\eta_i > 0$ on Ω_i, for $i = 1, 2$. The sub-domains Ω_1, Ω_2 are assumed to be Lipschitz domains such that $\Omega_1 \cap \Omega_2 = \emptyset$ and $\overline{\Omega} = \overline{\Omega_1} \cup \overline{\Omega_2}$. By Γ, we denote the interface between the sub-domains $\Gamma = \partial\Omega_1 \cap \partial\Omega_2$ and is a closed C^3 curve. Others notation are standard, $\vec{\mathbf{n}}$ is an unit normal vector to Γ from Ω_1 to Ω_2 and $[a]_{|\Gamma} = (a_{|\Omega_1} - a_{|\Omega_2})_{|\Gamma}$ denoted the jump of a across Γ.

1.1.1. Regularity of solution.

We note $\underline{\mathrm{C}}_\eta$=$\min(\eta_1, \eta_2)$ and $\overline{\mathrm{C}}_\eta$=$\max(\eta_1, \eta_2)$, we get

$$0 < \underline{\mathrm{C}}_\eta \leq \eta(x) \leq \overline{\mathrm{C}}_\eta, \quad \text{for a.e. } x \in \Omega. \tag{1.2}$$

The well-posedness of the problem (1.1) is studied in [4, 14, 25] with a constant viscosity and in [23] with a piecewise constant viscosity. In order to study the rates of convergence of our approximate solution, we need to make some assumptions on the regularity of a solution $(\mathbf{u}, p)$. Firstly, if Γ is a closed C^3 curve and Ω is a convex polygon, we have $\mathbf{u}_{|\Omega_i} \in (H^2(\Omega_i))^2$ and $p_{|\Omega_i} \in H^1(\Omega_i)$, for $i = 1, 2$. On the other hand, we cannot expect this regularity, when $\partial\Omega \cap \bar{\Gamma} \neq \emptyset$ and the maximum interior angle of $\partial\Omega_1$, $\partial\Omega_2$ at $\partial\Omega \cap \bar{\Gamma}$ is large enough. For more details of these facts, we refer to [22]. Since the viscosity is discontinuous, the pressure can have jumps. More precisely, we have $[p] = [2\eta \mathrm{D}\mathbf{u}\vec{\mathbf{n}} \cdot \vec{\mathbf{n}}]$, on Γ in [18]. We assume that $(\mathbf{u}, p)$ the solution of the problem (1.1) lies in $(H^2(\omega))^2 \times H^1(\omega)$, that is:

$$(\mathbf{u}, p) \in (H^1(\Omega))^2 \times L^2(\Omega), \quad \mathbf{u}_{|\Omega_i} \in (H^2(\Omega_i))^2, \; p_{|\Omega_i} \in H^1(\Omega_i), \text{ for } i = 1, 2. \tag{1.3}$$

*Université de Provence, Laboratoire d'Analyse, Topologie et Probabilités, 39 rue F. Joliot-Curie, 13453 Marseille Cedex 13, France (krell@cmi.univ-mrs.fr)

1.1.2. Notation. We denote by $\mathcal{M}_{m,n}(\mathbb{R})$ the set of real $m \times n$ matrices (we note $\mathcal{M}_n(\mathbb{R})$ when $m = n$). In the sequel, $\|\cdot\|_2$ stands for the natural L^2-norm when we consider scalar valued and vector valued functions and for the Frobenius norm when we consider matrix valued functions:

$$
\begin{aligned}
&\|\!|\xi|\!\|_2^2 = \int_\Omega \|\!|\xi(x)|\!\|_{\mathcal{F}}^2 \mathrm{d}x, \text{ with } \|\!|\xi|\!\|_{\mathcal{F}}^2 = (\xi : \xi), \quad \forall \xi \in L^2(\Omega, \mathcal{M}_2(\mathbb{R})),\\
&\|\mathbf{v}\|_{(H^2(\omega))^2}^2 = \|\mathbf{v}\|_{H^1(\Omega)}^2 + \|\nabla^2 \mathbf{v}\|_{L^2(\Omega_1)}^2 + \|\nabla^2 \mathbf{v}\|_{L^2(\Omega_2)}^2, \quad \forall \mathbf{v} \in (H^2(\omega))^2,\\
&\|q\|_{H^1(\omega)}^2 = \|q\|_{L^2(\Omega)}^2 + \|\nabla q\|_{L^2(\Omega_1)}^2 + \|\nabla q\|_{L^2(\Omega_2)}^2, \quad \forall q \in H^1(\omega),
\end{aligned}
$$

where $(\xi : \widetilde{\xi}) = \sum_{1 \le i,j \le 2} \xi_{i,j} \widetilde{\xi}_{i,j} = \mathrm{Tr}({}^t\xi \widetilde{\xi}), \quad \forall \xi, \widetilde{\xi} \in \mathcal{M}_2(\mathbb{R})$.

REMARK 1.1. *The matrix norm $\|\!|\cdot|\!\|_{\mathcal{F}}$ satisfies $\left\|\!\left| \frac{A + {}^tA}{2} \right|\!\right\|_{\mathcal{F}} \le \|\!|A|\!\|_{\mathcal{F}}$, for all $A \in \mathcal{M}_2(\mathbb{R})$.*

1.2. The DDFV method. Finite volume approximation of Stokes problems with constant viscosity on the whole domain is a current research topic, we refer to [7, 10, 13] for the description and the analysis of the main available schemes up to now. We propose here a staggered method: the discrete unknowns, the components of the velocity and the pressure, are located on different nodes. The most celebrated staggered scheme is the MAC scheme [15, 21] on cartesian grids. Actually, for a cartesian grid and constant viscosity, the scheme we propose here is equivalent (except on the boundary) to two uncoupled MAC schemes written on two different staggered meshes.

The presence of the symmetric part of the gradient $\mathrm{D}\mathbf{u}$ imposes to address the problem of the reconstruction of the full velocity gradient on the whole domain. Different methods of gradient reconstruction for cell-centered finite volume have been proposed since the last ten years, one can refer to [6], [9], [12] and [8, 17]. In all cases, the crucial feature is that the summation-by-parts procedure permits to reconstruct the whole two dimensional discrete gradient, starting from two point finite differences. Many of them have been compared in the benchmark of the FVCA5 conference [16], for scalar diffusion problems.

We consider here the class of finite volume schemes called "Discrete Duality Finite Volume" (DDFV for short). The DDFV method reconstitutes gradients from finite differences in two independent directions. Therefore, two finite volumes meshes are needed, noted $\mathcal{T}$. The second mesh of dual control volumes is built around the vertices of the primal mesh. Thus the dual mesh is thorough defined thanks the primal control volumes and their "centers". The primal mesh and the dual one play a symmetric role. Finally, these two meshes give a new mesh called diamond mesh $\mathfrak{D}$, on which the discrete gradient operator $\nabla^{\mathfrak{D}}$ is computed. Then, a discrete divergence operator $\mathbf{div}^{\boldsymbol{\mathcal{T}}}$ is built in order to be in duality with the discrete gradient in a discrete sense (see Theorem 4.3).

1.2.1. Previous work. Let us comeback to the Stokes problem. Our strategy is to approximate the velocity $\mathbf{u}$ on the mesh $\mathcal{T}$, denoted $\mathbf{u}^{\boldsymbol{\mathcal{T}}}$ and the pressure on the diamond mesh $\mathfrak{D}$, denoted $p^{\mathfrak{D}}$. The principle to obtain the DDFV scheme in [19] is the following. We integrate the momentum conservation law of the problem (1.1) on the primal mesh $\mathfrak{M}$ and the interior dual mesh $\mathfrak{M}^*$. The mass conservation equation is directly approached on the diamond mesh using the discrete operator $\mathrm{div}^{\mathfrak{D}}$ and a stabilized term $-\lambda h_{\mathfrak{D}}^2 \Delta^{\mathfrak{D}} p^{\mathfrak{D}}$ inspired by the well known Brezzi-Pitkäranta method [5]. The velocity is imposed to be equal to zero on boundary domain, that is denoted by $\mathbf{u}^{\boldsymbol{\mathcal{T}}} \in \mathbb{E}_0$. Finally, the integral of the pressure is

imposed to be equal to zero. The stabilized DDFV scheme can be written as follows:

$$\begin{array}{ll} \text{Find } \mathbf{u}^{\boldsymbol{\tau}} \in \mathbb{E}_0 \text{ and } p^{\mathfrak{D}} \in \mathbb{R}^{\mathfrak{D}} \text{ such that,} & \\ \mathbf{div}^{\mathfrak{M}}(-2\eta^{\mathfrak{D}}\mathrm{D}^{\mathfrak{D}}\mathbf{u}^{\boldsymbol{\tau}} + p^{\mathfrak{D}}\mathrm{Id}) = \mathbf{f}^{\mathfrak{M}}, & \mathbf{div}^{\mathfrak{M}^*}(-2\eta^{\mathfrak{D}}\mathrm{D}^{\mathfrak{D}}\mathbf{u}^{\boldsymbol{\tau}} + p^{\mathfrak{D}}\mathrm{Id}) = \mathbf{f}^{\mathfrak{M}^*}, \\ \mathrm{div}^{\mathfrak{D}}(\mathbf{u}^{\boldsymbol{\tau}}) - \lambda h_{\mathfrak{D}}^2 \Delta^{\mathfrak{D}} p^{\mathfrak{D}} = 0, & \sum_{\mathcal{D}\in\mathfrak{D}} m_{\mathcal{D}} p^{\mathcal{D}} = 0, \end{array} \tag{1.4}$$

where $\lambda > 0$ given. This stabilized DDFV scheme is proved to be well-posed for general 2D meshes. Hence, we succeeded in showing the convergence of such schemes and error estimates in the case where the viscosity and the exact solution are assumed to be smooth.

THEOREM 1.1. *We assume that the solution* $(\mathbf{u}, p)$ *of the Stokes problem* (1.1) *belongs to* $(H^2(\Omega))^2 \times H^1(\Omega)$ *and that* η *belongs to* $C^2(\bar{\Omega})$. *Let* $(\mathbf{u}^{\boldsymbol{\tau}}, p^{\mathfrak{D}}) \in \mathbb{E}_0 \times \mathbb{R}^{\mathfrak{D}}$ *be the solution of the scheme* (1.4). *There exists a constant* $C > 0$ *depending only on the regularity of the mesh, on the viscosity and of the couple* $(\mathbf{u}, p)$, *such that:*

$$\|\mathbf{u} - \mathbf{u}^{\boldsymbol{\tau}}\|_2 + \|\!|\nabla \mathbf{u} - \nabla^{\mathfrak{D}}\mathbf{u}^{\boldsymbol{\tau}}|\!\|_2 \leq C\mathrm{size}(\mathcal{T}) \quad \textit{and} \quad \|p - p^{\mathfrak{D}}\|_2 \leq C\mathrm{size}(\mathcal{T}).$$

We have extended this framework in 3D in [20]. In the case where η has discontinuous coefficients, our results in [19] show that the scheme is still convergent but the error analysis is no more valid.

1.2.2. Consideration of the discontinuities. Actually, it is known that such discontinuities in the coefficients imply a consistency defect in the numerical fluxes of usual finite volume schemes. It is needed to modify the scheme in order to take into account the jumps of the coefficients of the problem and then to recover a better convergence rate. As in in the scalar case [3], we need to introduce a new gradient operator and finally define a new approximate viscous stress tensor on each diamond cell. The new gradient operator $\nabla_{\mathfrak{Q}}^{\mathcal{N}}$ we propose to consider is built upon the usual DDFV gradient $\nabla^{\mathfrak{D}}$. It is chosen to be constant on all the quarter diamond cells $\mathcal{Q} \in \mathfrak{Q}$. Thanks to the modified discrete gradient, we can define a new symmetric operator a modified discrete strain rate tensor $\mathrm{D}_{\mathfrak{Q}}^{\mathcal{N}}$ as follows $\mathrm{D}_{\mathcal{Q}}^{\mathcal{N}}\mathbf{u}^{\boldsymbol{\tau}} = \frac{1}{2}\left(\nabla_{\mathcal{Q}}^{\mathcal{N}}\mathbf{u}^{\boldsymbol{\tau}} + {}^t(\nabla_{\mathcal{Q}}^{\mathcal{N}}\mathbf{u}^{\boldsymbol{\tau}})\right)$. We also introduce an artificial pressure unknowns on the quarter diamond cells, denoted $p^{\mathfrak{Q}}$, that will depend on the symmetric part of the velocity gradient $\mathrm{D}^{\mathfrak{D}}\mathbf{u}^{\boldsymbol{\tau}}$ and the pressure unknowns $p^{\mathfrak{D}}$. Then, we propose a modified DDFV scheme -that we called m-DDFV. The only difference with the scheme (1.4) introduced in [19] is in the viscous stress tensor and the stabilization term, to take into account the jumps of the pressure. We replace $\eta^{\mathfrak{D}}\mathrm{D}^{\mathfrak{D}}\mathbf{u}^{\boldsymbol{\tau}}$ (respectively $-\lambda h_{\mathfrak{D}}^2 \Delta^{\mathfrak{D}} p^{\mathfrak{D}}$) by a modified viscous stress tensor, denoted by $\mathrm{D}_{\mathfrak{D}}^{\eta,\mathcal{N}}\mathbf{u}^{\boldsymbol{\tau}}$, (respectively $-\lambda h_{\mathfrak{D}}^2 \underline{\Delta}^{\mathfrak{D}} p^{\mathfrak{Q}}$) as follows:

$$\begin{array}{ll} \text{Find } \mathbf{u}^{\boldsymbol{\tau}} \in \mathbb{E}_0 \text{ and } p^{\mathfrak{D}} \in \mathbb{R}^{\mathfrak{D}} \text{ such that,} & \\ \mathbf{div}^{\mathfrak{M}}(-2\mathrm{D}_{\mathfrak{D}}^{\eta,\mathcal{N}}\mathbf{u}^{\boldsymbol{\tau}} + p^{\mathfrak{D}}\mathrm{Id}) = \mathbf{f}^{\mathfrak{M}}, & \mathbf{div}^{\mathfrak{M}^*}(-2\mathrm{D}_{\mathfrak{D}}^{\eta,\mathcal{N}}\mathbf{u}^{\boldsymbol{\tau}} + p^{\mathfrak{D}}\mathrm{Id}) = \mathbf{f}^{\mathfrak{M}^*}, \\ \mathrm{div}^{\mathfrak{D}}(\mathbf{u}^{\boldsymbol{\tau}}) - \lambda h_{\mathfrak{D}}^2 \underline{\Delta}^{\mathfrak{D}} p^{\mathfrak{Q}} = 0, & \sum_{\mathcal{D}\in\mathfrak{D}} m_{\mathcal{D}} p^{\mathcal{D}} = 0. \end{array} \tag{1.5}$$

This m-DDFV scheme is proved to be well-posed provided a new discrete Korn inequality for the modified discrete operators (see Theorem 4.2) which is not just an extension of the discrete Korn inequality proved in [19].

The aim of this work is to present a modified DDFV scheme which recovers the first order convergence even if the viscosity is discontinuous. We provide an error estimate in case where the exact solution of the problem (1.1) lies in the space $(H^2(\omega))^2 \times H^1(\omega)$, we prove a first order convergence of the scheme (1.5) in the L^2-norm for the velocity gradient, as well as for the velocity and for the pressure. Hence, our analysis provides a theoretical confirmation of the behavior numerically observed in [19].

1.3. Outline. This paper is organized as follows. In Section 2, we recall the DDFV framework for the finite volume approximation of Stokes problems on unstructured grids. In Subsection 2.4, we introduce the modified new discrete operators. In Section 3, we describe the m-DDFV stabilized scheme. In Section 4, we present the main results of discrete functional analysis necessary for the theoretical study of the finite volume method. These results include properties of discrete operators proved in [19] but also properties of the modified discrete operators, including a new discrete Korn inequality (see Theorem 4.2). In Section 5, we prove the stability and well-posedness of the scheme. Then, in Section 6, we prove error estimates in the case where the exact solution lies in $(H^2(\omega))^2 \times H^1(\omega)$ (see Theorem 6.1). Finally, in Section 7, theoretical error estimates are illustrated with numerical results.

2. The DDFV framework.

2.1. The meshes and notation.

The meshes. We recall here the main notation and definitions taken from [1]. A DDFV mesh $\mathcal{T}$ is constituted by a primal mesh $\mathfrak{M} \cup \partial\mathfrak{M}$ and a dual mesh $\mathfrak{M}^* \cup \partial\mathfrak{M}^*$. An example for square locally refined primal mesh is on Figure 2.1.

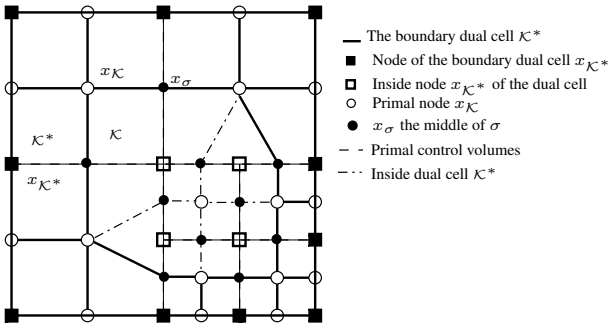


FIG. 2.1. *The mesh $\mathcal{T}$.*

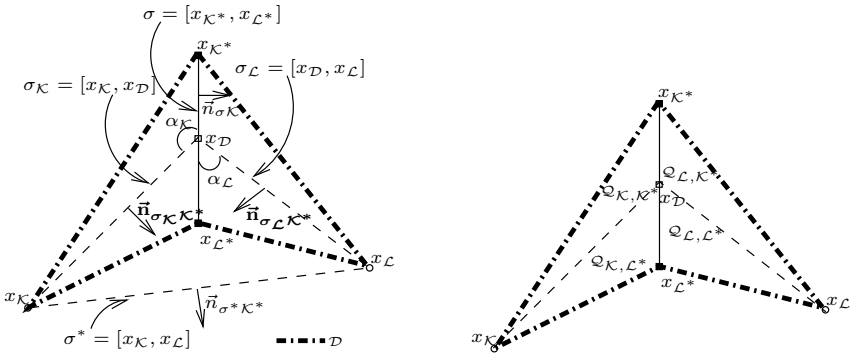
The primal mesh $\mathfrak{M}$ is a set of disjoint open polygonal control volumes $\kappa \subset \Omega$ such that $\cup\overline{\kappa} = \overline{\Omega}$. We denote by $\partial\mathfrak{M}$ the set of edges of the control volumes in $\mathfrak{M}$ included in $\partial\Omega$, which we consider as degenerate control volumes. To each control volume and degenerate control volume $\kappa \in \mathfrak{M} \cup \partial\mathfrak{M}$, we associate a point x_κ. For each degenerate control volume $\kappa \in \partial\mathfrak{M}$, we choose the point x_κ equal to the midpoint of the control volume κ. This family of points is denoted by $X = \{x_\kappa,\ \kappa \in \mathfrak{M} \cup \partial\mathfrak{M}\}$.

Let X^* denote the set of the vertices of the primal control volumes in $\mathfrak{M}$ that we split into $X^* = X^*_{int} \cup X^*_{ext}$ where $X^*_{int} \cap \partial\Omega = \emptyset$ and $X^*_{ext} \subset \partial\Omega$. With any point $x_{\kappa^*} \in X^*_{int}$ (resp. $x_{\kappa^*} \in X^*_{ext}$), we associate the polygon $\kappa^* \in \mathfrak{M}^*$ (resp. $\kappa^* \in \partial\mathfrak{M}^*$) whose sides are $\{[x_\kappa, x_\sigma]$ such that $x_\kappa \in X, x_{\kappa^*} \in \overline{\kappa} \cap \overline{\sigma},\ \kappa \in \mathfrak{M},\ \sigma \in \mathcal{E}\}$ (resp. $\{[x_{\kappa^*}, x_\sigma]$ such that $\sigma \in \partial\mathfrak{M}$ and $x_{\kappa^*} \in \overline{\sigma}\} \cup \{[x_\kappa, x_\sigma]$ such that $x_\kappa \in X, x_{\kappa^*} \in \overline{\kappa} \cap \overline{\sigma},\ \kappa \in \mathfrak{M},\ \sigma \in \mathcal{E}\}$) sorted with respect to the clockwise order of the corresponding control volumes. This defines the set $\mathfrak{M}^* \cup \partial\mathfrak{M}^*$ of dual control volumes. It is usually called the *barycentric dual mesh*.

REMARK 2.1. *Remark that our dual control volumes differ from the classic one proposed in [8] or [1, 3, 19]. The classic dual cells are build by joining only the centers x_κ associated to the elements of the primal mesh of which x_{κ^*} is a vertex. Barycentric dual cells never overlap, that is not always the case for classic dual cells.*

For all control volumes κ and $\mathcal{L}$, we assume that $\partial\kappa \cap \partial\mathcal{L}$ is either empty or a common vertex or an edge of the primal mesh denoted by $\sigma = \kappa|\mathcal{L}$. We note by $\mathcal{E}$ the set of such edges. We also note $\sigma^* = \kappa^*|\mathcal{L}^*$ and $\mathcal{E}^*$ for the corresponding dual definitions.

Given the primal and dual control volumes, we define the diamond cells $\mathcal{D}_{\sigma,\sigma^*}$ being the quadrangles whose diagonals are a primal edge $\sigma = \mathcal{K}|\mathcal{L} = (x_{\mathcal{K}^*}, x_{\mathcal{L}^*})$ and a corresponding dual edge $\sigma^* = \mathcal{K}^*|\mathcal{L}^* = (x_{\mathcal{K}}, x_{\mathcal{L}})$, (see Fig. 2.2(a)). Note that the diamond cells are not necessarily convex. If $\sigma \in \mathcal{E} \cap \partial\overline{\Omega}$, the quadrangle $\mathcal{D}_{\sigma,\sigma^*}$ degenerates into a triangle. The set of the diamond cells is denoted by $\mathfrak{D}$ and we have $\overline{\Omega} = \bigcup_{\mathcal{D}\in\mathfrak{D}} \overline{\mathcal{D}}$.



(a) Notation in the diamond cells. (b) Quarter diamond cells.

FIG. 2.2. *Diamond cells*

REMARK 2.2. *An important assumption for our analysis is that each DDFV mesh $\mathcal{T}$ is conforming with respect to the two sub-domains Ω_1, Ω_2 in the following sense:*

$$\exists\, \mathfrak{M}_i \subset \mathfrak{M} \ : \qquad \cup \{\mathcal{K}|\mathcal{K} \in \mathfrak{M}_i\} = \bar{\Omega}_i,\ i = 1,2$$
$$\exists\, (\mathfrak{M}^* \cup \partial\mathfrak{M}^*)_i \subset \mathfrak{M}^* \cup \partial\mathfrak{M}^* \ : \qquad \cup \{\mathcal{K}^*|\mathcal{K}^* \in (\mathfrak{M}^* \cup \partial\mathfrak{M}^*)_i\} = \bar{\Omega}_i,\ i = 1,2$$

This assumption is easily fulfilled if Ω_1 and Ω_2 are polyhedral sub-domains.

Notation. We recall here the main notation taken from [19]. For any primal control volume $\mathcal{K} \in \mathfrak{M} \cup \partial\mathfrak{M}$, we note $m_{\mathcal{K}}$ its Lebesgue measure, $d_{\mathcal{K}}$ its diameter, $\mathcal{E}_{\mathcal{K}}$ the set of its edges (if $\mathcal{K} \in \mathfrak{M}$), or the one-element set $\{\mathcal{K}\}$ if $\mathcal{K} \in \partial\mathfrak{M}$, $\mathfrak{D}_{\mathcal{K}} = \{\mathcal{D}_{\sigma,\sigma^*} \in \mathfrak{D},\ \sigma \in \mathcal{E}_{\mathcal{K}}\}$, $B_{\mathcal{K}} := B(x_{\mathcal{K}}, \rho_{\mathcal{K}}) \cap \partial\Omega \subset \overline{\kappa}$ the open ball of radius $\rho_{\mathcal{K}} > 0$ for $\mathcal{K} \in \partial\mathfrak{M}$, $m_{B_{\mathcal{K}}}$ its measure, the value $\rho_{\mathcal{K}}$ is chosen such that the inclusion is verified. We will also use corresponding dual notation for any dual cells $\mathcal{K}^* \in \mathfrak{M}^* \cup \partial\mathfrak{M}^*$: $m_{\mathcal{K}^*}$, $\mathcal{E}_{\mathcal{K}^*}$, $\mathfrak{D}_{\mathcal{K}^*}$, $d_{\mathcal{K}^*}$, $B_{\mathcal{K}^*}$, $m_{B_{\mathcal{K}^*}}$, $\rho_{\mathcal{K}^*}$.

For a diamond cell $\mathcal{D} = \mathcal{D}_{\sigma,\sigma^*}$ whose vertices are $(x_{\mathcal{K}}, x_{\mathcal{K}^*}, x_{\mathcal{L}}, x_{\mathcal{L}^*})$, we note $x_{\mathcal{D}}$ the center of the diamond cell $\mathcal{D}$, that is the middle point of the primal edge σ, $h_{\mathcal{D}}$ its diameter, m_σ the length of the primal edge σ, m_{σ^*} the length of the dual edge σ^*, $\vec{\mathbf{n}}_{\boldsymbol{\sigma\mathcal{K}}}$ the unit vector normal to σ oriented from $x_{\mathcal{K}}$ to $x_{\mathcal{L}}$, $\vec{\mathbf{n}}_{\boldsymbol{\sigma^*\mathcal{K}^*}}$ the unit vector normal to σ^* oriented from $x_{\mathcal{K}^*}$ to $x_{\mathcal{L}^*}$, $\vec{\boldsymbol{\tau}}_{\boldsymbol{\mathcal{K},\mathcal{L}}}$ the unit vector parallel to σ^* oriented from $x_{\mathcal{K}}$ to $x_{\mathcal{L}}$, $\vec{\boldsymbol{\tau}}_{\boldsymbol{\mathcal{K}^*,\mathcal{L}^*}}$ the unit vector parallel to σ oriented from $x_{\mathcal{K}^*}$ to $x_{\mathcal{L}^*}$, $\alpha_{\mathcal{D}}$ the angle between $\vec{\boldsymbol{\tau}}_{\boldsymbol{\mathcal{K},\mathcal{L}}}$ and $\vec{\boldsymbol{\tau}}_{\boldsymbol{\mathcal{K}^*,\mathcal{L}^*}}$, $m_{\mathcal{D}}$ its measure, equal to $\frac{1}{2}\sin(\alpha_{\mathcal{D}}) m_\sigma m_{\sigma^*}$, and $d_{\mathcal{K}^*,\mathcal{L}}$ (respectively $d_{\mathcal{L}^*,\mathcal{L}}$) the length between $x_{\mathcal{K}^*}$ (respectively $x_{\mathcal{L}^*}$) and $x_{\mathcal{L}}$ for any boundary degenerate diamond cell, $\mathfrak{s}$ its edges (for example $\mathfrak{s} = [x_{\mathcal{K}}, x_{\mathcal{K}^*}]$), $\mathcal{E}_{\mathcal{D}} = \{\mathfrak{s},\ \mathfrak{s} \in \partial\mathcal{D} \text{ and } \mathfrak{s} \not\subset \partial\Omega\}$ the set of interior edges of $\mathcal{D}$, $\mathfrak{S} = \{\mathfrak{s} \in \mathcal{E}_{\mathcal{D}},\ \forall\, \mathcal{D} \in \mathfrak{D}\}$ the set of interior edges of all diamond cells $\mathcal{D} \in \mathfrak{D}$, $m_{\mathfrak{s}}$ the length of a diamond edge $\mathfrak{s}$, $\vec{\mathbf{n}}_{\boldsymbol{\mathfrak{s}\mathcal{D}}}$ the unit vector normal to $\mathfrak{s} = \mathcal{D}|\mathcal{D}'$ oriented from $\mathcal{D}$ to $\mathcal{D}'$.

Since we use here the barycentric dual mesh, we introduce new notation in comparison to the notation in [19]. For a diamond cell $\mathcal{D} = \mathcal{D}_{\sigma,\sigma^*}$, we note $\sigma_{\mathcal{K}}$ (respectively $\sigma_{\mathcal{L}}$) the segment $[x_{\mathcal{K}}, x_{\mathcal{D}}]$ (respectively $[x_{\mathcal{D}}, x_{\mathcal{L}}]$), $\sigma_{\mathcal{K}^*}$ (respectively $\sigma_{\mathcal{L}^*}$) the segment $[x_{\mathcal{K}^*}, x_{\mathcal{D}}]$ (respectively $[x_{\mathcal{D}}, x_{\mathcal{L}^*}]$), $\vec{\mathbf{n}}_{\boldsymbol{\sigma_{\mathcal{K}}\mathcal{K}^*}}$ the unit vector normal to $\sigma_{\mathcal{K}}$ oriented from $x_{\mathcal{K}^*}$ to $x_{\mathcal{L}^*}$, $\vec{\mathbf{n}}_{\boldsymbol{\sigma_{\mathcal{L}}\mathcal{K}^*}}$ the unit vector normal to $\sigma_{\mathcal{L}}$ oriented from $x_{\mathcal{K}^*}$ to $x_{\mathcal{L}^*}$, $\vec{\boldsymbol{\tau}}_{\boldsymbol{\sigma_{\mathcal{K}}\mathcal{K}^*}}$ the unit vector parallel to $\sigma_{\mathcal{K}}$ oriented

from $x_{\mathcal{K}}$ to $x_{\mathcal{D}}$, $\vec{\boldsymbol{\tau}}_{\boldsymbol{\sigma_{\mathcal{L}}\mathcal{K}^*}}$ the unit vector parallel to $\sigma_{\mathcal{L}}$ oriented from $x_{\mathcal{D}}$ to $x_{\mathcal{L}}$, $\alpha_{\mathcal{K}}$ the angle between $\overrightarrow{x_{\mathcal{K}}x_{\mathcal{D}}}$ and $\vec{\boldsymbol{\tau}}_{\boldsymbol{\mathcal{K}^*,\mathcal{L}^*}}$, $\alpha_{\mathcal{L}}$ the angle between $\overrightarrow{x_{\mathcal{L}}x_{\mathcal{D}}}$ and $\vec{\boldsymbol{\tau}}_{\boldsymbol{\mathcal{K}^*,\mathcal{L}^*}}$, $m_{\sigma_{\mathcal{K}}}$ (respectively $m_{\sigma_{\mathcal{L}}}$) the length of $\sigma_{\mathcal{K}}$ (respectively $\sigma_{\mathcal{L}}$), $m_{\sigma_{\mathcal{K}^*}}$ (respectively $m_{\sigma_{\mathcal{L}^*}}$) the length of $\sigma_{\mathcal{K}^*}$ (respectively $\sigma_{\mathcal{L}^*}$), x_σ the middle point of the segment σ for each $\sigma \in \{\sigma_{\mathcal{K}}, \sigma_{\mathcal{L}}, \sigma_{\mathcal{K}^*}, \sigma_{\mathcal{L}^*}\}$.

For any $\mathcal{D} \in \mathfrak{D}$, we have $m_{\sigma^*}\vec{\mathbf{n}}_{\boldsymbol{\sigma^*\mathcal{K}^*}} = m_{\sigma_{\mathcal{K}}}\vec{\mathbf{n}}_{\boldsymbol{\sigma_{\mathcal{K}}\mathcal{K}^*}} + m_{\sigma_{\mathcal{L}}}\vec{\mathbf{n}}_{\boldsymbol{\sigma_{\mathcal{L}}\mathcal{K}^*}}$. In a diamond cell $\mathcal{D} \in \mathfrak{D}$, we have different direct orthonormal basis: $(\vec{\boldsymbol{\tau}}_{\boldsymbol{\mathcal{K}^*,\mathcal{L}^*}}, \vec{\mathbf{n}}_{\boldsymbol{\sigma\mathcal{K}}})$, $(\vec{\mathbf{n}}_{\boldsymbol{\sigma^*\mathcal{K}^*}}, \vec{\boldsymbol{\tau}}_{\boldsymbol{\mathcal{K},\mathcal{L}}})$, $(\vec{\mathbf{n}}_{\boldsymbol{\sigma_{\mathcal{K}}\mathcal{K}^*}}, \vec{\boldsymbol{\tau}}_{\boldsymbol{\sigma_{\mathcal{K}}\mathcal{K}^*}})$ and $(\vec{\mathbf{n}}_{\boldsymbol{\sigma_{\mathcal{L}}\mathcal{K}^*}}, \vec{\boldsymbol{\tau}}_{\boldsymbol{\sigma_{\mathcal{L}}\mathcal{K}^*}})$. We distinguish the interior diamond cells and the boundary diamond cells: $\mathfrak{D}_{ext} = \{\mathcal{D} \in \mathfrak{D},\ \mathcal{D} \cap \partial\Omega \neq \emptyset\}$, $\mathfrak{D}_{int} = \mathfrak{D} \backslash \mathfrak{D}_{ext}$. For all $\mathcal{D} \in \mathfrak{D}_{ext}$, we have $m_{\sigma_{\mathcal{K}^*}} = d_{\mathcal{K}^*,\mathcal{L}}$ and $m_{\sigma_{\mathcal{L}^*}} = d_{\mathcal{L}^*,\mathcal{L}}$.

Each diamond cell $\mathcal{D} \in \mathfrak{D}_{int}$ (resp. $\mathcal{D} \in \mathfrak{D}_{ext}$) can naturally be split into a set $\mathfrak{Q}_{\mathcal{D}}$ of four triangles (resp. two triangles), denoted by $\mathcal{Q}_{\mathcal{K},\mathcal{K}^*}$, $\mathcal{Q}_{\mathcal{K},\mathcal{L}^*}$, $\mathcal{Q}_{\mathcal{L},\mathcal{K}^*}$ and $\mathcal{Q}_{\mathcal{L},\mathcal{L}^*}$ $\mathcal{Q} \in \mathfrak{Q}_{\mathcal{D}}$, as shown in Figure 2.2(b), and satisfying $\mathcal{Q}_{\mathcal{K},\mathcal{K}^*} \subset \overline{\mathcal{K}} \cap \overline{\mathcal{K}^*}, \cdots$. If $\mathcal{D} \in \mathfrak{D}_{int}$, we have $\bar{\mathcal{D}} = \overline{\mathcal{Q}_{\mathcal{K},\mathcal{K}^*}} \cup \overline{\mathcal{Q}_{\mathcal{K},\mathcal{L}^*}} \cup \overline{\mathcal{Q}_{\mathcal{L},\mathcal{K}^*}} \cup \overline{\mathcal{Q}_{\mathcal{L},\mathcal{L}^*}}$ and if $\mathcal{D} \in \mathfrak{D}_{ext}$, we have $\bar{\mathcal{D}} = \overline{\mathcal{Q}_{\mathcal{K},\mathcal{K}^*}} \cup \overline{\mathcal{Q}_{\mathcal{K},\mathcal{L}^*}}$. The set of the quarter diamonds in the domain is denoted by $\mathfrak{Q} = \underset{\mathcal{D}\in\mathfrak{D}}{\cup} \mathfrak{Q}_{\mathcal{D}}$. For a quarter diamond cell $\mathcal{Q} \in \mathfrak{Q}$, we note by $m_{\mathcal{Q}}$ its measure, $h_{\mathcal{Q}}$ its diameter.

REMARK 2.3. *In order to right define the quarter diamond cells, we have chosen the barycentric dual mesh. In the classic dual mesh, when the diamond cells are non-convex the definition of the quarter diamond cells is not possible.*

The presence of the two angles $\alpha_{\mathcal{K}}$, $\alpha_{\mathcal{L}}$ takes a role in Definition 2.5 of the modified discrete gradient, in the new discrete Korn inequality (see Theorem 4.2) and in the consistency errors (see Lemma 6.4 and Corollary 6.1). The three last results rely on Lemma 2.2. In order to prove this Corollary, we need to introduce the following criterion:

Criterion: For each $\mathcal{D} \in \mathfrak{D}$, if $|\alpha_{\mathcal{K}} - \alpha_{\mathcal{L}}| < \epsilon_0$, we choose $x_{\mathcal{D}}$ to be the intersection of the primal edge σ and the dual edge σ^* instead of the middle point of the edge σ.

REMARK 2.4. *Now, for each $\mathcal{D} \in \mathfrak{D}$, we have either $|\alpha_{\mathcal{K}} - \alpha_{\mathcal{L}}| > \epsilon_0$ or $\alpha_{\mathcal{K}} = \alpha_{\mathcal{L}}$.*

It is needed since the proof of the estimate is divided into two cases: when $\alpha_{\mathcal{K}} = \alpha_{\mathcal{L}}$ or $\alpha_{\mathcal{K}} \neq \alpha_{\mathcal{L}}$. We can not generalize the case $|\alpha_{\mathcal{K}} - \alpha_{\mathcal{L}}| > \epsilon_0$ when ϵ_0 tends to 0. One hand, when $\alpha_{\mathcal{K}} \neq \alpha_{\mathcal{L}}$, we obtain estimate constant depending on ϵ_0 which explode when ϵ_0 tends to 0. On the other hand, when $\alpha_{\mathcal{K}} = \alpha_{\mathcal{L}}$, we can prove in an other way the same estimate with a finite constant.

Mesh regularity measurement. Set $\mathrm{size}(\mathcal{T})$ the maximum of the diameters of the diamond cells in $\mathfrak{D}$. To measure how flat the diamond cells can be, we note $\alpha_{\mathcal{T}}$ the unique real in $]0, \frac{\pi}{2}]$ such that $\sin(\alpha_{\mathcal{T}}) := \min_{\mathcal{D}\in\mathfrak{D}}(|\sin(\alpha_{\mathcal{K}})|, |\sin(\alpha_{\mathcal{L}})|)$. We introduce a positive number $\mathrm{reg}(\mathcal{T})$ that quantifies the regularity of a given mesh and is useful to perform the convergence analysis of finite volume schemes:

$$\mathrm{reg}(\mathcal{T}) := \max\left(\frac{1}{\sin(\alpha_{\mathcal{T}})}, \max_{\mathcal{D}\in\mathfrak{D}} \max_{\mathcal{Q}\in\mathfrak{Q}_{\mathcal{D}}} \frac{h_{\mathcal{D}}}{\min_{\sigma\in\partial\mathcal{Q}} m_\sigma}, \max_{\mathcal{K}\in\mathfrak{M}} \max_{\mathcal{D}\in\mathfrak{D}_{\mathcal{K}}} \frac{d_{\mathcal{K}}}{h_{\mathcal{D}}}, \max_{\mathcal{K}^*\in\mathfrak{M}^*\cup\partial\mathfrak{M}^*} \max_{\mathcal{D}\in\mathfrak{D}_{\mathcal{K}^*}} \frac{d_{\mathcal{K}^*}}{h_{\mathcal{D}}}\right). \quad (2.1)$$

The number $\mathrm{reg}(\mathcal{T})$ should be uniformly bounded when $\mathrm{size}(\mathcal{T}) \to 0$ for the convergence to hold. For instance, there exists a constant C depending on $\mathrm{reg}(\mathcal{T})$ such that

$$\frac{h_{\mathcal{D}}}{\sqrt{m_{\mathcal{D}}}} \leq \frac{h_{\mathcal{D}}}{\sqrt{m_{\mathcal{Q}}}} \leq C,\ \forall \mathcal{Q} \in \mathfrak{Q}_{\mathcal{D}}, \qquad h_{\mathcal{D}} \leq C\min(m_\sigma, m_{\sigma^*}),\ \forall \mathcal{D}_{\sigma,\sigma^*} \in \mathfrak{D}.$$

2.2. Unknowns and discrete projections. The DDFV method associates to any primal control volume $\mathcal{K} \in \mathfrak{M} \cup \partial\mathfrak{M}$ an unknown value $\mathbf{u}_{\boldsymbol{\mathcal{K}}} \in \mathbb{R}^2$ for the velocity, to any dual control volume $\mathcal{K}^* \in \mathfrak{M}^* \cup \partial\mathfrak{M}^*$ an unknown value $\mathbf{u}_{\boldsymbol{\mathcal{K}^*}} \in \mathbb{R}^2$ for the velocity and to any diamond cell $\mathcal{D} \in \mathfrak{D}$ an unknown value $p^{\mathcal{D}} \in \mathbb{R}$ for the pressure. These unknowns are collected in the

families :

$$\mathbf{u}^{\boldsymbol{\mathcal{T}}} = \left(\left(\mathbf{u}_{\boldsymbol{\mathcal{K}}}\right)_{\mathcal{K}\in(\mathfrak{M}\cup\partial\mathfrak{M})}, \left(\mathbf{u}_{\boldsymbol{\mathcal{K}^*}}\right)_{\mathcal{K}^*\in(\mathfrak{M}^*\cup\partial\mathfrak{M}^*)} \right) \in \left(\mathbb{R}^2\right)^{\mathcal{T}}, \quad p^{\mathfrak{D}} = \left(\left(p^{\mathcal{D}}\right)_{\mathcal{D}\in\mathfrak{D}}\right) \in \mathbb{R}^{\mathfrak{D}}.$$

We specify a discrete subset of $\left(\mathbb{R}^2\right)^{\mathcal{T}}$ needed to take into account the Dirichlet boundary conditions: $\mathbb{E}_0 = \{\mathbf{v}^{\boldsymbol{\mathcal{T}}} \in \left(\mathbb{R}^2\right)^{\mathcal{T}} \text{ s. t. } \mathbf{v}_{\boldsymbol{\mathcal{K}}} = 0,\ \forall \mathcal{K} \in \partial\mathfrak{M} \text{ and } \mathbf{v}_{\boldsymbol{\mathcal{K}^*}} = 0,\ \forall \mathcal{K}^* \in \partial\mathfrak{M}^*\}$.

We define a interior mean-value projection for any integrable vector function $\mathbf{v}$ on Ω:

$$\widetilde{\mathbb{P}}^{\mathfrak{M}}_{\boldsymbol{m}}\mathbf{v} = \left(\left(\frac{1}{m_{\mathcal{K}}} \int_{\mathcal{K}} \mathbf{v}(x)\mathrm{d}x \right)_{\mathcal{K}\in\mathfrak{M}} \right), \widetilde{\mathbb{P}}^{\mathfrak{M}^*}_{\boldsymbol{m}}\mathbf{v} = \left(\left(\frac{1}{m_{\mathcal{K}^*}} \int_{\mathcal{K}^*} \mathbf{v}(x)\mathrm{d}x \right)_{\mathcal{K}^*\in\mathfrak{M}^*} \right). \tag{2.2}$$

We also note the mean-value projection for any integrable vector function $\mathbf{v}$ on $\bar{\Omega}$ as follows

$$\mathbb{P}^{\boldsymbol{\mathcal{T}}}_{\boldsymbol{m}}\mathbf{v} = \left(\widetilde{\mathbb{P}}^{\mathfrak{M}}_{\boldsymbol{m}}\mathbf{v}, \left(\frac{1}{m_{B_{\mathcal{K}}}} \int_{B_{\mathcal{K}}} \mathbf{v}(x)\mathrm{d}x \right)_{\mathcal{K}\in\partial\mathfrak{M}}, \widetilde{\mathbb{P}}^{\mathfrak{M}^*}_{\boldsymbol{m}}\mathbf{v}, \left(\frac{1}{m_{B_{\mathcal{K}^*}}} \int_{B_{\mathcal{K}^*}} \mathbf{v}(x)\mathrm{d}x \right)_{\mathcal{K}^*\in\partial\mathfrak{M}^*} \right). \tag{2.3}$$

In particular, the mean-value projection is well defined for any vector $\mathbf{v}$ lying in $(H^1(\Omega))^2$.

2.3. Discrete operators. We recall the discrete operators introduced in [19].

DEFINITION 2.1 (Discrete gradient). *We define a consistent approximation of the gradient operator denoted by* $\nabla^{\mathfrak{D}} : \mathbf{u}^{\boldsymbol{\mathcal{T}}} \in \left(\mathbb{R}^2\right)^{\mathcal{T}} \mapsto \left(\nabla^{\mathcal{D}}\mathbf{u}^{\boldsymbol{\mathcal{T}}}\right)_{\mathcal{D}\in\mathfrak{D}} \in (\mathcal{M}_2(\mathbb{R}))^{\mathfrak{D}}$*, as follows:*

$$\nabla^{\mathcal{D}}\mathbf{u}^{\boldsymbol{\mathcal{T}}} = \frac{1}{\sin(\alpha_{\mathcal{D}})} \left[\frac{\mathbf{u}_{\boldsymbol{\mathcal{L}}} - \mathbf{u}_{\boldsymbol{\mathcal{K}}}}{m_{\sigma^*}} \otimes \vec{\mathbf{n}}_{\boldsymbol{\sigma\mathcal{K}}} + \frac{\mathbf{u}_{\boldsymbol{\mathcal{L}^*}} - \mathbf{u}_{\boldsymbol{\mathcal{K}^*}}}{m_{\sigma}} \otimes \vec{\mathbf{n}}_{\boldsymbol{\sigma^*\mathcal{K}^*}} \right].$$

where $\otimes$ *represents the tensor product.*

DEFINITION 2.2 (Discrete divergence). *We define a consistent approximation of the divergence operator applied to discrete tensor fields denoted by* $\mathbf{div}^{\boldsymbol{\mathcal{T}}} : \xi^{\mathfrak{D}} = (\xi^{\mathcal{D}})_{\mathcal{D}\in\mathfrak{D}} \in (\mathcal{M}_2(\mathbb{R}))^{\mathfrak{D}} \mapsto \mathbf{div}^{\boldsymbol{\mathcal{T}}}\xi^{\mathfrak{D}} \in \left(\mathbb{R}^2\right)^{\mathcal{T}}$*, as follows:*

$$\mathbf{div}^{\boldsymbol{\mathcal{K}}}\xi^{\mathfrak{D}} = \frac{1}{m_{\mathcal{K}}} \sum_{\sigma\in\partial\mathcal{K}} m_{\sigma}\xi^{\mathcal{D}}\vec{\mathbf{n}}_{\boldsymbol{\sigma\mathcal{K}}},\ \forall \mathcal{K} \in \mathfrak{M}, \text{ and } \mathbf{div}^{\boldsymbol{\mathcal{K}}}\xi^{\mathfrak{D}} = 0,\ \forall \mathcal{K} \in \partial\mathfrak{M},$$

$$\mathbf{div}^{\boldsymbol{\mathcal{K}^*}}\xi^{\mathfrak{D}} = \frac{1}{m_{\mathcal{K}^*}} \sum_{\sigma^*\in\partial\mathcal{K}^*} m_{\sigma^*}\xi^{\mathcal{D}}\vec{\mathbf{n}}_{\boldsymbol{\sigma^*\mathcal{K}^*}},\ \forall \mathcal{K}^* \in \mathfrak{M}^*,$$

$$\mathbf{div}^{\boldsymbol{\mathcal{K}^*}}\xi^{\mathfrak{D}} = \frac{1}{m_{\mathcal{K}^*}} \left(\sum_{\mathcal{D}_{\sigma,\sigma^*}\in\mathfrak{D}_{\mathcal{K}^*}} m_{\sigma^*}\xi^{\mathcal{D}}\vec{\mathbf{n}}_{\boldsymbol{\sigma^*\mathcal{K}^*}} + \sum_{\mathcal{D}_{\sigma,\sigma^*}\in\mathfrak{D}_{\mathcal{K}^*}\cap\mathfrak{D}_{ext}} d_{\mathcal{K}^*,\mathcal{L}}\xi^{\mathcal{D}}\vec{\mathbf{n}}_{\boldsymbol{\sigma\mathcal{K}}} \right),\ \forall \mathcal{K}^* \in \partial\mathfrak{M}^*.$$

To write the DDFV scheme, we also need to denote the discrete divergence on the primal mesh and on the interior dual mesh as follows:

$$\mathbf{div}^{\mathfrak{M}}\xi^{\mathfrak{D}} = \left(\left(\mathbf{div}^{\boldsymbol{\mathcal{K}}}\xi^{\mathfrak{D}}\right)_{\mathcal{K}\in\mathfrak{M}} \right), \qquad \mathbf{div}^{\mathfrak{M}^*}\xi^{\mathfrak{D}} = \left(\left(\mathbf{div}^{\boldsymbol{\mathcal{K}^*}}\xi^{\mathfrak{D}}\right)_{\mathcal{K}^*\in\mathfrak{M}^*} \right).$$

Using the barycentric dual mesh, we also can write the discrete divergence like in [7]

$$\mathbf{div}^{\boldsymbol{\mathcal{K}^*}}\xi^{\mathfrak{D}} = \frac{1}{m_{\mathcal{K}^*}} \sum_{\sigma^*\in\partial\mathcal{K}^*} \left(m_{\sigma_{\mathcal{K}}}\xi^{\mathcal{D}}\vec{\mathbf{n}}_{\boldsymbol{\sigma_{\mathcal{K}}\mathcal{K}^*}} + m_{\sigma_{\mathcal{L}}}\xi^{\mathcal{D}}\vec{\mathbf{n}}_{\boldsymbol{\sigma_{\mathcal{L}}\mathcal{K}^*}} \right),\ \forall \mathcal{K}^* \in \mathfrak{M}^*.$$

Thanks to the discrete gradient we can define a discrete strain rate tensor and a discrete divergence of a vector field in $\left(\mathbb{R}^2\right)^{\mathcal{T}}$.

DEFINITION 2.3 (Discrete strain rate tensor). *We define a discrete strain rate tensor of a vector field in* $\left(\mathbb{R}^2\right)^{\mathcal{T}}$*, denoted by:* $\mathrm{D}^{\mathfrak{D}} : \mathbf{u}^{\boldsymbol{\mathcal{T}}} \in \left(\mathbb{R}^2\right)^{\mathcal{T}} \mapsto \left(\mathrm{D}^{\mathcal{D}}\mathbf{u}^{\boldsymbol{\mathcal{T}}}\right)_{\mathcal{D}\in\mathfrak{D}} \in (\mathcal{M}_2(\mathbb{R}))^{\mathfrak{D}}$*, with* $\mathrm{D}^{\mathcal{D}}\mathbf{u}^{\boldsymbol{\mathcal{T}}} = \frac{\nabla^{\mathcal{D}}\mathbf{u}^{\boldsymbol{\mathcal{T}}} + {}^t\left(\nabla^{\mathcal{D}}\mathbf{u}^{\boldsymbol{\mathcal{T}}}\right)}{2}$*, for all* $\mathcal{D} \in \mathfrak{D}$.

DEFINITION 2.4. *We define a discrete divergence of a vector field in* $\left(\mathbb{R}^2\right)^{\mathcal{T}}$*, denoted by:* $\mathrm{div}^{\mathfrak{D}} : \mathbf{u}^{\boldsymbol{\mathcal{T}}} \in \left(\mathbb{R}^2\right)^{\mathcal{T}} \mapsto \left(\mathrm{div}^{\mathcal{D}}\mathbf{u}^{\boldsymbol{\mathcal{T}}}\right)_{\mathcal{D}\in\mathfrak{D}} \in \mathbb{R}^{\mathfrak{D}}$*, with* $\mathrm{div}^{\mathcal{D}}\mathbf{u}^{\boldsymbol{\mathcal{T}}} = \mathrm{Tr}(\nabla^{\mathcal{D}}\mathbf{u}^{\boldsymbol{\mathcal{T}}})$*, for all* $\mathcal{D} \in \mathfrak{D}$.

2.4. Local modification of the discrete strain rate tensor. Assume that the viscosity η is Lipschitz continuous on each quarter diamond cell: there exists $\mathrm{C}_\eta > 0$ such that:

$$|\eta(x) - \eta(x')| \leq \mathrm{C}_\eta |x - x'|, \quad \forall x, x' \in \bar{\mathcal{Q}}, \qquad \text{for all } \mathcal{Q} \in \mathfrak{Q}. \tag{2.4}$$

We note, for all $\mathcal{Q} \in \mathfrak{Q}$, $\eta_\mathcal{Q} = \int_{\bar{\mathcal{Q}}} \eta(s) \mathrm{d}\mu_{\bar{\mathcal{Q}}}(s)$ where $\mu_{\bar{\mathcal{Q}}}$ is a measure of probability. Furthermore, we always have $\underline{\mathrm{C}}_\eta \leq \eta_\mathcal{Q} \leq \overline{\mathrm{C}}_\eta$, for all $\mathcal{Q} \in \mathfrak{Q}$.

The point we are concerned with in this paper is that the DDFV scheme (1.4) suffers from a lost of consistency in the case where η presents discontinuities. More precisely, we present a way to recover the consistency of the fluxes even when η presents jumps. The normal component of the stress tensor $\sigma(\mathbf{u}, p) = 2\eta \mathrm{D}\mathbf{u} - p\mathrm{Id}$ is continuous across all primal and dual edges. For instance, we have

$$\int_{\sigma_\mathcal{K}} \sigma_{|\overline{\mathcal{Q}}_{\mathcal{K},\mathcal{K}^*}}(\mathbf{u}, p) \vec{\mathbf{n}}_{\boldsymbol{\sigma\mathcal{K}}} \mathrm{d}s = \int_{\sigma_\mathcal{K}} \sigma_{|\overline{\mathcal{Q}}_{\mathcal{L},\mathcal{K}^*}}(\mathbf{u}, p) \vec{\mathbf{n}}_{\boldsymbol{\sigma\mathcal{K}}} \mathrm{d}s.$$

We need to ensure this consistency at the discret level. We introduce a discret stress tensor $\varphi_\mathcal{Q}$ on quarter diamond cells. Thus we add additional unknowns $p^{\mathfrak{Q}_\mathcal{D}} = (p^\mathcal{Q})_{\mathcal{Q} \in \mathfrak{Q}_\mathcal{D}}$ on each diamond cell $\mathcal{D}$ and we must define a new discret strain rate tensor $\mathrm{D}^\mathcal{N}_\mathcal{Q}$ on each quarter diamond cell in order to define the discret stress tensor $\varphi_\mathcal{Q}$ as $2\eta_\mathcal{Q} \mathrm{D}^\mathcal{N}_\mathcal{Q} \mathbf{u}^{\boldsymbol{\mathcal{T}}} - p^\mathcal{Q} \mathrm{Id}$.

We first recall the scalar case propose in [3]. The discrete gradient operator $\nabla^\mathcal{D} u^\mathcal{T}$ can be understood as the gradient of the unique affine function $\Pi_\mathcal{D} u^\mathcal{T}$ on $\mathcal{D}$ whose value at the middle of each side of the diamond $\mathcal{D}$ is the mean value between the two unknowns associated to the extremities of this segment (sum up in Figure 2.3). We introduce the middle point $x_{\sigma_\mathcal{K}}, x_{\sigma_\mathcal{L}}, x_{\sigma_{\mathcal{K}^*}}, x_{\sigma_{\mathcal{L}^*}}$ of the segments $\sigma_\mathcal{K}, \sigma_\mathcal{L}, \sigma_{\mathcal{K}^*}, \sigma_{\mathcal{L}^*}$. The new gradient operator $\nabla^\mathcal{N}_\mathcal{Q}$ is

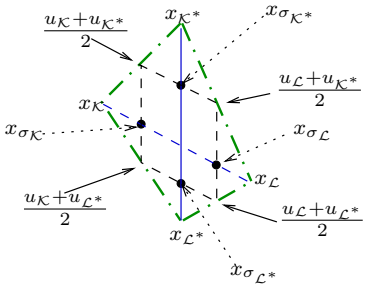


FIG. 2.3. .

chosen to be constant on all the quarter diamond cells $\mathcal{Q} \in \mathfrak{Q}$. It is the gradient of a function $\widetilde{\Pi}_\mathcal{D} u^\mathcal{T}$ whose is a affine function on each $\mathcal{Q} \in \mathfrak{Q}_\mathcal{D}$ which coincides with $\Pi_\mathcal{D} u^\mathcal{T}$ in the middle of each side of $\mathcal{D}$ and which is continuous at each point $x_{\sigma_\mathcal{K}}, x_{\sigma_\mathcal{L}}, x_{\sigma_{\mathcal{K}^*}}, x_{\sigma_{\mathcal{L}^*}}$. We add four new unknowns $\delta^\mathcal{D} = (\delta_\mathcal{K}, \delta_\mathcal{L}, \delta_{\mathcal{K}^*}, \delta_{\mathcal{L}^*})$ defined to be the differences $\widetilde{\Pi}_\mathcal{D} u^\mathcal{T}(y) - \Pi_\mathcal{D} u^\mathcal{T}(y)$ for each $y \in \{x_{\sigma_\mathcal{K}}, x_{\sigma_\mathcal{L}}, x_{\sigma_{\mathcal{K}^*}}, x_{\sigma_{\mathcal{L}^*}}\}$. Now we can write $\nabla^\mathcal{N}_\mathcal{Q} u^\mathcal{T} = \nabla^\mathcal{D} u^\mathcal{T} + B_\mathcal{Q} \delta^\mathcal{D}$, where $(B_\mathcal{Q})_{\mathcal{Q} \in \mathfrak{Q}_\mathcal{D}}$ is a family of matrices which can be explicitly compute. We have present the case where $\mathcal{D}$ is a interior diamond, we can do the same when $\mathcal{D} \in \mathfrak{D}_{ext}$, thus we note $n_\mathcal{D} = 4$ if $\mathcal{D} \in \mathfrak{D}_{int}$ and $n_\mathcal{D} = 1$ if $\mathcal{D} \in \mathfrak{D}_{ext}$. Here, we propose to adapt this framework to the vector case. We will work now with $\delta^\mathcal{D} = (\delta_\mathcal{K}, \delta_\mathcal{L}, \delta_{\mathcal{K}^*}, \delta_{\mathcal{L}^*})$ lying in $(\mathcal{M}_{n_\mathcal{D},2}(\mathbb{R}))^\mathfrak{D}$ and the family of matrix $B_\mathcal{Q}$ are the same.

2.4.1. Discrete operators on quarter diamond cells. DEFINITION 2.5 (Discrete gradient on quarter diamonds). *We define a discrete gradient of a vector field of* $\left(\mathbb{R}^2\right)^\mathcal{T}$ *on quarter diamond cells:* $\nabla^\mathcal{N}_\mathfrak{Q} : \left(\mathbb{R}^2\right)^\mathcal{T} \to (\nabla^\mathcal{N}_\mathcal{Q} \mathbf{u}^{\boldsymbol{\mathcal{T}}})_{\mathcal{Q} \in \mathfrak{Q}} \in (\mathcal{M}_2(\mathbb{R}))^\mathfrak{Q}$, *such that for any* $\mathcal{D} \in \mathfrak{D}$ *and*

for any $\mathcal{Q} \in \mathfrak{Q}_\mathcal{D}$: $\nabla^\mathcal{N}_\mathcal{Q}\mathbf{u}^{\boldsymbol{\mathcal{T}}} = \nabla^\mathcal{D}\mathbf{u}^{\boldsymbol{\mathcal{T}}} + {}^t\delta^\mathcal{D}\,{}^tB_\mathcal{Q}$, *where* $\delta^\mathfrak{D} = (\delta^\mathfrak{D}_1, \delta^\mathfrak{D}_2) \in (\mathcal{M}_{n_\mathcal{D},2}(\mathbb{R}))^\mathfrak{D}$ *is an artificial set of unknowns and* $(B_\mathcal{Q})_{\mathcal{Q}\in\mathfrak{Q}_\mathcal{D}}$ *is a set of* $\mathcal{M}_{2,n_\mathcal{D}}(\mathbb{R})$ *defined as follows:*

- $\forall \mathcal{D} \in \mathfrak{D}_{int}$, *we take* $\delta^\mathcal{D} = {}^t(\delta_\mathcal{K}, \delta_\mathcal{L}, \delta_{\mathcal{K}^*}, \delta_{\mathcal{L}^*}) \in \mathcal{M}_{n_\mathcal{D},2}(\mathbb{R})$ *and four matrices* $B_\mathcal{Q}$:

$$B_{\mathcal{Q}_{\mathcal{K},\mathcal{K}^*}}=\left(\frac{m_{\sigma_\mathcal{K}}\vec{\mathbf{n}}_{\boldsymbol{\sigma_{\mathcal{K}\mathcal{K}^*}}}}{m_{\mathcal{Q}_{\mathcal{K},\mathcal{K}^*}}}, 0, \frac{m_{\sigma_{\mathcal{K}^*}}\vec{\mathbf{n}}_{\boldsymbol{\sigma_\mathcal{K}}}}{m_{\mathcal{Q}_{\mathcal{K},\mathcal{K}^*}}}, 0\right),\ B_{\mathcal{Q}_{\mathcal{K},\mathcal{L}^*}}=\left(-\frac{m_{\sigma_\mathcal{K}}\vec{\mathbf{n}}_{\boldsymbol{\sigma_{\mathcal{K}\mathcal{K}^*}}}}{m_{\mathcal{Q}_{\mathcal{K},\mathcal{L}^*}}}, 0, 0, \frac{m_{\sigma_{\mathcal{L}^*}}\vec{\mathbf{n}}_{\boldsymbol{\sigma_\mathcal{K}}}}{m_{\mathcal{Q}_{\mathcal{K},\mathcal{L}^*}}}\right),$$

$$B_{\mathcal{Q}_{\mathcal{L},\mathcal{L}^*}}=\left(0, -\frac{m_{\sigma_\mathcal{L}}\vec{\mathbf{n}}_{\boldsymbol{\sigma_{\mathcal{L}\mathcal{K}^*}}}}{m_{\mathcal{Q}_{\mathcal{L},\mathcal{L}^*}}}, 0, -\frac{m_{\sigma_{\mathcal{L}^*}}\vec{\mathbf{n}}_{\boldsymbol{\sigma_\mathcal{K}}}}{m_{\mathcal{Q}_{\mathcal{L},\mathcal{L}^*}}}\right),\ B_{\mathcal{Q}_{\mathcal{L},\mathcal{K}^*}}=\left(0, \frac{m_{\sigma_\mathcal{L}}\vec{\mathbf{n}}_{\boldsymbol{\sigma_{\mathcal{L}\mathcal{K}^*}}}}{m_{\mathcal{Q}_{\mathcal{L},\mathcal{K}^*}}}, -\frac{m_{\sigma_{\mathcal{K}^*}}\vec{\mathbf{n}}_{\boldsymbol{\sigma_\mathcal{K}}}}{m_{\mathcal{Q}_{\mathcal{L},\mathcal{K}^*}}}, 0\right).$$

- $\forall \mathcal{D} \in \mathfrak{D}_{ext}$, *there is only two non-degenerate quarter diamonds in* $\mathfrak{Q}_\mathcal{D}$, *we take* $\delta^\mathcal{D} = \left(\delta_\mathcal{K}\right) \in \mathcal{M}_{n_\mathcal{D},2}(\mathbb{R})$ *and the two corresponding matrices* $B_\mathcal{Q}$ *are given by:*

$$B_{\mathcal{Q}_{\mathcal{K},\mathcal{K}^*}} = \left(\frac{m_{\sigma_\mathcal{K}}\vec{\mathbf{n}}_{\boldsymbol{\sigma_{\mathcal{K}\mathcal{K}^*}}}}{m_{\mathcal{Q}_{\mathcal{K},\mathcal{K}^*}}}\right), \quad B_{\mathcal{Q}_{\mathcal{K},\mathcal{L}^*}} = \left(-\frac{m_{\sigma_\mathcal{K}}\vec{\mathbf{n}}_{\boldsymbol{\sigma_{\mathcal{K}\mathcal{K}^*}}}}{m_{\mathcal{Q}_{\mathcal{K},\mathcal{L}^*}}}\right).$$

In [3], they have introduced a new discrete gradient for a scalar field. Here we extend this definition for a vector field. We use the same matrix $B_\mathcal{Q}$, the artificial set of unknowns is now a matrix in $\mathcal{M}_{n_\mathcal{D},2}$ instead of a vector in $\mathbb{R}^{n_\mathcal{D}}$. Thanks to the modified discrete gradient, we can define a new symmetric operator a modified discrete strain rate tensor as follows.

DEFINITION 2.6 (Discrete strain rate tensor on quarter diamonds). *We define a discrete strain rate tensor of a vector field of* $\left(\mathbb{R}^2\right)^\mathcal{T}$ *on the quarter diamonds:* $\mathrm{D}^\mathcal{N}_\mathfrak{Q} : \left(\mathbb{R}^2\right)^\mathcal{T} \to \left(\mathrm{D}^\mathcal{N}_\mathcal{Q}\mathbf{u}^{\boldsymbol{\mathcal{T}}}\right)_{\mathcal{Q}\in\mathfrak{Q}} \in (\mathcal{M}_2(\mathbb{R}))^\mathfrak{Q}$, *such that for any* $\mathcal{Q} \in \mathfrak{Q}$: $\mathrm{D}^\mathcal{N}_\mathcal{Q}\mathbf{u}^{\boldsymbol{\mathcal{T}}} = \frac{1}{2}\left(\nabla^\mathcal{N}_\mathcal{Q}\mathbf{u}^{\boldsymbol{\mathcal{T}}} + {}^t(\nabla^\mathcal{N}_\mathcal{Q}\mathbf{u}^{\boldsymbol{\mathcal{T}}})\right)$. *The discrete strain rate tensor on quarter diamonds can be written as follows, for any* $\mathcal{D} \in \mathfrak{D}$ *and for any* $\mathcal{Q} \in \mathfrak{Q}_\mathcal{D}$, $\mathrm{D}^\mathcal{N}_\mathcal{Q}\mathbf{u}^{\boldsymbol{\mathcal{T}}} = \mathrm{D}^\mathcal{D}\mathbf{u}^{\boldsymbol{\mathcal{T}}} + \frac{1}{2}\left({}^t\delta^\mathcal{D}\,{}^tB_\mathcal{Q} + B_\mathcal{Q}\delta^\mathcal{D}\right)$.

Furthermore, we easily see from the formulas above that $\sum_{\mathcal{Q}\in\mathfrak{Q}_\mathcal{D}} m_\mathcal{Q}B_\mathcal{Q} = 0$ for any diamond cell $\mathcal{D}$. Hence the following straightforward result holds

LEMMA 2.1. *For any* $\xi \in \mathcal{M}_2(\mathbb{R})$, *for any* $\delta \in (\mathcal{M}_{n_\mathcal{D},2}(\mathbb{R}))^\mathfrak{D}$, *we have for all* $\mathcal{D} \in \mathfrak{D}$

$$\xi = \frac{1}{m_\mathcal{D}} \sum_{\mathcal{Q}\in\mathfrak{Q}_\mathcal{D}} m_\mathcal{Q}\left(\xi + \frac{1}{2}(B_\mathcal{Q}\delta + {}^t\delta^t B_\mathcal{Q})\right).$$

This Lemma implies that the new strain rate tensor satisfied the following equality:

$$\mathrm{D}^\mathcal{D}\mathbf{u}^{\boldsymbol{\mathcal{T}}} = \frac{1}{m_\mathcal{D}} \sum_{\mathcal{Q}\in\mathfrak{Q}_\mathcal{D}} m_\mathcal{Q}\mathrm{D}^\mathcal{N}_\mathcal{Q}\mathbf{u}^{\boldsymbol{\mathcal{T}}}, \qquad \nabla^\mathcal{D}\mathbf{u}^{\boldsymbol{\mathcal{T}}} = \frac{1}{m_\mathcal{D}} \sum_{\mathcal{Q}\in\mathfrak{Q}_\mathcal{D}} m_\mathcal{Q}\nabla^\mathcal{N}_\mathcal{Q}\mathbf{u}^{\boldsymbol{\mathcal{T}}}, \qquad \forall\, \mathcal{D} \in \mathfrak{D}. \tag{2.5}$$

Thanks to the modified discrete strain rate tensor, we can define a new viscous stress tensor as follows.

DEFINITION 2.7 (Discrete viscous stress tensor on quarter diamonds). *We define a discrete viscous stress tensor of a vector field of* $\left(\mathbb{R}^2\right)^\mathcal{T}$ *on diamond cells:* $\mathrm{D}^{\eta,\mathcal{N}}_\mathfrak{D} : \left(\mathbb{R}^2\right)^\mathcal{T} \to \left(\mathrm{D}^{\eta,\mathcal{N}}_\mathcal{D}\mathbf{u}^{\boldsymbol{\mathcal{T}}}\right)_{\mathcal{D}\in\mathfrak{D}} \in (\mathcal{M}_2(\mathbb{R}))^\mathfrak{D}$, *by* $\mathrm{D}^{\eta,\mathcal{N}}_\mathcal{D}\mathbf{u}^{\boldsymbol{\mathcal{T}}} = \dfrac{1}{m_\mathcal{D}} \displaystyle\sum_{\mathcal{Q}\in\mathfrak{Q}_\mathcal{D}} m_\mathcal{Q}\eta_\mathcal{Q}\mathrm{D}^\mathcal{N}_\mathcal{Q}\mathbf{u}^{\boldsymbol{\mathcal{T}}}$, *for any* $\mathcal{D} \in \mathfrak{D}$.

We define a stress tensor on quarter diamond cells as follows.

DEFINITION 2.8. *We define a discrete stress tensor* $\varphi_\mathcal{Q}$, *for all* $\mathcal{Q} \in \mathfrak{Q}_\mathcal{D}$,
$\varphi_\mathcal{Q}(\mathrm{D}^\mathcal{D}\mathbf{u}^{\boldsymbol{\mathcal{T}}}, \delta^\mathcal{D}, p^{\mathfrak{Q}_\mathcal{D}}) = 2\eta_\mathcal{Q}\mathrm{D}^\mathcal{D}\mathbf{u}^{\boldsymbol{\mathcal{T}}} + \eta_\mathcal{Q}(B_\mathcal{Q}\delta^\mathcal{D} + {}^t\delta^\mathcal{D}\,{}^tB_\mathcal{Q}) - p^\mathcal{Q}\mathrm{Id}$.

We want to eliminate the additional unknowns $(\delta^\mathcal{D}, p^{\mathfrak{Q}_\mathcal{D}})$ on each diamond cell $\mathcal{D}$ in such a way that the conservativity of the numerical fluxes on all diagonals of $\mathcal{D}$ is ensured. More precisely, we want to choose $(\delta^\mathcal{D}, p^{\mathfrak{Q}_\mathcal{D}})$ such that, we have $\forall\, \mathcal{D} \in \mathfrak{D}_{int},\ \mathcal{D} = \mathcal{Q}_{\mathcal{K},\mathcal{K}^*} \cup \mathcal{Q}_{\mathcal{K},\mathcal{L}^*} \cup$

$\mathcal{Q}_{\mathcal{L},\mathcal{K}^*} \cup \mathcal{Q}_{\mathcal{L},\mathcal{L}^*}$,

$$\begin{cases} \varphi_{\mathcal{Q}_{\mathcal{K},\mathcal{K}^*}}(\mathrm{D}^{\mathcal{D}}\mathbf{u}^{\boldsymbol{\mathcal{T}}}, \delta^{\mathcal{D}}, p_{\mathcal{Q}_{\mathcal{K},\mathcal{K}^*}})\vec{\mathbf{n}}_{\boldsymbol{\sigma\mathcal{K}}} = \varphi_{\mathcal{Q}_{\mathcal{L},\mathcal{K}^*}}(\mathrm{D}^{\mathcal{D}}\mathbf{u}^{\boldsymbol{\mathcal{T}}}, \delta^{\mathcal{D}}, p_{\mathcal{Q}_{\mathcal{L},\mathcal{K}^*}})\vec{\mathbf{n}}_{\boldsymbol{\sigma\mathcal{K}}}, \\ \varphi_{\mathcal{Q}_{\mathcal{K},\mathcal{L}^*}}(\mathrm{D}^{\mathcal{D}}\mathbf{u}^{\boldsymbol{\mathcal{T}}}, \delta^{\mathcal{D}}, p_{\mathcal{Q}_{\mathcal{K},\mathcal{L}^*}})\vec{\mathbf{n}}_{\boldsymbol{\sigma\mathcal{K}}} = \varphi_{\mathcal{Q}_{\mathcal{L},\mathcal{L}^*}}(\mathrm{D}^{\mathcal{D}}\mathbf{u}^{\boldsymbol{\mathcal{T}}}, \delta^{\mathcal{D}}, p_{\mathcal{Q}_{\mathcal{L},\mathcal{L}^*}})\vec{\mathbf{n}}_{\boldsymbol{\sigma\mathcal{K}}}, \\ \varphi_{\mathcal{Q}_{\mathcal{K},\mathcal{K}^*}}(\mathrm{D}^{\mathcal{D}}\mathbf{u}^{\boldsymbol{\mathcal{T}}}, \delta^{\mathcal{D}}, p_{\mathcal{Q}_{\mathcal{K},\mathcal{K}^*}})\vec{\mathbf{n}}_{\boldsymbol{\sigma_{\mathcal{K}}\mathcal{K}^*}} = \varphi_{\mathcal{Q}_{\mathcal{K},\mathcal{L}^*}}(\mathrm{D}^{\mathcal{D}}\mathbf{u}^{\boldsymbol{\mathcal{T}}}, \delta^{\mathcal{D}}, p_{\mathcal{Q}_{\mathcal{K},\mathcal{L}^*}})\vec{\mathbf{n}}_{\boldsymbol{\sigma_{\mathcal{K}}\mathcal{K}^*}}, \\ \varphi_{\mathcal{Q}_{\mathcal{L},\mathcal{K}^*}}(\mathrm{D}^{\mathcal{D}}\mathbf{u}^{\boldsymbol{\mathcal{T}}}, \delta^{\mathcal{D}}, p_{\mathcal{Q}_{\mathcal{L},\mathcal{K}^*}})\vec{\mathbf{n}}_{\boldsymbol{\sigma_{\mathcal{L}}\mathcal{K}^*}} = \varphi_{\mathcal{Q}_{\mathcal{L},\mathcal{L}^*}}(\mathrm{D}^{\mathcal{D}}\mathbf{u}^{\boldsymbol{\mathcal{T}}}, \delta^{\mathcal{D}}, p_{\mathcal{Q}_{\mathcal{L},\mathcal{L}^*}})\vec{\mathbf{n}}_{\boldsymbol{\sigma_{\mathcal{L}}\mathcal{K}^*}}, \\ \mathrm{Tr}({}^t\delta^{\mathcal{D}\,t}B_{\mathcal{Q}}) = 0, \quad \forall \mathcal{Q} \in \mathfrak{Q}_{\mathcal{D}}, \quad \sum\limits_{\mathcal{Q}\in\mathfrak{Q}_{\mathcal{D}}} m_{\mathcal{Q}}p^{\mathcal{Q}} = m_{\mathcal{D}}p^{\mathcal{D}}, \end{cases} \tag{2.6}$$

and $\forall\, \mathcal{D} \in \mathfrak{D}_{ext}, \quad \mathcal{D} = \mathcal{Q}_{\mathcal{K},\mathcal{K}^*} \cup \mathcal{Q}_{\mathcal{K},\mathcal{L}^*}$,

$$\begin{cases} \varphi_{\mathcal{Q}_{\mathcal{K},\mathcal{K}^*}}(\mathrm{D}^{\mathcal{D}}\mathbf{u}^{\boldsymbol{\mathcal{T}}}, \delta^{\mathcal{D}}, p_{\mathcal{Q}_{\mathcal{K},\mathcal{K}^*}})\vec{\mathbf{n}}_{\boldsymbol{\sigma_{\mathcal{K}}\mathcal{K}^*}} = \varphi_{\mathcal{Q}_{\mathcal{K},\mathcal{L}^*}}(\mathrm{D}^{\mathcal{D}}\mathbf{u}^{\boldsymbol{\mathcal{T}}}, \delta^{\mathcal{D}}, p_{\mathcal{Q}_{\mathcal{K},\mathcal{L}^*}})\vec{\mathbf{n}}_{\boldsymbol{\sigma_{\mathcal{K}}\mathcal{K}^*}}, \\ \mathrm{Tr}({}^t\delta^{\mathcal{D}\,t}B_{\mathcal{Q}}) = 0, \quad \forall \mathcal{Q} \in \mathfrak{Q}_{\mathcal{D}}, \quad \sum\limits_{\mathcal{Q}\in\mathfrak{Q}_{\mathcal{D}}} m_{\mathcal{Q}}p^{\mathcal{Q}} = m_{\mathcal{D}}p^{\mathcal{D}}. \end{cases} \tag{2.7}$$

We are going to show that (2.6) or (2.7) uniquely define $(\delta^{\mathcal{D}}, p^{\mathfrak{Q}_{\mathcal{D}}}) \in \mathcal{M}_{n_{\mathcal{D}},2}(\mathbb{R}) \times \mathbb{R}^{n_{\mathcal{D}}}$ as a function of $\mathrm{D}^{\mathfrak{D}}\mathbf{u}^{\boldsymbol{\mathcal{T}}}$ and $p^{\mathfrak{D}}$. For any $\mathcal{D} \in \mathfrak{D}$, the existence of $(\delta^{\mathcal{D}}, p^{\mathfrak{Q}_{\mathcal{D}}})$ is not only a generalization of the work of [3], since we use the discrete strain rate tensor and not the discrete gradient, we have $B_{\mathcal{Q}}\delta^{\mathcal{D}} + {}^t\delta^{\mathcal{D}\,t}B_{\mathcal{Q}}$ instead of $B_{\mathcal{Q}}\delta^{\mathcal{D}}$. Then we have to differentiate two cases: the first one is when $\alpha_{\mathcal{K}} \neq \alpha_{\mathcal{L}}$ where we can prove the existence and uniqueness and the second one is when $\alpha_{\mathcal{K}} = \alpha_{\mathcal{L}}$ where we need to add an other condition.

PROPOSITION 2.1. *The condition* (2.6) *and* (2.7) *are equivalent to for any* $\mathcal{D} \in \mathfrak{D}$

$$\sum_{\mathcal{Q}\in\mathfrak{Q}_{\mathcal{D}}} m_{\mathcal{Q}}\varphi_{\mathcal{Q}}(\mathrm{D}^{\mathcal{D}}\mathbf{u}^{\boldsymbol{\mathcal{T}}}, \delta^{\mathcal{D}}, p^{\mathcal{Q}})B_{\mathcal{Q}} = 0, \tag{2.8a}$$

$$\mathrm{Tr}({}^t\delta^{\mathcal{D}\,t}B_{\mathcal{Q}}) = 0,\ \forall \mathcal{Q} \in \mathfrak{Q}_{\mathcal{D}}, \quad \sum_{\mathcal{Q}\in\mathfrak{Q}_{\mathcal{D}}} m_{\mathcal{Q}}p^{\mathcal{Q}} = m_{\mathcal{D}}p^{\mathcal{D}}. \tag{2.8b}$$

Since we work with the symmetric part of the gradient, we have to study the overdetermined system, for all $\mathcal{D} \in \mathfrak{D}$, ${}^t\delta^{\mathcal{D}\,t}B_{\mathcal{Q}} + B_{\mathcal{Q}}\delta^{\mathcal{D}} = F_{\mathcal{Q}}, \forall \mathcal{Q} \in \mathfrak{Q}_{\mathcal{D}}$. We determine its kernel and range distinguishing the case where $\alpha_{\mathcal{K}} \neq \alpha_{\mathcal{L}}$ and $\alpha_{\mathcal{K}} = \alpha_{\mathcal{L}}$ as follows.

PROPOSITION 2.2. *Let* $\mathcal{D} \in \mathfrak{D}$. *The kernel of* ${}^t\delta^{\mathcal{D}\,t}B_{\mathcal{Q}} + B_{\mathcal{Q}}\delta^{\mathcal{D}} = F_{\mathcal{Q}}, \forall \mathcal{Q} \in \mathfrak{Q}_{\mathcal{D}}$, *is:*

- *zero when* $\alpha_{\mathcal{K}} \neq \alpha_{\mathcal{L}}$,
- *generated by* $\delta_0 = {}^t\left(-\frac{{}^t\vec{\mathbf{n}}_{\boldsymbol{\sigma\mathcal{K}}}}{m_{\sigma_{\mathcal{K}}}}, \frac{{}^t\vec{\mathbf{n}}_{\boldsymbol{\sigma\mathcal{K}}}}{m_{\sigma_{\mathcal{L}}}}, \frac{{}^t\vec{\mathbf{n}}_{\boldsymbol{\sigma^*\mathcal{K}^*}}}{m_{\sigma_{\mathcal{K}^*}}}, -\frac{{}^t\vec{\mathbf{n}}_{\boldsymbol{\sigma^*\mathcal{K}^*}}}{m_{\sigma_{\mathcal{L}^*}}}\right)$ *when* $\alpha_{\mathcal{K}} = \alpha_{\mathcal{L}}$.

To have the existence of $\delta^{\mathcal{D}}$, *the second member have to satisfy the following conditions:*

- *when* $\alpha_{\mathcal{K}} \neq \alpha_{\mathcal{L}}$,

$$\begin{cases} F_{\mathcal{Q}} \textit{ have to be symmetric }, \quad \forall \mathcal{Q} \in \mathfrak{Q}_{\mathcal{D}}, \quad \sum\limits_{\mathcal{Q}\in\mathfrak{Q}_{\mathcal{D}}} m_{\mathcal{Q}}F_{\mathcal{Q}} = 0, \\ m_{\mathcal{Q}_{\mathcal{K},\mathcal{K}^*}}\,{}^t\vec{\boldsymbol{\tau}}_{\boldsymbol{\mathcal{K},\mathcal{L}}}F_{\mathcal{Q}_{\mathcal{K},\mathcal{K}^*}}\vec{\boldsymbol{\tau}}_{\boldsymbol{\mathcal{K},\mathcal{L}}} + m_{\mathcal{Q}_{\mathcal{K},\mathcal{L}^*}}\,{}^t\vec{\boldsymbol{\tau}}_{\boldsymbol{\mathcal{K},\mathcal{L}}}F_{\mathcal{Q}_{\mathcal{K},\mathcal{L}^*}}\vec{\boldsymbol{\tau}}_{\boldsymbol{\mathcal{K},\mathcal{L}}} = 0, \end{cases} \tag{2.9}$$

- *when* $\alpha_{\mathcal{K}} = \alpha_{\mathcal{L}}$,

$$\begin{cases} F_{\mathcal{Q}} \textit{ have to be symmetric }, \quad \forall \mathcal{Q} \in \mathfrak{Q}_{\mathcal{D}}, \quad \sum\limits_{\mathcal{Q}\in\mathfrak{Q}_{\mathcal{D}}} m_{\mathcal{Q}}F_{\mathcal{Q}} = 0, \\ m_{\mathcal{Q}_{\mathcal{K},\mathcal{K}^*}}\,{}^t\vec{\boldsymbol{\tau}}_{\boldsymbol{\mathcal{K},\mathcal{L}}}F_{\mathcal{Q}_{\mathcal{K},\mathcal{K}^*}}\vec{\boldsymbol{\tau}}_{\boldsymbol{\mathcal{K},\mathcal{L}}} + m_{\mathcal{Q}_{\mathcal{K},\mathcal{L}^*}}\,{}^t\vec{\boldsymbol{\tau}}_{\boldsymbol{\mathcal{K},\mathcal{L}}}F_{\mathcal{Q}_{\mathcal{K},\mathcal{L}^*}}\vec{\boldsymbol{\tau}}_{\boldsymbol{\mathcal{K},\mathcal{L}}} = 0, \\ m_{\mathcal{Q}_{\mathcal{K},\mathcal{K}^*}}\,{}^t\vec{\boldsymbol{\tau}}_{\boldsymbol{\mathcal{K}^*,\mathcal{L}^*}}F_{\mathcal{Q}_{\mathcal{K},\mathcal{K}^*}}\vec{\boldsymbol{\tau}}_{\boldsymbol{\mathcal{K}^*,\mathcal{L}^*}} + m_{\mathcal{Q}_{\mathcal{L},\mathcal{K}^*}}\,{}^t\vec{\boldsymbol{\tau}}_{\boldsymbol{\mathcal{K}^*,\mathcal{L}^*}}F_{\mathcal{Q}_{\mathcal{L},\mathcal{K}^*}}\vec{\boldsymbol{\tau}}_{\boldsymbol{\mathcal{K}^*,\mathcal{L}^*}} = 0. \end{cases} \tag{2.10}$$

To sum up, if the second member satisfies the above conditions, there exists a unique $\delta^{\mathcal{D}} \in \mathcal{M}_{n_{\mathcal{D}},2}(\mathbb{R})$ *satisfying* ${}^t\delta^{\mathcal{D}}{}^t B_{\mathcal{Q}} + B_{\mathcal{Q}}\delta^{\mathcal{D}} = F_{\mathcal{Q}}$, $\forall \mathcal{Q} \in \mathfrak{Q}_{\mathcal{D}}$, *and if* $\alpha_{\mathcal{K}} = \alpha_{\mathcal{L}}$, *we add* $(\delta^{\mathcal{D}}, \delta_0) = 0$.

PROPOSITION 2.3. *We have* $\forall \widehat{\delta^{\mathcal{D}}} \in \mathcal{M}_{n_{\mathcal{D}},2}(\mathbb{R})$ *such that* $\mathrm{Tr}(B_{\mathcal{Q}}\widehat{\delta^{\mathcal{D}}}) = 0$, $\forall \mathcal{Q} \in \mathfrak{Q}_{\mathcal{D}}$,

$$2\sum_{\mathcal{Q}\in\mathfrak{Q}_{\mathcal{D}}} m_{\mathcal{Q}}\eta_{\mathcal{Q}}(\mathrm{D}^{\mathcal{N}}_{\mathcal{Q}}\mathbf{u}^{\boldsymbol{\tau}} : B_{\mathcal{Q}}\widehat{\delta^{\mathcal{D}}}) = \sum_{\mathcal{Q}\in\mathfrak{Q}_{\mathcal{D}}} m_{\mathcal{Q}}\eta_{\mathcal{Q}}(\mathrm{D}^{\mathcal{N}}_{\mathcal{Q}}\mathbf{u}^{\boldsymbol{\tau}} : B_{\mathcal{Q}}\widehat{\delta^{\mathcal{D}}} + {}^t\widehat{\delta^{\mathcal{D}}}{}^t B_{\mathcal{Q}}) = 0.$$

Proof. Multiplying (2.8a) by $\widehat{\delta^{\mathcal{D}}}$ and applying the trace operatorsince we have $(\mathrm{Id} : B_{\mathcal{Q}}\widehat{\delta^{\mathcal{D}}}) = \mathrm{Tr}(B_{\mathcal{Q}}\widehat{\delta^{\mathcal{D}}})$, it implies the result. □

In particular, Proposition 2.3 implies that

$$\sum_{\mathcal{Q}\in\mathfrak{Q}_{\mathcal{D}}} m_{\mathcal{Q}}\eta_{\mathcal{Q}}(\mathrm{D}^{\mathcal{N}}_{\mathcal{Q}}\mathbf{u}^{\boldsymbol{\tau}} : B_{\mathcal{Q}}\delta^{\mathcal{D}} + {}^t\delta^{\mathcal{D}}{}^t B_{\mathcal{Q}}) = 0. \tag{2.11}$$

Now we are able to prove the existence and uniqueness of $(\delta^{\mathcal{D}}, p^{\mathfrak{Q}_{\mathcal{D}}}) \in \mathcal{M}_{n_{\mathcal{D}},2}(\mathbb{R}) \times \mathbb{R}^{n_{\mathcal{D}}}$.

PROPOSITION 2.4. *For any* $(\mathrm{D}^{\mathfrak{D}}\mathbf{u}^{\boldsymbol{\tau}}, p^{\mathfrak{D}}) \in \left(\mathbb{R}^2\right)^{\mathcal{T}} \times \mathbb{R}^{\mathfrak{D}}$, *and for any* $\mathcal{D} \in \mathfrak{D}$, *there exists a unique* $(\delta^{\mathcal{D}}, p^{\mathfrak{Q}_{\mathcal{D}}}) \in \mathcal{M}_{n_{\mathcal{D}},2}(\mathbb{R}) \times \mathbb{R}^{n_{\mathcal{D}}}$ *satisfying* (2.8) *and if* $\alpha_{\mathcal{K}} = \alpha_{\mathcal{L}}$ *we impose* $(\delta^{\mathcal{D}}, \delta_0) = 0$, *where* $\delta_0 = {}^t\left(-\frac{{}^t\vec{\mathbf{n}}_{\boldsymbol{\sigma}\mathcal{K}}}{m_{\sigma_{\mathcal{K}}}}, \frac{{}^t\vec{\mathbf{n}}_{\boldsymbol{\sigma}\mathcal{K}}}{m_{\sigma_{\mathcal{L}}}}, \frac{{}^t\vec{\mathbf{n}}_{\boldsymbol{\sigma}^*\mathcal{K}^*}}{m_{\sigma_{\mathcal{K}^*}}}, -\frac{{}^t\vec{\mathbf{n}}_{\boldsymbol{\sigma}^*\mathcal{K}^*}}{m_{\sigma_{\mathcal{L}^*}}}\right)$.

Proof. We only give the proof for $\mathcal{D} \in \mathfrak{D}_{int}$ (so that $n_{\mathcal{D}} = 4$), since the case of boundary diamond cells can be treated in the same way. We can write the systems (2.8) like a linear rectangle system $A\delta = b$ with $\delta = (\delta^{\mathcal{D}}_1; \delta^{\mathcal{D}}_2; p^{\mathfrak{Q}_{\mathcal{D}}}) \in \mathbb{R}^{3n_{\mathcal{D}}}$ and $A \in \mathcal{M}_{3n_{\mathcal{D}}+1,3n_{\mathcal{D}}}(\mathbb{R})$.

We are interesting in the injectivity of the matrix A. We assume that $\mathrm{D}^{\mathcal{D}}\mathbf{u}^{\boldsymbol{\tau}}$ and $p^{\mathcal{D}}$ are zero, thus the second member b is zero. Thanks to Proposition 2.3 and to $\mathrm{D}^{\mathcal{D}}\mathbf{u}^{\boldsymbol{\tau}} = 0$, we get

$$\sum_{\mathcal{Q}\in\mathfrak{Q}_{\mathcal{D}}} m_{\mathcal{Q}}\eta_{\mathcal{Q}}({}^t\delta^{\mathcal{D}}{}^t B_{\mathcal{Q}} + B_{\mathcal{Q}}\delta^{\mathcal{D}} : B_{\mathcal{Q}}\delta^{\mathcal{D}}) = 0.$$

Remark that the Frobenius scalar product of a symmetric and antisymmetric tensor is equal to zero, we get $\sum_{\mathcal{Q}\in\mathfrak{Q}_{\mathcal{D}}} m_{\mathcal{Q}}\eta_{\mathcal{Q}}\|{}^t\delta^{\mathcal{D}}{}^t B_{\mathcal{Q}} + B_{\mathcal{Q}}\delta^{\mathcal{D}}\|^2_{\mathcal{F}} = 0$. Therefore, it implies $B_{\mathcal{Q}}\delta^{\mathcal{D}} + {}^t\delta^{\mathcal{D}}{}^t B_{\mathcal{Q}} = 0$, for all $\mathcal{Q} \in \mathfrak{Q}_{\mathcal{D}}$. Using the last equality in (2.6), we get that $p_{\mathcal{Q}_{\mathcal{K},\mathcal{K}^*}}$, $p_{\mathcal{Q}_{\mathcal{L},\mathcal{K}^*}}$, $p_{\mathcal{Q}_{\mathcal{K},\mathcal{L}^*}}$ and $p_{\mathcal{Q}_{\mathcal{L},\mathcal{L}^*}}$ are equal and thanks to $\sum_{\mathcal{Q}\in\mathfrak{Q}_{\mathcal{D}}} m_{\mathcal{Q}}p^{\mathcal{Q}} = 0$, we get $p^{\mathfrak{Q}_{\mathcal{D}}} = 0$.

•*Case* $\alpha_{\mathcal{K}} \neq \alpha_{\mathcal{L}}$. Thanks to Proposition 2.2, we must have $\delta^{\mathcal{D}} = 0$ and $p^{\mathfrak{Q}_{\mathcal{D}}} = 0$. The matrix A is injective. We determine the kernel of its transpose $\mathrm{Ker}\,{}^t A$:

$$\mathrm{Ker}\,{}^t A = \mathrm{Span}\,{}^t(0, \cdots, 0, m_{\mathcal{Q}_{\mathcal{K},\mathcal{K}^*}}, m_{\mathcal{Q}_{\mathcal{K},\mathcal{L}^*}}, m_{\mathcal{Q}_{\mathcal{L},\mathcal{K}^*}}, m_{\mathcal{Q}_{\mathcal{L},\mathcal{L}^*}}, 0).$$

For the existence of δ, we have to prove that $b \in \mathrm{Im}A = \left(\mathrm{Ker}\,{}^t A\right)^o$. Let $X \in \left(\mathrm{Ker}\,{}^t A\right)^o$. We immediately get that $(X, b) = 0$.

•*Case* $\alpha_{\mathcal{K}} = \alpha_{\mathcal{L}}$. In this case, thanks to Proposition 2.2, we must have $\delta^{\mathcal{D}} = \lambda\delta_0$, $\lambda \in \mathbb{R}$, dim(Ker$A$)=1. The matrix A is not injective. Furthermore, we impose $(\delta^{\mathcal{D}}, \delta_0) = 0$ thus λ is equal to zero. We determine the kernel of its transpose $\mathrm{Ker}\,{}^t A$. We have

$$\mathrm{Ker}\,{}^t A = \mathrm{Span}\Big({}^t(-\vec{\mathbf{n}}_{\boldsymbol{\sigma}_{\mathcal{K}}\mathcal{K}^*}, \vec{\mathbf{n}}_{\boldsymbol{\sigma}_{\mathcal{K}}\mathcal{K}^*}, \vec{\mathbf{n}}_{\boldsymbol{\sigma}\mathcal{K}}, -\vec{\mathbf{n}}_{\boldsymbol{\sigma}\mathcal{K}}, 0 \cdots, 0),$$
$${}^t(0, \cdots, 0, m_{\mathcal{Q}_{\mathcal{K},\mathcal{K}^*}}, m_{\mathcal{Q}_{\mathcal{K},\mathcal{L}^*}}, m_{\mathcal{Q}_{\mathcal{L},\mathcal{K}^*}}, m_{\mathcal{Q}_{\mathcal{L},\mathcal{L}^*}}, 0)\Big).$$

For the existence of δ, we have to prove that $b \in \mathrm{Im}A = \left(\mathrm{Ker}\,{}^t A\right)^o$. Let $X \in \left(\mathrm{Ker}\,{}^t A\right)^o$, we have prove in the case $\alpha_{\mathcal{K}} \neq \alpha_{\mathcal{L}}$ that for X equal to the second span vector, we have $(X, b) = 0$. We just have to prove that for $X = {}^t\big(-\vec{\mathbf{n}}_{\boldsymbol{\sigma}_{\mathcal{K}}\mathcal{K}^*}, \vec{\mathbf{n}}_{\boldsymbol{\sigma}_{\mathcal{K}}\mathcal{K}^*}, \vec{\mathbf{n}}_{\boldsymbol{\sigma}\mathcal{K}}, -\vec{\mathbf{n}}_{\boldsymbol{\sigma}\mathcal{K}}, 0, 0, 0, 0, 0\big)$,

we also have $(X,b)=0$. Using the fact that $\mathrm{D}^{\mathcal{D}}\mathbf{u}^{\boldsymbol{\mathcal{T}}}$ is symmetric, ie $(\mathrm{D}^{\mathcal{D}}\mathbf{u}^{\boldsymbol{\mathcal{T}}}\vec{\mathbf{n}}_{\boldsymbol{\sigma\mathcal{K}}},\vec{\mathbf{n}}_{\boldsymbol{\sigma_{\mathcal{K}}\mathcal{K}^*}}) = (\mathrm{D}^{\mathcal{D}}\mathbf{u}^{\boldsymbol{\mathcal{T}}}\vec{\mathbf{n}}_{\boldsymbol{\sigma_{\mathcal{K}}\mathcal{K}^*}},\vec{\mathbf{n}}_{\boldsymbol{\sigma\mathcal{K}}})$, we get

$$\begin{aligned}(X,b) =& -2(\eta_{\mathcal{Q}_{\mathcal{K},\mathcal{K}^*}}-\eta_{\mathcal{Q}_{\mathcal{L},\mathcal{K}^*}})(\mathrm{D}^{\mathcal{D}}\mathbf{u}^{\boldsymbol{\mathcal{T}}}\vec{\mathbf{n}}_{\boldsymbol{\sigma\mathcal{K}}},\vec{\mathbf{n}}_{\boldsymbol{\sigma_{\mathcal{K}}\mathcal{K}^*}}) + 2(\eta_{\mathcal{Q}_{\mathcal{K},\mathcal{L}^*}}-\eta_{\mathcal{Q}_{\mathcal{L},\mathcal{L}^*}})(\mathrm{D}^{\mathcal{D}}\mathbf{u}^{\boldsymbol{\mathcal{T}}}\vec{\mathbf{n}}_{\boldsymbol{\sigma\mathcal{K}}},\vec{\mathbf{n}}_{\boldsymbol{\sigma_{\mathcal{K}}\mathcal{K}^*}})\\ &+2(\eta_{\mathcal{Q}_{\mathcal{K},\mathcal{K}^*}}-\eta_{\mathcal{Q}_{\mathcal{K},\mathcal{L}^*}})(\mathrm{D}^{\mathcal{D}}\mathbf{u}^{\boldsymbol{\mathcal{T}}}\vec{\mathbf{n}}_{\boldsymbol{\sigma_{\mathcal{K}}\mathcal{K}^*}},\vec{\mathbf{n}}_{\boldsymbol{\sigma\mathcal{K}}}) - 2(\eta_{\mathcal{Q}_{\mathcal{L},\mathcal{K}^*}}-\eta_{\mathcal{Q}_{\mathcal{L},\mathcal{L}^*}})(\mathrm{D}^{\mathcal{D}}\mathbf{u}^{\boldsymbol{\mathcal{T}}}\vec{\mathbf{n}}_{\boldsymbol{\sigma_{\mathcal{K}}\mathcal{K}^*}},\vec{\mathbf{n}}_{\boldsymbol{\sigma\mathcal{K}}})\end{aligned}$$

We deduce $(X,b)=0$. Therefore $b\in \mathrm{Im}A$, we deduce the existence of δ. □

2.4.2. Properties of the artificial unknowns. Since we have the presence of the full velocity tensor $\mathrm{D}\mathbf{u}$, we have to prove discrete Korn inequalities. We want now to prove one for the new discrete operators $\mathrm{D}^{\mathcal{N}}_{\Omega}$ and $\nabla^{\mathcal{N}}_{\Omega}$ (see Theorem 4.2). The extension to this new operator is far from straight forward. The difficulty leads on the new artificial unknowns $\delta^{\mathfrak{D}}$. First of all, we prove estimates between $B_{\mathcal{Q}}\delta^{\mathcal{D}}$ and $B_{\mathcal{Q}}\delta^{\mathcal{D}}+{}^t\delta^{\mathcal{D}}{}^tB_{\mathcal{Q}}$. Like in the proof of the existence of $\delta^{\mathcal{D}}$, the two cases $\alpha_{\mathcal{K}}=\alpha_{\mathcal{L}}$ and $\alpha_{\mathcal{K}}\neq\alpha_{\mathcal{L}}$ have to been investigate. The following Lemma is proved in Subsection 9.1.

LEMMA 2.2. *For all $\mathcal{D}\in\mathfrak{D}$, for all $\delta^{\mathcal{D}}\in\mathcal{M}_{n_{\mathcal{D}},2}(\mathbb{R})$ such that if $\alpha_{\mathcal{K}}=\alpha_{\mathcal{L}}$ we have $(\delta^{\mathcal{D}},\delta_0)=0$, there exists $C_1>0$, depending only on $\mathrm{reg}(\mathcal{T})$ and $\sin(\epsilon_0)$, such that*

$$\sum_{\mathcal{Q}\in\Omega_{\mathcal{D}}} m_{\mathcal{Q}}\|B_{\mathcal{Q}}\delta^{\mathcal{D}}\|^2_{\mathcal{F}} \leq C_1\sum_{\mathcal{Q}\in\Omega_{\mathcal{D}}} m_{\mathcal{Q}}\|B_{\mathcal{Q}}\delta^{\mathcal{D}}+{}^t\delta^{\mathcal{D}}{}^tB_{\mathcal{Q}}\|^2_{\mathcal{F}},$$

where $\delta_0 = {}^t\Big(-\frac{{}^t\vec{\mathbf{n}}_{\boldsymbol{\sigma\mathcal{K}}}}{m_{\sigma_{\mathcal{K}}}},\frac{{}^t\vec{\mathbf{n}}_{\boldsymbol{\sigma\mathcal{K}}}}{m_{\sigma_{\mathcal{L}}}},\frac{{}^t\vec{\mathbf{n}}_{\boldsymbol{\sigma^*\mathcal{K}^*}}}{m_{\sigma_{\mathcal{K}^*}}},-\frac{{}^t\vec{\mathbf{n}}_{\boldsymbol{\sigma^*\mathcal{K}^*}}}{m_{\sigma_{\mathcal{L}^*}}}\Big)$.

We want to bring out the form of the artificial unknowns $(\delta^{\mathcal{D}},p^{\Omega_{\mathcal{D}}})$ on the diamond cells.

REMARK 2.5. *If η is constant per sub-domains, equal to η_1 on Ω_1 and to η_2 on Ω_2. For*

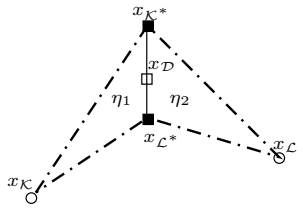


FIG. 2.4. *The viscosity on $\mathcal{D}\cap\Gamma\neq\emptyset$.*

$\mathcal{D}\cap\Gamma\neq\emptyset$ (see Figure 2.4), the solution $(\delta^{\mathcal{D}},p^{\Omega_{\mathcal{D}}})$ is equal to

$$\begin{aligned}\delta_{\mathcal{K}}=\delta_{\mathcal{L}}=0,\quad \delta_{\mathcal{K}^*}=\delta_{\mathcal{L}^*} &= -\frac{m_{\mathcal{Q}_{\mathcal{K},\mathcal{K}^*}}m_{\mathcal{Q}_{\mathcal{L},\mathcal{K}^*}}(\eta_1-\eta_2)\mathrm{D}^{\mathcal{D}}\mathbf{u}^{\boldsymbol{\mathcal{T}}}\vec{\mathbf{n}}_{\boldsymbol{\sigma\mathcal{K}}}\cdot\vec{\boldsymbol{\tau}}_{\boldsymbol{\mathcal{K},\mathcal{L}}}}{\eta_2 m_{\mathcal{Q}_{\mathcal{K},\mathcal{K}^*}}+\eta_1 m_{\mathcal{Q}_{\mathcal{L},\mathcal{K}^*}}}\vec{\boldsymbol{\tau}}_{\boldsymbol{\mathcal{K},\mathcal{L}}}\\ p_{\mathcal{Q}_{\mathcal{K},\mathcal{K}^*}} &= p^{\mathcal{D}}+2(\eta_1-\eta_2)\mathrm{D}^{\mathcal{D}}\mathbf{u}^{\boldsymbol{\mathcal{T}}}\vec{\mathbf{n}}_{\boldsymbol{\sigma\mathcal{K}}}\cdot\vec{\mathbf{n}}_{\boldsymbol{\sigma\mathcal{K}}}\frac{m_{\mathcal{Q}_{\mathcal{L},\mathcal{K}^*}}}{m_{\mathcal{Q}_{\mathcal{K},\mathcal{K}^*}}+m_{\mathcal{Q}_{\mathcal{L},\mathcal{K}^*}}},\quad p_{\mathcal{Q}_{\mathcal{K},\mathcal{L}^*}}=p_{\mathcal{Q}_{\mathcal{K},\mathcal{K}^*}},\\ p_{\mathcal{Q}_{\mathcal{L},\mathcal{K}^*}} &= p^{\mathcal{D}}+2(\eta_2-\eta_1)\mathrm{D}^{\mathcal{D}}\mathbf{u}^{\boldsymbol{\mathcal{T}}}\vec{\mathbf{n}}_{\boldsymbol{\sigma\mathcal{K}}}\cdot\vec{\mathbf{n}}_{\boldsymbol{\sigma\mathcal{K}}}\frac{m_{\mathcal{Q}_{\mathcal{K},\mathcal{K}^*}}}{m_{\mathcal{Q}_{\mathcal{K},\mathcal{K}^*}}+m_{\mathcal{Q}_{\mathcal{L},\mathcal{K}^*}}},\quad p_{\mathcal{Q}_{\mathcal{L},\mathcal{L}^*}}=p_{\mathcal{Q}_{\mathcal{L},\mathcal{K}^*}}.\end{aligned}$$

We generalize the form of $p^{\Omega_{\mathcal{D}}}$ as follows. The result is proved in Subsection 9.2.

LEMMA 2.3. *For any $\mathcal{D}\in\mathfrak{D}$, any $(\mathrm{D}^{\mathcal{D}}\mathbf{u}^{\boldsymbol{\mathcal{T}}},p^{\mathcal{D}})\in\mathcal{M}_2(\mathbb{R})\times\mathbb{R}$, there exists $C_2>0$, depending only on $\mathrm{reg}(\mathcal{T})$ and $\overline{\mathrm{C}}_{\eta}$, and a function $\alpha_{\mathfrak{s},\mathcal{D}}$ such that the solution $(\delta^{\mathcal{D}},p^{\Omega_{\mathcal{D}}})$ of (2.8) with $(\mathrm{D}^{\mathcal{D}}\mathbf{u}^{\boldsymbol{\mathcal{T}}},p^{\mathcal{D}})$ as the following form*

$$p^{\mathcal{Q}}=p^{\mathcal{D}}+\alpha_{\mathfrak{s},\mathcal{D}}(\mathrm{D}^{\mathcal{D}}\mathbf{u}^{\boldsymbol{\mathcal{T}}}),\qquad \text{where } \mathfrak{s}=\partial\mathcal{D}\cap\partial\mathcal{Q},$$

with $|\alpha_{\mathfrak{s},\mathcal{D}}(q^{\mathcal{D}})|^2\leq C_2\|q^{\mathcal{D}}\|^2_{\mathcal{F}}$, for any $q^{\mathcal{D}}\in\mathcal{M}_2(\mathbb{R})$.

2.5. Inner products and norms. We define the four following inner products

$$\begin{aligned}
&[\![\mathbf{v}^{\boldsymbol{\mathcal{T}}},\mathbf{u}^{\boldsymbol{\mathcal{T}}}]\!]_{\mathcal{T}}=\frac{1}{2}\left(\sum_{\mathcal{K}\in\mathfrak{M}}m_{\mathcal{K}}\mathbf{u}_{\boldsymbol{\mathcal{K}}}\cdot\mathbf{v}_{\boldsymbol{\mathcal{K}}}+\sum_{\mathcal{K}^*\in\mathfrak{M}^*\cup\partial\mathfrak{M}^*}m_{\mathcal{K}^*}\mathbf{u}_{\boldsymbol{\mathcal{K}}^*}\cdot\mathbf{v}_{\boldsymbol{\mathcal{K}}^*}\right),\quad\forall\mathbf{u}^{\boldsymbol{\mathcal{T}}},\mathbf{v}^{\boldsymbol{\mathcal{T}}}\in\left(\mathbb{R}^2\right)^{\mathcal{T}},\\
&(p^{\mathfrak{D}},q^{\mathfrak{D}})_{\mathfrak{D}}=\sum_{\mathcal{D}\in\mathfrak{D}}m_{\mathcal{D}}p^{\mathcal{D}}q^{\mathcal{D}},\quad\forall p^{\mathfrak{D}},q^{\mathfrak{D}}\in\mathbb{R}^{\mathfrak{D}},\\
&(\xi^{\mathfrak{D}}:\phi^{\mathfrak{D}})_{\mathfrak{D}}=\sum_{\mathcal{D}\in\mathfrak{D}}m_{\mathcal{D}}(\xi^{\mathcal{D}}:\phi^{\mathcal{D}}),\quad\forall\xi^{\mathfrak{D}},\phi^{\mathfrak{D}}\in(\mathcal{M}_2(\mathbb{R}))^{\mathfrak{D}},\\
&(\xi^{\mathfrak{Q}}:\phi^{\mathfrak{Q}})_{\mathfrak{Q}}=\sum_{\mathcal{Q}\in\mathfrak{Q}}m_{\mathcal{Q}}(\xi^{\mathcal{Q}}:\phi^{\mathcal{Q}}),\quad\forall\xi^{\mathfrak{Q}},\phi^{\mathfrak{Q}}\in(\mathcal{M}_2(\mathbb{R}))^{\mathfrak{Q}}.
\end{aligned}$$

We define the corresponding norms as follows

$$\begin{aligned}
&\|\mathbf{u}^{\boldsymbol{\mathcal{T}}}\|_2=[\![\mathbf{u}^{\boldsymbol{\mathcal{T}}},\mathbf{u}^{\boldsymbol{\mathcal{T}}}]\!]_{\mathcal{T}}^{\frac{1}{2}},\quad\forall\mathbf{u}^{\boldsymbol{\mathcal{T}}}\in\left(\mathbb{R}^2\right)^{\mathcal{T}},\\
&\|p^{\mathfrak{D}}\|_2=(p^{\mathfrak{D}},p^{\mathfrak{D}})_{\mathfrak{D}}^{\frac{1}{2}},\quad\forall p^{\mathfrak{D}}\in\mathbb{R}^{\mathfrak{D}},\qquad[\![\xi^{\mathfrak{D}}]\!]_2=(\xi^{\mathfrak{D}}:\xi^{\mathfrak{D}})_{\mathfrak{D}}^{\frac{1}{2}},\quad\forall\xi^{\mathfrak{D}}\in(\mathcal{M}_2(\mathbb{R}))^{\mathfrak{D}},\\
&\|q^{\mathfrak{Q}}\|_2=(q^{\mathfrak{Q}},q^{\mathfrak{Q}})_{\mathfrak{Q}}^{\frac{1}{2}},\quad\forall q^{\mathfrak{Q}}\in\mathbb{R}^{\mathfrak{Q}},\qquad[\![\xi^{\mathfrak{Q}}]\!]_2=(\xi^{\mathfrak{Q}}:\xi^{\mathfrak{Q}})_{\mathfrak{Q}}^{\frac{1}{2}},\quad\forall\xi^{\mathfrak{Q}}\in(\mathcal{M}_2(\mathbb{R}))^{\mathfrak{Q}}.
\end{aligned}$$

For any $\mathcal{D}\in\mathfrak{D}$, we define a norm over $\mathfrak{Q}_{\mathcal{D}}$ as follows

$$[\![A]\!]_{\mathfrak{Q}_{\mathcal{D}}}^2=\sum_{\mathcal{Q}\in\mathfrak{Q}_{\mathcal{D}}}\int_{\mathcal{Q}}[\![A(z)]\!]_{\mathcal{F}}^2\mathrm{d}z,\qquad\forall A\in L^2(\mathfrak{Q}_{\mathcal{D}},\mathcal{M}_2(\mathbb{R})).$$

2.6. Preparation of the stabilization procedure. Now we can define the new stabilization term, that considers the jumps of the pressure on quarter diamond cells.

DEFINITION 2.9. *The new stabilization term is a non consistent discrete approximation of the operator Δp, denoted by $\underline{\Delta}^{\mathfrak{D}}:p^{\mathfrak{Q}}\in\mathbb{R}^{\mathfrak{Q}}\mapsto\underline{\Delta}^{\mathfrak{D}}p^{\mathfrak{Q}}\in\mathbb{R}^{\mathfrak{D}}$, and defined as follows:*

$$\underline{\Delta}^{\mathcal{D}}p^{\mathfrak{Q}}=\frac{1}{m_{\mathcal{D}}}\sum_{\substack{\mathfrak{s}=\mathcal{Q}|\mathcal{Q}'\\=\mathcal{D}|\mathcal{D}'\in\mathcal{E}_{\mathcal{D}}}}\frac{h_{\mathcal{D}}^2+h_{\mathcal{D}'}^2}{h_{\mathcal{D}}^2}(p^{\mathcal{Q}'}-p^{\mathcal{Q}}),\qquad\forall\,\mathcal{D}\in\mathfrak{D}.$$

Note that we do not need a consistent approximation of the laplacian operator. In fact, a consistent approximation based on a two-point flux formula would require the diamond mesh to verify an orthogonality constraint as, for instance, in the case of *admissible* meshes [11], which has no reason to hold here. Then, we define two other second order discrete difference operators. The first one is the following.

DEFINITION 2.10. *We define a non consistent discrete approximation of the Laplace operator Δp, denoted by $\Delta^{\mathfrak{D}}:p^{\mathfrak{D}}\in\mathbb{R}^{\mathfrak{D}}\mapsto\Delta^{\mathfrak{D}}p^{\mathfrak{D}}\in\mathbb{R}^{\mathfrak{D}}$, and defined as follows:*

$$\Delta^{\mathcal{D}}p^{\mathfrak{D}}=\frac{1}{m_{\mathcal{D}}}\sum_{\mathfrak{s}=\mathcal{D}|\mathcal{D}'\in\mathcal{E}_{\mathcal{D}}}\frac{h_{\mathcal{D}}^2+h_{\mathcal{D}'}^2}{h_{\mathcal{D}}^2}(p^{\mathcal{D}'}-p^{\mathcal{D}}),\qquad\forall\,\mathcal{D}\in\mathfrak{D}.$$

Related to this operator, we define a mesh dependent semi-norm $|\cdot|_h$ over $\mathbb{R}^{\mathfrak{D}}$ by:

DEFINITION 2.11. *We define a discrete semi-norm for any $p^{\mathfrak{D}}\in\mathbb{R}^{\mathfrak{D}}$:*

$$|p^{\mathfrak{D}}|_h^2=\sum_{\mathfrak{s}=\mathcal{D}|\mathcal{D}'\in\mathfrak{S}}(h_{\mathcal{D}}^2+h_{\mathcal{D}'}^2)(p^{\mathcal{D}'}-p^{\mathcal{D}})^2.$$

The semi-norm $|p|_h$ is the discrete counterpart of $\mathrm{size}(\mathcal{T})|\nabla p|_2$. We have that

$$-(h_{\mathfrak{D}}^2\Delta^{\mathfrak{D}}p^{\mathfrak{D}},p^{\mathfrak{D}})_{\mathfrak{D}}=|p^{\mathfrak{D}}|_h^2,\qquad\forall\,p^{\mathfrak{D}}\in\mathbb{R}^{\mathfrak{D}}.\tag{2.12}$$

The second one uses the function $\alpha_{\mathfrak{s},\mathcal{D}}$ as follows.

DEFINITION 2.12. *We define a new discrete laplacian for any* $q^{\mathfrak{D}} \in (\mathcal{M}_2(\mathbb{R}))^{\mathfrak{D}}$*:*

$$\Delta_\alpha^{\mathcal{D}}(q^{\mathfrak{D}}) = \frac{1}{m_{\mathcal{D}}} \sum_{\mathfrak{s}=\mathcal{D}|\mathcal{D}'\in\mathcal{E}_{\mathcal{D}}} \frac{h_{\mathcal{D}}^2 + h_{\mathcal{D}'}^2}{h_{\mathcal{D}}^2}(\alpha_{\mathfrak{s},\mathcal{D}'}(q^{\mathcal{D}'}) - \alpha_{\mathfrak{s},\mathcal{D}}(q^{\mathcal{D}})),\ \forall \mathcal{D} \in \mathfrak{D},$$

where $\alpha_{\mathfrak{s},\mathcal{D}}$ *is the function defined in Lemma 2.3.*

We introduce the corresponding semi-norm.

DEFINITION 2.13. *We define a new discrete semi-norm for any* $q^{\mathfrak{D}} \in (\mathcal{M}_2(\mathbb{R}))^{\mathfrak{D}}$*:*

$$|q^{\mathfrak{D}}|_{\alpha,h}^2 := \sum_{\mathfrak{s}=\mathcal{D}|\mathcal{D}'\in\mathfrak{S}} (h_{\mathcal{D}}^2 + h_{\mathcal{D}'}^2)(\alpha_{\mathfrak{s},\mathcal{D}'}(q^{\mathcal{D}'}) - \alpha_{\mathfrak{s},\mathcal{D}}(q^{\mathcal{D}}))^2,$$

where $\alpha_{\mathfrak{s},\mathcal{D}}$ *is the function defined in Lemma 2.3.*

Thanks to $|\alpha_{\mathfrak{s},\mathcal{D}}(q^{\mathcal{D}})|^2 \leq C_2 |\!|\!| q^{\mathcal{D}} |\!|\!|_{\mathcal{F}}^2$ and relation (2.1), Cauchy-Schwarz inequality imply that

$$|q^{\mathfrak{D}}|_{\alpha,h}^2 \leq C_3 |\!|\!| q^{\mathfrak{D}} |\!|\!|_2^2, \qquad \forall\, q^{\mathfrak{D}} \in (\mathcal{M}_2(\mathbb{R}))^{\mathfrak{D}}, \tag{2.13}$$

with $C_3 = 8C_2 \mathrm{reg}(\mathcal{T})^2(1 + \mathrm{reg}(\mathcal{T})^2)$. Lemma 2.3, Definition 2.10 and 2.12 imply that

$$h_{\mathcal{D}}^2 m_{\mathcal{D}} \underline{\Delta}^{\mathcal{D}} p^{\mathfrak{Q}} = h_{\mathcal{D}}^2 m_{\mathcal{D}} (\Delta^{\mathcal{D}}(p^{\mathfrak{D}}) + \mu \Delta_\alpha^{\mathcal{D}}(\mathrm{D}^{\mathfrak{D}} \mathbf{u}^{\boldsymbol{\tau}})), \qquad \text{with } \mu = 1. \tag{2.14}$$

3. DDFV schemes for the Stokes equation. As claimed in introduction, we integrate the momentum conservation law of the problem (1.1) on the primal mesh $\mathfrak{M}$ and the interior dual mesh $\mathfrak{M}^*$. The mass conservation equation is directly approached on the diamond mesh using the discrete operator $\mathrm{div}^{\mathfrak{D}}$ and a stabilized term inspired by the well known Brezzi-Pitkäranta method [5]. We impose on $\partial\mathfrak{M}$ and on $\partial\mathfrak{M}^*$ the Dirichlet boundary conditions. Finally, the integral of the pressure is imposed to be equal to zero. The only difference with the scheme 1.4 introduced in [19] is in the viscous stress tensor and the stabilization term, to take into account the jumps of the pressure. We replace $\eta^{\mathfrak{D}} \mathrm{D}^{\mathfrak{D}} \mathbf{u}^{\boldsymbol{\tau}}$ (respectively $-\lambda h_{\mathfrak{D}}^2 \Delta^{\mathfrak{D}} p^{\mathfrak{D}}$) by $\mathrm{D}_{\mathfrak{D}}^{\eta,\mathcal{N}} \mathbf{u}^{\boldsymbol{\tau}}$, defined by Definition 2.7, (respectively $-\lambda h_{\mathfrak{D}}^2 \underline{\Delta}^{\mathfrak{D}} p^{\mathfrak{Q}}$) as follows:

$$\begin{cases} \text{Find } \mathbf{u}^{\boldsymbol{\tau}} \in \mathbb{E}_0 \text{ and } p^{\mathfrak{D}} \in \mathbb{R}^{\mathfrak{D}} \text{ such that,} \\ \mathbf{div}^{\mathfrak{M}}(-2\mathrm{D}_{\mathfrak{D}}^{\eta,\mathcal{N}} \mathbf{u}^{\boldsymbol{\tau}} + p^{\mathfrak{D}} \mathrm{Id}) = \mathbf{f}^{\mathfrak{M}}, \qquad \mathbf{div}^{\mathfrak{M}^*}(-2\mathrm{D}_{\mathfrak{D}}^{\eta,\mathcal{N}} \mathbf{u}^{\boldsymbol{\tau}} + p^{\mathfrak{D}} \mathrm{Id}) = \mathbf{f}^{\mathfrak{M}^*}, \\ \mathrm{div}^{\mathfrak{D}}(\mathbf{u}^{\boldsymbol{\tau}}) - \lambda h_{\mathfrak{D}}^2 \underline{\Delta}^{\mathfrak{D}} p^{\mathfrak{Q}} = 0, \qquad \sum_{\mathcal{D}\in\mathfrak{D}} m_{\mathcal{D}} p^{\mathcal{D}} = 0, \end{cases} \tag{3.1}$$

where $\lambda > 0$ given, $\mathbf{f}^{\mathfrak{M}} = \widetilde{\mathbb{P}}_{\boldsymbol{m}}^{\mathfrak{M}} \mathbf{f}$ and $\mathbf{f}^{\mathfrak{M}^*} = \widetilde{\mathbb{P}}_{\boldsymbol{m}}^{\mathfrak{M}^*} \mathbf{f}$ where the projections are defined by (2.2), and $(\delta^{\mathcal{D}}, p^{\mathfrak{Q}_{\mathcal{D}}}) \in \mathcal{M}_{n_{\mathcal{D}},2}(\mathbb{R}) \times \mathbb{R}^{n_{\mathcal{D}}}$ satisfying (2.8). If we take $\mu = 0$ in (2.14), we recover the old stabilization term $-\lambda h_{\mathfrak{D}}^2 \Delta^{\mathfrak{D}} p^{\mathfrak{D}}$. In this case, the scheme is well-posed nevertheless the error estimates is an open problem, since we have take into account the jumps of pressure. The numerical tests also bring out the role of the new stabilization term ($\mu = 1$ in (2.14)).

4. Results on discrete operators. In this section, we present some several results on the discrete operators. In Section 4.1.1, we begin with estimates between the modified and old discrete strain rate tensor. Then, we want to obtain two discrete Korn inequality for the old and modified discrete strain rate tensor. We show that the discrete Korn inequality for the modified operators is not only an extension of the old one, it is obtained thanks to the old one proved in [19] and Lemma 2.2. This Lemma proves estimate between $B_{\mathcal{Q}}\delta^{\mathcal{D}}$ and $B_{\mathcal{Q}}\delta^{\mathcal{D}} + {}^t\delta^{\mathcal{D}\,t} B_{\mathcal{Q}}$. This is the main difference between the present study and the work of [3]. In other subsections, we sum up results of [19] and adapt them in the case of $\mathbf{u} \in (H^2(\omega))^2$.

4.1. Discrete strain rate tensor.

4.1.1. Estimations of the discrete strain rate tensor. We recall results proved in [19], and extend them on the quarter diamond cells. The first one is a consequence of Remark 1.1.

PROPOSITION 4.1. *For all* $\mathbf{u}^{\boldsymbol{\mathcal{T}}} \in \left(\mathbb{R}^2\right)^{\mathcal{T}}$*, we get*

$$\|\mathrm{D}^{\mathfrak{D}}\mathbf{u}^{\boldsymbol{\mathcal{T}}}\|_2 \leq \|\nabla^{\mathfrak{D}}\mathbf{u}^{\boldsymbol{\mathcal{T}}}\|_2 \qquad \textit{and} \qquad \|\mathrm{D}^{\mathcal{N}}_{\Omega}\mathbf{u}^{\boldsymbol{\mathcal{T}}}\|_2 \leq \|\nabla^{\mathcal{N}}_{\Omega}\mathbf{u}^{\boldsymbol{\mathcal{T}}}\|_2.$$

The discrete strain rate tensor and the modified one can be compared as follows.

LEMMA 4.1. *Assume that* η *satisfies* (1.2)*. There exists a constant* $C_4 > 0$*, depending only on* $\underline{\mathrm{C}}_\eta$ *and* $\overline{\mathrm{C}}_\eta$*, such that for all* $\mathbf{u}^{\boldsymbol{\mathcal{T}}} \in \left(\mathbb{R}^2\right)^{\mathcal{T}}$*:*

$$\|\mathrm{D}^{\mathfrak{D}}\mathbf{u}^{\boldsymbol{\mathcal{T}}}\|_2 \leq \|\mathrm{D}^{\mathcal{N}}_{\Omega}\mathbf{u}^{\boldsymbol{\mathcal{T}}}\|_2 \leq C_4\|\mathrm{D}^{\mathfrak{D}}\mathbf{u}^{\boldsymbol{\mathcal{T}}}\|_2.$$

Proof. **First estimate.** Let $\mathcal{D} \in \mathfrak{D}$. The relation 2.5 and Cauchy-Schwarz inequality give

$$m_{\mathcal{D}}\|\mathrm{D}^{\mathcal{D}}\mathbf{u}^{\boldsymbol{\mathcal{T}}}\|^2_{\mathcal{F}} = \sum_{\mathcal{Q}\in\mathfrak{Q}_{\mathcal{D}}} m_{\mathcal{Q}}(\mathrm{D}^{\mathcal{N}}_{\mathcal{Q}}\mathbf{u}^{\boldsymbol{\mathcal{T}}} : \mathrm{D}^{\mathcal{D}}\mathbf{u}^{\boldsymbol{\mathcal{T}}}) \leq \sum_{\mathcal{Q}\in\mathfrak{Q}_{\mathcal{D}}} m_{\mathcal{Q}}\|\mathrm{D}^{\mathcal{N}}_{\mathcal{Q}}\mathbf{u}^{\boldsymbol{\mathcal{T}}}\|_{\mathcal{F}}\|\mathrm{D}^{\mathcal{D}}\mathbf{u}^{\boldsymbol{\mathcal{T}}}\|_{\mathcal{F}}.$$

We apply once more Cauchy-Schwarz inequality $m_{\mathcal{D}}\|\mathrm{D}^{\mathcal{D}}\mathbf{u}^{\boldsymbol{\mathcal{T}}}\|^2_{\mathcal{F}} \leq \sum_{\mathcal{Q}\in\mathfrak{Q}_{\mathcal{D}}} m_{\mathcal{Q}}\|\mathrm{D}^{\mathcal{N}}_{\mathcal{Q}}\mathbf{u}^{\boldsymbol{\mathcal{T}}}\|^2_{\mathcal{F}}$, which concludes the first estimate.

Second estimate. Let $\mathcal{D} \in \mathfrak{D}$. Definition 2.6 of $\mathrm{D}^{\mathcal{N}}_{\mathcal{Q}}\mathbf{u}^{\boldsymbol{\mathcal{T}}}$ and (2.11) imply that

$$\sum_{\mathcal{Q}\in\mathfrak{Q}_{\mathcal{D}}} m_{\mathcal{Q}}\eta_{\mathcal{Q}}\|\mathrm{D}^{\mathcal{N}}_{\mathcal{Q}}\mathbf{u}^{\boldsymbol{\mathcal{T}}}\|^2_{\mathcal{F}} = \sum_{\mathcal{Q}\in\mathfrak{Q}_{\mathcal{D}}} m_{\mathcal{Q}}\eta_{\mathcal{Q}}(\mathrm{D}^{\mathcal{N}}_{\mathcal{Q}}\mathbf{u}^{\boldsymbol{\mathcal{T}}} : \mathrm{D}^{\mathcal{D}}\mathbf{u}^{\boldsymbol{\mathcal{T}}}).$$

Using (1.2) and applying Cauchy-Schwarz inequality, we obtain the result noting $C_4 = \frac{\overline{\mathrm{C}}_\eta}{\underline{\mathrm{C}}_\eta}$. □

4.1.2. Discrete Korn inequality. In this section, we recall the discrete Korn inequality proved in [19], and prove one for the new operators defined on quarter diamond cells.

THEOREM 4.1 (Discrete Korn inequality). *For all* $\mathbf{u}^{\boldsymbol{\mathcal{T}}} \in \mathbb{E}_0$*, we have*

$$\|\nabla^{\mathfrak{D}}\mathbf{u}^{\boldsymbol{\mathcal{T}}}\|_2 \leq \sqrt{2}\|\mathrm{D}^{\mathfrak{D}}\mathbf{u}^{\boldsymbol{\mathcal{T}}}\|_2.$$

THEOREM 4.2 (Discrete Korn inequality). *Assume that* η *satisfies* (1.2)*. There exists* $C_5 > 0$ *depending only on* $\underline{\mathrm{C}}_\eta$*,* $\overline{\mathrm{C}}_\eta$*,* $\mathrm{reg}(\mathcal{T})$ *and* $\sin(\epsilon_0)$ *such that for all* $\mathbf{u}^{\boldsymbol{\mathcal{T}}} \in \mathbb{E}_0$*:*

$$\|\nabla^{\mathcal{N}}_{\Omega}\mathbf{u}^{\boldsymbol{\mathcal{T}}}\|_2 \leq C_5\|\mathrm{D}^{\mathcal{N}}_{\Omega}\mathbf{u}^{\boldsymbol{\mathcal{T}}}\|_2.$$

Proof. The equality (2.11), Cauchy-Schwarz inequality, (1.2) and Lemma 2.2 imply that

$$\sum_{\mathcal{Q}\in\mathfrak{Q}_{\mathcal{D}}} m_{\mathcal{Q}}\|B_{\mathcal{Q}}\delta^{\mathcal{D}}\|^2_{\mathcal{F}} \leq C_1\frac{\overline{\mathrm{C}}^2_\eta}{\underline{\mathrm{C}}^2_\eta}m_{\mathcal{D}}\|\mathrm{D}^{\mathcal{D}}\mathbf{u}^{\boldsymbol{\mathcal{T}}}\|^2_{\mathcal{F}}.$$

Furthermore, the relation $\sum_{\mathcal{Q}\in\mathfrak{Q}_{\mathcal{D}}} m_{\mathcal{Q}}B_{\mathcal{Q}} = 0$ gives

$$\sum_{\mathcal{Q}\in\mathfrak{Q}_{\mathcal{D}}} m_{\mathcal{Q}}\|\nabla^{\mathcal{N}}_{\mathcal{Q}}\mathbf{u}^{\boldsymbol{\mathcal{T}}}\|^2_{\mathcal{F}} \leq m_{\mathcal{D}}\|\nabla^{\mathcal{D}}\mathbf{u}^{\boldsymbol{\mathcal{T}}}\|^2_{\mathcal{F}} + C_1\frac{\overline{\mathrm{C}}^2_\eta}{\underline{\mathrm{C}}^2_\eta}m_{\mathcal{D}}\|\mathrm{D}^{\mathcal{D}}\mathbf{u}^{\boldsymbol{\mathcal{T}}}\|^2_{\mathcal{F}}.$$

Using the discrete Korn inequality Theorem 4.1 and than Lemma 4.1, we conclude

$$\|\nabla^{\mathcal{N}}_{\Omega}\mathbf{u}^{\boldsymbol{\mathcal{T}}}\|^2_2 \leq \left(2 + C_1\frac{\overline{\mathrm{C}}^2_\eta}{\underline{\mathrm{C}}^2_\eta}\right)\|\mathrm{D}^{\mathfrak{D}}\mathbf{u}^{\boldsymbol{\mathcal{T}}}\|^2_2 \leq \left(2 + C_1\frac{\overline{\mathrm{C}}^2_\eta}{\underline{\mathrm{C}}^2_\eta}\right)\|\mathrm{D}^{\mathcal{N}}_{\Omega}\mathbf{u}^{\boldsymbol{\mathcal{T}}}\|^2_2.$$

□

Using Lemma 4.1, these two discrete Korn inequalities imply the comparison between the discrete gradient and the modified one, as follows.

LEMMA 4.2. *Assume that η satisfies* (1.2)*. There exists two constants $C_6, C_7 > 0$, depending only on $\underline{\mathrm{C}}_\eta$, $\overline{\mathrm{C}}_\eta$, $\mathrm{reg}(\mathcal{T})$ and $\sin(\epsilon_0)$ such that for all $\mathbf{u}^{\boldsymbol{\mathcal{T}}} \in \mathbb{E}_0$:*

$$C_6 \|\nabla^{\mathfrak{D}} \mathbf{u}^{\boldsymbol{\mathcal{T}}}\|_2 \leq \|\nabla^{\mathcal{N}}_{\mathfrak{Q}} \mathbf{u}^{\boldsymbol{\mathcal{T}}}\|_2 \leq C_7 \|\nabla^{\mathfrak{D}} \mathbf{u}^{\boldsymbol{\mathcal{T}}}\|_2.$$

4.2. Discrete Stokes formula. In [19], the discrete gradient and discrete divergence for a vector-value function are linked by a discrete Stokes formula, as follows.

THEOREM 4.3 (Discrete Stokes formula). *For all $\xi^{\mathfrak{D}} \in (\mathcal{M}_2(\mathbb{R}))^{\mathfrak{D}}$, $\mathbf{v}^{\boldsymbol{\mathcal{T}}} \in \mathbb{E}_0$:*

$$[\![\mathbf{div}^{\boldsymbol{\mathcal{T}}} \xi^{\mathfrak{D}}, \mathbf{v}^{\boldsymbol{\mathcal{T}}}]\!]_{\mathcal{T}} = -(\xi^{\mathfrak{D}} : \nabla^{\mathfrak{D}} \mathbf{v}^{\boldsymbol{\mathcal{T}}})_{\mathfrak{D}}.$$

Since we have introduced new discrete operators on the quarter diamond cells, we want to rewrite the discrete Stokes formula for a specific tensor $\mathrm{D}^{\eta,\mathcal{N}}_{\mathfrak{D}} \mathbf{u}^{\boldsymbol{\mathcal{T}}}$ on the quarter diamond cells.

THEOREM 4.4 (Formule de Green). *For all $\mathbf{u}^{\boldsymbol{\mathcal{T}}}, \mathbf{v}^{\boldsymbol{\mathcal{T}}} \in \left(\mathbb{R}^2\right)^{\mathcal{T}} \times \mathbb{E}_0$*

$$[\![\mathbf{div}^{\boldsymbol{\mathcal{T}}}(\mathrm{D}^{\eta,\mathcal{N}}_{\mathfrak{D}} \mathbf{u}^{\boldsymbol{\mathcal{T}}}), \mathbf{v}^{\boldsymbol{\mathcal{T}}}]\!]_{\mathcal{T}} = -(\eta^{\mathfrak{Q}} \mathrm{D}^{\mathcal{N}}_{\mathfrak{Q}} \mathbf{u}^{\boldsymbol{\mathcal{T}}} : \nabla^{\mathcal{N}}_{\mathfrak{Q}} \mathbf{v}^{\boldsymbol{\mathcal{T}}})_{\mathfrak{Q}}.$$

Proof. Thanks to Proposition 2.4, there exists a unique $(\widehat{\delta^{\mathcal{D}}}, \widehat{p^{\mathfrak{Q}_{\mathcal{D}}}}) \in \mathcal{M}_{n_{\mathcal{D}},2}(\mathbb{R}) \times \mathbb{R}^{n_{\mathcal{D}}}$ satisfied the conditions (2.8), with $\mathrm{D}^{\mathcal{D}} \mathbf{v}^{\boldsymbol{\mathcal{T}}}$ and $p^{\mathfrak{D}}$. Using the symmetry of $\mathrm{D}^{\mathcal{D}} \mathbf{v}^{\boldsymbol{\mathcal{T}}}$, we have

$$m_{\mathcal{D}}(\mathrm{D}^{\eta,\mathcal{N}}_{\mathcal{D}} \mathbf{u}^{\boldsymbol{\mathcal{T}}} : 2\mathrm{D}^{\mathcal{D}} \mathbf{v}^{\boldsymbol{\mathcal{T}}}) = \sum_{\mathcal{Q} \in \mathfrak{Q}_{\mathcal{D}}} m_{\mathcal{Q}} \eta_{\mathcal{Q}} (\mathrm{D}^{\mathcal{N}}_{\mathcal{Q}} \mathbf{u}^{\boldsymbol{\mathcal{T}}} : 2\mathrm{D}^{\mathcal{N}}_{\mathcal{Q}} \mathbf{v}^{\boldsymbol{\mathcal{T}}} - {}^t\widehat{\delta^{\mathcal{D}}}^t B_{\mathcal{Q}} - {}^t\widehat{\delta^{\mathcal{D}}}^t B_{\mathcal{Q}}).$$

Furthermore, since for all $\mathcal{Q} \in \mathfrak{Q}_{\mathcal{D}}$, $\mathrm{Tr}(B_{\mathcal{Q}} \widehat{\delta^{\mathcal{D}}}) = 0$, Proposition 2.3 implies that

$$\sum_{\mathcal{Q} \in \mathfrak{Q}_{\mathcal{D}}} m_{\mathcal{Q}} \eta_{\mathcal{Q}} (\mathrm{D}^{\mathcal{N}}_{\mathcal{Q}} \mathbf{u}^{\boldsymbol{\mathcal{T}}} : B_{\mathcal{Q}} \widehat{\delta^{\mathcal{D}}} + {}^t\widehat{\delta^{\mathcal{D}}}^t B_{\mathcal{Q}}) = 0.$$

Finally, we deduce that $m_{\mathcal{D}}(\mathrm{D}^{\eta,\mathcal{N}}_{\mathcal{D}} \mathbf{u}^{\boldsymbol{\mathcal{T}}} : \nabla^{\mathcal{D}} \mathbf{v}^{\boldsymbol{\mathcal{T}}}) = \sum_{\mathcal{Q} \in \mathfrak{Q}_{\mathcal{D}}} m_{\mathcal{Q}} \eta_{\mathcal{Q}} (\mathrm{D}^{\mathcal{N}}_{\mathcal{Q}} \mathbf{u}^{\boldsymbol{\mathcal{T}}} : \mathrm{D}^{\mathcal{N}}_{\mathcal{Q}} \mathbf{v}^{\boldsymbol{\mathcal{T}}})$. The symmetry of $\mathrm{D}^{\mathcal{N}}_{\mathcal{Q}} \mathbf{u}^{\boldsymbol{\mathcal{T}}}$ and the discrete Stokes formula 4.3 imply the result. □

4.3. Poincaré inequality. Properties of the mean-value projection operator. We recall results which are proved in [19] for the first four results or [1] for the last one. We begin with the discrete Poincaré inequality.

THEOREM 4.5 (Discrete Poincaré inequality). *Let $\mathcal{T}$ be a mesh of Ω. There exists a constant $C_8 > 0$, depending only on the diameter of Ω and $\mathrm{reg}(\mathcal{T})$, such that*

$$\|\mathbf{u}^{\boldsymbol{\mathcal{T}}}\|_2 \leq C_8 \|\nabla^{\mathfrak{D}} \mathbf{u}^{\boldsymbol{\mathcal{T}}}\|_2, \quad \forall \mathbf{u}^{\boldsymbol{\mathcal{T}}} \in \mathbb{E}_0.$$

We give below the main property of the mean-value projection onto the set of discrete functions in our framework.

LEMMA 4.3. *Let $\mathcal{T}$ be a mesh of Ω. There exists $C_9, C_{10} > 0$ depending only on $\mathrm{reg}(\mathcal{T})$, such that for any function $\mathbf{v}$ in $(H^1(\Omega))^2$, we have*

$$\|\nabla^{\mathfrak{D}} \mathbb{P}^{\boldsymbol{\mathcal{T}}}_{\boldsymbol{m}} \mathbf{v}\|_2 \leq C_9 \|\nabla \mathbf{v}\|_2 \text{ and } \sum_{\mathcal{D} \in \mathfrak{D}} \int_{\mathcal{D}} p^{\mathcal{D}} \left(\mathrm{div}^{\mathcal{D}}(\mathbb{P}^{\boldsymbol{\mathcal{T}}}_{\boldsymbol{m}} \mathbf{v}) - \mathrm{div}(\mathbf{v})\right) \mathrm{d}z \leq C_{10} |p^{\mathfrak{D}}|_h \|\mathbf{v}\|_{H^1}.$$

LEMMA 4.4. *Let $\mathcal{T}$ be a mesh of Ω. There exists $C_{11} > 0$ depending only on* $\mathrm{reg}(\mathcal{T})$*, such that for any $p^{\mathfrak{D}} \in \mathbb{R}^{\mathfrak{D}}$, we have*

$$|p^{\mathfrak{D}}|_h \leq C_{11}\|p^{\mathfrak{D}}\|_2.$$

Finally, the following result is similar to [1, Lemma 3.4].

LEMMA 4.5. *There exists a number $C_{12} > 0$ such that for any bounded set $\mathcal{P} \subset \mathbb{R}^2$ with positive measure, any segment $\sigma \subset \mathbb{R}^2$ and any $v \in H^1(\mathcal{P})$, we have*

$$|v_{\mathcal{P}} - v_\sigma|^2 \leq \frac{1}{m_\sigma m_{\mathcal{P}}} \int_\sigma \int_{\mathcal{P}} |v(x) - v(y)|^2 \mathrm{d}x\mathrm{d}y \leq C_{12} \frac{\mathrm{diam}(\widehat{\mathcal{P}_\sigma})^3}{m_\sigma m_{\mathcal{P}}} \int_{\widehat{\mathcal{P}_\sigma}} |\nabla v(z)|^2 \mathrm{d}z,$$

where $v_{\mathcal{P}}$ denotes the mean value of v on $\mathcal{P}$, v_σ the mean value of v on the segment σ, and $\widehat{\mathcal{P}_\sigma}$ is the convex hull of $\mathcal{P} \cup \sigma$.

4.4. Properties of the center-value projection operator. By usual Taylor inside each quarter diamond $\mathcal{Q}$ (see [3], for instance), we can easily show the main properties of the center-value projection onto the set of functions in $(H^2(\omega))^2$.

LEMMA 4.6. *Let $\mathcal{T}$ be a mesh of Ω. There exists a number $C_{13}, C_{14} > 0$, depending only on* $\mathrm{reg}(\mathcal{T})$*, such that for any function $\mathbf{v}$ in $(H^2(\omega))^2$, we have*

$$\|\mathbf{v} - \mathbb{P}_{\boldsymbol{c}}^{\boldsymbol{\mathcal{T}}}\mathbf{v}\|_2 \leq C_{13}\mathrm{size}(\mathcal{T})\|\mathbf{v}\|_{(H^2(\omega))^2},$$
$$m_{\mathcal{D}}\|\nabla^{\mathcal{D}}\mathbb{P}_{\boldsymbol{c}}^{\boldsymbol{\mathcal{T}}}\mathbf{v}\|_{\mathcal{F}}^2 \leq C_{14} \sum_{\mathcal{Q}\in\mathfrak{Q}_{\mathcal{D}}} \int_{\mathcal{Q}} \left(\|\nabla\mathbf{v}(z)\|_{\mathcal{F}}^2 + |\nabla^2\mathbf{v}(z)|^2\right) \mathrm{d}z.$$

We immediately have the inequality on the norm,

$$\|\nabla^{\mathfrak{D}}\mathbb{P}_{\boldsymbol{c}}^{\boldsymbol{\mathcal{T}}}\mathbf{v}\|_2 \leq C_{14}\|\mathbf{v}\|_{(H^2(\omega))^2}, \quad \forall \mathbf{v} \in (H^2(\omega))^2. \tag{4.1}$$

5. Stability of the scheme. In this section, we prove the uniform stability of our finite volume scheme and its well-posedness. The proof of the uniform stability result relies on an appropriate choice of the stabilization term. Let us introduce the bilinear form associated to our DDFV scheme:

DEFINITION 5.1. *We define the bilinear form associated to our DDFV scheme* (3.1)*:*

$$\forall\, (\mathbf{u}^{\boldsymbol{\mathcal{T}}}, p^{\mathfrak{D}}), (\widetilde{\mathbf{u}}^{\boldsymbol{\mathcal{T}}}, \widetilde{p}^{\mathfrak{D}}) \in \left(\mathbb{R}^2\right)^{\mathcal{T}} \times \mathbb{R}^{\mathfrak{D}},$$
$$B(\mathbf{u}^{\boldsymbol{\mathcal{T}}}, p^{\mathfrak{D}}; \widetilde{\mathbf{u}}^{\boldsymbol{\mathcal{T}}}, \widetilde{p}^{\mathfrak{D}}) = [\![\mathbf{div}^{\boldsymbol{\mathcal{T}}}(-2\mathrm{D}_{\mathfrak{D}}^{\eta,\mathcal{N}}\mathbf{u}^{\boldsymbol{\mathcal{T}}} + p^{\mathfrak{D}}\mathrm{Id}), \widetilde{\mathbf{u}}^{\boldsymbol{\mathcal{T}}}]\!]_{\mathcal{T}} + (\mathrm{div}^{\mathfrak{D}}(\mathbf{u}^{\boldsymbol{\mathcal{T}}}) - \lambda h_{\mathfrak{D}}^2 \underline{\Delta}^{\mathfrak{D}}(p^{\mathfrak{Q}}), \widetilde{p}^{\mathfrak{D}})_{\mathfrak{D}},$$

where the stabilization parameter λ is a positive number.

THEOREM 5.1 (Stability of the scheme). *Assume that η satisfies* (1.2) *and $\lambda < \frac{4\underline{\mathrm{C}}_\eta}{C_3}$. Then there exists $C_{15}, C_{16} > 0$, depending only on the diameter of Ω, λ, $\underline{\mathrm{C}}_\eta$, $\overline{\mathrm{C}}_\eta$,* $\mathrm{reg}(\mathcal{T})$ *and* $\sin(\epsilon_0)$*, for each pair $(\mathbf{u}^{\boldsymbol{\mathcal{T}}}, p^{\mathfrak{D}}) \in \mathbb{E}_0 \times \mathbb{R}^{\mathfrak{D}}$ such that $\sum_{\mathcal{D}\in\mathfrak{D}} m_{\mathcal{D}} p^{\mathcal{D}} = 0$, there exists $(\widetilde{\mathbf{u}}^{\boldsymbol{\mathcal{T}}}, \widetilde{p}^{\mathfrak{D}}) \in \mathbb{E}_0 \times \mathbb{R}^{\mathfrak{D}}$:*

$$\|\nabla_{\mathfrak{Q}}^{\mathcal{N}}\widetilde{\mathbf{u}}^{\boldsymbol{\mathcal{T}}}\|_2 + \|\widetilde{p}^{\mathfrak{Q}}\|_2 \leq C_{15}\left(\|\nabla_{\mathfrak{Q}}^{\mathcal{N}}\mathbf{u}^{\boldsymbol{\mathcal{T}}}\|_2 + \|p^{\mathfrak{Q}}\|_2\right), \tag{5.1}$$

and

$$\|\nabla_{\mathfrak{Q}}^{\mathcal{N}}\mathbf{u}^{\boldsymbol{\mathcal{T}}}\|_2^2 + \|p^{\mathfrak{Q}}\|_2^2 \leq C_{16}B(\mathbf{u}^{\boldsymbol{\mathcal{T}}}, p^{\mathfrak{D}}; \widetilde{\mathbf{u}}^{\boldsymbol{\mathcal{T}}}, \widetilde{p}^{\mathfrak{D}}). \tag{5.2}$$

with $(\delta^{\mathfrak{D}}, p^{\mathfrak{Q}})$ (resp. $(\widetilde{\delta}^{\mathfrak{D}}, \widetilde{p}^{\mathfrak{Q}})$) the solution of (2.8) *for $\mathrm{D}^{\mathfrak{D}}\mathbf{u}^{\boldsymbol{\mathcal{T}}}$ and $p^{\mathfrak{D}}$ (resp. $\mathrm{D}^{\mathfrak{D}}\widetilde{\mathbf{u}}^{\boldsymbol{\mathcal{T}}}$ and $\widetilde{p}^{\mathfrak{D}}$), thus we have $m_{\mathcal{D}}p^{\mathcal{D}} = \sum_{\mathcal{Q}\in\mathfrak{Q}_{\mathcal{D}}} m_{\mathcal{Q}}p^{\mathcal{Q}}$, for all $\mathcal{D} \in \mathfrak{D}$.*

Proof. Let $(\mathbf{u}^{\boldsymbol{\mathcal{T}}}, p^{\mathfrak{D}}) \in \mathbb{E}_0 \times \mathbb{R}^{\mathfrak{D}}$ such that $\sum_{\mathcal{D}\in\mathfrak{D}} m_{\mathcal{D}}p^{\mathcal{D}} = 0$. The proof of this Theorem is obtained by building explicitly $(\widetilde{\mathbf{u}}^{\boldsymbol{\mathcal{T}}}, \widetilde{p}^{\mathfrak{D}}) \in \mathbb{E}_0 \times \mathbb{R}^{\mathfrak{D}}$ such that (5.1) and (5.2) hold.

Step 1. We apply to B the two discrete Stokes formula 4.3 and 4.4, we get

$$B(\mathbf{u}^{\boldsymbol{\mathcal{T}}},p^{\mathfrak{D}};\mathbf{u}^{\boldsymbol{\mathcal{T}}},p^{\mathfrak{D}}) =(2\eta^{\mathfrak{Q}}\mathrm{D}_{\mathfrak{Q}}^{\mathcal{N}}\mathbf{u}^{\boldsymbol{\mathcal{T}}} : \nabla_{\mathfrak{Q}}^{\mathcal{N}}\mathbf{u}^{\boldsymbol{\mathcal{T}}})_{\mathfrak{Q}} - (\lambda h_{\mathfrak{D}}^2\underline{\Delta}^{\mathfrak{D}}(p^{\mathfrak{Q}}),p^{\mathfrak{D}})_{\mathfrak{D}}.$$

The symmetry of $\mathrm{D}_{\mathfrak{Q}}^{\mathcal{N}}\mathbf{u}^{\boldsymbol{\mathcal{T}}}$ and (2.14) imply that

$$B(\mathbf{u}^{\boldsymbol{\mathcal{T}}},p^{\mathfrak{D}};\mathbf{u}^{\boldsymbol{\mathcal{T}}},p^{\mathfrak{D}}) = (2\eta^{\mathfrak{Q}}\mathrm{D}_{\mathfrak{Q}}^{\mathcal{N}}\mathbf{u}^{\boldsymbol{\mathcal{T}}} : \mathrm{D}_{\mathfrak{Q}}^{\mathcal{N}}\mathbf{u}^{\boldsymbol{\mathcal{T}}})_{\mathfrak{Q}} - (\lambda h_{\mathfrak{D}}^2\Delta^{\mathfrak{D}}(p^{\mathfrak{D}}) + \lambda h_{\mathfrak{D}}^2\Delta_{\alpha}^{\mathfrak{D}}(\mathrm{D}^{\mathfrak{D}}\mathbf{u}^{\boldsymbol{\mathcal{T}}}),p^{\mathfrak{D}})_{\mathfrak{D}}.$$

Reorganizing the sum over $\mathfrak{s}\in\mathfrak{S}$, we have

$$-(h_{\mathfrak{D}}^2\Delta_{\alpha}^{\mathfrak{D}}(\mathrm{D}^{\mathfrak{D}}\mathbf{u}^{\boldsymbol{\mathcal{T}}}),p^{\mathfrak{D}})_{\mathfrak{D}} = \sum_{\mathfrak{s}=\mathcal{D}|\mathcal{D}'\in\mathfrak{S}}(h_{\mathcal{D}}^2+h_{\mathcal{D}'}^2)(\alpha_{\mathfrak{s},\mathcal{D}'}(\mathrm{D}^{\mathcal{D}'}\mathbf{u}^{\boldsymbol{\mathcal{T}}}) - \alpha_{\mathfrak{s},\mathcal{D}}(\mathrm{D}^{\mathcal{D}}\mathbf{u}^{\boldsymbol{\mathcal{T}}}))(p^{\mathcal{D}'}-p^{\mathcal{D}}).$$

Young inequality and (2.12) imply that

$$-(\lambda h_{\mathfrak{D}}^2\Delta^{\mathfrak{D}}(p^{\mathfrak{D}}) + \lambda h_{\mathfrak{D}}^2\Delta_{\alpha}^{\mathfrak{D}}(\mathrm{D}^{\mathfrak{D}}\mathbf{u}^{\boldsymbol{\mathcal{T}}}),p^{\mathfrak{D}})_{\mathfrak{D}} \geq \frac{\lambda}{2}|p^{\mathfrak{D}}|_h^2 - \frac{\lambda}{2}|\mathrm{D}^{\mathfrak{D}}\mathbf{u}^{\boldsymbol{\mathcal{T}}}|_{\alpha,h}^2.$$

Thanks to the inequality (1.2), we obtain

$$B(\mathbf{u}^{\boldsymbol{\mathcal{T}}},p^{\mathfrak{D}};\mathbf{u}^{\boldsymbol{\mathcal{T}}},p^{\mathfrak{D}}) \geq 2\underline{\mathrm{C}}_{\eta}\|\mathrm{D}_{\mathfrak{Q}}^{\mathcal{N}}\mathbf{u}^{\boldsymbol{\mathcal{T}}}\|_2^2 + \frac{\lambda}{2}|p^{\mathfrak{D}}|_h^2 - \frac{\lambda}{2}|\mathrm{D}^{\mathfrak{D}}\mathbf{u}^{\boldsymbol{\mathcal{T}}}|_{\alpha,h}^2.$$

Thanks to (2.13) and Lemma 4.1, we have $|\mathrm{D}^{\mathfrak{D}}\mathbf{u}^{\boldsymbol{\mathcal{T}}}|_{\alpha,h}^2 \leq C_3\|\mathrm{D}_{\mathfrak{Q}}^{\mathcal{N}}\mathbf{u}^{\boldsymbol{\mathcal{T}}}\|_2^2$. Finally we use the discrete Korn inequality (Theorem 4.2) in order to get

$$B(\mathbf{u}^{\boldsymbol{\mathcal{T}}},p^{\mathfrak{D}};\mathbf{u}^{\boldsymbol{\mathcal{T}}},p^{\mathfrak{D}}) \geq C_5\left(2\underline{\mathrm{C}}_{\eta} - C_3\frac{\lambda}{2}\right)\|\nabla_{\mathfrak{Q}}^{\mathcal{N}}\mathbf{u}^{\boldsymbol{\mathcal{T}}}\|_2^2 + \frac{\lambda}{2}|p^{\mathfrak{D}}|_h^2. \tag{5.3}$$

With $\lambda < \frac{4\underline{\mathrm{C}}_{\eta}}{C_3}$, constants in the above estimate are non negative. Note that the above estimate on the pressure is mesh dependent (the semi-norm $|.|_h$ is itself mesh dependent). That is why we could not bound uniformly the L^2-norm of the pressure by the semi-norm $|.|_h$.

Step 2. We use the Nečas Lemma (see [14, Corollary 2.4] or [2, Lemma III.1.17]): since $p^{\mathfrak{Q}} = \sum_{\mathcal{D}\in\mathfrak{D}}\sum_{\mathcal{Q}\in\mathfrak{Q}_{\mathcal{D}}} p^{\mathcal{Q}}\mathbf{1}_{\mathcal{Q}} \in L^2(\Omega)$ and its integral over Ω is zero, there exists a constant $C>0$ depending only on Ω, and $\mathbf{v}\in(H_0^1(\Omega))^2$ such that $\mathrm{div}(\mathbf{v}) = -p^{\mathfrak{Q}}$ and

$$\|\mathbf{v}\|_{H^1} \leq C\|p^{\mathfrak{Q}}\|_2. \tag{5.4}$$

Let us choose $\mathbf{v}^{\boldsymbol{\mathcal{T}}} = \mathbb{P}_{\boldsymbol{m}}^{\boldsymbol{\mathcal{T}}}\mathbf{v}$ the mean-value projection $\mathbb{P}_{\boldsymbol{m}}^{\boldsymbol{\mathcal{T}}}\mathbf{v}$, defined by (2.3). In particular, we have $\mathbf{v}^{\boldsymbol{\mathcal{T}}}\in\mathbb{E}_0$. Thanks to Lemma 4.1, Proposition 4.1 and Lemma 4.3, we deduce

$$\|\mathrm{D}_{\mathfrak{Q}}^{\mathcal{N}}\mathbf{v}^{\boldsymbol{\mathcal{T}}}\|_2 \leq C_4C_9C\|p^{\mathfrak{Q}}\|_2. \tag{5.5}$$

The discrete Stokes formula 4.4 implies

$$B(\mathbf{u}^{\boldsymbol{\mathcal{T}}},p^{\mathfrak{D}};\mathbf{v}^{\boldsymbol{\mathcal{T}}},0) = 2(\eta^{\mathfrak{Q}}\mathrm{D}_{\mathfrak{Q}}^{\mathcal{N}}\mathbf{u}^{\boldsymbol{\mathcal{T}}} : \nabla_{\mathfrak{Q}}^{\mathcal{N}}\mathbf{v}^{\boldsymbol{\mathcal{T}}})_{\mathfrak{Q}} - (p^{\mathfrak{D}},\mathrm{div}^{\mathfrak{D}}(\mathbf{v}^{\boldsymbol{\mathcal{T}}}))_{\mathfrak{D}}.$$

Using the fact that $(\eta^{\mathfrak{Q}}\mathrm{D}_{\mathfrak{Q}}^{\mathcal{N}}\mathbf{u}^{\boldsymbol{\mathcal{T}}} : \nabla_{\mathfrak{Q}}^{\mathcal{N}}\mathbf{v}^{\boldsymbol{\mathcal{T}}})_{\mathfrak{Q}} = (\eta^{\mathfrak{Q}}\nabla_{\mathfrak{Q}}^{\mathcal{N}}\mathbf{u}^{\boldsymbol{\mathcal{T}}} : \mathrm{D}_{\mathfrak{Q}}^{\mathcal{N}}\mathbf{v}^{\boldsymbol{\mathcal{T}}})_{\mathfrak{Q}}$ and the Cauchy-Schwarz inequality, we deduce

$$\begin{aligned}B(\mathbf{u}^{\boldsymbol{\mathcal{T}}},p^{\mathfrak{D}};\mathbf{v}^{\boldsymbol{\mathcal{T}}},0) \geq -\overline{\mathrm{C}}_{\eta}\|\nabla_{\mathfrak{Q}}^{\mathcal{N}}\mathbf{u}^{\boldsymbol{\mathcal{T}}}\|_2\|\mathrm{D}_{\mathfrak{Q}}^{\mathcal{N}}\mathbf{v}^{\boldsymbol{\mathcal{T}}}\|_2 - \sum_{\mathcal{D}\in\mathfrak{D}}\sum_{\mathcal{Q}\in\mathfrak{Q}_{\mathcal{D}}}\int_{\mathcal{Q}} p^{\mathcal{Q}}\mathrm{div}(\mathbf{v}(z))\mathrm{d}z\\ - \sum_{\mathcal{D}\in\mathfrak{D}}\int_{\mathcal{D}} p^{\mathcal{D}}\left(\mathrm{div}^{\mathcal{D}}(\mathbf{v}^{\boldsymbol{\mathcal{T}}}) - \mathrm{div}(\mathbf{v}(z))\right)\mathrm{d}z + \sum_{\mathcal{D}\in\mathfrak{D}}\sum_{\mathcal{Q}\in\mathfrak{Q}_{\mathcal{D}}}\int_{\mathcal{Q}}(p^{\mathcal{Q}}-p^{\mathcal{D}})\mathrm{div}(\mathbf{v}(z))\mathrm{d}z.\end{aligned}$$

Since we have $\mathrm{div}(\mathbf{v}) = -p^{\mathfrak{Q}}$ and the inequality (5.5) gives

$$B(\mathbf{u}^{\boldsymbol{\mathcal{T}}},p^{\mathfrak{D}};\mathbf{v}^{\boldsymbol{\mathcal{T}}},0) \geq -\overline{\mathrm{C}}_\eta C_4 C_9 C |\!|\!| \nabla_{\mathfrak{Q}}^{\mathcal{N}} \mathbf{u}^{\boldsymbol{\mathcal{T}}} |\!|\!|_2 \|p^{\mathfrak{Q}}\|_2 + \|p^{\mathfrak{Q}}\|_2^2 - \sum_{\mathcal{D}\in\mathfrak{D}} \int_{\mathcal{D}} p^{\mathcal{D}} \left(\mathrm{div}^{\mathcal{D}}(\mathbf{v}^{\boldsymbol{\mathcal{T}}}) - \mathrm{div}(\mathbf{v}(z))\right) \mathrm{d}z - \sum_{\mathcal{D}\in\mathfrak{D}} \sum_{\mathcal{Q}\in\mathfrak{Q}_{\mathcal{D}}} \int_{\mathcal{Q}} (p^{\mathcal{D}} - p^{\mathcal{Q}}) \mathrm{div}(\mathbf{v}(z)) \mathrm{d}z.$$

Thanks to Lemma 4.3 and to estimate (5.4), we obtain

$$\sum_{\mathcal{D}\in\mathfrak{D}} \int_{\mathcal{D}} p^{\mathcal{D}} \left(\mathrm{div}^{\mathcal{D}}(\mathbf{v}^{\boldsymbol{\mathcal{T}}}) - \mathrm{div}(\mathbf{v}(z))\right) \mathrm{d}z \leq C C_{10} |p^{\mathfrak{D}}|_h \|p^{\mathfrak{Q}}\|_2.$$

Thanks to Lemma 2.3, we have $p^{\mathcal{D}} - p^{\mathcal{Q}} = -\alpha_{\mathfrak{s},\mathcal{D}}(\mathrm{D}^{\mathcal{D}}\mathbf{u}^{\boldsymbol{\mathcal{T}}})$, with $\mathfrak{s} = \partial\mathcal{D} \cap \partial\mathcal{Q}$. Cauchy-Schwarz implies

$$\sum_{\mathcal{D}\in\mathfrak{D}} \sum_{\mathcal{Q}\in\mathfrak{Q}_{\mathcal{D}}} \int_{\mathcal{Q}} (p^{\mathcal{D}} - p^{\mathcal{Q}}) \mathrm{div}(\mathbf{v}(z)) \mathrm{d}z \leq \|\mathrm{div}(\mathbf{v})\|_2 \left(\sum_{\mathcal{D}\in\mathfrak{D}} \sum_{\mathfrak{s}\in\mathcal{E}_{\mathcal{D}}} m_{\mathcal{D}} |\alpha_{\mathfrak{s},\mathcal{D}}(\mathrm{D}^{\mathcal{D}}\mathbf{u}^{\boldsymbol{\mathcal{T}}})|^2 \right)^{\frac{1}{2}}.$$

Thanks to $\mathrm{div}(\mathbf{v}) = -p^{\mathfrak{Q}}$, Lemma 2.3, Lemma 4.1 and Remark 1.1 give

$$\sum_{\mathcal{D}\in\mathfrak{D}} \sum_{\mathcal{Q}\in\mathfrak{Q}_{\mathcal{D}}} \int_{\mathcal{Q}} (p^{\mathcal{D}} - p^{\mathcal{Q}}) \mathrm{div}(\mathbf{v}(z)) \mathrm{d}z \leq \sqrt{C_2} \|p^{\mathfrak{Q}}\|_2 |\!|\!| \nabla_{\mathfrak{Q}}^{\mathcal{N}} \mathbf{u}^{\boldsymbol{\mathcal{T}}} |\!|\!|_2.$$

We deduce that

$$B(\mathbf{u}^{\boldsymbol{\mathcal{T}}},p^{\mathfrak{D}};\mathbf{v}^{\boldsymbol{\mathcal{T}}},0) \geq \|p^{\mathfrak{Q}}\|_2^2 - \overline{\mathrm{C}}_\eta C_4 C_9 C \|p^{\mathfrak{Q}}\|_2 |\!|\!| \nabla_{\mathfrak{Q}}^{\mathcal{N}} \mathbf{u}^{\boldsymbol{\mathcal{T}}} |\!|\!|_2 - (\sqrt{C_2} + C C_{10}) |p^{\mathfrak{D}}|_h \|p^{\mathfrak{Q}}\|_2.$$

Using Young's inequality, we obtain the existence of three constants $\widetilde{C}_1, \widetilde{C}_2, \widetilde{C}_3 > 0$, depending only on Ω, $\overline{\mathrm{C}}_\eta$ and $\mathrm{reg}(\mathcal{T})$, such that

$$B(\mathbf{u}^{\boldsymbol{\mathcal{T}}},p^{\mathfrak{D}};\mathbf{v}^{\boldsymbol{\mathcal{T}}},0) \geq \widetilde{C}_1 \|p^{\mathfrak{Q}}\|_2^2 - \widetilde{C}_2 |\!|\!| \nabla_{\mathfrak{Q}}^{\mathcal{N}} \mathbf{u}^{\boldsymbol{\mathcal{T}}} |\!|\!|_2^2 - \widetilde{C}_3 |p^{\mathfrak{D}}|_h^2. \tag{5.6}$$

Step 3. By bilinearity of B,(5.3) and (5.6) give for each positive number $\xi > 0$:

$$B(\mathbf{u}^{\boldsymbol{\mathcal{T}}},p^{\mathfrak{D}};\mathbf{u}^{\boldsymbol{\mathcal{T}}} + \xi\mathbf{v}^{\boldsymbol{\mathcal{T}}},p^{\mathfrak{D}}) \geq \left(C_5 \left(2\underline{\mathrm{C}}_\eta - C_3 \frac{\lambda}{2} \right) - \xi \widetilde{C}_2 \right) |\!|\!| \nabla_{\mathfrak{Q}}^{\mathcal{N}} \mathbf{u}^{\boldsymbol{\mathcal{T}}} |\!|\!|_2^2 + \xi \widetilde{C}_1 \|p^{\mathfrak{Q}}\|_2^2 + \left(\frac{\lambda}{2} - \xi \widetilde{C}_3 \right) |p^{\mathfrak{D}}|_h^2.$$

Choosing a value of $\xi > 0$ small enough, this inequality yields an estimate of the form (5.2). As the relation (5.1) is clearly verified by the pair $\widetilde{\mathbf{u}}^{\boldsymbol{\mathcal{T}}} = \mathbf{u}^{\boldsymbol{\mathcal{T}}} + \xi\mathbf{v}^{\boldsymbol{\mathcal{T}}}$ and $\widetilde{p}^{\mathfrak{D}} = p^{\mathfrak{D}}$, this concludes the proof.

□

A consequence of this stability inequality is the well-posedness of the scheme (3.1).

THEOREM 5.2. *Assume that η satisfies* (1.2)*. For all mesh $\mathcal{T}$ as described in section 2, the finite volume scheme* (3.1) *with $0 < \lambda < \frac{4\underline{\mathrm{C}}_\eta}{C_3}$ admits a unique solution $(\mathbf{u}^{\boldsymbol{\mathcal{T}}}, p^{\mathfrak{D}}) \in \left(\mathbb{R}^2\right)^{\mathcal{T}} \times \mathbb{R}^{\mathfrak{D}}$.*

Proof. Let us consider the homogeneous discrete problem given by setting $\mathbf{f}^{\boldsymbol{\mathcal{T}}}$, the right-hand side of (3.1), to zero. Thanks to Theorem 5.1, there exists $\widetilde{\mathbf{u}}^{\boldsymbol{\mathcal{T}}} \in \mathbb{E}_0$, $\widetilde{p}^{\mathfrak{D}} \in \mathbb{R}^{\mathfrak{D}}$, such that

$$|\!|\!| \nabla_{\mathfrak{Q}}^{\mathcal{N}} \mathbf{u}^{\boldsymbol{\mathcal{T}}} |\!|\!|_2^2 + \|p^{\mathfrak{Q}}\|_2^2 \leq C_{16} B(\mathbf{u}^{\boldsymbol{\mathcal{T}}},p^{\mathfrak{D}};\widetilde{\mathbf{u}}^{\boldsymbol{\mathcal{T}}},\widetilde{p}^{\mathfrak{D}}).$$

Definition 5.1 of B implies that $B(\mathbf{u}^{\boldsymbol{\mathcal{T}}},p^{\mathfrak{D}};\widetilde{\mathbf{u}}^{\boldsymbol{\mathcal{T}}},\widetilde{p}^{\mathfrak{D}}) = 0$. It follows that $\nabla_{\mathfrak{Q}}^{\mathcal{N}} \mathbf{u}^{\boldsymbol{\mathcal{T}}} = 0$ and $p^{\mathfrak{Q}} = 0$, with $(\delta^{\mathcal{D}}, p^{\mathfrak{Q}})$ the solution of (2.8) with $\mathrm{D}^{\mathfrak{D}}\mathbf{u}^{\boldsymbol{\mathcal{T}}}$ and $p^{\mathfrak{D}}$. We deduce that $p^{\mathfrak{D}} = 0$. The former identity implies that the degrees of freedom of the velocity $\mathbf{u}^{\boldsymbol{\mathcal{T}}}$ are constant, since $\mathbf{u}^{\boldsymbol{\mathcal{T}}} \in \mathbb{E}_0$, we conclude that $\mathbf{u}^{\boldsymbol{\mathcal{T}}} = 0$. □

6. Error estimates.

6.1. Definitions. We define a projections of functions defined on Ω over the primal and dual meshes $\mathcal{T}$. We call the center-value projection for any continuous function $\mathbf{v}$ on $\overline{\Omega}$:

$$\mathbb{P}_{\boldsymbol{c}}^{\boldsymbol{\mathcal{T}}}\mathbf{v} = ((\mathbf{v}(x_{\mathcal{K}}))_{\mathcal{K}\in(\mathfrak{M}\cup\partial\mathfrak{M})}, (\mathbf{v}(x_{\mathcal{K}^*}))_{\mathcal{K}^*\in(\mathfrak{M}^*\cup\partial\mathfrak{M}^*)}), \qquad \forall \mathbf{v}\in(H^2(\omega))^2.$$

We also define a mean-value projection over the diamond mesh $\mathcal{D}$ and over the quarter diamond mesh $\mathcal{Q}$ for any integrable functions q on Ω:

$$\mathbb{P}_m^{\mathfrak{D}} q = \left(\left(\frac{1}{m_{\mathcal{D}}}\int_{\mathcal{D}} q(x)\mathrm{d}x\right)_{\mathcal{D}\in\mathfrak{D}}\right), \qquad \mathbb{P}_m^{\mathfrak{Q}} q = \left(\left(\frac{1}{m_{\mathcal{Q}}}\int_{\mathcal{Q}} q(x)\mathrm{d}x\right)_{\mathcal{Q}\in\mathfrak{Q}}\right).$$

The following proposition is a consequence of Proposition 2.4.

PROPOSITION 6.1. *Let $(\mathbf{u},p)$ the solution of the Stokes problem* (1.1)*. There exists $(\delta_{\mathrm{ex}}^{\mathcal{D}}, p_{\mathrm{ex}}^{\mathfrak{Q}_{\mathcal{D}}}) \in \mathcal{M}_{n_{\mathcal{D}},2}(\mathbb{R})\times\mathbb{R}^{n_{\mathcal{D}}}$ satisfying*

$$\sum_{\mathcal{Q}\in\mathfrak{Q}_{\mathcal{D}}} m_{\mathcal{Q}}\varphi_{\mathcal{Q}}(\mathrm{D}^{\mathcal{D}}\mathbb{P}_{\boldsymbol{c}}^{\boldsymbol{\mathcal{T}}}\mathbf{u}, \delta_{\mathrm{ex}}^{\mathcal{D}}, p_{\mathrm{ex}}^{\mathcal{Q}})B_{\mathcal{Q}} = 0,\ \mathrm{Tr}({}^t\delta_{\mathrm{ex}}^{\mathcal{D}}\,{}^tB_{\mathcal{Q}}) = 0,\ \forall\mathcal{Q}\in\mathfrak{Q}_{\mathcal{D}},\ \sum_{\mathcal{Q}\in\mathfrak{Q}_{\mathcal{D}}} m_{\mathcal{Q}}p_{\mathrm{ex}}^{\mathcal{Q}} = m_{\mathcal{D}}\mathbb{P}_m^{\mathcal{D}}p.$$

DEFINITION 6.1. *Let $\widetilde{p}_{\mathrm{ex}}^{\mathcal{Q}} = \mathbb{P}_m^{\mathcal{Q}}p - p_{\mathrm{ex}}^{\mathcal{Q}}$, $\forall\mathcal{Q}\in\mathfrak{Q}$, and $\sum_{\mathcal{Q}\in\mathfrak{Q}_{\mathcal{D}}} m_{\mathcal{Q}}\widetilde{p}_{\mathrm{ex}}^{\mathcal{Q}} = 0$.*

As usual for the error analysis of the finite volume methods, the consistency error which has to be studied is the error on the numerical fluxes across each of the edges and dual edges in the mesh. We first give the precise definition of these terms, then we state the various estimates need to prove the error estimates.

DEFINITION 6.2. *For any $\mathcal{Q}\in\mathfrak{Q}$, we define the consistency errors in $\mathcal{Q}$ by*

$$R_{\mathcal{Q}}^{\mathbf{v}}(z) = \eta_{|\overline{\mathcal{Q}}}(z)\mathrm{D}\mathbf{v}_{|\overline{\mathcal{Q}}}(z) - \eta_{\mathcal{Q}}\mathrm{D}_{\mathcal{Q}}^{\mathcal{N}}\mathbb{P}_{\boldsymbol{c}}^{\boldsymbol{\mathcal{T}}}\mathbf{v}, \quad R_{\mathcal{Q}}^{p}(z) = \mathbb{P}_m^{\mathcal{Q}}p - p_{|\overline{\mathcal{Q}}}(z), \qquad \forall\, z\in\mathcal{D}.$$

We introduce the following consistency errors on the numerical fluxes, for all $\mathfrak{s} = \mathcal{Q}|\mathcal{Q}'\in\mathcal{E}_{\mathcal{Q}}$:

$$\mathbf{R}_{\mathfrak{s},\mathcal{Q}}^{i} = -\mathbf{R}_{\mathfrak{s},\mathcal{Q}'}^{i} = \frac{1}{m_{\mathfrak{s}}}\int_{\mathfrak{s}} R_{\mathcal{Q}}^{i}(z)\vec{\mathbf{n}}_{\mathfrak{s}\boldsymbol{\mathcal{Q}}}\mathrm{d}z, \qquad i = \mathbf{v}, p.$$

We note the L^2-norm of the consistency error as follows:

$$\|\mathbf{R}_{\mathfrak{s},\mathfrak{Q}}^{i}\|_2^2 = \sum_{\mathcal{Q}\in\mathfrak{Q}}\sum_{\mathfrak{s}\in\mathcal{E}_{\mathcal{Q}}} m_{\mathcal{Q}}|\mathbf{R}_{\mathfrak{s},\mathcal{Q}}^{i}|^2, \qquad i = \mathbf{v}, p.$$

Thanks to Definition 6.1, we define for all $\mathfrak{s} = \mathcal{Q}|\mathcal{Q}'\in\mathcal{E}_{\mathcal{Q}}$:

$$\mathbf{R}_{\mathfrak{s},\mathcal{Q}} = 2\mathbf{R}_{\mathfrak{s},\mathcal{Q}}^{\mathbf{u}} + \mathbf{R}_{\mathfrak{s},\mathcal{Q}}^{p} - \widetilde{p}_{\mathrm{ex}}^{\mathcal{Q}}\vec{\mathbf{n}}_{\mathfrak{s}\boldsymbol{\mathcal{Q}}}. \tag{6.1}$$

6.2. Statement of the result and sketch of proof. We conclude by providing an error estimate in case where the exact solution of the problem (1.1) lies in the space $(H^2(\omega))^2\times H^1(\omega)$. Our main result is the following

THEOREM 6.1. *Assume that η satisfies* (1.2) *and* (2.4) *and $0<\lambda<\frac{4\underline{\mathrm{C}}_\eta}{C_3}$. We assume that the solution $(\mathbf{u},p)$ of the Stokes problem* (1.1) *belongs to $(H^2(\omega))^2\times H^1(\omega)$. Let $(\mathbf{u}^{\boldsymbol{\mathcal{T}}}, p^{\mathfrak{D}})\in\left(\mathbb{R}^2\right)^{\mathcal{T}}\times\mathbb{R}^{\mathfrak{D}}$ be the solution of the scheme* (3.1)*. There exists a constant $C_{17}>0$ depending only on $\mathrm{reg}(\mathcal{T})$, λ, $\sin(\epsilon_0)$, C_η, $\underline{\mathrm{C}}_\eta$, $\overline{\mathrm{C}}_\eta$, $\|\mathbf{u}\|_{(H^2(\omega))^2}$ and $\|p\|_{H^1(\omega)}$, such that:*

$$\|\mathbf{u}-\mathbf{u}^{\boldsymbol{\mathcal{T}}}\|_2 + |||\nabla\mathbf{u} - \nabla_{\mathfrak{Q}}^{\mathcal{N}}\mathbf{u}^{\boldsymbol{\mathcal{T}}}|||_2 \le C_{17}\mathrm{size}(\mathcal{T}) \quad \textit{and} \quad \|p-p^{\mathfrak{Q}}\|_2 \le C_{17}\mathrm{size}(\mathcal{T}).$$

with $(\delta^{\mathfrak{D}}, p^{\mathfrak{Q}})$ the solution of (2.8) *for $\mathrm{D}^{\mathfrak{D}}\mathbf{u}^{\boldsymbol{\mathcal{T}}}$ and $p^{\mathfrak{D}}$.*

Step 1. Let $\mathbf{e}^{\boldsymbol{\mathcal{T}}} = \mathbb{P}_{\mathbf{c}}^{\boldsymbol{\mathcal{T}}}\mathbf{u} - \mathbf{u}^{\boldsymbol{\mathcal{T}}} \in \mathbb{E}_0$ denote the approximation error for the velocity solution field and $e^{\mathfrak{D}} = \mathbb{P}_m^{\mathfrak{D}} p - p^{\mathfrak{D}} \in \mathbb{R}^{\mathfrak{D}}$ the approximation error for the pressure solution field. Thanks to (3.1) and (1.1), we have $\forall \kappa \in \mathfrak{M}$

$$\mathbf{div}^{\boldsymbol{\kappa}}(-2\mathrm{D}_{\mathfrak{D}}^{\eta,\mathcal{N}}\mathbf{u}^{\boldsymbol{\mathcal{T}}} + p^{\mathfrak{D}} Id) = \mathbf{f}_{\boldsymbol{\kappa}} = -\frac{1}{m_\kappa}\int_\kappa \mathrm{div}(2\eta(x)\mathrm{D}\mathbf{u}(x))\mathrm{d}x + \frac{1}{m_\kappa}\int_\kappa \nabla p(x)\mathrm{d}x.$$

Therefore, Definition 2.2 of $\mathbf{div}^{\boldsymbol{\kappa}}$ and the continuous Stokes formula imply that

$$\begin{aligned} m_\kappa \mathbf{div}^{\boldsymbol{\kappa}}(-2\mathrm{D}_{\mathfrak{D}}^{\eta,\mathcal{N}}\mathbf{e}^{\boldsymbol{\mathcal{T}}} + e^{\mathfrak{D}} Id) = &\sum_{\mathcal{Q}\subset\kappa}\sum_{\mathfrak{s}\in\mathcal{E}_{\mathcal{Q}}\cap\partial\kappa}\int_{\mathfrak{s}} 2\eta(z)\mathrm{D}\mathbf{u}(z)\vec{\mathbf{n}}_{\mathfrak{s}\mathcal{Q}}\mathrm{d}z \\ &- \sum_{\mathcal{D}\in\mathfrak{D}_\kappa} m_\sigma(2\mathrm{D}_{\mathcal{D}}^{\eta,\mathcal{N}}\mathbb{P}_{\mathbf{c}}^{\boldsymbol{\mathcal{T}}}\mathbf{u})\vec{\mathbf{n}}_{\boldsymbol{\sigma\kappa}} + \sum_{\mathcal{D}\in\mathfrak{D}_\kappa} m_\sigma \mathbb{P}_m^{\mathcal{D}} p\vec{\mathbf{n}}_{\boldsymbol{\sigma\kappa}} - \sum_{\mathcal{Q}\subset\kappa}\sum_{\mathfrak{s}\in\mathcal{E}_{\mathcal{Q}}\cap\partial\kappa}\int_{\mathfrak{s}} p(z)\vec{\mathbf{n}}_{\mathfrak{s}\mathcal{Q}}\mathrm{d}z. \end{aligned}$$

Using Definition 2.8 of $\varphi_{\mathcal{Q}}$ and Definition 6.1, we deduce for any $\mathcal{D}\in\mathfrak{D}_\kappa$,

$$\frac{m_\sigma}{m_{\mathcal{D}}}\sum_{\mathcal{Q}\in\mathfrak{Q}_{\mathcal{D}}} m_{\mathcal{Q}}\varphi_{\mathcal{Q}}(\mathrm{D}^{\mathcal{D}}\mathbb{P}_{\mathbf{c}}^{\boldsymbol{\mathcal{T}}}\mathbf{u}, \delta_{\mathrm{ex}}^{\mathcal{D}}, p_{\mathrm{ex}}^{\mathfrak{Q}_{\mathcal{D}}})\vec{\mathbf{n}}_{\boldsymbol{\sigma\kappa}} = m_{\sigma_{\kappa^*}}\varphi_{\mathcal{Q}_{\kappa,\kappa^*}}\vec{\mathbf{n}}_{\boldsymbol{\sigma\kappa}} + m_{\sigma_{\mathcal{L}^*}}\varphi_{\mathcal{Q}_{\kappa,\mathcal{L}^*}}\vec{\mathbf{n}}_{\boldsymbol{\sigma\kappa}}.$$

Thanks to Definition 6.2 of the consistency error and Definition 6.1, we deduce

$$m_\kappa \mathbf{div}^{\boldsymbol{\kappa}}(-2\mathrm{D}_{\mathfrak{D}}^{\eta,\mathcal{N}}\mathbf{e}^{\boldsymbol{\mathcal{T}}} + e^{\mathfrak{D}} Id) = \sum_{\mathcal{Q}\subset\kappa}\sum_{\mathfrak{s}\in\mathcal{E}_{\mathcal{Q}}\cap\partial\kappa} m_{\mathfrak{s}}\mathbf{R}_{\mathfrak{s},\mathcal{Q}}.$$

In the same way, for $\kappa^*\in\mathfrak{M}^*$. Finally, the couple $(\mathbf{e}^{\boldsymbol{\mathcal{T}}}, e^{\mathfrak{D}}) \in \mathbb{E}_0\times\mathbb{R}^{\mathfrak{D}}$ satisfies :

$$\begin{cases} \mathbf{div}^{\mathfrak{M}}(-2\mathrm{D}_{\mathfrak{D}}^{\eta,\mathcal{N}}\mathbf{e}^{\boldsymbol{\mathcal{T}}} + e^{\mathfrak{D}}\mathrm{Id}) = \mathbf{R}_{\mathfrak{M}}, & \mathbf{div}^{\mathfrak{M}^*}(-2\mathrm{D}_{\mathfrak{D}}^{\eta,\mathcal{N}}\mathbf{e}^{\boldsymbol{\mathcal{T}}} + e^{\mathfrak{D}}\mathrm{Id}) = \mathbf{R}_{\mathfrak{M}^*}, \\ \mathrm{div}^{\mathfrak{D}}(\mathbf{e}^{\boldsymbol{\mathcal{T}}}) - \lambda h_{\mathfrak{D}}^2 \underline{\Delta}^{\mathfrak{D}} e^{\mathfrak{Q}} = R_{\mathfrak{D}}, & \sum_{\mathcal{D}\in\mathfrak{D}} m_{\mathcal{D}} e^{\mathcal{D}} = 0, \end{cases} \tag{6.2}$$

where

$$\mathbf{R}_\kappa = \frac{1}{m_\kappa}\sum_{\mathcal{Q}\subset\kappa}\sum_{\mathfrak{s}\in\mathcal{E}_{\mathcal{Q}}\cap\partial\kappa} m_{\mathfrak{s}}\mathbf{R}_{\mathfrak{s},\mathcal{Q}},\ \forall\,\kappa\in\mathfrak{M}, \quad \mathbf{R}_{\kappa^*} = \frac{1}{m_{\kappa^*}}\sum_{\mathcal{Q}\subset\kappa^*}\sum_{\mathfrak{s}\in\mathcal{E}_{\mathcal{Q}}\cap\partial\kappa^*} m_{\mathfrak{s}}\mathbf{R}_{\mathfrak{s},\mathcal{Q}},\ \forall\,\kappa^*\in\mathfrak{M}^*,$$
$$R_{\mathcal{D}} = \mathrm{div}^{\mathcal{D}}(\mathbb{P}_{\mathbf{c}}^{\boldsymbol{\mathcal{T}}}\mathbf{u}) - \lambda h_{\mathcal{D}}^2 \underline{\Delta}^{\mathcal{D}} p_{\mathrm{ex}}^{\mathfrak{Q}},\ \forall\,\mathcal{D}\in\mathfrak{D}, \quad e^{\mathcal{Q}} = p_{\mathrm{ex}}^{\mathcal{Q}} - p^{\mathcal{Q}},\ \forall\,\mathcal{Q}\in\mathfrak{Q}.$$

Theorem 5.1 implies that there exists $\widetilde{\mathbf{e}}^{\boldsymbol{\mathcal{T}}}\in\mathbb{E}_0$, $\widetilde{e}^{\mathfrak{D}}\in\mathbb{R}^{\mathfrak{D}}$ such that :

$$|||\nabla_{\mathfrak{Q}}^{\mathcal{N}}\widetilde{\mathbf{e}}^{\boldsymbol{\mathcal{T}}}|||_2 + \|\widetilde{e}^{\mathfrak{Q}}\|_2 \le C_{15}\left(|||\nabla_{\mathfrak{Q}}^{\mathcal{N}}\mathbf{e}^{\boldsymbol{\mathcal{T}}}|||_2 + \|e^{\mathfrak{Q}}\|_2\right), \tag{6.3}$$

and

$$|||\nabla_{\mathfrak{Q}}^{\mathcal{N}}\mathbf{e}^{\boldsymbol{\mathcal{T}}}|||_2^2 + \|e^{\mathfrak{Q}}\|_2^2 \le C_{16} B(\mathbf{e}^{\boldsymbol{\mathcal{T}}}, e^{\mathfrak{D}}; \widetilde{\mathbf{e}}^{\boldsymbol{\mathcal{T}}}, \widetilde{e}^{\mathfrak{D}}). \tag{6.4}$$

Thanks to Definition 5.1 of B and to 6.2, we have $B(\mathbf{e}^{\boldsymbol{\mathcal{T}}}, e^{\mathfrak{D}}; \widetilde{\mathbf{e}}^{\boldsymbol{\mathcal{T}}}, \widetilde{e}^{\mathfrak{D}}) = I + T$, with $I := [\![\mathbf{div}^{\boldsymbol{\mathcal{T}}}(-2\mathrm{D}_{\mathfrak{D}}^{\eta,\mathcal{N}}\mathbf{e}^{\boldsymbol{\mathcal{T}}} + e^{\mathfrak{D}} Id), \widetilde{\mathbf{e}}^{\boldsymbol{\mathcal{T}}}]\!]$ and $T := (\mathrm{div}^{\mathfrak{D}}(\mathbf{e}^{\boldsymbol{\mathcal{T}}}) - \lambda h_{\mathfrak{D}}^2 \underline{\Delta}^{\mathfrak{D}} e^{\mathfrak{Q}}, \widetilde{e}^{\mathfrak{D}})_{\mathfrak{D}}$. Using the fact that $\widetilde{\mathbf{e}}_{\boldsymbol{\kappa^*}} = 0$ for any $\kappa^*\in\partial\mathfrak{M}^*$ and the definition of I, we have small

$$I = \sum_{\kappa\in\mathfrak{M}}\sum_{\mathcal{Q}\subset\kappa}\sum_{\mathfrak{s}\in\mathcal{E}_{\mathcal{Q}}\cap\partial\kappa} m_{\mathfrak{s}}(\mathbf{R}_{\mathfrak{s},\mathcal{Q}}, \widetilde{\mathbf{e}}_{\boldsymbol{\kappa}}) + \sum_{\kappa^*\in\mathfrak{M}^*\cup\partial\mathfrak{M}^*}\sum_{\mathcal{Q}\subset\kappa^*}\sum_{\mathfrak{s}\in\mathcal{E}_{\mathcal{Q}}\cap\partial\kappa^*} m_{\mathfrak{s}}(\mathbf{R}_{\mathfrak{s},\mathcal{Q}}, \widetilde{\mathbf{e}}_{\boldsymbol{\kappa^*}}).$$

Using the fact that $\mathbf{R}_{\sigma_{\kappa^*},\mathcal{Q}_{\kappa,\kappa^*}} = -\mathbf{R}_{\sigma_{\kappa^*},\mathcal{Q}_{\mathcal{L},\kappa^*}}$, we have

$$\begin{aligned} &m_{\sigma_{\kappa^*}}\left((\mathbf{R}_{\sigma_{\kappa^*},\mathcal{Q}_{\kappa,\kappa^*}}, \widetilde{\mathbf{e}}_{\boldsymbol{\kappa}}) + (\mathbf{R}_{\sigma_{\kappa^*},\mathcal{Q}_{\mathcal{L},\kappa^*}}, \widetilde{\mathbf{e}}_{\boldsymbol{\mathcal{L}}})\right) \\ &= -\frac{2}{\sin(\alpha_\kappa)} m_{\mathcal{Q}_{\kappa,\kappa^*}}\mathbf{R}_{\sigma_{\kappa^*},\mathcal{Q}_{\kappa,\kappa^*}}\nabla^{\mathcal{D}}\widetilde{\mathbf{e}}^{\boldsymbol{\mathcal{T}}}\vec{\boldsymbol{\tau}}_{\boldsymbol{\kappa},\boldsymbol{\mathcal{L}}} + \frac{2}{\sin(\alpha_{\mathcal{L}})} m_{\mathcal{Q}_{\mathcal{L},\kappa^*}}\mathbf{R}_{\sigma_{\kappa^*},\mathcal{Q}_{\mathcal{L},\kappa^*}}\nabla^{\mathcal{D}}\widetilde{\mathbf{e}}^{\boldsymbol{\mathcal{T}}}\vec{\boldsymbol{\tau}}_{\boldsymbol{\kappa},\boldsymbol{\mathcal{L}}}. \end{aligned}$$

Reorganizing the summation over the set of diamond cells, we deduce

$$I \leq \frac{4}{\sin(\alpha_{\mathcal{T}})} \sum_{\mathcal{D}\in\mathfrak{D}} \sum_{\mathcal{Q}\in\mathfrak{Q}_{\mathcal{D}}} \sum_{\mathfrak{s}\in\mathcal{E}_{\mathcal{Q}}} m_{\mathcal{Q}} |\mathbf{R}_{\mathfrak{s},\mathcal{Q}}| |\!|\!| \nabla^{\mathcal{D}} \widetilde{\mathbf{e}}^{\boldsymbol{\mathcal{T}}} |\!|\!|_{\mathcal{F}}.$$

Thanks to the Cauchy-Schwarz inequality and to Lemma 4.2, we obtain

$$I \leq \frac{4}{\sin(\alpha_{\mathcal{T}})} C_6 \|\mathbf{R}_{\mathfrak{s},\mathfrak{Q}}\|_2 |\!|\!| \nabla^{\mathcal{N}}_{\mathfrak{Q}} \widetilde{\mathbf{e}}^{\boldsymbol{\mathcal{T}}} |\!|\!|_2.$$

We note $T_1 := -\lambda(h^2_{\mathfrak{D}} \underline{\Delta}^{\mathfrak{D}} p^{\mathfrak{Q}}_{\mathrm{ex}}, \widetilde{e}^{\mathfrak{D}})_{\mathfrak{D}}$. Reordering the summation over $\mathfrak{s} \in \mathfrak{S}$, we have

$$T_1 = \lambda \sum_{\mathfrak{s}=\mathcal{Q}|\mathcal{Q}'=\mathcal{D}|\mathcal{D}'\in\mathfrak{S}} (h^2_{\mathcal{D}} + h^2_{\mathcal{D}'})(p^{\mathcal{Q}'}_{\mathrm{ex}} - p^{\mathcal{Q}}_{\mathrm{ex}})(\widetilde{e}^{\mathcal{D}'} - \widetilde{e}^{\mathcal{D}}).$$

The Cauchy-Schwarz inequality and Lemma 4.4 give

$$|T_1| \leq C_{11} 2\mathrm{size}(\mathcal{T}) \lambda \|\widetilde{e}^{\mathfrak{D}}\|_2 \left(\sum_{\mathfrak{s}=\mathcal{Q}|\mathcal{Q}'=\mathcal{D}|\mathcal{D}'\in\mathfrak{S}} (p^{\mathcal{Q}'}_{\mathrm{ex}} - p^{\mathcal{Q}}_{\mathrm{ex}})^2 \right)^{\frac{1}{2}}.$$

For the term T, we have the following estimate:

$$|T| \leq \|\widetilde{e}^{\mathfrak{D}}\|_2 \left(2\mathrm{size}(\mathcal{T}) \lambda C_{11} \left(\sum_{\mathfrak{s}=\mathcal{Q}|\mathcal{Q}'=\mathcal{D}|\mathcal{D}'\in\mathfrak{S}} (p^{\mathcal{Q}'}_{\mathrm{ex}} - p^{\mathcal{Q}}_{\mathrm{ex}})^2 \right)^{\frac{1}{2}} + \|\mathrm{div}^{\mathfrak{D}} \mathbb{P}^{\boldsymbol{\mathcal{T}}}_{\boldsymbol{c}} \mathbf{u}\|_2 \right).$$

To sum up, using the fact that $\|\widetilde{e}^{\mathfrak{D}}\|_2 \leq \|\widetilde{e}^{\mathfrak{Q}}\|_2$ and (6.3), (6.4) becomes

$$\begin{aligned} &|\!|\!| \nabla^{\mathcal{N}}_{\mathfrak{Q}} \mathbf{e}^{\boldsymbol{\mathcal{T}}} |\!|\!|_2^2 + \|e^{\mathfrak{Q}}\|_2^2 \\ &\quad \leq \widetilde{C}_1 (|\!|\!| \nabla^{\mathcal{N}}_{\mathfrak{Q}} \mathbf{e}^{\boldsymbol{\mathcal{T}}} |\!|\!|_2 + \|e^{\mathfrak{Q}}\|_2)(\|\mathbf{R}_{\mathfrak{s},\mathfrak{Q}}\|_2 + \|\mathrm{div}^{\mathfrak{D}} \mathbb{P}^{\boldsymbol{\mathcal{T}}}_{\boldsymbol{c}} \mathbf{u}\|_2) \\ &\qquad + \widetilde{C}_2 \mathrm{size}(\mathcal{T}) (|\!|\!| \nabla^{\mathcal{N}}_{\mathfrak{Q}} \mathbf{e}^{\boldsymbol{\mathcal{T}}} |\!|\!|_2 + \|e^{\mathfrak{Q}}\|_2) \left(\sum_{\mathfrak{s}=\mathcal{Q}|\mathcal{Q}'=\mathcal{D}|\mathcal{D}'\in\mathfrak{S}} (p^{\mathcal{Q}'}_{\mathrm{ex}} - p^{\mathcal{Q}}_{\mathrm{ex}})^2 \right)^{\frac{1}{2}}, \end{aligned} \tag{6.5}$$

where $\widetilde{C}_1 = C_{16} C_{15} \max\left(\frac{4}{\sin(\alpha_{\mathcal{T}})} C_6, 1 \right)$ and $\widetilde{C}_2 = 2 C_{16} C_{15} \lambda C_{11}$. It remains to estimate the consistency errors.

6.3. Consistency error for the pressure. LEMMA 6.1. *For any mesh $\mathcal{T}$ on Ω, there exists a constant $C_{18}, C_{19} > 0$, depending only on* $\mathrm{reg}(\mathcal{T})$*, such that for all $p \in H^1(\omega)$*

$$\|\mathbf{R}^p_{\mathfrak{s},\mathcal{Q}}\|_2^2 \leq C_{18} h^2_{\mathcal{D}} \sum_{\mathcal{Q}\in\mathfrak{Q}_{\mathcal{D}}} \int_{\mathcal{Q}} |\nabla p(z)|^2 \mathrm{d}z, \quad \textit{and} \quad \|R^p_{\mathcal{Q}}\|_2^2 \leq C_{19} h^2_{\mathcal{D}} \sum_{\mathcal{Q}\in\mathfrak{Q}_{\mathcal{D}}} \int_{\mathcal{Q}} |\nabla p(z)|^2 \mathrm{d}z.$$

Proof. Definition 6.2 gives

$$\sum_{\mathcal{Q}\in\mathfrak{Q}_{\mathcal{D}}} \sum_{\mathfrak{s}\in\mathcal{E}_{\mathcal{Q}}} m_{\mathcal{Q}} |\mathbf{R}^p_{\mathfrak{s},\mathcal{Q}}|^2 \leq \sum_{\mathcal{Q}\in\mathfrak{Q}_{\mathcal{D}}} \sum_{\mathfrak{s}\in\mathcal{E}_{\mathcal{Q}}} m_{\mathcal{Q}} \left| \frac{1}{m_{\mathfrak{s}}} \int_{\mathfrak{s}} (\mathbb{P}^{\mathcal{Q}}_m p - p(z)) \mathrm{d}z \right|^2.$$

Thanks to Lemma 4.5 and $\frac{h^3_{\mathcal{Q}}}{m_{\mathfrak{s}} m_{\mathcal{Q}}} \leq \mathrm{reg}(\mathcal{T})^3$, we have

$$\frac{1}{m_{\mathfrak{s}}} \int_{\mathfrak{s}} |\mathbb{P}^{\mathcal{Q}}_m p - p(z)|^2 \mathrm{d}z \leq C_{12} \frac{h^3_{\mathcal{Q}}}{m_{\mathfrak{s}} m_{\mathcal{Q}}} \int_{\mathcal{Q}} |\nabla p|^2 \mathrm{d}z \leq C_{12} \mathrm{reg}(\mathcal{T})^3 \int_{\mathcal{Q}} |\nabla p|^2 \mathrm{d}z. \tag{6.6}$$

Jensen inequality and (6.6) imply the first estimate. For the second estimate, we add and subtract $\frac{1}{m_{\mathfrak{s}}}\int_{\mathfrak{s}} p(x)\mathrm{d}x$ on $R_{\mathcal{Q}}^{p}(z)$. Cauchy-Schwarz inequality implies that

$$\int_{\mathcal{Q}} \left(p(z)-\mathbb{P}_m^{\mathcal{Q}}p\right)^2 \mathrm{d}z \leq 4m_{\mathcal{Q}}\frac{1}{m_{\mathcal{Q}}m_{\mathfrak{s}}}\int_{\mathcal{Q}}\int_{\mathfrak{s}}\left(p(z)-p(x)\right)^2\mathrm{d}x\mathrm{d}z.$$

Thanks to (6.6), we get the result with $C_{19}=4C_{12}\mathrm{reg}(\mathcal{T})^3$.

□

6.4. Properties of the velocity.

6.4.1. Definitions. DEFINITION 6.3. *The consistency error $R_{\mathcal{Q}}^{\mathbf{v}}$ can be split into three different contributions $R_{\mathcal{Q}}^{\mathbf{v},\eta}$, $R_{\mathcal{Q}}^{\mathbf{v},\mathrm{D}\mathbf{v}}$ and $R_{\mathcal{Q}}^{\mathbf{v},z}$. They originate, respectively, from the errors due to the approximation with respect to the space variable of the viscous stress tensor, to the approximation of the gradient and to the approximation of the viscosity: $R_{\mathcal{Q}}^{\mathbf{v}}(z)=R_{\mathcal{Q}}^{\mathbf{v},\eta}(z)+R_{\mathcal{Q}}^{\mathbf{v},\mathrm{D}\mathbf{v}}+R_{\mathcal{Q}}^{\mathbf{v},z}$, where*

$$\begin{aligned}
R_{\mathcal{Q}}^{\mathbf{v},\eta}(z) &= \eta_{|\overline{\mathcal{Q}}}(z)\mathrm{D}\mathbf{v}_{|\overline{\mathcal{Q}}}(z)-\frac{1}{m_{\mathcal{Q}}}\int_{\mathcal{Q}}\eta(x)\mathrm{D}\mathbf{v}(x)\mathrm{d}x,\\
R_{\mathcal{Q}}^{\mathbf{v},\mathrm{D}\mathbf{v}} &= \frac{1}{m_{\mathcal{Q}}}\int_{\mathcal{Q}}\eta(x)(\mathrm{D}\mathbf{v}(x)-\mathrm{D}_{\mathcal{Q}}^{\mathcal{N}}\mathbb{P}_{\boldsymbol{c}}^{\boldsymbol{\mathcal{T}}}\mathbf{v})\mathrm{d}x,\\
R_{\mathcal{Q}}^{\mathbf{v},z} &= \left(\frac{1}{m_{\mathcal{Q}}}\int_{\mathcal{Q}}\eta(x)\mathrm{d}x-\eta_{\mathcal{Q}}\right)\mathrm{D}_{\mathcal{Q}}^{\mathcal{N}}\mathbb{P}_{\boldsymbol{c}}^{\boldsymbol{\mathcal{T}}}\mathbf{v}.
\end{aligned}$$

We also introduce the quantity, for all $\mathfrak{s}=\mathcal{Q}|\mathcal{Q}'\in\mathcal{E}_{\mathcal{Q}}$:

$$\mathbf{R}_{\mathfrak{s},\mathcal{Q}}^{\mathbf{v},\eta}=-\mathbf{R}_{\mathfrak{s},\mathcal{Q}'}^{\mathbf{v},\eta}=\frac{1}{m_{\mathfrak{s}}}\int_{\mathfrak{s}}R_{\mathcal{Q}}^{\mathbf{v},\eta}(z)\vec{\mathbf{n}}_{\mathfrak{s}\mathcal{Q}}\mathrm{d}z.$$

DEFINITION 6.4. *We define the projection $\mathbb{P}_c^{\mathfrak{Q}}\mathbf{u}$ of $\mathbf{u}$ on the set of quarter diamond cells as follows. For each quarter diamond cell $\mathcal{Q}\in\mathfrak{Q}$, the restriction of $\mathbb{P}_c^{\mathfrak{Q}}\mathbf{u}$ to the triangle $\mathcal{Q}$ is the unique affine function $\mathbb{P}_c^{\mathcal{Q}}\mathbf{u}$ which coincides with $\mathbf{u}$ at the middle point of the semi-edges $\mathfrak{s}\in\mathcal{E}_{\mathcal{Q}}$ and whose value at the middle point of the third side of $\mathcal{Q}$ is the mean-value of the value $\mathbf{u}$ at the extremities of this side. Remarks that this definition makes sense since $\mathbf{u}_{|_{\mathcal{Q}}}\in(H^2(\mathcal{Q}))^2\subset(\mathcal{C}^0(\bar{\mathcal{Q}}))^2$.*

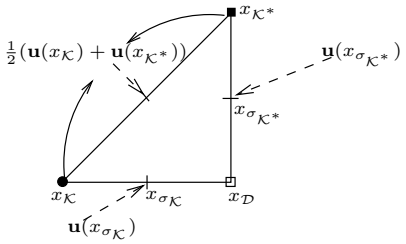


FIG. 6.1. *The affine interpolation $\mathbb{P}_c^{\mathcal{Q}}$ on the quarter diamond cell $\mathcal{Q}=\mathcal{Q}_{\mathcal{K},\mathcal{K}^*}$.*

For instance, in the case of the quarter diamond cell $\mathcal{Q}=\mathcal{Q}_{\mathcal{K},\mathcal{K}^*}$ (Figure 6.1), it reads

$$\mathbb{P}_c^{\mathcal{Q}}\mathbf{u}(x_{\sigma_{\mathcal{K}}})=\mathbf{u}(x_{\sigma_{\mathcal{K}}}),\ \mathbb{P}_c^{\mathcal{Q}}\mathbf{u}(x_{\sigma_{\mathcal{K}^*}})=\mathbf{u}(x_{\sigma_{\mathcal{K}^*}}),\ \mathbb{P}_c^{\mathcal{Q}}\mathbf{u}\left(\frac{x_{\mathcal{K}}+x_{\mathcal{K}^*}}{2}\right)=\frac{\mathbf{u}(x_{\mathcal{K}})+\mathbf{u}(x_{\mathcal{K}^*})}{2}.$$

The following proposition is the vector-valued version of [3, inequality (5.4)].

PROPOSITION 6.2. *Let $\mathcal{T}$ be a mesh of Ω. There exists a constant $C_{20} > 0$, depending only on $\mathrm{reg}(\mathcal{T})$, such that for any function $\mathbf{v}$ in $(H^2(\omega))^2$, we have for all $\mathcal{Q} \in \mathfrak{Q}$*

$$\int_{\mathcal{Q}} \|\nabla \mathbf{v}(z) - \nabla \mathbb{P}_c^{\mathcal{Q}} \mathbf{v}\|_{\mathcal{F}}^2 \mathrm{d}z + \int_{\mathcal{Q}} \|\mathrm{D}\mathbf{v}(z) - \mathrm{D}\mathbb{P}_c^{\mathcal{Q}} \mathbf{v}\|_{\mathcal{F}}^2 \mathrm{d}z \leq C_{20} h_{\mathcal{D}}^2 \int_{\mathcal{Q}} |\nabla^2 \mathbf{v}(z)|^2 \mathrm{d}z.$$

6.4.2. Approximation of the viscous stress tensor. LEMMA 6.2. *Assume that η satisfies* (1.2) *and* (2.4)*. Let $\mathcal{T}$ be a mesh of Ω. There exists a constant $C_{21} > 0$, depending only on C_η, $\overline{\mathrm{C}}_\eta$ and $\mathrm{reg}(\mathcal{T})$, such that for any function $\mathbf{v}$ in $(H^2(\omega))^2$, we have for all $\mathcal{D} \in \mathfrak{D}$*

$$m_{\mathcal{Q}} |\mathbf{R}_{\mathfrak{s},\mathcal{Q}}^{\mathbf{v},\eta}|^2 \leq C_{21} h_{\mathcal{D}}^2 \int_{\mathcal{Q}} (\|\nabla \mathbf{v}\|_{\mathcal{F}}^2 + |\nabla^2 \mathbf{v}|^2) \mathrm{d}z, \qquad \forall\, \mathcal{Q} \in \mathfrak{Q}_{\mathcal{D}},\ \forall\, \mathfrak{s} \in \mathcal{E}_{\mathcal{Q}}.$$

Proof. Applying the Jensen inequality, we add and subtract $\eta(s)\mathrm{D}\mathbf{v}(x)$, Cauchy-Schwarz inequality implies that

$$\|R_{\mathcal{Q}}^{\mathbf{v},\eta}(s)\|_{\mathcal{F}}^2 \leq \frac{2}{m_{\mathcal{Q}}} \int_{\mathcal{Q}} |\eta(s) - \eta(x)|^2 \|\mathrm{D}\mathbf{v}(x)\|_{\mathcal{F}}^2 \mathrm{d}x + \frac{2}{m_{\mathcal{Q}}} \int_{\mathcal{Q}} |\eta(s)|^2 \|\mathrm{D}\mathbf{v}(s) - \mathrm{D}\mathbf{v}(x)\|_{\mathcal{F}}^2 \mathrm{d}x.$$

Jensen inequality, (1.2) and (2.4) give

$$m_{\mathcal{Q}} |\mathbf{R}_{\mathfrak{s},\mathcal{Q}}^{\mathbf{v},\eta}|^2 \leq h_{\mathcal{D}}^2 2\mathrm{C}_\eta^2 \int_{\mathcal{Q}} \|\mathrm{D}\mathbf{v}(x)\|_{\mathcal{F}}^2 \mathrm{d}x + \frac{2\overline{\mathrm{C}}_\eta^2}{m_{\mathfrak{s}}} \int_{\mathfrak{s}} \int_{\mathcal{Q}} \|\mathrm{D}\mathbf{v}(s) - \mathrm{D}\mathbf{v}(x)\|_{\mathcal{F}}^2 \mathrm{d}x \mathrm{d}s.$$

For the second integral, we apply Lemma 4.5 on a edge $\mathfrak{s}$ and the quarter diamond cell $\mathcal{Q}$, since $\frac{h_{\mathcal{Q}}^3}{m_{\mathfrak{s}}} \leq \mathrm{reg}(\mathcal{T}) h_{\mathcal{D}}^2$:

$$\frac{1}{m_{\mathfrak{s}}} \int_{\mathfrak{s}} \int_{\mathcal{Q}} \|\mathrm{D}\mathbf{v}(s) - \mathrm{D}\mathbf{v}(x)\|_{\mathcal{F}}^2 \mathrm{d}x \mathrm{d}s \leq C_{12} \mathrm{reg}(\mathcal{T}) h_{\mathcal{D}}^2 \int_{\mathcal{Q}} |\nabla \mathrm{D}\mathbf{v}(y)|^2 \mathrm{d}y.$$

Finally, we deduce the result with $C_{21} = \max(2C_\eta^2, 2\overline{\mathrm{C}}_\eta^2 C_{12} \mathrm{reg}(\mathcal{T}))$. □

6.4.3. Approximation of the viscosity. LEMMA 6.3. *Assume that η satisfies* (2.4)*. Let $\mathcal{T}$ be a mesh of Ω. There exists a constant $C_{22} > 0$, depending only on C_η, such that for any function $\mathbf{v}$ in $(H^2(\omega))^2$, we have for all $\mathcal{D} \in \mathfrak{D}$*

$$m_{\mathcal{Q}} \|R_{\mathcal{Q}}^{\mathbf{v},z}\|_{\mathcal{F}}^2 \leq C_{22} h_{\mathcal{D}}^2 m_{\mathcal{Q}} \|\mathrm{D}_{\mathcal{Q}}^{\mathcal{N}} \mathbb{P}_{\boldsymbol{c}}^{\boldsymbol{\mathcal{T}}} \mathbf{v}\|_{\mathcal{F}}^2, \qquad \forall\, \mathcal{Q} \in \mathfrak{Q}_{\mathcal{D}},\ \forall\, \mathfrak{s} \in \mathcal{E}_{\mathcal{Q}}.$$

Proof. The Jensen inequality and (2.4) give the result with $C_{22} = \mathrm{C}_\eta^2$. □

6.4.4. Approximation of the gradient. DEFINITION 6.5. *We define $R \in \mathcal{M}_{2,4}(\mathbb{R})$ as follows*

$$\begin{aligned} R =& 2 \begin{pmatrix} m_{\sigma_{\mathcal{K}}} (\mathbf{R}_{\sigma_{\mathcal{K}},\mathcal{Q}_{\mathcal{K},\mathcal{L}^*}}^{\mathbf{u},\eta} + \mathbf{R}_{\sigma_{\mathcal{K}},\mathcal{Q}_{\mathcal{K},\mathcal{L}^*}}^{\mathbf{u},z} - \mathbf{R}_{\sigma_{\mathcal{K}},\mathcal{Q}_{\mathcal{K},\mathcal{K}^*}}^{\mathbf{u},\eta} - \mathbf{R}_{\sigma_{\mathcal{K}},\mathcal{Q}_{\mathcal{K},\mathcal{K}^*}}^{\mathbf{u},z}) \\ m_{\sigma_{\mathcal{L}}} (\mathbf{R}_{\sigma_{\mathcal{L}},\mathcal{Q}_{\mathcal{L},\mathcal{L}^*}}^{\mathbf{u},\eta} + \mathbf{R}_{\sigma_{\mathcal{L}},\mathcal{Q}_{\mathcal{L},\mathcal{L}^*}}^{\mathbf{u},z} - \mathbf{R}_{\sigma_{\mathcal{L}},\mathcal{Q}_{\mathcal{L},\mathcal{K}^*}}^{\mathbf{u},\eta} - \mathbf{R}_{\sigma_{\mathcal{L}},\mathcal{Q}_{\mathcal{L},\mathcal{K}^*}}^{\mathbf{u},z}) \\ m_{\sigma_{\mathcal{K}^*}} (\mathbf{R}_{\sigma_{\mathcal{K}^*},\mathcal{Q}_{\mathcal{L},\mathcal{K}^*}}^{\mathbf{u},\eta} + \mathbf{R}_{\sigma_{\mathcal{K}^*},\mathcal{Q}_{\mathcal{L},\mathcal{K}^*}}^{\mathbf{u},z} - \mathbf{R}_{\sigma_{\mathcal{K}^*},\mathcal{Q}_{\mathcal{K},\mathcal{K}^*}}^{\mathbf{u},\eta} - \mathbf{R}_{\sigma_{\mathcal{K}^*},\mathcal{Q}_{\mathcal{K},\mathcal{K}^*}}^{\mathbf{u},z}) \\ m_{\sigma_{\mathcal{L}^*}} (\mathbf{R}_{\sigma_{\mathcal{L}^*},\mathcal{Q}_{\mathcal{L},\mathcal{L}^*}}^{\mathbf{u},\eta} + \mathbf{R}_{\sigma_{\mathcal{L}^*},\mathcal{Q}_{\mathcal{L},\mathcal{L}^*}}^{\mathbf{u},z} - \mathbf{R}_{\sigma_{\mathcal{L}^*},\mathcal{Q}_{\mathcal{K},\mathcal{L}^*}}^{\mathbf{u},\eta} - \mathbf{R}_{\sigma_{\mathcal{L}^*},\mathcal{Q}_{\mathcal{K},\mathcal{L}^*}}^{\mathbf{u},z}) \end{pmatrix} \\ &+ \begin{pmatrix} m_{\sigma_{\mathcal{K}}} (\mathbf{R}_{\sigma_{\mathcal{K}},\mathcal{Q}_{\mathcal{K},\mathcal{L}^*}}^{p} - \mathbf{R}_{\sigma_{\mathcal{K}},\mathcal{Q}_{\mathcal{K},\mathcal{K}^*}}^{p}) \\ m_{\sigma_{\mathcal{L}}} (\mathbf{R}_{\sigma_{\mathcal{L}},\mathcal{Q}_{\mathcal{L},\mathcal{L}^*}}^{p} - \mathbf{R}_{\sigma_{\mathcal{L}},\mathcal{Q}_{\mathcal{L},\mathcal{K}^*}}^{p}) \\ m_{\sigma_{\mathcal{K}^*}} (\mathbf{R}_{\sigma_{\mathcal{K}^*},\mathcal{Q}_{\mathcal{L},\mathcal{K}^*}}^{p} - \mathbf{R}_{\sigma_{\mathcal{K}^*},\mathcal{Q}_{\mathcal{K},\mathcal{K}^*}}^{p}) \\ m_{\sigma_{\mathcal{L}^*}} (\mathbf{R}_{\sigma_{\mathcal{L}^*},\mathcal{Q}_{\mathcal{L},\mathcal{L}^*}}^{p} - \mathbf{R}_{\sigma_{\mathcal{L}^*},\mathcal{Q}_{\mathcal{K},\mathcal{L}^*}}^{p}) \end{pmatrix}. \end{aligned}$$

PROPOSITION 6.3. *Assume that η satisfies* (1.2) *and* (2.4)*. Let $\mathcal{T}$ be a mesh of Ω. There exists a constant $C_{23} > 0$, depending only on $\underline{\mathrm{C}}_\eta$, $\overline{\mathrm{C}}_\eta$, C_η and $\mathrm{reg}(\mathcal{T})$, such that for all $\mathcal{D} \in \mathfrak{D}$, such that for any $\delta^{\mathcal{D}}$ in $\mathcal{M}_{4,2}(\mathbb{R})$, we have*

$$m_{\mathcal{D}}|\mathrm{Tr}({}^t\delta^{\mathcal{D}}R)|^2 \leq C_{23}m_{\mathcal{D}}\|B_{\mathcal{Q}}\delta^{\mathcal{D}}\|^2_{\mathfrak{Q}_{\mathcal{D}}}h^2_{\mathcal{D}} \sum_{\mathcal{Q}\in\mathfrak{Q}_{\mathcal{D}}}\int_{\mathcal{Q}}(\|\nabla\mathbf{u}(z)\|^2_{\mathcal{F}} + |\nabla^2\mathbf{u}(z)|^2 + |\nabla p(z)|^2)\mathrm{d}z.$$

Proof. First, Lemma 6.2 and 6.1 give the estimates on $\mathbf{R}^{\mathbf{u},\eta}_{\mathfrak{s},\mathcal{Q}}$ and $\mathbf{R}^{p}_{\mathfrak{s},\mathcal{Q}}$. Then, Lemma 6.3, Lemma 4.1, Proposition 4.1 and Lemma 4.6 imply the estimate on $R^{\mathbf{u},z}_{\mathcal{Q}}$. To sum up, we get

$$\sum_{\mathcal{Q}\in\mathfrak{Q}_{\mathcal{D}}} m_{\mathcal{Q}} \sum_{\mathfrak{s}\in\mathcal{E}_{\mathcal{Q}}}(|\mathbf{R}^{\mathbf{u},\eta}_{\mathfrak{s},\mathcal{Q}}|^2 + |R^{\mathbf{u},z}_{\mathcal{Q}}|^2 + |\mathbf{R}^{p}_{\mathfrak{s},\mathcal{Q}}|^2) \leq Ch^2_{\mathcal{D}} \sum_{\mathcal{Q}\in\mathfrak{Q}_{\mathcal{D}}}\int_{\mathcal{Q}}\big(\|\nabla\mathbf{u}(z)\|^2_{\mathcal{F}} + |\nabla^2\mathbf{u}(z)|^2 + |\nabla p(z)|^2\big)\mathrm{d}z. \tag{6.7}$$

Using (6.7), Cauchy-Schwarz inequality conclude the result. □

The following proposition is proved in Subsection 9.3.

PROPOSITION 6.4. *Assume that η satisfies* (1.2)*. There exists a constant $C_{24} > 0$, depending only on $\underline{\mathrm{C}}_\eta$, $\overline{\mathrm{C}}_\eta$ and $\mathrm{reg}(\mathcal{T})$, and a function $\mathbf{v}$ such that $\widetilde{\mathbf{v}} \in (H^1(\mathcal{D}))^2 \cap (H^2(\mathcal{Q}))^2$, for all $\mathcal{Q} \in \mathfrak{Q}_{\mathcal{D}}$, such that $\nabla^2\widetilde{\mathbf{v}} = 0$ on $\mathcal{Q} \in \mathfrak{Q}_{\mathcal{D}}$,*

$$\|\mathrm{D}\widetilde{\mathbf{v}} - \mathrm{D}^{\mathcal{N}}_{\mathfrak{Q}}\mathbb{P}^{\boldsymbol{\mathcal{T}}}_{\boldsymbol{c}}\widetilde{\mathbf{v}}\|_{\mathfrak{Q}_{\mathcal{D}}} \leq C_{24}(\|\widetilde{p}^{\mathfrak{Q}_{\mathcal{D}}}_{\mathrm{ex}}\|_2 + \|\mathrm{D}\mathbf{u} - \mathrm{D}^{\mathcal{N}}_{\mathcal{Q}}\mathbb{P}^{\boldsymbol{\mathcal{T}}}_{\boldsymbol{c}}\mathbf{u}\|_{\mathfrak{Q}_{\mathcal{D}}} + h_{\mathcal{D}}),$$

and

$$-\sum_{\mathcal{Q}\in\mathfrak{Q}_{\mathcal{D}}}\int_{\mathcal{Q}}\mathrm{div}\widetilde{\mathbf{v}}\widetilde{p}^{\mathcal{Q}}_{\mathrm{ex}}\mathrm{d}z \geq -C_{24}\|\mathrm{D}\mathbf{u} - \mathrm{D}^{\mathcal{N}}_{\mathcal{Q}}\mathbb{P}^{\boldsymbol{\mathcal{T}}}_{\boldsymbol{c}}\mathbf{u}\|^2_{\mathfrak{Q}_{\mathcal{D}}} + \frac{1}{2}\|\widetilde{p}^{\mathfrak{Q}_{\mathcal{D}}}_{\mathrm{ex}}\|^2_2 - C_{24}\mathrm{size}(\mathcal{T}).$$

We prove a consistency estimate for the new strain rate tensor $\mathrm{D}^{\mathcal{N}}_{\mathcal{Q}}$ that we have introduced. This is the main difference between the present study and our previous work since the definition of the new discrete strain rate tensor depends on the jumps of η in each diamond cell. Hence, the consistency estimate for this operator cannot be obtained as in the usual way, that is, only by applying well-chosen Taylor formula. Note that here we use the fact that the pair $(\mathbf{u}, p)$ is a smooth solution of the problem (1.1) and the estimate of Lemma 2.2.

LEMMA 6.4. *Assume that η satisfies* (1.2) *and* (2.4)*. We assume that $(\mathbf{u}, p)$, the solution of the problem* (1.1)*, lies in $(H^2(\omega))^2 \times H^1(\omega)$ and $0 < \lambda < \frac{4\underline{\mathrm{C}}_\eta}{C_3}$. Let $\mathcal{T}$ be a mesh of Ω. There exists a constant $C_{25} > 0$, depending only on $\underline{\mathrm{C}}_\eta$, $\overline{\mathrm{C}}_\eta$, C_η, $\mathrm{reg}(\mathcal{T})$ and $\sin(\epsilon_0)$, such that for all $\mathcal{D} \in \mathfrak{D}$, we have*

$$\|\mathrm{D}\mathbf{u} - \mathrm{D}^{\mathcal{N}}_{\mathfrak{Q}}\mathbb{P}^{\boldsymbol{\mathcal{T}}}_{\boldsymbol{c}}\mathbf{u}\|^2_{\mathfrak{Q}_{\mathcal{D}}} + \|\widetilde{p}^{\mathfrak{Q}_{\mathcal{D}}}_{\mathrm{ex}}\|^2_2 \leq C_{25}h^2_{\mathcal{D}} \sum_{\mathcal{Q}\in\mathfrak{Q}_{\mathcal{D}}}\int_{\mathcal{Q}}(\|\nabla\mathbf{u}\|^2_{\mathcal{F}} + |\nabla^2\mathbf{u}|^2 + |\nabla p(z)|^2)\mathrm{d}z.$$

Proof. We note $\mathbf{u} = \begin{pmatrix} u^1 \\ u^2 \end{pmatrix}$. Let us give the proof in the case where the diamond cell $\mathcal{D}$ is an interior diamond cell. The case $\mathcal{D} \in \mathfrak{D}_{ext}$ can be treated in a same way.

Step 1. Since $(\mathbf{u}, p)$ solves (1.1), we have the conservativity of the fluxes through $\mathfrak{s} = \mathcal{Q}|\mathcal{Q}'$ as follows

$$\int_{\mathfrak{s}}(2\eta_{|\overline{\mathcal{Q}}}(z)\mathrm{D}\mathbf{u}_{|\overline{\mathcal{Q}}}(z) - p_{|\overline{\mathcal{Q}}}(z)\mathrm{Id})\vec{\mathbf{n}}_{\mathfrak{s}\mathcal{Q}}\mathrm{d}z = \int_{\mathfrak{s}}(2\eta_{|\overline{\mathcal{Q}'}}(z)\mathrm{D}\mathbf{u}_{|\overline{\mathcal{Q}'}}(z) - p_{|\overline{\mathcal{Q}'}}(z)\mathrm{Id})\vec{\mathbf{n}}_{\mathfrak{s}\mathcal{Q}}\mathrm{d}z.$$

We recall that the discrete strain rate tensor satisfies Proposition 6.1, we can deduce that

$$\left(\frac{1}{m_{\mathfrak{s}}}\int_{\mathfrak{s}}(2\eta_{|\overline{\mathcal{Q}}}(z)\mathrm{D}\mathbf{u}_{|\overline{\mathcal{Q}}}(z) - p_{|\overline{\mathcal{Q}}}(z)\mathrm{Id})\mathrm{d}z - (2\eta_{\mathcal{Q}}\mathrm{D}^{\mathcal{N}}_{\mathcal{Q}}\mathbb{P}^{\boldsymbol{\mathcal{T}}}_{\boldsymbol{c}}\mathbf{u} - p^{\mathcal{Q}}_{\mathrm{ex}}\mathrm{Id})\right)\vec{\mathbf{n}}_{\mathfrak{s}\mathcal{Q}}$$
$$-\left(\frac{1}{m_{\mathfrak{s}}}\int_{\mathfrak{s}}(2\eta_{|\overline{\mathcal{Q}'}}(z)\mathrm{D}\mathbf{u}_{|\overline{\mathcal{Q}'}}(z) - p_{|\overline{\mathcal{Q}'}}(z)\mathrm{Id})\mathrm{d}z - (2\eta_{\mathcal{Q}'}\mathrm{D}^{\mathcal{N}}_{\mathcal{Q}'}\mathbb{P}^{\boldsymbol{\mathcal{T}}}_{\boldsymbol{c}}\mathbf{u} - p^{\mathcal{Q}'}_{\mathrm{ex}}\mathrm{Id})\right)\vec{\mathbf{n}}_{\mathfrak{s}\mathcal{Q}} = 0.$$

Using Definition 6.2 and the last equality, we have

$$\begin{aligned}\frac{2}{m_{\mathcal{Q}}}\int_{\mathcal{Q}}\eta(z)\left(\mathrm{D}\mathbf{u}(z)-\mathrm{D}_{\mathcal{Q}}^{\mathcal{N}}\mathbb{P}_{\boldsymbol{c}}^{\boldsymbol{\mathcal{T}}}\mathbf{u}\right)\vec{\mathbf{n}}_{\mathfrak{s}_{\mathcal{Q}}}\mathrm{d}z-\frac{2}{m_{\mathcal{Q}'}}\int_{\mathcal{Q}'}\eta(z)\left(\mathrm{D}\mathbf{u}(z)-\mathrm{D}_{\mathcal{Q}'}^{\mathcal{N}}\mathbb{P}_{\boldsymbol{c}}^{\boldsymbol{\mathcal{T}}}\mathbf{u}\right)\vec{\mathbf{n}}_{\mathfrak{s}_{\mathcal{Q}}}\mathrm{d}z\\ -\frac{1}{m_{\mathcal{Q}}}\int_{\mathcal{Q}}\left(p(z)-p_{\mathrm{ex}}^{\mathcal{Q}}\right)\vec{\mathbf{n}}_{\mathfrak{s}_{\mathcal{Q}}}\mathrm{d}z+\frac{1}{m_{\mathcal{Q}'}}\int_{\mathcal{Q}'}\left(p(z)-p_{\mathrm{ex}}^{\mathcal{Q}'}\right)\vec{\mathbf{n}}_{\mathfrak{s}_{\mathcal{Q}}}\mathrm{d}z\\ =2\mathbf{R}_{\mathfrak{s},\mathcal{Q}'}^{\mathbf{u},\eta}+2\mathbf{R}_{\mathfrak{s},\mathcal{Q}'}^{\mathbf{u},z}-2\mathbf{R}_{\mathfrak{s},\mathcal{Q}}^{\mathbf{u},\eta}-2\mathbf{R}_{\mathfrak{s},\mathcal{Q}}^{\mathbf{u},z}-\mathbf{R}_{\mathfrak{s},\mathcal{Q}}^{p}+\mathbf{R}_{\mathfrak{s},\mathcal{Q}'}^{p}.\end{aligned}$$

We sum over the quarter diamond cells $\mathcal{Q}\in\mathfrak{Q}_{\mathcal{D}}$

$$\sum_{\mathcal{Q}\in\mathfrak{Q}_{\mathcal{D}}}\int_{\mathcal{Q}}2\eta(z)\left(\mathrm{D}\mathbf{u}(z)-\mathrm{D}_{\mathcal{Q}}^{\mathcal{N}}\mathbb{P}_{\boldsymbol{c}}^{\boldsymbol{\mathcal{T}}}\mathbf{u}\right)\mathrm{d}zB_{\mathcal{Q}}-\sum_{\mathcal{Q}\in\mathfrak{Q}_{\mathcal{D}}}\int_{\mathcal{Q}}\left(p(z)-p_{\mathrm{ex}}^{\mathcal{Q}}\right)\mathrm{d}zB_{\mathcal{Q}}={}^{t}R,\quad(6.8)$$

with R defined by Definition 6.5. We multiply (6.8) by ${}^{t}\delta^{\mathcal{D}}$, thanks to the symmetry of $\eta(z)\mathrm{D}\mathbf{u}(z)-\eta(z)\mathrm{D}_{\mathcal{Q}}^{\mathcal{N}}\mathbb{P}_{\boldsymbol{c}}^{\boldsymbol{\mathcal{T}}}\mathbf{u}$, we obtain

$$\sum_{\mathcal{Q}\in\mathfrak{Q}_{\mathcal{D}}}\int_{\mathcal{Q}}{}^{t}\delta^{\mathcal{D}}{}^{t}B_{\mathcal{Q}}2\eta(z)\left(\mathrm{D}\mathbf{u}(z)-\mathrm{D}_{\mathcal{Q}}^{\mathcal{N}}\mathbb{P}_{\boldsymbol{c}}^{\boldsymbol{\mathcal{T}}}\mathbf{u}\right)\mathrm{d}z-\sum_{\mathcal{Q}\in\mathfrak{Q}_{\mathcal{D}}}m_{\mathcal{Q}}{}^{t}\delta^{\mathcal{D}}{}^{t}B_{\mathcal{Q}}\widetilde{p}_{\mathrm{ex}}^{\mathcal{Q}}={}^{t}\delta^{\mathcal{D}}R$$

Using the trace operator and the symmetry of the matrix $\eta(z)\mathrm{D}\mathbf{u}(z)-\eta(z)\mathrm{D}_{\mathcal{Q}}^{\mathcal{N}}\mathbb{P}_{\boldsymbol{c}}^{\boldsymbol{\mathcal{T}}}\mathbf{u}$, it implies that for all $\delta^{\mathcal{D}}\in\mathcal{M}_{n_{\mathcal{D}},2}(\mathbb{R})$ we have

$$\begin{aligned}\sum_{\mathcal{Q}\in\mathfrak{Q}_{\mathcal{D}}}\int_{\mathcal{Q}}\eta(z)\left(B_{\mathcal{Q}}\delta^{\mathcal{D}}+{}^{t}\delta^{\mathcal{D}}{}^{t}B_{\mathcal{Q}}:\mathrm{D}\mathbf{u}(z)-\mathrm{D}_{\mathcal{Q}}^{\mathcal{N}}\mathbb{P}_{\boldsymbol{c}}^{\boldsymbol{\mathcal{T}}}\mathbf{u}\right)\mathrm{d}z\\ -\sum_{\mathcal{Q}\in\mathfrak{Q}_{\mathcal{D}}}m_{\mathcal{Q}}\mathrm{Tr}\left({}^{t}\delta^{\mathcal{D}}{}^{t}B_{\mathcal{Q}}\right)\widetilde{p}_{\mathrm{ex}}^{\mathcal{Q}}=\mathrm{Tr}\left({}^{t}\delta^{\mathcal{D}}R\right).\end{aligned}\quad(6.9)$$

Step 2. For $\mathbf{u},\mathbf{v}\in(H^2(\omega))^2$, and $p^{\mathfrak{Q}},q^{\mathfrak{Q}}\in\mathbb{R}^{\mathfrak{Q}}$, we define a new bilinear form $\mathcal{B}$ as follows

$$\begin{aligned}\mathcal{B}(\mathbf{u},p^{\mathfrak{Q}},\mathbf{v},q^{\mathfrak{Q}})=\sum_{\mathcal{Q}\in\mathfrak{Q}_{\mathcal{D}}}2\int_{\mathcal{Q}}\eta(z)(\mathrm{D}\mathbf{u}-\mathrm{D}_{\mathcal{Q}}^{\mathcal{N}}\mathbb{P}_{\boldsymbol{c}}^{\boldsymbol{\mathcal{T}}}\mathbf{u}:\mathrm{D}\mathbf{v}-\mathrm{D}_{\mathcal{Q}}^{\mathcal{N}}\mathbb{P}_{\boldsymbol{c}}^{\boldsymbol{\mathcal{T}}}\mathbf{v})\mathrm{d}z\\ -\sum_{\mathcal{Q}\in\mathfrak{Q}_{\mathcal{D}}}\int_{\mathcal{Q}}\mathrm{Tr}(\mathrm{D}\mathbf{v}-\mathrm{D}_{\mathcal{Q}}^{\mathcal{N}}\mathbb{P}_{\boldsymbol{c}}^{\boldsymbol{\mathcal{T}}}\mathbf{v})p^{\mathcal{Q}}\mathrm{d}z+\sum_{\mathcal{Q}\in\mathfrak{Q}_{\mathcal{D}}}\int_{\mathcal{Q}}\mathrm{Tr}(\mathrm{D}\mathbf{u}-\mathrm{D}_{\mathcal{Q}}^{\mathcal{N}}\mathbb{P}_{\boldsymbol{c}}^{\boldsymbol{\mathcal{T}}}\mathbf{u})q^{\mathcal{Q}}\mathrm{d}z.\end{aligned}$$

We easily have that

$$\mathcal{B}(\mathbf{u},\widetilde{p}_{\mathrm{ex}}^{\mathfrak{Q}_{\mathcal{D}}},\mathbf{u},\widetilde{p}_{\mathrm{ex}}^{\mathfrak{Q}_{\mathcal{D}}})\geq 2\underline{\mathrm{C}}_{\eta}|||\mathrm{D}\mathbf{u}-\mathrm{D}_{\mathcal{Q}}^{\mathcal{N}}\mathbb{P}_{\boldsymbol{c}}^{\boldsymbol{\mathcal{T}}}\mathbf{u}|||_{\mathfrak{Q}_{\mathcal{D}}}^{2}.\quad(6.10)$$

Thanks to Cauchy-Schwarz, using the function $\widetilde{\mathbf{v}}$ obtained in Proposition 6.4, we have

$$\begin{aligned}\mathcal{B}(\mathbf{u},\widetilde{p}_{\mathrm{ex}}^{\mathfrak{Q}_{\mathcal{D}}},\widetilde{\mathbf{v}},0)\geq&-2\overline{\mathrm{C}}_{\eta}|||\mathrm{D}\mathbf{u}-\mathrm{D}_{\mathcal{Q}}^{\mathcal{N}}\mathbb{P}_{\boldsymbol{c}}^{\boldsymbol{\mathcal{T}}}\mathbf{u}|||_{\mathfrak{Q}_{\mathcal{D}}}|||\mathrm{D}\widetilde{\mathbf{v}}-\mathrm{D}_{\mathcal{Q}}^{\mathcal{N}}\mathbb{P}_{\boldsymbol{c}}^{\boldsymbol{\mathcal{T}}}\widetilde{\mathbf{v}}|||_{\mathfrak{Q}_{\mathcal{D}}}\\ &-\sum_{\mathcal{Q}\in\mathfrak{Q}_{\mathcal{D}}}\int_{\mathcal{Q}}\mathrm{div}\widetilde{\mathbf{v}}\widetilde{p}_{\mathrm{ex}}^{\mathcal{Q}}\mathrm{d}z+\sum_{\mathcal{Q}\in\mathfrak{Q}_{\mathcal{D}}}m_{\mathcal{Q}}\mathrm{Tr}(\mathrm{D}_{\mathcal{Q}}^{\mathcal{N}}\mathbb{P}_{\boldsymbol{c}}^{\boldsymbol{\mathcal{T}}}\widetilde{\mathbf{v}})\widetilde{p}_{\mathrm{ex}}^{\mathcal{Q}}.\end{aligned}$$

Applying Proposition 2.4 with $\mathbb{P}_{\boldsymbol{c}}^{\boldsymbol{\mathcal{T}}}\widetilde{\mathbf{v}}$ and $\mathbb{P}_{m}^{\mathcal{D}}p$, there exists $(\delta^{\mathcal{D}},p^{\mathfrak{Q}_{\mathcal{D}}})\in\mathcal{M}_{n_{\mathcal{D}},2}(\mathbb{R})\times\mathbb{R}^{n_{\mathcal{D}}}$ and $\mathrm{Tr}(\mathrm{D}_{\mathcal{Q}}^{\mathcal{N}}\mathbb{P}_{\boldsymbol{c}}^{\boldsymbol{\mathcal{T}}}\widetilde{\mathbf{v}})=\mathrm{Tr}(\mathrm{D}^{\mathcal{D}}\mathbb{P}_{\boldsymbol{c}}^{\boldsymbol{\mathcal{T}}}\widetilde{\mathbf{v}})$. Finally, using Definition 6.1, Proposition 6.4 and Young inequality, we conclude that

$$\mathcal{B}(\mathbf{u},\widetilde{p}_{\mathrm{ex}}^{\mathfrak{Q}_{\mathcal{D}}},\widetilde{\mathbf{v}},0)\geq-C|||\mathrm{D}\mathbf{u}-\mathrm{D}_{\mathcal{Q}}^{\mathcal{N}}\mathbb{P}_{\boldsymbol{c}}^{\boldsymbol{\mathcal{T}}}\mathbf{u}|||_{\mathfrak{Q}_{\mathcal{D}}}^{2}+\frac{1}{4}\|\widetilde{p}_{\mathrm{ex}}^{\mathfrak{Q}_{\mathcal{D}}}\|_{2}^{2}-C\mathrm{size}(\mathcal{T})^{2}.\quad(6.11)$$

By bilinearity of $\mathcal{B}$, the inequalities (6.10) and (6.11) give for each positive number $\xi > 0$:

$$\mathcal{B}(\mathbf{u}, \widetilde{p}_{\rm ex}^{\mathfrak{O}_\mathcal{D}}; \mathbf{u} + \xi\widetilde{\mathbf{v}}, \widetilde{p}_{\rm ex}^{\mathfrak{O}_\mathcal{D}}) + \xi C{\rm size}(\mathcal{T})^2 \geq \left(2\underline{\rm C}_\eta - \xi C\right) \|{\rm D}\mathbf{u} - {\rm D}_\mathfrak{Q}^\mathcal{N}\mathbb{P}_{\boldsymbol{c}}^{\boldsymbol{\mathcal{T}}}\mathbf{u}\|_{\mathfrak{O}_\mathcal{D}}^2 + \frac{\xi}{2}\|\widetilde{p}_{\rm ex}^{\mathfrak{O}_\mathcal{D}}\|_2^2.$$

Choosing a value of $\xi > 0$ small enough (depending only on $\underline{\rm C}_\eta$ and $\overline{\rm C}_\eta$), this inequality yields the following estimates such that $\bar{\mathbf{u}} = \mathbf{u} + \xi\widetilde{\mathbf{v}} \in (H^2(\omega))^2$

$$\|{\rm D}\bar{\mathbf{u}} - {\rm D}_\mathfrak{Q}^\mathcal{N}\mathbb{P}_{\boldsymbol{c}}^{\boldsymbol{\mathcal{T}}}\bar{\mathbf{u}}\|_{\mathfrak{O}_\mathcal{D}} \leq C\left(\|{\rm D}\mathbf{u} - {\rm D}_\mathfrak{Q}^\mathcal{N}\mathbb{P}_{\boldsymbol{c}}^{\boldsymbol{\mathcal{T}}}\mathbf{u}\|_{\mathfrak{O}_\mathcal{D}}^2 + \|\widetilde{p}_{\rm ex}^{\mathfrak{O}_\mathcal{D}}\|_2 + h_\mathcal{D}\right), \tag{6.12}$$

and

$$\|{\rm D}\mathbf{u} - {\rm D}_\mathfrak{Q}^\mathcal{N}\mathbb{P}_{\boldsymbol{c}}^{\boldsymbol{\mathcal{T}}}\mathbf{u}\|_{\mathfrak{O}_\mathcal{D}}^2 + \|\widetilde{p}_{\rm ex}^{\mathfrak{O}_\mathcal{D}}\|_2^2 \leq C\mathcal{B}(\mathbf{u}, \widetilde{p}_{\rm ex}^{\mathfrak{O}_\mathcal{D}}; \bar{\mathbf{u}}, \widetilde{p}_{\rm ex}^{\mathfrak{O}_\mathcal{D}}) + C_3{\rm size}(\mathcal{T})^2. \tag{6.13}$$

Step 3. We define now the consistency error for the projection $\mathbb{P}_c^\mathfrak{O}$ as follows

$$T_{\bar{\mathfrak{Q}},\bar{\mathbf{u}}}(z) = {\rm D}\bar{\mathbf{u}}(z) - {\rm D}\mathbb{P}_c^\mathfrak{Q}\bar{\mathbf{u}}, \quad \forall\, z \in \mathfrak{q}, \quad \forall\, \mathfrak{q} \in \mathfrak{Q}.$$

We want the existence of $\widetilde{\delta^\mathcal{D}} \in \mathcal{M}_{n_\mathcal{D},2}(\mathbb{R})$ such that

$${\rm D}\mathbb{P}_c^\mathfrak{Q}\bar{\mathbf{u}} - {\rm D}^\mathcal{D}\mathbb{P}_{\boldsymbol{c}}^{\boldsymbol{\mathcal{T}}}\bar{\mathbf{u}} = \frac{1}{2}(B_\mathfrak{Q}\widetilde{\delta^\mathcal{D}} + {}^t\widetilde{\delta^\mathcal{D}}{}^t B_\mathfrak{Q}), \quad \forall\, \mathfrak{q} \in \mathfrak{Q}_\mathcal{D}. \tag{6.14}$$

We verified that $2{\rm D}\mathbb{P}_c^\mathfrak{Q}\bar{\mathbf{u}} - 2{\rm D}^\mathcal{D}\mathbb{P}_{\boldsymbol{c}}^{\boldsymbol{\mathcal{T}}}\bar{\mathbf{u}}$ satisfied the conditions (2.9) if $\alpha_\mathcal{K} \neq \alpha_\mathcal{L}$ and (2.10) if $\alpha_\mathcal{K} = \alpha_\mathcal{L}$. Thanks to Proposition 2.2, there exists a unique $\widetilde{\delta^\mathcal{D}} \in \mathcal{M}_{n_\mathcal{D},2}(\mathbb{R})$ such that $(\widetilde{\delta^\mathcal{D}}, \delta_0) = 0$ if $\alpha_\mathcal{K} = \alpha_\mathcal{L}$ and satisfy (6.14). Thus with Definition 2.6 of the discrete strain rate tensor ${\rm D}_\mathfrak{Q}^\mathcal{N}$ and Definition 6.1, we have ${\rm D}_\mathfrak{Q}^\mathcal{N}\mathbb{P}_{\boldsymbol{c}}^{\boldsymbol{\mathcal{T}}}\bar{\mathbf{u}} = {\rm D}^\mathcal{D}\mathbb{P}_{\boldsymbol{c}}^{\boldsymbol{\mathcal{T}}}\bar{\mathbf{u}} + \frac{1}{2}(B_\mathfrak{Q}\delta_{\rm ex}^\mathfrak{D} + {}^t\delta_{\rm ex}^\mathfrak{D}{}^t B_\mathfrak{Q})$, we note $\widehat{\delta^\mathcal{D}} = \widetilde{\delta^\mathcal{D}} - \delta_{\rm ex}^\mathcal{D} \in (\mathcal{M}_{n_\mathcal{D},2}(\mathbb{R}))^\mathfrak{D}$ which satisfies

$${\rm D}\mathbb{P}_c^\mathfrak{Q}\bar{\mathbf{u}} - {\rm D}_\mathfrak{Q}^\mathcal{N}\mathbb{P}_{\boldsymbol{c}}^{\boldsymbol{\mathcal{T}}}\bar{\mathbf{u}} = \frac{1}{2}(B_\mathfrak{Q}\widehat{\delta^\mathcal{D}} + {}^t\widehat{\delta^\mathcal{D}}{}^t B_\mathfrak{Q}), \quad \forall\, \mathfrak{q} \in \mathfrak{Q}_\mathcal{D}, \tag{6.15}$$

and such that $(\widehat{\delta^\mathcal{D}}, \delta_0) = 0$ if $\alpha_\mathcal{K} = \alpha_\mathcal{L}$. Replacing $\delta^\mathcal{D}$ by $\widehat{\delta^\mathcal{D}}$ in (6.9) and using the fact that

$$\frac{1}{2}(B_\mathfrak{Q}\widehat{\delta^\mathcal{D}} + {}^t\widehat{\delta^\mathcal{D}}{}^t B_\mathfrak{Q}) = {\rm D}\mathbb{P}_c^\mathfrak{Q}\bar{\mathbf{u}} - {\rm D}_\mathfrak{Q}^\mathcal{N}\mathbb{P}_{\boldsymbol{c}}^{\boldsymbol{\mathcal{T}}}\bar{\mathbf{u}} = {\rm D}\bar{\mathbf{u}}(z) - {\rm D}_\mathfrak{Q}^\mathcal{N}\mathbb{P}_{\boldsymbol{c}}^{\boldsymbol{\mathcal{T}}}\bar{\mathbf{u}} - T_{\bar{\mathfrak{Q}},\bar{\mathbf{u}}}(z), \tag{6.16}$$

we deduce that

$$\begin{aligned}{\rm Tr}\left({}^t\widehat{\delta^\mathcal{D}}R\right) &= \sum_{\mathfrak{q}\in\mathfrak{Q}_\mathcal{D}}\int_\mathfrak{q} 2\eta(z)\left({\rm D}\bar{\mathbf{u}}(z) - {\rm D}_\mathfrak{Q}^\mathcal{N}\mathbb{P}_{\boldsymbol{c}}^{\boldsymbol{\mathcal{T}}}\bar{\mathbf{u}} : {\rm D}\mathbf{u}(z) - {\rm D}_\mathfrak{Q}^\mathcal{N}\mathbb{P}_{\boldsymbol{c}}^{\boldsymbol{\mathcal{T}}}\mathbf{u}\right){\rm d}z + \sum_{\mathfrak{q}\in\mathfrak{Q}_\mathcal{D}} m_\mathfrak{q}{\rm Tr}\left(T_{\bar{\mathfrak{Q}},\bar{\mathbf{u}}}(z)\right)\widetilde{p}_{\rm ex}^\mathfrak{Q} \\ &- \sum_{\mathfrak{q}\in\mathfrak{Q}_\mathcal{D}}\int_\mathfrak{q} 2\eta(z)\left(T_{\bar{\mathfrak{Q}},\bar{\mathbf{u}}}(z) : {\rm D}\mathbf{u}(z) - {\rm D}_\mathfrak{Q}^\mathcal{N}\mathbb{P}_{\boldsymbol{c}}^{\boldsymbol{\mathcal{T}}}\mathbf{u}\right){\rm d}z - \sum_{\mathfrak{q}\in\mathfrak{Q}_\mathcal{D}} m_\mathfrak{q}{\rm Tr}\left({\rm D}\bar{\mathbf{u}}(z) - {\rm D}_\mathfrak{Q}^\mathcal{N}\mathbb{P}_{\boldsymbol{c}}^{\boldsymbol{\mathcal{T}}}\bar{\mathbf{u}}\right)\widetilde{p}_{\rm ex}^\mathfrak{Q}.\end{aligned}$$

Now we can link R and $\mathcal{B}$ as follows

$$\begin{aligned}\mathcal{B}(\mathbf{u}, \widetilde{p}_{\rm ex}^{\mathfrak{O}_\mathcal{D}}; \bar{\mathbf{u}}, \widetilde{p}_{\rm ex}^{\mathfrak{O}_\mathcal{D}}) &= {\rm Tr}\left({}^t\widehat{\delta^\mathcal{D}}R\right) - \sum_{\mathfrak{q}\in\mathfrak{Q}_\mathcal{D}} m_\mathfrak{q}{\rm Tr}\left(T_{\bar{\mathfrak{Q}},\bar{\mathbf{u}}}(z)\right)\widetilde{p}_{\rm ex}^\mathfrak{Q} \\ &+ \sum_{\mathfrak{q}\in\mathfrak{Q}_\mathcal{D}}\int_\mathfrak{q} 2\eta(z)\left(T_{\bar{\mathfrak{Q}},\bar{\mathbf{u}}}(z) : {\rm D}\mathbf{u}(z) - {\rm D}_\mathfrak{Q}^\mathcal{N}\mathbb{P}_{\boldsymbol{c}}^{\boldsymbol{\mathcal{T}}}\mathbf{u}\right){\rm d}z + \sum_{\mathfrak{q}\in\mathfrak{Q}_\mathcal{D}} m_\mathfrak{q}{\rm Tr}\left({\rm D}\mathbf{u}(z) - {\rm D}_\mathfrak{Q}^\mathcal{N}\mathbb{P}_{\boldsymbol{c}}^{\boldsymbol{\mathcal{T}}}\mathbf{u}\right)\widetilde{p}_{\rm ex}^\mathfrak{Q}.\end{aligned}$$

Thanks to (1.2) and the Cauchy-Schwarz inequality, Proposition 6.3 implies that

$$\begin{aligned}\mathcal{B}(\mathbf{u}, \widetilde{p}_{\rm ex}^{\mathfrak{O}_\mathcal{D}}; \bar{\mathbf{u}}, \widetilde{p}_{\rm ex}^{\mathfrak{O}_\mathcal{D}}) &\leq Ch_\mathcal{D}\|B_\mathfrak{Q}\widehat{\delta^\mathcal{D}}\|_{\mathfrak{O}_\mathcal{D}}\left(\sum_{\mathfrak{q}\in\mathfrak{Q}_\mathcal{D}}\int_\mathfrak{q}(\|\nabla\mathbf{u}(z)\|_\mathcal{F}^2 + |\nabla^2\mathbf{u}(z)|^2 + |\nabla p(z)|^2){\rm d}z\right)^{\frac{1}{2}} \\ &+ \|T_{\bar{\mathfrak{Q}},\bar{\mathbf{u}}}\|_{\mathfrak{O}_\mathcal{D}}\|\widetilde{p}_{\rm ex}^{\mathfrak{O}_\mathcal{D}}\|_2 + 2\overline{\rm C}_\eta\|{\rm D}\mathbf{u} - {\rm D}_\Omega^\mathcal{N}\mathbb{P}_{\boldsymbol{c}}^{\boldsymbol{\mathcal{T}}}\mathbf{u}\|_{\mathfrak{O}_\mathcal{D}}\|T_{\bar{\mathfrak{Q}},\bar{\mathbf{u}}}\|_{\mathfrak{O}_\mathcal{D}} + \|{\rm D}\mathbf{u} - {\rm D}_\mathfrak{Q}^\mathcal{N}\mathbb{P}_{\boldsymbol{c}}^{\boldsymbol{\mathcal{T}}}\mathbf{u}\|_{\mathfrak{O}_\mathcal{D}}\|\widetilde{p}_{\rm ex}^{\mathfrak{O}_\mathcal{D}}\|_2.\end{aligned}$$

Thanks to Lemma 2.2 and the estimate (6.16), we have

$$\sum_{\mathcal{Q}\in\mathfrak{Q}_{\mathcal{D}}} m_{\mathcal{Q}}\|B_{\mathcal{Q}}\widehat{\delta^{\mathcal{D}}}\|_{\mathcal{F}}^2 \leq C\left(\|\mathrm{D}\bar{\mathbf{u}} - \mathrm{D}_{\mathfrak{Q}}^{\mathcal{N}}\mathbb{P}_{\boldsymbol{c}}^{\boldsymbol{\mathcal{T}}}\bar{\mathbf{u}}\|_{\mathfrak{Q}_{\mathcal{D}}}^2 + \|T_{\bar{\Omega},\bar{\mathbf{u}}}\|_{\mathfrak{Q}_{\mathcal{D}}}^2\right).$$

Using (6.12) and (6.13), Proposition 6.2 implies

$$\begin{aligned}\|\mathrm{D}\mathbf{u} - \mathrm{D}_{\mathfrak{Q}}^{\mathcal{N}}\mathbb{P}_{\boldsymbol{c}}^{\boldsymbol{\mathcal{T}}}\mathbf{u}\|_{\mathfrak{Q}_{\mathcal{D}}}^2 + \|\widetilde{p}_{\mathrm{ex}}^{\mathfrak{Q}_{\mathcal{D}}}\|_2^2 \leq Ch_{\mathcal{D}}^2\sum_{\mathcal{Q}\in\mathfrak{Q}_{\mathcal{D}}}\int_{\mathcal{Q}}(\|\nabla\mathbf{u}(z)\|_{\mathcal{F}}^2 + |\nabla^2\mathbf{u}(z)|^2 + |\nabla p(z)|^2)\mathrm{d}z\\ +Ch_{\mathcal{D}}\Big(\|\mathrm{D}\mathbf{u} - \mathrm{D}_{\mathfrak{Q}}^{\mathcal{N}}\mathbb{P}_{\boldsymbol{c}}^{\boldsymbol{\mathcal{T}}}\mathbf{u}\|_{\mathfrak{Q}_{\mathcal{D}}} + \|\widetilde{p}_{\mathrm{ex}}^{\mathfrak{Q}_{\mathcal{D}}}\|_2\Big)\left(\sum_{\mathcal{Q}\in\mathfrak{Q}_{\mathcal{D}}}\int_{\mathcal{Q}}(\|\nabla\mathbf{u}(z)\|_{\mathcal{F}}^2 + |\nabla^2\mathbf{u}(z)|^2 + |\nabla p(z)|^2)\mathrm{d}z\right)^{\frac{1}{2}}\\ +\|\mathrm{D}\mathbf{u} - \mathrm{D}_{\mathfrak{Q}}^{\mathcal{N}}\mathbb{P}_{\boldsymbol{c}}^{\boldsymbol{\mathcal{T}}}\mathbf{u}\|_{\mathfrak{Q}_{\mathcal{D}}}\|\widetilde{p}_{\mathrm{ex}}^{\mathfrak{Q}_{\mathcal{D}}}\|_2.\end{aligned}$$

Finally, Young inequality gives the result. □

REMARK 6.1. *We immediately have the estimate on the whole norm for* $(\mathbf{u}, p)$

$$\|\mathrm{D}\mathbf{u} - \mathrm{D}_{\mathfrak{Q}}^{\mathcal{N}}\mathbb{P}_{\boldsymbol{c}}^{\boldsymbol{\mathcal{T}}}\mathbf{u}\|_2^2 + \|\widetilde{p}_{\mathrm{ex}}^{\mathfrak{Q}_{\mathfrak{D}}}\|_2^2 \leq C_{25}\mathrm{size}(\mathcal{T})(\|\mathbf{u}\|_{(H^2(\omega))^2} + \|p\|_{H^1(\omega)}). \tag{6.17}$$

The following Corollary is obtained thanks to Lemma 2.2 and Lemma 6.4.

COROLLARY 6.1. *Assume that* η *satisfies* (1.2) *and* (2.4)*. We assume that* $(\mathbf{u}, p)$*, the solution of the problem* (1.1)*, lies in* $(H^2(\omega))^2 \times H^1(\omega)$*. Let* $\mathcal{T}$ *be a mesh of* Ω*. There exists a constant* $C_{26} > 0$*, depending only on* $\underline{\mathrm{C}}_\eta$*,* $\overline{\mathrm{C}}_\eta$*,* C_η*,* $\mathrm{reg}(\mathcal{T})$ *and* $\sin(\epsilon_0)$*, such that*

$$\|\nabla\mathbf{u} - \nabla_{\mathfrak{Q}}^{\mathcal{N}}\mathbb{P}_{\boldsymbol{c}}^{\boldsymbol{\mathcal{T}}}\mathbf{u}\|_2 \leq C_{26}\mathrm{size}(\mathcal{T})(\|\mathbf{u}\|_{(H^2(\omega))^2} + \|p\|_{H^1(\omega)}).$$

Proof. Like in the beginning of Step 3, (see (6.15)), we prove the existence of $\widehat{\delta^{\mathcal{D}}}$, such that $\mathrm{D}\mathbb{P}_c^{\mathcal{Q}}\mathbf{u} - \mathrm{D}_{\mathcal{Q}}^{\mathcal{N}}\mathbb{P}_{\boldsymbol{c}}^{\boldsymbol{\mathcal{T}}}\mathbf{u} = \frac{1}{2}(B_{\mathcal{Q}}\widehat{\delta^{\mathcal{D}}} + {}^t\widehat{\delta^{\mathcal{D}}}{}^tB_{\mathcal{Q}})$. We deduce that $\nabla\mathbb{P}_c^{\mathcal{Q}}\mathbf{u} - \nabla_{\mathcal{Q}}^{\mathcal{N}}\mathbb{P}_{\boldsymbol{c}}^{\boldsymbol{\mathcal{T}}}\mathbf{u} = {}^t\widehat{\delta^{\mathcal{D}}}{}^tB_{\mathcal{Q}}$. Lemma 2.2, Lemma 6.4 and Proposition 6.2 give

$$\sum_{\mathcal{Q}\in\mathfrak{Q}_{\mathcal{D}}} m_{\mathcal{Q}}\|B_{\mathcal{Q}}\widehat{\delta^{\mathcal{D}}}\|_{\mathcal{F}}^2 \leq \frac{C_1}{2}(C_{25} + C_{20})h_{\mathcal{D}}^2\int_{\mathcal{Q}}(\|\nabla\mathbf{u}(z)\|_{\mathcal{F}}^2 + |\nabla^2\mathbf{u}(z)|^2 + |\nabla p(z)|^2)\mathrm{d}z. \tag{6.18}$$

Proposition 6.2 and (6.18) conclude the proof. □

LEMMA 6.5. *Assume that* η *satisfies* (1.2) *and* (2.4)*. We assume that* $(\mathbf{u}, p)$*, the solution of the problem* (1.1)*, lies in* $(H^2(\omega))^2 \times H^1(\omega)$ *with* $\mathrm{div}\ \mathbf{u} = 0$*. Let* $\mathcal{T}$ *be a mesh of* Ω*. There exists a constant* $C_{27} > 0$*, depending only on* $\underline{\mathrm{C}}_\eta$*,* $\overline{\mathrm{C}}_\eta$*,* C_η*,* $\mathrm{reg}(\mathcal{T})$ *and* $\sin(\epsilon_0)$*, such that*

$$\begin{aligned}&|\mathrm{div}^{\mathcal{D}}\mathbb{P}_{\boldsymbol{c}}^{\boldsymbol{\mathcal{T}}}\mathbf{u}| \leq 2\|\mathrm{D}_{\mathcal{Q}}^{\mathcal{N}}\mathbb{P}_{\boldsymbol{c}}^{\boldsymbol{\mathcal{T}}}\mathbf{u} - \mathrm{D}\mathbf{u}\|_{\mathfrak{Q}_{\mathcal{D}}}, \quad \forall\,\mathcal{D}\in\mathfrak{D},\\ &\|\mathrm{div}^{\mathfrak{D}}\mathbb{P}_{\boldsymbol{c}}^{\boldsymbol{\mathcal{T}}}\mathbf{u}\|_2 \leq C_{27}\mathrm{size}(\mathcal{T})(\|\mathbf{u}\|_{(H^2(\omega))^2} + \|p\|_{H^1(\omega)}).\end{aligned}$$

Proof. Thanks to $\mathrm{div}^{\mathcal{D}}(\mathbb{P}_{\boldsymbol{c}}^{\boldsymbol{\mathcal{T}}}\mathbf{u}) = \mathrm{Tr}(\mathrm{D}^{\mathcal{D}}\mathbb{P}_{\boldsymbol{c}}^{\boldsymbol{\mathcal{T}}}\mathbf{u})$ and $\mathrm{div}\ \mathbf{u} = 0$, the equality (2.5) gives

$$\mathrm{div}^{\mathcal{D}}(\mathbb{P}_{\boldsymbol{c}}^{\boldsymbol{\mathcal{T}}}\mathbf{u}) = \mathrm{div}^{\mathcal{D}}(\mathbb{P}_{\boldsymbol{c}}^{\boldsymbol{\mathcal{T}}}\mathbf{u}) - \mathrm{div}\ \mathbf{u}(z) = \frac{1}{m_{\mathcal{D}}}\sum_{\mathcal{Q}\in\mathfrak{Q}_{\mathcal{D}}} m_{\mathcal{Q}}\mathrm{Tr}(\mathrm{D}_{\mathcal{Q}}^{\mathcal{N}}\mathbb{P}_{\boldsymbol{c}}^{\boldsymbol{\mathcal{T}}}\mathbf{u} - \mathrm{D}\mathbf{u}(z)).$$

Cauchy-Schwarz inequality implies the first estimate. Lemma 6.5 and 6.17 imply the second estimate with $C_{27} = 2\sqrt{C_{25}}$. □

6.5. Consistency error for the velocity. Now, we can control $\mathbf{R}^{\mathbf{u}}_{\mathfrak{s},\mathfrak{Q}}$, as follows

LEMMA 6.6. *Assume that η satisfies* (1.2)*,* (2.4) *and that* $(\mathbf{u},p)$*, the solution of the problem* (1.1)*, lies in* $(H^2(\omega))^2\times H^1(\omega)$*. For any mesh $\mathcal{T}$ on Ω, there exists a constant $C_{28}>0$, depending only on* C_η*,* $\underline{\mathrm{C}}_\eta$*,* $\overline{\mathrm{C}}_\eta$*,* $\mathrm{reg}(\mathcal{T})$ *and* $\sin(\epsilon_0)$*, such that*

$$\|\mathbf{R}^{\mathbf{u}}_{\mathfrak{s},\mathfrak{Q}}\|_2\leq C_{28}\mathrm{size}(\mathcal{T})(\|\mathbf{u}\|_{(H^2(\omega))^2}+\|p\|_{H^1(\omega)}).$$

Proof. Definition 6.3 implies that

$$\|\mathbf{R}^{\mathbf{u}}_{\mathfrak{s},\mathfrak{Q}}\|_2^2\leq\sum_{\mathcal{Q}\in\mathfrak{Q}}m_{\mathcal{Q}}\sum_{\mathfrak{s}\in\mathcal{E}_{\mathcal{Q}}}|\mathbf{R}^{\mathbf{u},\eta}_{\mathfrak{s},\mathcal{Q}}|^2+\|\!|R^{\mathbf{u},\mathrm{D}\mathbf{u}}_{\mathfrak{Q}}|\!\|_2^2+\|\!|R^{\mathbf{u},z}_{\mathfrak{Q}}|\!\|_2^2. \tag{6.19}$$

First, the inequality (1.2) and Lemma 6.4 imply

$$\|\!|R^{\mathbf{u},\mathrm{D}\mathbf{u}}_{\mathfrak{Q}}|\!\|_2^2\leq\overline{\mathrm{C}}_\eta^2C_{25}\mathrm{size}(\mathcal{T})^2\sum_{\mathcal{Q}\in\mathfrak{Q}}\int_{\mathcal{Q}}\left(\|\!|\nabla\mathbf{u}(x)|\!\|_{\mathcal{F}}^2+|\nabla^2\mathbf{u}(x)|^2+|\nabla p|^2\right)\mathrm{d}x. \tag{6.20}$$

Substituting (6.20), and (6.7) into (6.19), we get the result.

□

6.6. Jump of pressure. LEMMA 6.7. *Assume that η satisfies* (1.2)*,* (2.4) *and that* $(\mathbf{u},p)$*, the solution of the problem* (1.1)*, lies in* $(H^2(\omega))^2\times H^1(\omega)$*. Let $\mathcal{T}$ be a mesh of Ω, there exists $C_{29}>0$, depending only on* $\underline{\mathrm{C}}_\eta$*,* $\overline{\mathrm{C}}_\eta$*,* C_η*,* $\mathrm{reg}(\mathcal{T})$ *and* $\sin(\epsilon_0)$*, such that*

$$\left(\sum_{\mathfrak{s}=\mathcal{Q}|\mathcal{Q}'=\mathcal{D}|\mathcal{D}'\in\mathfrak{S}}(p^{\mathcal{Q}'}_{\mathrm{ex}}-p^{\mathcal{Q}}_{\mathrm{ex}})^2\right)^{\frac12}\leq C_{29}(\|\mathbf{u}\|_{(H^2(\omega))^2}+\|p\|_{H^1(\omega)}).$$

Proof. We note $\mathbb{P}^{\mathfrak{s}}_m p:=\dfrac{1}{m_{\mathfrak{s}}}\displaystyle\int_{\mathfrak{s}}p(y)\mathrm{d}y$, for any $\mathfrak{s}\in\mathfrak{S}$, adding and subtracting $\mathbb{P}^{\mathfrak{s}}_m p$, Cauchy-Schwarz inequality implies

$$\sum_{\mathfrak{s}=\mathcal{Q}|\mathcal{Q}'=\mathcal{D}|\mathcal{D}'\in\mathfrak{S}}(p^{\mathcal{Q}'}_{\mathrm{ex}}-p^{\mathcal{Q}}_{\mathrm{ex}})^2\leq4\sum_{\mathcal{D}\in\mathfrak{D}}\sum_{\mathcal{Q}\in\mathfrak{Q}_{\mathcal{D}}}\sum_{\mathfrak{s}\in\partial\mathcal{Q}\cap\partial\mathcal{D}}|p^{\mathcal{Q}}_{\mathrm{ex}}-\mathbb{P}^{\mathfrak{s}}_m p|^2.$$

Then adding and subtracting $\mathbb{P}^{\mathcal{Q}}_m p$, Cauchy-Schwarz inequality implies

$$\sum_{\mathfrak{s}=\mathcal{Q}|\mathcal{Q}'=\mathcal{D}|\mathcal{D}'\in\mathfrak{S}}(p^{\mathcal{Q}'}_{\mathrm{ex}}-p^{\mathcal{Q}}_{\mathrm{ex}})^2\leq8\sum_{\mathcal{D}\in\mathfrak{D}}\sum_{\mathcal{Q}\in\mathfrak{Q}_{\mathcal{D}}}|p^{\mathcal{Q}}_{\mathrm{ex}}-\mathbb{P}^{\mathcal{Q}}_m p|^2+8\sum_{\mathcal{D}\in\mathfrak{D}}\sum_{\mathcal{Q}\in\mathfrak{Q}_{\mathcal{D}}}\sum_{\mathfrak{s}\in\partial\mathcal{Q}\cap\partial\mathcal{D}}|\mathbb{P}^{\mathcal{Q}}_m p-\mathbb{P}^{\mathfrak{s}}_m p|^2. \tag{6.21}$$

Since $p\in H^1(\mathcal{Q})$, thanks to Lemma 4.5, we get

$$|\mathbb{P}^{\mathcal{Q}}_m p-\mathbb{P}^{\mathfrak{s}}_m p|^2\leq C_{12}\mathrm{reg}(\mathcal{T})^3\int_{\mathcal{Q}}|\nabla p(z)|^2\mathrm{d}z. \tag{6.22}$$

Lemma 6.4 and (6.22) conclude the proof with $C_{29}=8(C_{25}+C_{12}\mathrm{reg}(\mathcal{T})^3)$.

□

6.7. Proof of Theorem 6.1. We may now collect all the previous results in order to conclude the proof of Theorem 6.1, that we started in Subsection 6.2.

Proof. Having denoted by $\mathbf{e}^{\boldsymbol{\mathcal{T}}}=\mathbb{P}^{\boldsymbol{\mathcal{T}}}_{\boldsymbol{c}}\mathbf{u}-\mathbf{u}^{\boldsymbol{\mathcal{T}}}$ and $e^{\mathfrak{D}}=\mathbb{P}^{\mathfrak{D}}_m p-p^{\mathfrak{D}}$, we have obtained the inequality (6.5)

$$\begin{aligned}\|\!|\nabla^{\mathcal{N}}_{\mathfrak{Q}}\mathbf{e}^{\boldsymbol{\mathcal{T}}}|\!\|_2^2+\|e^{\mathfrak{Q}}\|_2^2\leq&\widetilde{C}_1(\|\!|\nabla^{\mathcal{N}}_{\mathfrak{Q}}\mathbf{e}^{\boldsymbol{\mathcal{T}}}|\!\|_2+\|e^{\mathfrak{Q}}\|_2)(\|\mathbf{R}_{\mathfrak{s},\mathfrak{Q}}\|_2+\|\mathrm{div}^{\mathfrak{D}}\mathbb{P}^{\boldsymbol{\mathcal{T}}}_{\boldsymbol{c}}\mathbf{u}\|_2)\\&+\widetilde{C}_2\mathrm{size}(\mathcal{T})(\|\!|\nabla^{\mathcal{N}}_{\mathfrak{Q}}\mathbf{e}^{\boldsymbol{\mathcal{T}}}|\!\|_2+\|e^{\mathfrak{Q}}\|_2)\left(\sum_{\mathfrak{s}=\mathcal{Q}|\mathcal{Q}'=\mathcal{D}|\mathcal{D}'\in\mathfrak{S}}(p^{\mathcal{Q}'}_{\mathrm{ex}}-p^{\mathcal{Q}}_{\mathrm{ex}})^2\right)^{\frac12}.\end{aligned}$$

Using the estimate $\|\mathbf{R}_{\mathfrak{s},\mathfrak{Q}}\|_2 \leq 2\|\mathbf{R}^{\mathbf{u}}_{\mathfrak{s},\mathfrak{Q}}\|_2 + \|\mathbf{R}^{p}_{\mathfrak{s},\mathfrak{Q}}\|_2 + \|\widetilde{p}^{\mathfrak{Q}_{\mathfrak{D}}}_{\mathrm{ex}}\|_2$, Lemmas 6.1, 6.17, 6.6 and Lemma 6.5 imply $\|\mathbf{R}_{\mathfrak{s},\mathfrak{Q}}\|_2 + \|\mathrm{div}^{\mathfrak{D}}\mathbb{P}^{\boldsymbol{\mathcal{T}}}_{\boldsymbol{c}}\mathbf{u}\|_2 \leq C\mathrm{size}(\mathcal{T})$. Finally, Lemma 6.7 gives

$$|||\nabla^{\mathcal{N}}_{\mathfrak{Q}}\mathbf{e}^{\boldsymbol{\mathcal{T}}}|||_2 \leq C\mathrm{size}(\mathcal{T}) \quad \text{and} \quad \|e^{\mathfrak{Q}}\|_2 \leq C\mathrm{size}(\mathcal{T}). \tag{6.23}$$

Estimate of $\|\mathbf{u} - \mathbf{u}^{\boldsymbol{\mathcal{T}}}\|_2$. We have $\|\mathbf{u} - \mathbf{u}^{\boldsymbol{\mathcal{T}}}\|_2 \leq \|\mathbf{u} - \mathbb{P}^{\boldsymbol{\mathcal{T}}}_{\boldsymbol{c}}\mathbf{u}\|_2 + \|\mathbb{P}^{\boldsymbol{\mathcal{T}}}_{\boldsymbol{c}}\mathbf{u} - \mathbf{u}^{\boldsymbol{\mathcal{T}}}\|_2$. Lemma 4.6 and the discrete Poincaré inequality Theorem 4.5 imply

$$\|\mathbf{u} - \mathbf{u}^{\boldsymbol{\mathcal{T}}}\|_2 \leq C\mathrm{size}(\mathcal{T}) + C|||\nabla^{\mathfrak{D}}\mathbb{P}^{\boldsymbol{\mathcal{T}}}_{\boldsymbol{c}}\mathbf{u} - \nabla^{\mathfrak{D}}\mathbf{u}^{\boldsymbol{\mathcal{T}}}|||_2.$$

Lemma 4.2 and (6.23) gives the estimate of $\|\mathbf{u} - \mathbf{u}^{\boldsymbol{\mathcal{T}}}\|_2$.

Estimate of $|||\nabla\mathbf{u} - \nabla^{\mathcal{N}}_{\mathfrak{Q}}\mathbf{u}^{\boldsymbol{\mathcal{T}}}|||_2$. We have $|||\nabla\mathbf{u} - \nabla^{\mathcal{N}}_{\mathfrak{Q}}\mathbf{u}^{\boldsymbol{\mathcal{T}}}|||_2 \leq |||\nabla\mathbf{u} - \nabla^{\mathcal{N}}_{\mathfrak{Q}}\mathbb{P}^{\boldsymbol{\mathcal{T}}}_{\boldsymbol{c}}\mathbf{u}|||_2 + |||\nabla^{\mathcal{N}}_{\mathfrak{Q}}\mathbb{P}^{\boldsymbol{\mathcal{T}}}_{\boldsymbol{c}}\mathbf{u} - \nabla^{\mathcal{N}}_{\mathfrak{Q}}\mathbf{u}^{\boldsymbol{\mathcal{T}}}|||_2$. Finally, Lemma 6.1 and (6.23) imply the estimate of $|||\nabla\mathbf{u} - \nabla^{\mathcal{N}}_{\mathfrak{Q}}\mathbf{u}^{\boldsymbol{\mathcal{T}}}|||_2$.

Estimate of $\|p - p^{\mathfrak{Q}}\|_2$. Using (6.23), we obtain $\|\mathbb{P}^{\mathfrak{Q}}_{m}p - p^{\mathfrak{Q}}\|_2 \leq C\mathrm{size}(\mathcal{T})$. We conclude thanks to Lemma 6.1. □

7. Numerical results. We show here some numerical results obtained on a rectangular domain $\Omega =]0,1[^2$. Error estimates are given for two different tests with a stabilization coefficient $\lambda = 10^{-3}$.

(a) Non conformal quadrangle mesh. (b) Quadrangle and triangle mesh.

FIG. 7.1. *Family of meshes.*

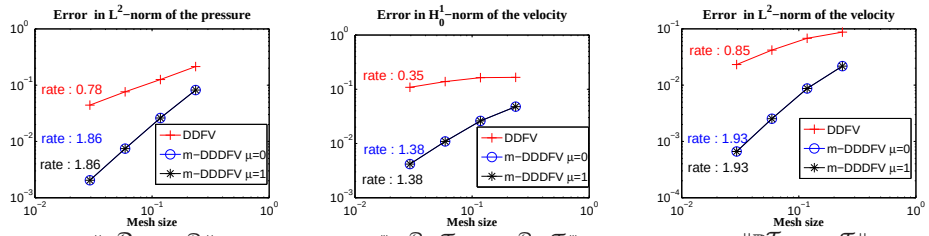
In order to illustrate error estimates, the family of meshes (see Figure 7.1) are obtained by successive global refinement of the original mesh.

The first one is performed using a discontinuous viscosity function. The exact solution $(\mathbf{u}, p)$ and the viscosity η being chosen, we define the source term $\mathbf{f}$ and the boundary data $\mathbf{g}$ in such a way that (1.1) is satisfied. In all tests, in Figures 7.2 and 7.3, we compare the errors in the three different norms the L^2 norm of the error obtained with the DDFV scheme, for the pressure $\dfrac{\|\mathbb{P}^{\mathfrak{Q}}_{c}p - p^{\mathfrak{Q}}\|_2}{\|\mathbb{P}^{\mathfrak{Q}}_{c}p\|_2}$, for the velocity gradient $\dfrac{\|\mathbb{P}^{\mathfrak{Q}}_{c}\nabla\mathbf{u} - \nabla^{\mathcal{N}}_{\mathfrak{Q}}\mathbf{u}^{\boldsymbol{\mathcal{T}}}\|_2}{\|\mathbb{P}^{\mathfrak{Q}}_{c}\nabla\mathbf{u}\|_2}$ and for the velocity $\dfrac{\|\mathbb{P}^{\boldsymbol{\mathcal{T}}}_{\boldsymbol{c}}\mathbf{u} - \mathbf{u}^{\boldsymbol{\mathcal{T}}}\|_2}{\|\mathbb{P}^{\boldsymbol{\mathcal{T}}}_{\boldsymbol{c}}\mathbf{u}\|_2}$ respectively as a function of the mesh size, in a logarithmic scale, for the original DDFV scheme (1.4), for the m-DDFV scheme (3.1) with $\mu = 0$ in (2.14) and for the m-DDFV scheme (3.1) with $\mu = 1$ in (2.14).In the numerical tests, we have smooth pressure on quarter diamond cells, thus we use the center-value projection on $\mathfrak{Q}$: $\mathbb{P}^{\mathfrak{Q}}_{c}p = \left((p(x_{\mathfrak{Q}}))_{\mathfrak{Q}\in\mathfrak{Q}}\right)$.

7.1. Test 1 - Discontinuous viscosity. The interface Γ is located at $\{y = 0.5\}$. Let us consider the following exact solution $\mathbf{u}(x,y) = \begin{pmatrix} u(x,y) \\ 0 \end{pmatrix}$:

$$u(x,y) = \begin{cases} y^2 - 0.5y & \text{for } y > 0.5 \\ 10^4(y^2 - 0.5y) & \text{else.} \end{cases}, \qquad p(x,y) = 2x - 1,$$

and the discontinuous viscosity: $\eta_1 = 1$, $\eta_2 = 10^{-4}$, which gives that $\mathrm{D}u$ is discontinuous across Γ. We show the comparison between (1.4) and (3.1) schemes with a discontinuous viscosity. We use the non conformal quadrangle mesh, locally refined where the discontinuity occurs, shown on Figure 7.1(a). In this case, we only have convex diamond cells, thus



(a) $\dfrac{\|\mathbb{P}_c^{\mathfrak{D}} p - p^{\mathfrak{D}}\|_2}{\|\mathbb{P}_c^{\mathfrak{D}} p\|_2}$ (b) $\dfrac{\|\nabla^{\mathfrak{D}} \mathbb{P}_c^{\mathcal{T}} \mathbf{u} - \nabla^{\mathfrak{D}} \mathbf{u}^{\mathcal{T}}\|_2}{\|\nabla^{\mathfrak{D}} \mathbb{P}_c^{\mathcal{T}} \mathbf{u}\|_2}$ (c) $\dfrac{\|\mathbb{P}_c^{\mathcal{T}} \mathbf{u} - \mathbf{u}^{\mathcal{T}}\|_2}{\|\mathbb{P}_c^{\mathcal{T}} \mathbf{u}\|_2}$

FIG. 7.2. *Test2, discontinuous viscosity on a non conformal quadrangle mesh Figure 7.1(a).*

the (3.1) scheme behaviour is essentially the same with the classic dual mesh than with the barycentric one. As predicted by the theory, the S-m-DDFV scheme provides a much better convergence rate than the original S-DDFV scheme. Furthermore, the error (in any of the three norms we consider) obtained by the S-m-DDFV scheme is better even in the case of coarse meshes. Note that the convergence rates obtained with the (3.1) scheme are greater than the theoretical one. This is related to some uniformity of the mesh away from the refinement area. Furthermore, let us emphasize that the convergence rate is not sensitive to the presence of non conformal control volumes.

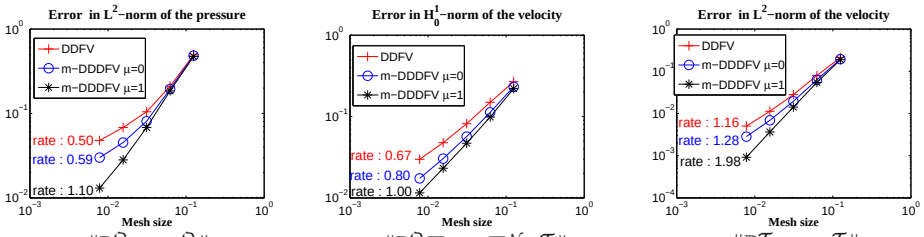
7.2. Test 2 - Discontinuous viscosity and discontinuous pressure. The interface Γ is located at $\{x = 0.5\}$. We take the discontinuous viscosity: $\eta_1 = 10^2$, $\eta_2 = 10^{-2}$. We note $c = -\frac{\eta_2 \pi}{\eta_1 + 0.5\eta_2 \pi}$. The exact solution is the following

$$\mathbf{u}(x,y) = \begin{pmatrix} \begin{cases} (x-0.5)(cx + \sin(5.0\pi x))\dfrac{4.0\pi\cos(4.0\pi y)}{0.5c+1}, & \text{for } x \le 0.5 \\ (x-0.5)(\cos(\pi x)+1)4.0\pi\cos(4.0\pi y), & \text{elsewhere.} \end{cases} \\ \begin{cases} -(cx + \sin(5.0\pi x) + (x-0.5)(c + 5.0\pi\cos(5.0\pi x)))\dfrac{\sin(4.0\pi y)}{0.5c+1}, & \text{for } x \le 0.5 \\ -(\cos(\pi x) + 1 - \pi(x-0.5)\sin(\pi x))\sin(4.0\pi y), & \text{elsewhere.} \end{cases} \end{pmatrix},$$

$$p(x,y) = \begin{cases} 8.0\pi(\eta_1 - \eta_2)\cos(4\pi y) + \cos(4\pi x)\sin(4\pi y), & \text{for } x \le 0.5 \\ \cos(4\pi x)\sin(4\pi y), & \text{elsewhere.} \end{cases}$$

As predicted in Theorem 6.1, we observe a first order convergence for the L^2-norm of the velocity gradient and of the pressure, which seems to be optimal. We obtain a second order convergence for the L^2-norm of the velocity. This super-convergence of the L^2-norm is classical for finite volume method, however its proof still remains an open problem see [24]. Figure 7.3 brings out the role of the new stabilization term. With the old stabilization term, using Definition 2.10, we observe that the scheme is still convergent even if we have lost the first order convergence, as expected.

8. Conclusion. In this paper, we provide a modification of the stabilized DDFV scheme with Dirichlet boundary conditions for the interface Stokes problem on general 2D grids in order to take into account discontinuities in the viscosity. The S-m-DDFV scheme we obtained is proved to present a better consistency of the fluxes at the discontinuities. We



(a) $\dfrac{\|\mathbb{P}_c^{\mathfrak{Q}} p - p^{\mathfrak{Q}}\|_2}{\|\mathbb{P}_c^{\mathfrak{Q}} p\|_2}$ (b) $\dfrac{\|\mathbb{P}_c^{\mathfrak{Q}} \nabla \mathbf{u} - \nabla_{\mathfrak{Q}}^{\mathcal{N}} \mathbf{u}^{\boldsymbol{\mathcal{T}}}\|_2}{\|\mathbb{P}_c^{\mathfrak{Q}} \nabla \mathbf{u}\|_2}$ (c) $\dfrac{\|\mathbb{P}_c^{\boldsymbol{\mathcal{T}}} \mathbf{u} - \mathbf{u}^{\boldsymbol{\mathcal{T}}}\|_2}{\|\mathbb{P}_c^{\boldsymbol{\mathcal{T}}} \mathbf{u}\|_2}$

FIG. 7.3. *Test2, discontinuous viscosity on a quadrangle mesh Figure 7.1(b).*

prove a first order convergence of the DDFV scheme in the L^2 norm for the velocity gradient $\|\nabla \mathbf{u} - \nabla^{\mathfrak{D}} \mathbf{u}^{\boldsymbol{\mathcal{T}}}\|_2$, for the velocity and for the pressure. The performance of the scheme is illustrated by numerical results.

9. Appendix.

9.1. Proof of Lemma 2.2. *Proof.* **Estimate between $\delta^{\mathcal{D}}$ and the symmetric part of $B_{\mathcal{Q}}\delta^{\mathcal{D}}$.** Let us explicit the components of $\delta^{\mathcal{D}}$ in the local basis of the diamond cell.

$$
\begin{aligned}
{}^t\delta_{\mathcal{K}} &= \mu_{\mathcal{K}} \vec{\boldsymbol{\tau}}_{\boldsymbol{\mathcal{K}^*,\mathcal{L}^*}} + \lambda_{\mathcal{K}} \vec{\mathbf{n}}_{\boldsymbol{\sigma_{\mathcal{K}}}}, \qquad {}^t\delta_{\mathcal{L}} = \mu_{\mathcal{L}} \vec{\boldsymbol{\tau}}_{\boldsymbol{\mathcal{K}^*,\mathcal{L}^*}} + \lambda_{\mathcal{L}} \vec{\mathbf{n}}_{\boldsymbol{\sigma_{\mathcal{K}}}},\\
{}^t\delta_{\mathcal{K}^*} &= \mu_{\mathcal{K}^*} \vec{\boldsymbol{\tau}}_{\boldsymbol{\sigma_{\mathcal{K}}\mathcal{K}^*}} + \lambda_{\mathcal{K}^*} \vec{\mathbf{n}}_{\boldsymbol{\sigma_{\mathcal{K}}\mathcal{K}^*}}, \qquad {}^t\delta_{\mathcal{L}^*} = \mu_{\mathcal{L}^*} \vec{\boldsymbol{\tau}}_{\boldsymbol{\sigma_{\mathcal{K}}\mathcal{K}^*}} + \lambda_{\mathcal{L}^*} \vec{\mathbf{n}}_{\boldsymbol{\sigma_{\mathcal{K}}\mathcal{K}^*}},
\end{aligned}
$$

where $\mu_{\cdot}, \lambda_{\cdot}$ lie in $\mathbb{R}$. Using notation $S_{\mathcal{Q}} = B_{\mathcal{Q}}\delta^{\mathcal{D}} + {}^t\delta^{\mathcal{D}}\,{}^tB_{\mathcal{Q}}$, for all $\mathcal{Q} \in \mathfrak{Q}_{\mathcal{D}}$, and the fact that ${}^tXB_{\mathcal{Q}}\delta^{\mathcal{D}}X = {}^tX\,{}^t\delta^{\mathcal{D}}\,{}^tB_{\mathcal{Q}}X$ for any $X \in \mathbb{R}^2$, the definition of $B_{\mathcal{Q}}$ and the decomposition of $\delta^{\mathcal{D}}$ imply four equations:

$$
\begin{aligned}
&\mu_{\mathcal{K}} m_{\sigma_{\mathcal{K}}}(X, \vec{\boldsymbol{\tau}}_{\boldsymbol{\mathcal{K}^*,\mathcal{L}^*}})(\vec{\mathbf{n}}_{\boldsymbol{\sigma_{\mathcal{K}}\mathcal{K}^*}}, X) + \lambda_{\mathcal{K}} m_{\sigma_{\mathcal{K}}}(X, \vec{\mathbf{n}}_{\boldsymbol{\sigma_{\mathcal{K}}}})(\vec{\mathbf{n}}_{\boldsymbol{\sigma_{\mathcal{K}}\mathcal{K}^*}}, X)\\
&\quad + m_{\sigma_{\mathcal{K}^*}} \mu_{\mathcal{K}^*}(X, \vec{\boldsymbol{\tau}}_{\boldsymbol{\sigma_{\mathcal{K}}\mathcal{K}^*}})(\vec{\mathbf{n}}_{\boldsymbol{\sigma_{\mathcal{K}}}}, X) + \lambda_{\mathcal{K}^*} m_{\sigma_{\mathcal{K}^*}}(X, \vec{\mathbf{n}}_{\boldsymbol{\sigma_{\mathcal{K}}\mathcal{K}^*}})(\vec{\mathbf{n}}_{\boldsymbol{\sigma_{\mathcal{K}}}}, X)\\
&\quad = \frac{m_{\mathcal{Q}_{\mathcal{K},\mathcal{K}^*}}}{2}\,{}^tXS_{\mathcal{Q}_{\mathcal{K},\mathcal{K}^*}}X,
\end{aligned}
\tag{9.1a}
$$

$$
\begin{aligned}
&- m_{\sigma_{\mathcal{K}}} \mu_{\mathcal{K}}(X, \vec{\boldsymbol{\tau}}_{\boldsymbol{\mathcal{K}^*,\mathcal{L}^*}})(\vec{\mathbf{n}}_{\boldsymbol{\sigma_{\mathcal{K}}\mathcal{K}^*}}, X) - \lambda_{\mathcal{K}} m_{\sigma_{\mathcal{K}}}(X, \vec{\mathbf{n}}_{\boldsymbol{\sigma_{\mathcal{K}}}})(\vec{\mathbf{n}}_{\boldsymbol{\sigma_{\mathcal{K}}\mathcal{K}^*}}, X)\\
&\quad + m_{\sigma_{\mathcal{L}^*}} \mu_{\mathcal{L}^*}(X, \vec{\boldsymbol{\tau}}_{\boldsymbol{\sigma_{\mathcal{K}}\mathcal{K}^*}})(\vec{\mathbf{n}}_{\boldsymbol{\sigma_{\mathcal{K}}}}, X) + \lambda_{\mathcal{L}^*} m_{\sigma_{\mathcal{L}^*}}(X, \vec{\mathbf{n}}_{\boldsymbol{\sigma_{\mathcal{K}}\mathcal{K}^*}})(\vec{\mathbf{n}}_{\boldsymbol{\sigma_{\mathcal{K}}}}, X)\\
&\quad = \frac{m_{\mathcal{Q}_{\mathcal{K},\mathcal{L}^*}}}{2}\,{}^tXS_{\mathcal{Q}_{\mathcal{K},\mathcal{L}^*}}X,
\end{aligned}
\tag{9.1b}
$$

$$
\begin{aligned}
&m_{\sigma_{\mathcal{L}}} \mu_{\mathcal{L}}(X, \vec{\boldsymbol{\tau}}_{\boldsymbol{\mathcal{K}^*,\mathcal{L}^*}})(\vec{\mathbf{n}}_{\boldsymbol{\sigma_{\mathcal{L}}\mathcal{K}^*}}, X) + \lambda_{\mathcal{L}} m_{\sigma_{\mathcal{L}}}(X, \vec{\mathbf{n}}_{\boldsymbol{\sigma_{\mathcal{K}}}})(\vec{\mathbf{n}}_{\boldsymbol{\sigma_{\mathcal{L}}\mathcal{K}^*}}, X)\\
&\quad - m_{\sigma_{\mathcal{K}^*}} \mu_{\mathcal{K}^*}(X, \vec{\boldsymbol{\tau}}_{\boldsymbol{\sigma_{\mathcal{K}}\mathcal{K}^*}})(\vec{\mathbf{n}}_{\boldsymbol{\sigma_{\mathcal{K}}}}, X) - \lambda_{\mathcal{K}^*} m_{\sigma_{\mathcal{K}^*}}(X, \vec{\mathbf{n}}_{\boldsymbol{\sigma_{\mathcal{K}}\mathcal{K}^*}})(\vec{\mathbf{n}}_{\boldsymbol{\sigma_{\mathcal{K}}}}, X)\\
&\quad = \frac{m_{\mathcal{Q}_{\mathcal{L},\mathcal{K}^*}}}{2}\,{}^tXS_{\mathcal{Q}_{\mathcal{L},\mathcal{K}^*}}X,
\end{aligned}
\tag{9.1c}
$$

$$
\begin{aligned}
&- m_{\sigma_{\mathcal{L}}} \mu_{\mathcal{L}}(X, \vec{\boldsymbol{\tau}}_{\boldsymbol{\mathcal{K}^*,\mathcal{L}^*}})(\vec{\mathbf{n}}_{\boldsymbol{\sigma_{\mathcal{L}}\mathcal{K}^*}}, X) - \lambda_{\mathcal{L}} m_{\sigma_{\mathcal{L}}}(X, \vec{\mathbf{n}}_{\boldsymbol{\sigma_{\mathcal{K}}}})(\vec{\mathbf{n}}_{\boldsymbol{\sigma_{\mathcal{L}}\mathcal{K}^*}}, X)\\
&\quad - m_{\sigma_{\mathcal{L}^*}} \mu_{\mathcal{L}^*}(X, \vec{\boldsymbol{\tau}}_{\boldsymbol{\sigma_{\mathcal{K}}\mathcal{K}^*}})(\vec{\mathbf{n}}_{\boldsymbol{\sigma_{\mathcal{K}}}}, X) - \lambda_{\mathcal{L}^*} m_{\sigma_{\mathcal{L}^*}}(X, \vec{\mathbf{n}}_{\boldsymbol{\sigma_{\mathcal{K}}\mathcal{K}^*}})(\vec{\mathbf{n}}_{\boldsymbol{\sigma_{\mathcal{K}}}}, X)\\
&\quad = \frac{m_{\mathcal{Q}_{\mathcal{L},\mathcal{L}^*}}}{2}\,{}^tXS_{\mathcal{Q}_{\mathcal{L},\mathcal{L}^*}}X.
\end{aligned}
\tag{9.1d}
$$

We deduce the different value of $\mu_{\cdot}$ by taking $X = \vec{\boldsymbol{\tau}}_{\boldsymbol{\sigma_{\mathcal{K}\mathcal{K}^*}}}$ in (9.1a)-(9.1b) and by taking $X = \vec{\boldsymbol{\tau}}_{\boldsymbol{\mathcal{K}^*,\mathcal{L}^*}}$ in (9.1c)-(9.1d). Thanks to the relation (2.1), we have the following estimate

$$\mu_{\cdot}^2 \leq C(\mathrm{reg}(\mathcal{T})) \sum_{\mathcal{Q}\in\mathfrak{Q}_{\mathcal{D}}} m_{\mathcal{Q}} \|S_{\mathcal{Q}}\|_{\mathcal{F}}^2. \tag{9.2}$$

•*Case* $\alpha_{\mathcal{K}} \neq \alpha_{\mathcal{L}}$. We have $(\vec{\mathbf{n}}_{\boldsymbol{\sigma_{\mathcal{L}\mathcal{K}^*}}}, \vec{\boldsymbol{\tau}}_{\boldsymbol{\sigma_{\mathcal{K}\mathcal{K}^*}}}) \neq 0$. We deduce the different value of $\lambda_{\cdot}$ by taking in (9.1c) respectively $X = \vec{\boldsymbol{\tau}}_{\boldsymbol{\sigma_{\mathcal{K}\mathcal{K}^*}}}$ and $X = \vec{\boldsymbol{\tau}}_{\boldsymbol{\sigma_{\mathcal{L}\mathcal{K}^*}}}$, and in (9.1d) $X = \vec{\boldsymbol{\tau}}_{\boldsymbol{\sigma_{\mathcal{L}\mathcal{K}^*}}}$. The value $\lambda_{\mathcal{K}}$ is deduced from (9.1a). Using the criterion ϵ_0 and the estimate (9.2), we obtain

$$\lambda_{\cdot}^2 \leq C(\mathrm{reg}(\mathcal{T}), \sin(\epsilon_0)) \sum_{\mathcal{Q}\in\mathfrak{Q}_{\mathcal{D}}} m_{\mathcal{Q}} \|S_{\mathcal{Q}}\|_{\mathcal{F}}^2. \tag{9.3}$$

Finally, we deduce thanks to (9.2) and (9.3) that

$$\|\delta^{\mathcal{D}}\|_2^2 \leq C \sum_{\mathcal{Q}\in\mathfrak{Q}_{\mathcal{D}}} m_{\mathcal{Q}} \|S_{\mathcal{Q}}\|_{\mathcal{F}}^2 = C(\mathrm{reg}(\mathcal{T}), \sin(\epsilon_0)) \sum_{\mathcal{Q}\in\mathfrak{Q}_{\mathcal{D}}} m_{\mathcal{Q}} \|B_{\mathcal{Q}}\delta^{\mathcal{D}} + {}^t\delta^{\mathcal{D}\,t}B_{\mathcal{Q}}\|_{\mathcal{F}}^2.$$

•*Case* $\alpha_{\mathcal{K}} = \alpha_{\mathcal{L}}$. We have chosen $\delta^{\mathcal{D}}$ such that $(\delta^{\mathcal{D}}, \delta_0) = 0$. We write the system on $\lambda_{\cdot}$ as follows $B\lambda = F$, where B is a following matrix in $\mathcal{M}_{5,4}(\mathbb{R})$

$$B = \begin{pmatrix} m_{\sigma_{\mathcal{K}}} & 0 & m_{\sigma_{\mathcal{K}^*}} & 0 \\ -m_{\sigma_{\mathcal{K}}} & 0 & 0 & m_{\sigma_{\mathcal{L}^*}} \\ 0 & m_{\sigma_{\mathcal{L}}} & -m_{\sigma_{\mathcal{K}^*}} & 0 \\ 0 & -m_{\sigma_{\mathcal{L}}} & 0 & -m_{\sigma_{\mathcal{L}^*}} \\ -\frac{1}{m_{\sigma_{\mathcal{K}}}} & \frac{1}{m_{\sigma_{\mathcal{L}}}} & \frac{1}{m_{\sigma_{\mathcal{K}^*}}} & -\frac{1}{m_{\sigma_{\mathcal{L}^*}}} \end{pmatrix},$$

$\lambda = {}^t(\lambda_{\mathcal{K}}, \lambda_{\mathcal{L}}, \lambda_{\mathcal{K}^*}, \lambda_{\mathcal{L}^*})$ is a vector in $\mathbb{R}^4$ and $F = {}^t(F_{\mathcal{Q}_{\mathcal{K},\mathcal{K}^*}}, F_{\mathcal{Q}_{\mathcal{K},\mathcal{L}^*}}, F_{\mathcal{Q}_{\mathcal{L},\mathcal{K}^*}}, F_{\mathcal{Q}_{\mathcal{L},\mathcal{L}^*}}, 0)$ is a vector in $\mathbb{R}^5$. We have $\sum_{\mathcal{Q}\in\mathfrak{Q}_{\mathcal{D}}} F_{\mathcal{Q}} = 0$ and using the estimate (9.2), for all $\mathcal{Q} \in \mathfrak{Q}_{\mathcal{D}}$

$$|F_{\mathcal{Q}}|^2 \leq C(\mathrm{reg}(\mathcal{T}))h_{\mathcal{D}}^2 \sum_{\mathcal{Q}\in\mathfrak{Q}_{\mathcal{D}}} m_{\mathcal{Q}} \|S_{\mathcal{Q}}\|_{\mathcal{F}}^2. \tag{9.4}$$

The solution of $B\lambda = F$ is

$$\begin{aligned} \lambda_{\mathcal{L}^*} &= \frac{b}{b^*}, \qquad \lambda_{\mathcal{K}^*} = \frac{1}{m_{\sigma_{\mathcal{K}^*}}} \left(F_{\mathcal{Q}_{\mathcal{K},\mathcal{K}^*}} + F_{\mathcal{Q}_{\mathcal{K},\mathcal{L}^*}} - m_{\sigma_{\mathcal{L}^*}} \frac{b}{b^*} \right), \\ \lambda_{\mathcal{L}} &= \frac{1}{m_{\sigma_{\mathcal{L}}}} \left(-F_{\mathcal{Q}_{\mathcal{L},\mathcal{L}^*}} - m_{\sigma_{\mathcal{L}^*}} \frac{b}{b^*} \right), \qquad \lambda_{\mathcal{K}} = \frac{1}{m_{\sigma_{\mathcal{K}}}} \left(-F_{\mathcal{Q}_{\mathcal{L},\mathcal{K}^*}} + m_{\sigma_{\mathcal{L}^*}} \frac{b}{b^*} \right), \end{aligned} \tag{9.5}$$

where

$$|b|^2 \leq C(\mathrm{reg}(\mathcal{T})) \frac{1}{h_{\mathcal{D}}^2} \sum_{\mathcal{Q}\in\mathfrak{Q}_{\mathcal{D}}} m_{\mathcal{Q}} \|S_{\mathcal{Q}}\|_{\mathcal{F}}^2 \quad \text{and} \quad |b^*| = -b^* \geq C(\mathrm{reg}(\mathcal{T})) \frac{1}{h_{\mathcal{D}}}. \tag{9.6}$$

We deduce thanks to (9.5), (9.4) and (9.6) that

$$\lambda_{\cdot}^2 \leq C(\mathrm{reg}(\mathcal{T})) \sum_{\mathcal{Q}\in\mathfrak{Q}_{\mathcal{D}}} m_{\mathcal{Q}} \|S_{\mathcal{Q}}\|_{\mathcal{F}}^2. \tag{9.7}$$

Finally, we deduce thanks to (9.2) and (9.7) that

$$\|\delta^{\mathcal{D}}\|_2^2 \leq C(\mathrm{reg}(\mathcal{T})) \sum_{\mathcal{Q}\in\mathfrak{Q}_{\mathcal{D}}} m_{\mathcal{Q}} \|S_{\mathcal{Q}}\|_{\mathcal{F}}^2 = C(\mathrm{reg}(\mathcal{T})) \sum_{\mathcal{Q}\in\mathfrak{Q}_{\mathcal{D}}} m_{\mathcal{Q}} \|B_{\mathcal{Q}}\delta^{\mathcal{D}} + {}^t\delta^{\mathcal{D}\,t}B_{\mathcal{Q}}\|_{\mathcal{F}}^2.$$

Estimate between $B_{\mathcal{Q}}\delta^{\mathcal{D}}$ and $\delta^{\mathcal{D}}$. Thanks to $h_{\mathcal{D}} \leq C \min(m_{\sigma_{\mathcal{K}}}, m_{\sigma_{\mathcal{K}^*}})$, we deduce $\|B_{\mathcal{Q}}\delta^{\mathcal{D}}\|_{\mathcal{F}}^2 \leq C\frac{1}{h_{\mathcal{D}}^2}\|\delta^{\mathcal{D}}\|_2^2$. Thanks to $\mathrm{reg}(\mathcal{T})$, we obtain $\sum_{\mathcal{Q}\in\mathfrak{Q}_{\mathcal{D}}} m_{\mathcal{Q}} \|B_{\mathcal{Q}}\delta^{\mathcal{D}}\|_{\mathcal{F}}^2 \leq C\|\delta^{\mathcal{D}}\|_2^2$, that concludes the proof. □

9.2. Proof of Lemma 2.3. *Proof.* We improperly note $\alpha_{\mathfrak{s},\mathcal{D}}(\mathrm{D}^{\mathcal{D}}\mathbf{u}^{\boldsymbol{\mathcal{T}}}) = \alpha_{\mathcal{Q}} = p^{\mathcal{Q}} - p^{\mathcal{D}}$ when $\mathfrak{s} = \partial\mathcal{D} \cap \partial\mathcal{Q}$. We have that $(\delta^{\mathcal{D}}, \alpha_{\mathfrak{Q}_{\mathcal{D}}})$ satisfied the following system

$$\sum_{\mathcal{Q}\in\mathfrak{Q}_{\mathcal{D}}} m_{\mathcal{Q}}\varphi_{\mathcal{Q}}(\mathrm{D}^{\mathcal{D}}\mathbf{u}^{\boldsymbol{\mathcal{T}}}, \delta^{\mathcal{D}}, \alpha_{\mathcal{Q}})B_{\mathcal{Q}} = 0,\ \mathrm{Tr}({}^t\delta^{\mathcal{D}\,t}B_{\mathcal{Q}}) = 0,\ \forall \mathcal{Q} \in \mathfrak{Q}_{\mathcal{D}},\ \sum_{\mathcal{Q}\in\mathfrak{Q}_{\mathcal{D}}} m_{\mathcal{Q}}\alpha_{\mathcal{Q}} = 0. \quad (9.8)$$

Using the value of $\varphi_{\mathcal{Q}}$ in (9.8), we deduce that

$$\sum_{\mathcal{Q}\in\mathfrak{Q}_{\mathcal{D}}} m_{\mathcal{Q}}\alpha_{\mathcal{Q}}B_{\mathcal{Q}} = 2\sum_{\mathcal{Q}\in\mathfrak{Q}_{\mathcal{D}}} m_{\mathcal{Q}}\eta_{\mathcal{Q}}\mathrm{D}^{\mathcal{N}}_{\mathcal{Q}}\mathbf{u}^{\boldsymbol{\mathcal{T}}}B_{\mathcal{Q}}. \quad (9.9)$$

We have that $\|m_{\mathcal{Q}_{\mathcal{K},\mathcal{K}^*}}B_{\mathcal{Q}_{\mathcal{K},\mathcal{K}^*}}\|^2_{\mathcal{F}} = m^2_{\sigma_{\mathcal{K}}} + m^2_{\sigma_{\mathcal{K}^*}} \leq 2h^2_{\mathcal{D}}$. The same estimate holds for all $\mathcal{Q} \in \mathfrak{Q}_{\mathcal{D}}$. We estimate the right hand side of (9.9) thanks to Cauchy-Schwarz inequality

$$\|2\sum_{\mathcal{Q}\in\mathfrak{Q}_{\mathcal{D}}} m_{\mathcal{Q}}\eta_{\mathcal{Q}}\mathrm{D}^{\mathcal{N}}_{\mathcal{Q}}\mathbf{u}^{\boldsymbol{\mathcal{T}}}B_{\mathcal{Q}}\|_{\mathcal{F}} \leq 2\overline{\mathrm{C}}_{\eta}h_{\mathcal{D}}\left(\sum_{\mathcal{Q}\in\mathfrak{Q}_{\mathcal{D}}} \|\mathrm{D}^{\mathcal{N}}_{\mathcal{Q}}\mathbf{u}^{\boldsymbol{\mathcal{T}}}\|^2_{\mathcal{F}}\right)^{\frac{1}{2}}. \quad (9.10)$$

Then we have that the norm of the left-hand side of (9.9) is

$$\begin{aligned}\|\sum_{\mathcal{Q}\in\mathfrak{Q}_{\mathcal{D}}} m_{\mathcal{Q}}\alpha_{\mathcal{Q}}B_{\mathcal{Q}}\|^2_{\mathcal{F}} =& m^2_{\sigma_{\mathcal{K}}}(\alpha_{\mathcal{Q}_{\mathcal{K},\mathcal{K}^*}} - \alpha_{\mathcal{Q}_{\mathcal{K},\mathcal{L}^*}})^2 + m^2_{\sigma_{\mathcal{L}}}(\alpha_{\mathcal{Q}_{\mathcal{L},\mathcal{K}^*}} - \alpha_{\mathcal{Q}_{\mathcal{L},\mathcal{L}^*}})^2 \\ &+ m^2_{\sigma_{\mathcal{K}^*}}(\alpha_{\mathcal{Q}_{\mathcal{K},\mathcal{K}^*}} - \alpha_{\mathcal{Q}_{\mathcal{L},\mathcal{K}^*}})^2 + m^2_{\sigma_{\mathcal{L}^*}}(\alpha_{\mathcal{Q}_{\mathcal{K},\mathcal{L}^*}} - \alpha_{\mathcal{Q}_{\mathcal{L},\mathcal{L}^*}})^2\end{aligned} \quad (9.11)$$

Using (9.10)-(9.11), (2.1) and Lemma 4.1, $\forall \mathcal{Q}, \mathcal{Q}' \in \mathfrak{Q}_{\mathcal{D}}$ such that $\bar{\mathcal{Q}} \cap \bar{\mathcal{Q}}' \neq \emptyset$, we obtain

$$|\alpha_{\mathcal{Q}} - \alpha_{\mathcal{Q}'}| \leq 2\overline{\mathrm{C}}_{\eta}\mathrm{reg}(\mathcal{T})\frac{\mathrm{reg}(\mathcal{T})\overline{\mathrm{C}}_{\eta}}{\sqrt{2}\underline{\mathrm{C}}_{\eta}}\|\mathrm{D}^{\mathcal{D}}\mathbf{u}^{\boldsymbol{\mathcal{T}}}\|_{\mathcal{F}}. \quad (9.12)$$

Now we can estimate $|\alpha_{\mathcal{Q}}|$ with differences like $\alpha_{\mathcal{Q}'} - \alpha_{\mathcal{Q}''}$, using (9.8). Thanks to (9.12), we obtain the result with $C_2 = 6\frac{\mathrm{reg}(\mathcal{T})^2\overline{\mathrm{C}}^2_{\eta}}{\sqrt{2}\underline{\mathrm{C}}_{\eta}}$.

□

9.3. Proof of Proposition 6.4. *Proof.* We define $F_{\mathcal{Q}}$ as follows

$$F_{\mathcal{Q}_{\mathcal{K},\mathcal{K}^*}} = \frac{\mathrm{Id}}{m_{\mathcal{Q}_{\mathcal{K},\mathcal{K}^*}}},\quad F_{\mathcal{Q}_{\mathcal{K},\mathcal{L}^*}} = -\frac{\mathrm{Id}}{m_{\mathcal{Q}_{\mathcal{K},\mathcal{L}^*}}},\quad F_{\mathcal{Q}_{\mathcal{L},\mathcal{K}^*}} = -\frac{\mathrm{Id}}{m_{\mathcal{Q}_{\mathcal{L},\mathcal{K}^*}}},\quad F_{\mathcal{Q}_{\mathcal{K},\mathcal{L}^*}} = \frac{\mathrm{Id}}{m_{\mathcal{Q}_{\mathcal{L},\mathcal{L}^*}}}.$$

We verified that $F_{\mathcal{Q}}$ satisfied the conditions (2.9) if $\alpha_{\mathcal{K}} \neq \alpha_{\mathcal{L}}$ and (2.10) if $\alpha_{\mathcal{K}} = \alpha_{\mathcal{L}}$. Thus, there exists $\bar{\delta}^{\mathcal{D}} \in \mathcal{M}_{n_{\mathcal{D}},2}(\mathbb{R})$ such that $(B_{\mathcal{Q}}\bar{\delta}^{\mathcal{D}} + {}^t\bar{\delta}^{\mathcal{D}\,t}B_{\mathcal{Q}}) = F_{\mathcal{Q}}, \forall\, \mathcal{Q} \in \mathfrak{Q}_{\mathcal{D}}$. Taken $\delta^{\mathcal{D}}$ equal to $\bar{\delta}^{\mathcal{D}}$ in (6.9), we deduce that

$$\sum_{\mathcal{Q}\in\mathfrak{Q}_{\mathcal{D}}}\int_{\mathcal{Q}}\eta(z)\,(F_{\mathcal{Q}} : \mathrm{D}\mathbf{u}(z) - \mathrm{D}^{\mathcal{N}}_{\mathcal{Q}}\mathbb{P}^{\boldsymbol{\mathcal{T}}}_{\boldsymbol{c}}\mathbf{u})\,\mathrm{d}z - \sum_{\mathcal{Q}\in\mathfrak{Q}_{\mathcal{D}}} m_{\mathcal{Q}}\mathrm{Tr}\,(F_{\mathcal{Q}})\,\widetilde{p}^{\mathcal{Q}}_{\mathrm{ex}} = \mathrm{Tr}\left({}^t\bar{\delta}^{\mathcal{D}}R\right). \quad (9.13)$$

We construct a function $\widetilde{\mathbf{v}}$ such that $\widetilde{\mathbf{v}} \in (H^1(\mathcal{D}))^2 \cap (H^2(\mathcal{Q}))^2$, for all $\mathcal{Q} \in \mathfrak{Q}_{\mathcal{D}}$, such that

$$\mathrm{div}(\widetilde{\mathbf{v}})(x) = \begin{cases} -\widetilde{p}^{\mathcal{Q}}_{\mathrm{ex}} + \dfrac{\mathrm{div}^{\mathcal{D}}(\mathbb{P}^{\boldsymbol{\mathcal{T}}}_{\boldsymbol{c}}\mathbf{u})}{2m_{\mathcal{Q}}}\displaystyle\int_{\mathcal{Q}}\eta(z)\mathrm{d}z - \frac{1}{2}\mathrm{Tr}({}^t\bar{\delta}^{\mathcal{D}}R),\ \text{ if } x \in \mathcal{Q}_{\mathcal{K},\mathcal{K}^*}, \\ -\widetilde{p}^{\mathcal{Q}}_{\mathrm{ex}} + \dfrac{\mathrm{div}^{\mathcal{D}}(\mathbb{P}^{\boldsymbol{\mathcal{T}}}_{\boldsymbol{c}}\mathbf{u})}{2m_{\mathcal{Q}}}\displaystyle\int_{\mathcal{Q}}\eta(z)\mathrm{d}z,\ \text{ if } x \in \mathcal{Q},\ \mathcal{Q} \cap \mathcal{Q}_{\mathcal{K},\mathcal{K}^*} = \emptyset, \end{cases} \quad (9.14)$$

with R defined by Definition 6.5. We choose $\widetilde{\mathbf{v}}$ linear per quarter diamond cells of this form

$$\widetilde{\mathbf{v}}(x) = \begin{pmatrix} \alpha_{\mathcal{Q}} & 0 \\ 0 & \beta_{\mathcal{Q}} \end{pmatrix} x, \qquad \text{if } x \in \mathcal{Q},$$

it implies that $\nabla^2\widetilde{\mathbf{v}}=0$ on $\mathcal{Q}$. The continuity over the edge of $\mathcal{Q}$ imposes that

$$\beta_{\mathcal{Q}_{\mathcal{K},\mathcal{K}^*}}=\beta_{\mathcal{Q}_{\mathcal{L},\mathcal{K}^*}}:=\beta_1,\ \alpha_{\mathcal{Q}_{\mathcal{L},\mathcal{K}^*}}=\alpha_{\mathcal{Q}_{\mathcal{L},\mathcal{L}^*}}:=\alpha_2,\ \beta_{\mathcal{Q}_{\mathcal{K},\mathcal{L}^*}}=\beta_{\mathcal{Q}_{\mathcal{L},\mathcal{L}^*}}:=\beta_2,\ \alpha_{\mathcal{Q}_{\mathcal{K},\mathcal{K}^*}}=\alpha_{\mathcal{Q}_{\mathcal{K},\mathcal{L}^*}}:=\alpha_1.$$

We obtain four unknowns $\alpha_i,\ \beta_i$ which are solutions of

$$\begin{aligned}
\alpha_1+\beta_1 &= -\widetilde{p}^{\rm ex}_{\mathcal{Q}_{\mathcal{K},\mathcal{K}^*}}+\frac{{\rm div}^{\mathcal{D}}(\mathbb{P}^{\boldsymbol{\mathcal{T}}}_{\boldsymbol{c}}\mathbf{u})}{2m_{\mathcal{Q}_{\mathcal{K},\mathcal{K}^*}}}\int_{\mathcal{Q}_{\mathcal{K},\mathcal{K}^*}}\eta(z){\rm d}z-\frac{1}{2}{\rm Tr}({}^t\bar{\delta}^{\mathcal{D}}R):=b_1\\
\alpha_2+\beta_1 &= -\widetilde{p}^{\rm ex}_{\mathcal{Q}_{\mathcal{L},\mathcal{K}^*}}+\frac{{\rm div}^{\mathcal{D}}(\mathbb{P}^{\boldsymbol{\mathcal{T}}}_{\boldsymbol{c}}\mathbf{u})}{2m_{\mathcal{Q}_{\mathcal{L},\mathcal{K}^*}}}\int_{\mathcal{Q}_{\mathcal{L},\mathcal{K}^*}}\eta(z){\rm d}z:=b_2\\
\alpha_1+\beta_2 &= -\widetilde{p}^{\rm ex}_{\mathcal{Q}_{\mathcal{K},\mathcal{L}^*}}+\frac{{\rm div}^{\mathcal{D}}(\mathbb{P}^{\boldsymbol{\mathcal{T}}}_{\boldsymbol{c}}\mathbf{u})}{2m_{\mathcal{Q}_{\mathcal{K},\mathcal{L}^*}}}\int_{\mathcal{Q}_{\mathcal{K},\mathcal{L}^*}}\eta(z){\rm d}z:=b_3\\
\alpha_2+\beta_2 &= -\widetilde{p}^{\rm ex}_{\mathcal{Q}_{\mathcal{L},\mathcal{L}^*}}+\frac{{\rm div}^{\mathcal{D}}(\mathbb{P}^{\boldsymbol{\mathcal{T}}}_{\boldsymbol{c}}\mathbf{u})}{2m_{\mathcal{Q}_{\mathcal{L},\mathcal{L}^*}}}\int_{\mathcal{Q}_{\mathcal{L},\mathcal{L}^*}}\eta(z){\rm d}z:=b_4
\end{aligned}$$

Thanks to (9.13) ($b_1-b_2-b_3+b_4=0$), we get

$$4\alpha_1=b_3-b_1-b_2,\ 4\alpha_2=b_3-5b_1+3b_2,\ 4\beta_1=5b_1+b_2-b_3,\ 4\beta_2=b_1+b_2+3b_3.$$

We get using (1.2) and Lemma 6.5

$$\|\widetilde{\mathbf{v}}\|_{H^1(\mathcal{D})}\le C(\|\widetilde{p}^{\Omega_{\mathcal{D}}}_{\rm ex}\|_2+\|\!|{\rm D}\mathbf{u}-{\rm D}^{\mathcal{N}}_{\mathcal{Q}}\mathbb{P}^{\boldsymbol{\mathcal{T}}}_{\boldsymbol{c}}\mathbf{u}|\!\|_{\Omega_{\mathcal{D}}}+\sqrt{m_{\mathcal{D}}}|{\rm Tr}({}^t\bar{\delta}^{\mathcal{D}}R)|). \tag{9.15}$$

We have $m_{\mathcal{D}}\|\!|B_{\mathcal{Q}}\bar{\delta}^{\mathcal{D}}|\!\|^2_{\Omega_{\mathcal{D}}}\le C({\rm reg}(\mathcal{T}))$. Thanks to Proposition 6.3, we deduce

$$m_{\mathcal{D}}|{\rm Tr}({}^t\bar{\delta}^{\mathcal{D}}R)|^2\le Ch^2_{\mathcal{D}}\sum_{\mathcal{Q}\in\mathfrak{Q}_{\mathcal{D}}}\int_{\mathcal{Q}}(\|\!|\nabla\mathbf{u}(z)|\!\|^2_{\mathcal{F}}+|\nabla^2\mathbf{u}(z)|^2+|\nabla p(z)|^2){\rm d}z. \tag{9.16}$$

Proposition 4.1 gives that $\|\!|{\rm D}\widetilde{\mathbf{v}}-{\rm D}^{\mathcal{N}}_{\mathfrak{Q}}\mathbb{P}^{\boldsymbol{\mathcal{T}}}_{\boldsymbol{c}}\widetilde{\mathbf{v}}|\!\|_{\Omega_{\mathcal{D}}}\le\|\widetilde{\mathbf{v}}\|_{H^1(\mathcal{D})}+\|\!|\nabla^{\mathcal{N}}_{\mathfrak{Q}}\mathbb{P}^{\boldsymbol{\mathcal{T}}}_{\boldsymbol{c}}\widetilde{\mathbf{v}}|\!\|_{\Omega_{\mathcal{D}}}$. Lemma 4.2, (4.1), (9.15) and (9.16) imply that

$$\|\!|{\rm D}\widetilde{\mathbf{v}}-{\rm D}^{\mathcal{N}}_{\mathfrak{Q}}\mathbb{P}^{\boldsymbol{\mathcal{T}}}_{\boldsymbol{c}}\widetilde{\mathbf{v}}|\!\|_{\Omega_{\mathcal{D}}}\le C(\|\widetilde{p}^{\Omega_{\mathcal{D}}}_{\rm ex}\|_2+\|\!|{\rm D}\mathbf{u}-{\rm D}^{\mathcal{N}}_{\mathcal{Q}}\mathbb{P}^{\boldsymbol{\mathcal{T}}}_{\boldsymbol{c}}\mathbf{u}|\!\|_{\Omega_{\mathcal{D}}}+Ch_{\mathcal{D}}).$$

Furthermore, we have

$$-\sum_{\mathcal{Q}\in\mathfrak{Q}_{\mathcal{D}}}\int_{\mathcal{Q}}{\rm div}\widetilde{\mathbf{v}}\widetilde{p}^{\mathcal{Q}}_{\rm ex}{\rm d}z\ge\|\widetilde{p}^{\Omega_{\mathcal{D}}}_{\rm ex}\|^2_2-{\rm div}^{\mathcal{D}}\mathbb{P}^{\boldsymbol{\mathcal{T}}}_{\boldsymbol{c}}\mathbf{u}\sum_{\mathcal{Q}\in\mathfrak{Q}_{\mathcal{D}}}\int_{\mathcal{Q}}\eta(z){\rm d}z\widetilde{p}^{\mathcal{Q}}_{\rm ex}-m_{\mathcal{Q}_{\mathcal{K},\mathcal{K}^*}}{\rm Tr}({}^t\bar{\delta}^{\mathcal{D}}R)\widetilde{p}^{\rm ex}_{\mathcal{Q}_{\mathcal{K},\mathcal{K}^*}}.$$

Furthermore, thanks to ${\rm div}\mathbf{u}=0$, Proposition 6.3 gives

$$-\sum_{\mathcal{Q}\in\mathfrak{Q}_{\mathcal{D}}}\int_{\mathcal{Q}}{\rm div}\widetilde{\mathbf{v}}\widetilde{p}^{\mathcal{Q}}_{\rm ex}{\rm d}z\ge\|\widetilde{p}^{\Omega_{\mathcal{D}}}_{\rm ex}\|^2_2-C\|\!|{\rm D}\mathbf{u}-{\rm D}^{\mathcal{N}}_{\mathcal{Q}}\mathbb{P}^{\boldsymbol{\mathcal{T}}}_{\boldsymbol{c}}\mathbf{u}|\!\|_{\Omega_{\mathcal{D}}}\|\widetilde{p}^{\Omega_{\mathcal{D}}}_{\rm ex}\|_2-C{\rm size}(\mathcal{T})|\|\widetilde{p}^{\Omega_{\mathcal{D}}}_{\rm ex}\|_2.$$

Young inequality concludes the proof. □

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