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# Full and convex linear subcategories are incompressible

Claude Cibils, Maria Julia Redondo and Andrea Solotar \*

## Abstract

Consider the intrinsic fundamental group *à la* Grothendieck of a linear category, introduced in [5] and [6] using connected gradings. In this article we prove that a full convex subcategory is incompressible, in the sense that the group map between the corresponding fundamental groups is injective. The proof makes essential use of the functoriality of the intrinsic fundamental group, and it is based on the study of the restriction of connected gradings to full subcategories. Moreover, we study in detail the fibre product of coverings and of Galois coverings.

2010 MSC: 16W50, 18G55, 55Q05, 16B50

## 1 Introduction

In two recent papers [5, 6] we have considered a new intrinsic fundamental group attached to a linear category. We have obtained this group using methods inspired by the definition of the fundamental group in other mathematical contexts. Note that in [6] we made explicit computations of this new intrinsic fundamental group for several families of algebras.

In this paper we first recall the main tools in order to provide the definition *à la* Grothendieck, in the sense that the intrinsic fundamental group is the automorphism group of a fibre functor.

Previously, a non canonical fundamental group has been introduced by R. Martínez-Villa and J.A. de la Peña in [11] and K. Bongartz and P. Gabriel in [2] and [8]. This group is not canonical since it varies considerably according to the presentation of the linear category, see for instance [1, 3], as well as a first approach to solve this problem [9, 10].

The main tool we use are connected gradings of linear categories. The fundamental group that we consider is a group which is derived from all the groups grading the linear category in a connected way. More precisely each connected

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grading provides a Galois covering through the smash product, see [4]. Considering the category of Galois coverings of this type we define the intrinsic fundamental group as the automorphism group of the fibre functor over a chosen object.

In this paper we face a difficulty for this theory which makes it non standard with respect to other similar theories, hence, some usual tools are not available here. In fact the fibre product of two Galois coverings is not even a covering in general. Nevertheless other properties hold, for instance the fibre product of a Galois covering with a fully faithful functor is a Galois covering. Also, the square fibre product of a Galois covering is a trivial covering and this characterizes Galois coverings.

The main purpose of this paper is to prove that full and convex subcategories are incompressible, in the sense used in algebraic topology where a subvariety is called incompressible if the group map between the corresponding fundamental groups is injective.

In order to do so, we first need to prove that the intrinsic fundamental group is functorial, answering in this way a question by Alain Bruguières. Note that this is not automatic due to the already quoted difficulty on fibre products of Galois coverings.

We provide a description of elements of the intrinsic fundamental group as *coherent* families of elements lying in each group which grades the linear category in a connected way. The other main ingredient for proving the functoriality is a procedure for constructing a connected grading out of a non-connected one. This depends on some choices which appear to be irrelevant at the intrinsic fundamental group level.

Finally we consider convex subcategories. Recall that a linear subcategory of a linear category is convex if morphisms of the subcategory can only be factorized through morphisms in the subcategory. We prove that given a full and convex subcategory, the group map obtained by functoriality between the intrinsic fundamental groups is injective.

## 2 Coverings

In this section we recall some definitions and results from [5] that we will use throughout this paper.

Let  $k$  be a commutative ring. A  $k$ -category is a small category  $\mathcal{B}$  such that each morphism set  ${}_y\mathcal{B}_x$  from an object  $x \in \mathcal{B}_0$  to an object  $y \in \mathcal{B}_0$  is a  $k$ -module, the composition of morphisms is  $k$ -bilinear and the identity at each object is central in its endomorphism ring. In particular each endomorphism set is a  $k$ -algebra, and  ${}_y\mathcal{B}_x$  is a  ${}_y\mathcal{B}_y - {}_x\mathcal{B}_x$ -bimodule.

Note that each  $k$ -algebra  $A$  provides a single object  $k$ -category  $\mathcal{B}_A$  with endomorphism ring  $A$ . The structure of  $A$  can be described more precisely using  $k$ -categories as follows: for each choice of a finite set  $E$  of orthogonal idempotents of  $A$  such that  $\sum_{e \in A} e = 1$ , consider the  $k$ -category  $\mathcal{B}_{A,E}$  whose set of objects is  $E$ , and the set of morphisms from  $e$  to  $f$  is the  $k$ -module  $fAe$ . Composition of morphisms is given by the product in  $A$ . Note that in this way  $\mathcal{B}_A = \mathcal{B}_{A,\{1\}}$ . The direct

sum the  $k$ -modules of morphisms of  $\mathcal{B}_{A,E}$  can be equipped with a matrix product combined with composition of morphisms, the resulting algebra is isomorphic to  $A$ .

**Definition 2.1** The star  $\text{St}_b\mathcal{B}$  of a  $k$ -category  $\mathcal{B}$  at an object  $b$  is the direct sum of all  $k$ -modules of morphisms with source or target  $b$ :

$$\text{St}_b\mathcal{B} = \left( \bigoplus_{y \in \mathcal{B}_0} {}_y\mathcal{B}_b \right) \oplus \left( \bigoplus_{y \in \mathcal{B}_0} {}_b\mathcal{B}_y \right)$$

Note that this  $k$ -module counts twice the endomorphism algebra at  $b$ .

**Definition 2.2** Let  $\mathcal{C}$  and  $\mathcal{B}$  be  $k$ -categories. A  $k$ -functor  $F : \mathcal{C} \rightarrow \mathcal{B}$  is a covering of  $\mathcal{B}$  if it is surjective on objects and if  $F$  induces  $k$ -isomorphisms between all corresponding stars. More precisely, for each  $b \in \mathcal{B}_0$  and each  $x$  in the non-empty fibre  $F^{-1}(b)$ , the map

$$F_b^x : \text{St}_x\mathcal{C} \longrightarrow \text{St}_b\mathcal{B}.$$

induced by  $F$  is a  $k$ -isomorphism.

**Remark 2.3** Each star is the direct sum of the source star  $\text{St}_b^-\mathcal{B} = \bigoplus_{y \in \mathcal{B}_0} {}_y\mathcal{B}_b$  and the target star  $\text{St}_b^+\mathcal{B} = \bigoplus_{y \in \mathcal{B}_0} {}_b\mathcal{B}_y$ . Since  $\text{St}^-$  and  $\text{St}^+$  are preserved by any  $k$ -functor, the condition of the definition is equivalent to the requirement that the corresponding target and source stars are isomorphic through  $F$ . Moreover this splitting goes further: the restriction of  $F$  to  $\bigoplus_{y \in F^{-1}(c)} {}_y\mathcal{C}_x$  is  $k$ -isomorphic to the corresponding  $k$ -module  ${}_c\mathcal{B}_b$ . The same holds with respect to the target star and morphisms starting at all objects in a single fibre.

**Remark 2.4** The previous facts show that Definition 2.2 coincides with the one given by K. Bongartz and P. Gabriel in [2]. In particular a covering is a faithful functor.

**Definition 2.5** Given  $k$ -categories  $\mathcal{B}, \mathcal{C}, \mathcal{D}$ , the set of morphisms  $\text{Mor}(F, G)$  from a covering  $F : \mathcal{C} \rightarrow \mathcal{B}$  to a covering  $G : \mathcal{D} \rightarrow \mathcal{B}$  is the set of pairs of  $k$ -linear functors  $(H, J)$  where  $H : \mathcal{C} \rightarrow \mathcal{D}$ ,  $J : \mathcal{B} \rightarrow \mathcal{B}$  are such that  $J$  is an isomorphism,  $J$  is the identity on objects and  $GH = JF$ .

We will consider within the group of automorphisms of a covering  $F : \mathcal{C} \rightarrow \mathcal{B}$ , the subgroup  $\text{Aut}_1 F$  of invertible endofunctors  $G$  of  $\mathcal{C}$  such that  $FG = F$ .

Next we recall the definition of a connected  $k$ -category. We use the following notation: given a morphism  $f$ , its source and target object are denoted  $s(f)$  and  $t(f)$  respectively. We will also make use of walks. For this purpose we consider the set of formal pairs  $(f, \epsilon)$  as **virtual morphisms**, where  $f$  is a morphism in  $\mathcal{B}$  and  $\epsilon \in \{-1, 1\}$ . We extend source and target maps to this set as follows:

$$s(f, 1) = s(f), \quad s(f, -1) = t(f), \quad t(f, 1) = t(f), \quad t(f, -1) = s(f).$$

**Definition 2.6** Let  $\mathcal{B}$  be a  $k$ -category. A non-zero **walk** in  $\mathcal{B}$  is a sequence of non-zero virtual morphisms  $(f_n, \epsilon_n) \dots (f_1, \epsilon_1)$  such that  $s(f_{i+1}, \epsilon_{i+1}) = t(f_i, \epsilon_i)$ . We say that this walk goes from  $s(f_1, \epsilon_1)$  to  $t(f_n, \epsilon_n)$ . A  $k$ -category  $\mathcal{B}$  is **connected** if any two objects  $b$  and  $c$  of  $\mathcal{B}$  can be joined by a non-zero walk.

**Remark 2.7** Let  $F : \mathcal{C} \longrightarrow \mathcal{B}$  be a covering of  $k$ -categories, and let  $\mathcal{C}'$  be a full connected component of  $\mathcal{C}$  such that  $F$  restricted to  $\mathcal{C}'$  is still surjective on objects. Then this restriction functor is a covering of  $\mathcal{B}$ .

We recall the following known results.

**Proposition 2.8** [9, 5] Let  $F : \mathcal{C} \longrightarrow \mathcal{B}$  be a covering of  $k$ -categories. If  $\mathcal{C}$  is connected, then  $\mathcal{B}$  is connected.

**Proposition 2.9** [9, 5] Let  $F : \mathcal{C} \longrightarrow \mathcal{B}$  and  $G : \mathcal{D} \longrightarrow \mathcal{B}$  be coverings of  $k$ -linear categories. Assume  $\mathcal{C}$  is connected. Two morphisms  $(H_1, J), (H_2, J)$  from  $F$  to  $G$  such that  $H_1$  and  $H_2$  coincide on some object are equal.

**Corollary 2.10** Let  $F : \mathcal{C} \rightarrow \mathcal{B}$  be a connected covering of a  $k$ -linear category  $\mathcal{B}$ . The group  $\text{Aut}_1(F) = \{s : F \rightarrow F \mid s \text{ is an isomorphism of } \mathcal{C} \text{ verifying } Fs = F\}$  acts freely on each fibre.

Next we recall the definition of a Galois covering:

**Definition 2.11** A covering  $F : \mathcal{C} \longrightarrow \mathcal{B}$  of  $k$ -categories is a Galois covering if  $\mathcal{C}$  is connected and  $\text{Aut}_1 F$  acts transitively on some fibre.

As expected the automorphism group acts transitively at every fibre as soon as it acts transitively on a particular fibre, see [9, 10, 5]. A quotient category by the action of a group exists in case the action is free on objects. The following is a structure theorem providing an explicit description of Galois coverings.

**Theorem 2.12** [9, 5] Let  $F : \mathcal{C} \longrightarrow \mathcal{B}$  be a Galois covering. Then there exists a unique isomorphism of categories  $F' : \mathcal{C}/\text{Aut}_1 F \longrightarrow \mathcal{B}$  such that  $F'P = F$ , where  $P : \mathcal{C} \longrightarrow \mathcal{C}/\text{Aut}_1 F$  is the Galois covering given by the categorical quotient.

### 3 Fibre products

**Definition 3.1** Let  $F : \mathcal{C} \longrightarrow \mathcal{B}$  and  $G : \mathcal{D} \longrightarrow \mathcal{B}$  be  $k$ -functors of  $k$ -categories. The fibre product  $\mathcal{C} \times_{\mathcal{B}} \mathcal{D}$  is the category defined as follows: objects are pairs  $(c, d) \in \mathcal{C}_0 \times \mathcal{D}_0$  such that  $F(c) = G(d)$ ; the set of morphisms from  $(c, d)$  to  $(c', d')$  is the  $k$ -submodule of  ${}_c \mathcal{C}_c \oplus {}_d \mathcal{D}_d$  given by pairs of morphisms  $(f, g)$  verifying  $Ff = Gg$ . Composition of morphisms is defined componentwise.

**Remark 3.2** The fibre product defined above is the categorical fibre product defined in the category of  $k$ -linear categories, satisfying the usual universal property that makes commutative the following diagram

$$\begin{array}{ccc}
\mathcal{C} \times_{\mathcal{B}} \mathcal{D} & \xrightarrow{\text{pr}_G} & \mathcal{D} \\
\text{pr}_F \downarrow & & \downarrow G \\
\mathcal{C} & \xrightarrow{F} & \mathcal{B}.
\end{array}$$

As already quoted in the introduction, the next example shows that the behaviour of the fibre product is unusual, since the functors  $\text{pr}_G$  nor  $\text{pr}_F$  are not even coverings in general.

**Example 3.3** Let  $\mathcal{B}$  be the  $k$ -category associated to the Kronecker quiver given by two parallel arrows  $a$  and  $b$  with common source object  $s$  and common target object  $t$ . More precisely,  ${}_s\mathcal{B}_s = {}_t\mathcal{B}_t = k$ , the  $k$ -module  ${}_t\mathcal{B}_s$  is free with basis  $\{a, b\}$  and  ${}_s\mathcal{B}_t = 0$ . Let  $\mathcal{C}$  be the  $k$ -category associated to the quiver which has four vertices  $\{s_0, s_1, t_0, t_1\}$  and four arrows:  $a_i : s_i \rightarrow t_i$  and  $b_i : s_i \rightarrow t_{i+1}$  for  $i \in \mathbb{Z}/2\mathbb{Z}$ .

$$\begin{array}{ccc}
\mathcal{C} : s_1 & \xrightarrow{a_1} & t_1, \\
& \searrow^{b_1} & \nearrow \\
& & s_0 \\
& \nearrow_{b_0} & \searrow \\
s_0 & \xrightarrow{a_0} & t_0
\end{array}
\quad , \quad
\mathcal{B} : s \begin{array}{c} \xrightarrow{a} \\ \xrightarrow{b} \end{array} t$$

We consider two Galois coverings  $F, G : \mathcal{C} \rightarrow \mathcal{B}$ , both functors sending  $s_i$  to  $s$  and  $t_i$  to  $t$ . The functor  $F$  respects the given basis, namely  $F(a_i) = a$  and  $F(b_i) = b$ . The functor  $G$  do the same for the arrows  $a_0$  and  $a_1$ , while  $G(b_i) = a + b$ . One can check that both  $F$  and  $G$  are Galois coverings of  $\mathcal{B}$ .

We assert that the fibre product  $\mathcal{C} \times_{\mathcal{B}} \mathcal{C}$  is not a covering of  $\mathcal{C}$  through the second component projection corresponding to  $G$ . Indeed, consider the objects  $s_0$  and  $t_1$ , and choose the object  $(s_0, s_0)$  in the fibre of  $s_0$ . Morphisms in  $\mathcal{C}$  from  $s_0$  to  $t_1$  are scalar multiples of  $b_0$ . If  $\mathcal{C} \times_{\mathcal{B}} \mathcal{C}$  was a covering,  ${}_{t_1}\mathcal{C}_{s_0}$  should be isomorphic to the direct sum

$${}_{(t_1, t_0)}(\mathcal{C} \times_{\mathcal{B}} \mathcal{C})_{(s_0, s_0)} \oplus {}_{(t_1, t_1)}(\mathcal{C} \times_{\mathcal{B}} \mathcal{C})_{(s_0, s_0)}.$$

Now  ${}_{(t_1, t_0)}(\mathcal{C} \times_{\mathcal{B}} \mathcal{C})_{(s_0, s_0)} = 0$  since the only candidate is  $(b_0, a_0)$  (and its scalar multiples) but this element is not in the fibre product since  $F(a_0) \neq G(b_0)$ . On the other hand,  ${}_{(t_1, t_1)}(\mathcal{C} \times_{\mathcal{B}} \mathcal{C})_{(s_0, s_0)} = 0$  also since the only candidate is  $(b_0, b_0)$ , but  $F(b_0) \neq G(b_0)$ . Hence the direct sum to be considered in the fibre product is zero, which is not isomorphic to  $k b_0$ .

**Remark 3.4** An easy example of a covering which is not Galois can be given using the previous one. Consider  $F'$ , which coincides with  $F$  above, except that  $F'(b_1) = a + b$ . Clearly  $F'$  is a covering but  $\text{Aut}_1 F'$  is trivial, its action on the fibre of an object cannot be transitive since each fibre has two objects.

**Lemma 3.5** Let  $F : \mathcal{C} \rightarrow \mathcal{B}$  be a covering. Let  $\mathcal{D}$  be a  $k$ -subcategory of  $\mathcal{B}$  and let  $G : \mathcal{D} \rightarrow \mathcal{B}$  be the inclusion functor. Then  $\mathcal{C} \times_{\mathcal{B}} \mathcal{D}$  is canonically identified with  $F^{-1}\mathcal{D}$ , where  $(F^{-1}\mathcal{D})_0 = \{c \in \mathcal{C}_0 \mid Fc \in \mathcal{D}_0\}$  and

$${}_{c'}(F^{-1}\mathcal{D})_c = \{f \in {}_{c'}\mathcal{C}_c \mid Ff \in {}_{Fc'}\mathcal{D}_{Fc}\}.$$

**Proposition 3.6** Let  $F : \mathcal{C} \rightarrow \mathcal{B}$  be a covering and let  $\mathcal{D}$  be a full subcategory of  $\mathcal{B}$ ,  $G : \mathcal{D} \rightarrow \mathcal{B}$  the inclusion functor. The functor  $\text{pr}_{\mathcal{C}} : \mathcal{C} \times_{\mathcal{B}} \mathcal{D} \rightarrow \mathcal{C}$  is a covering.

**Proof.** Since  $\mathcal{D}$  is a full subcategory of  $\mathcal{B}$ , then  $F^{-1}\mathcal{D}$  is also full in  $\mathcal{C}$ . As a consequence the defining property of coverings concerning isomorphisms between stars is preserved.  $\diamond$

Note however that in general  $F^{-1}\mathcal{D}$  is not connected even if  $\mathcal{D}$  is connected.

Next we want to prove an analogous result for a fully faithful functor  $G : \mathcal{D} \rightarrow \mathcal{B}$ . We will first provide a description of such functors (see for instance [7]). We recall the definition of an *inflated* category  $\mathcal{B}^\alpha$  where  $\mathcal{B}$  is a  $k$ -category and  $\alpha : I \rightarrow \mathcal{B}_0$  is a surjective map assigning *inflating* sets  $I_b = \alpha^{-1}b$  to each object of  $\mathcal{B}$ . The set of objects of  $\mathcal{B}^\alpha$  is  $I$ , while  ${}_j(\mathcal{B}^\alpha)_i$  is a copy of  ${}_{\alpha(j)}\mathcal{B}_{\alpha(i)}$ . Composition is the evident one.

Two objects of  $\mathcal{B}^\alpha$  which belong to the same inflating set are isomorphic. Clearly there exists a *deflating* functor which sends all the objects in  $I_b$  to the object  $b$ . Any choice of exactly one object in each inflating set provides a full subcategory of  $\mathcal{B}^\alpha$  which is isomorphic to  $\mathcal{B}$ . In this way  $\mathcal{B}^\alpha$  is equivalent to  $\mathcal{B}$ .

**Proposition 3.7** Let  $F : \mathcal{C} \rightarrow \mathcal{B}$  be a covering of  $k$ -categories, and let  $\alpha : I \rightarrow \mathcal{B}_0$  be a surjective map. Let  $G : \mathcal{B}^\alpha \rightarrow \mathcal{B}$  be the deflating functor. The fibre product  $\mathcal{C} \times_{\mathcal{B}} \mathcal{B}^\alpha$  is identified with  $\mathcal{C}^\beta$  where  $J$  is the set fibre product  $\mathcal{C}_0 \times_{\mathcal{B}_0} I$  and  $\beta : J \rightarrow \mathcal{C}_0$  is the projection map. The functor  $\mathcal{C}^\beta \rightarrow \mathcal{B}^\alpha$  is a covering.

The proof follows by a direct computation.

**Theorem 3.8** Let  $F : \mathcal{C} \rightarrow \mathcal{B}$  be a covering and let  $G$  be a fully faithful functor  $G : \mathcal{D} \rightarrow \mathcal{B}$ . Then the projection  $\mathcal{C} \times_{\mathcal{B}} \mathcal{D} \rightarrow \mathcal{C}$  is a covering.

**Proof.** The image of  $G$  is a full subcategory  $G\mathcal{D}$  of  $\mathcal{B}$ . We consider the set of objects  $\mathcal{D}_0$  as an inflating set for  $G\mathcal{D}$ , the required surjective map  $\alpha$  is given by  $G$ . A straightforward computation shows that the inflated category  $(G\mathcal{D})^\alpha$  is isomorphic to  $\mathcal{D}$ .

The original functor  $G$  decomposes as a composition of functors  $\mathcal{D} \simeq (G\mathcal{D})^\alpha \rightarrow G\mathcal{D} \subset \mathcal{B}$  where the last inclusion corresponds to the inclusion of the full subcategory  $G\mathcal{D}$  of  $\mathcal{B}$  and the middle functor is the deflating functor. Using the previous results we get that all the vertical maps in the following diagram are covering functors:

$$\begin{array}{ccccc} (\mathcal{C} \times_{\mathcal{B}} G\mathcal{B}) \times_{G\mathcal{B}} \mathcal{D} & \longrightarrow & \mathcal{C} \times_{\mathcal{B}} G\mathcal{B} & \longrightarrow & \mathcal{C} \\ \downarrow & & \downarrow & & \downarrow F \\ \mathcal{D} & \xrightarrow{G} & G\mathcal{D} & \longrightarrow & \mathcal{B} \end{array}$$

Since both squares are fibre squares, the same is true for the big one, and  $(\mathcal{C} \times_{\mathcal{B}} G\mathcal{B}) \times_{G\mathcal{B}} \mathcal{D} \cong \mathcal{C} \times_{\mathcal{B}} \mathcal{D}$ .  $\diamond$

Next we prove several results concerning fibre products of Galois coverings.

**Theorem 3.9** *If  $F : \mathcal{C} \rightarrow \mathcal{B}$  is a Galois covering then the fibre square of  $F$  is a disjoint union of copies of  $\mathcal{C}$  indexed by  $\text{Aut}_1 F$ .*

**Proof.** Let  $(c, c')$  be an object of  $\mathcal{C} \times_{\mathcal{B}} \mathcal{C}$ , which means that  $c$  and  $c'$  are in the same  $F$ -fibre. Since  $F$  is a Galois covering there exists a unique  $s \in \text{Aut}_1 F$  such that  $c' = sc$ . Consequently objects of  $\mathcal{C} \times_{\mathcal{B}} \mathcal{C}$  are precisely the pairs  $(c, sc)$  where  $c \in \mathcal{C}_0$  and  $s \in \text{Aut}_1 F$ .

We assert that if  $s$  and  $t$  are different elements of  $\text{Aut}_1 F$ , then the  $k$ -module

$${}_{(c_1, tc_1)}(\mathcal{C} \times_{\mathcal{B}} \mathcal{C})_{(c_0, sc_0)}$$

is zero while it is identified with  ${}_{c_1}\mathcal{C}_{c_0}$  if  $s = t$ . Indeed, let  $({}_{c_1}f_{c_0}, {}_{tc_1}g_{sc_0})$  be a non-zero morphism in the fibre product, hence verifying  $F(f) = F(g)$ . Since  $s^{-1} \in \text{Aut}_1 F$  we infer that  $F(g) = F(s^{-1}g)$ . The point is that  $f$  and  $s^{-1}g$  share a common source  $c_0$ . Since  $F$  is a covering, it induces an isomorphism between stars, then  $f$  and  $s^{-1}g$  have also a common target, namely  $c_1 = s^{-1}tc_1$ , and  $f = s^{-1}g$ . Since the action of  $\text{Aut}_1 F$  is free on objects,  $s^{-1}t = 1$  and  $s = t$ . Moreover the morphism is of the form  $(f, sf)$ , which provides the identification with a disjoint union of copies of  $\mathcal{C}$  indexed by  $\text{Aut}_1 F$ .  $\diamond$

**Definition 3.10** *A trivial covering of a  $k$ -category  $\mathcal{B}$  is a covering which is isomorphic to the the product  $\mathcal{B} \times E$  of  $\mathcal{B}$  by a set  $E$ , with objects  $\mathcal{B}_0 \times E$  and where the morphisms are  ${}_{(y,e)}(\mathcal{B} \times E)_{(x,e)} = {}_y\mathcal{B}_x$  while  ${}_{(y,f)}(\mathcal{B} \times E)_{(x,e)} = 0$  if  $e \neq f$ . The covering functor is the projection functor to the first factor.*

**Remark 3.11** *We have proved that if a covering is Galois, then its fibre square is trivial. In order to prove the converse, we need the following lemmas.*

**Lemma 3.12** *Let  $F : \mathcal{C} \rightarrow \mathcal{B}$  be a connected covering of  $k$ -categories. Let  $S$  be a section of  $F$ , namely a  $k$ -functor  $S : \mathcal{B} \rightarrow \mathcal{C}$  such that  $FS = 1_{\mathcal{B}}$ . Then the subcategory  $S\mathcal{B}$  is full and it is a connected component of  $\mathcal{C}$ .*

**Proof.** Let  $f$  be a non-zero morphism of  $\mathcal{C}$  having one of its extreme objects in  $S\mathcal{B}$ . Let  $x_0 \in S\mathcal{B}_0$  and let  $f \in {}_y\mathcal{C}_{x_0}$ . Let  $b_0 \in \mathcal{B}_0$  be such that  $x_0 = Sb_0$ . In order to prove that  $y$  is also in the image of  $S$ , consider  $\text{St}_{x_0}\mathcal{C}$  which is isomorphic through  $F$  to  $\text{St}_{b_0}\mathcal{B}$ . Since  $FS = 1_{\mathcal{B}}$ , we have that  $S : \text{St}_{b_0}\mathcal{B} \rightarrow \text{St}_{x_0}\mathcal{C}$  is the inverse of  $F$  at the star level. Hence  $f = SFf$  and  $y = SFy$ . We have also shown that a morphism between objects in the image of  $S$  is in the image of  $S$ , which shows that  $S\mathcal{B}$  is full in  $\mathcal{C}$ . Since  $S\mathcal{B}$  is isomorphic to  $\mathcal{B}$ , the category  $S\mathcal{B}$  is connected.  $\diamond$



**Lemma 3.13** *Let  $F : \mathcal{C} \rightarrow \mathcal{B}$  be a covering of connected  $k$ -categories. The covering is trivial if and only if for each object  $x \in \mathcal{C}_0$  there exists a section  $S : \mathcal{B} \rightarrow \mathcal{C}$  such that  $FS$  is the identity functor of  $\mathcal{B}$  and  $SFx = x$ .*

**Proof.** Assume that for each  $x \in \mathcal{C}_0$  there exists a section  $S : \mathcal{B} \rightarrow \mathcal{C}$  such that  $FS$  is the identity functor of  $\mathcal{B}$  and  $SFx = x$ . Let  $\Sigma$  be the set of all the sections of  $F$  and consider the functor  $\alpha : \mathcal{B} \times \Sigma \rightarrow \mathcal{C}$  given by  $\alpha(b, S) = Sb$  and  $\alpha_{(c, S)}(f_{(b, S)}) = Sf$ . This functor is clearly faithful and surjective on objects. It is also full by the previous lemma. The other implication is immediate.  $\diamond$

**Theorem 3.14** *Let  $F : \mathcal{C} \rightarrow \mathcal{B}$  be a covering of connected  $k$ -categories and assume that the fibre square of  $F$  is a trivial covering of  $\mathcal{C}$ . Then  $F$  is a Galois covering.*

**Proof.** Let  $c$  and  $c'$  be in the same  $F$ -fibre. In order to define an automorphism  $s$  carrying  $c$  to  $c'$ , let  $S$  be the section of the projection functor through  $(c, c')$  obtained using Lemma 3.13. Let  $s$  be defined by  $Sy = (y, sy)$  and  $Sf = (f, sf)$ . By definition of the fibre product it is clear that  $Fs = F$ . Note that  $s$  is invertible: consider the morphism  $s'$  sending  $c'$  to  $c$ , then  $s's(c) = c$ . Since there is at most one morphism of coverings sending an object to another fix one, and the identity already carries  $c$  to  $c$ , we infer  $s's = 1$ .  $\diamond$

**Definition 3.15** *A universal covering  $U : \mathcal{U} \rightarrow \mathcal{B}$  is a Galois covering such that for any Galois covering  $F : \mathcal{C} \rightarrow \mathcal{B}$ , and for any  $u \in \mathcal{U}_0$ ,  $c \in \mathcal{C}_0$  with  $U(u) = F(c)$ , there exists a unique morphism  $(H, 1)$  from  $U$  to  $F$  verifying  $H(u) = c$ .*

We already know that the fibre square of a universal covering  $U$  is trivial, since  $U$  is Galois. We shall prove that for any Galois covering  $F : \mathcal{C} \rightarrow \mathcal{B}$ , the fibre product  $\mathcal{U} \times_{\mathcal{B}} \mathcal{C}$  is trivial. This property characterizes universal coverings.

**Theorem 3.16** *A connected covering  $U : \mathcal{U} \rightarrow \mathcal{B}$  is universal if and only if the fibre product of  $U$  with any Galois covering  $F : \mathcal{C} \rightarrow \mathcal{B}$  provides a trivial covering of  $\mathcal{U}$ .*

**Proof.** In case  $U$  is universal, we assert that objects of the fibre product  $\mathcal{U} \times_{\mathcal{B}} \mathcal{C}$  can be written uniquely as pairs  $(u, Hu)$  where  $u$  is an object of  $\mathcal{U}$  and  $H$  is a morphism from  $\mathcal{U}$  to  $\mathcal{C}$  verifying  $FH = U$ . Indeed if  $(u, c)$  is an object of the fibre product then  $U(u) = F(c)$  and there exists a unique morphism  $H$  such that  $H(u) = c$ . Moreover if  $H_1$  and  $H_2$  are different morphisms we assert that

$${}_{(v, H_2v)}(\mathcal{U} \times_{\mathcal{B}} \mathcal{C})_{(u, H_1u)} = 0.$$

In order to prove this assertion, let  $(f, g)$  be a non-zero morphism in this fibre product, hence  $U(f) = F(g)$ . Since  $FH_1 = U$  we infer  $FH_1(f) = U(f) = F(g)$ . Note that  $H_1(f)$  and  $g$  share the same source object  $H_1(u)$ . Since  $F$  is a covering the star property implies  $g = H_1(f)$ , consequently they share the same target

object, namely  $H_2(v) = H_1(v)$  which implies  $H_2 = H_1$ . Note that in this case the  $k$ -module of morphisms in the fibre product is identified with  ${}_v\mathcal{U}_u$ . This shows that the category  $\mathcal{U} \times_{\mathcal{B}} \mathcal{C}$  is a product of copies of  $\mathcal{U}$ , using any fibre of  $F$  as an indexing set. The converse can be proved using the same methods.  $\diamond$

## 4 Elements of the intrinsic fundamental group

The intrinsic fundamental group of a  $k$ -category is obtained from its connected gradings, see [5, 6]. The main purpose of this section is to prove that this group is functorial with respect to full subcategories. In order to do so we will use results on fibre products as well as a concrete interpretation of elements of the intrinsic fundamental group that we provide below.

Recall that a grading  $X$  of a  $k$ -category  $\mathcal{B}$  by a group  $\Gamma_X$  is a direct sum decomposition of each  $k$ -module of morphisms

$${}_c\mathcal{B}_b = \bigoplus_{s \in \Gamma_X} X^s {}_c\mathcal{B}_b$$

such that for  $s, t \in \Gamma_X$

$$X^t {}_d\mathcal{B}_c \ X^s {}_c\mathcal{B}_b \subset X^{ts} {}_d\mathcal{B}_b.$$

A morphism is called homogeneous of degree  $s$  from  $b$  to  $b'$  if it belongs to  $X^s {}_{b'}\mathcal{B}_b$ . A walk is homogeneous if its virtual morphisms are homogeneous. The degree of a non-zero homogeneous walk is the ordered product of the degrees of the non-zero virtual morphisms, where the degree of a homogeneous virtual morphism  $(f, -1)$  is the inverse in  $\Gamma_X$  of the degree of  $f$ , see [6]. Of course the degree of  $(f, 1)$  is the degree of  $f$ . The grading is connected if given any two objects they can be joined by a non-zero homogeneous walk of arbitrary degree.

**Definition 4.1** [4] *Let  $\mathcal{B}$  be a  $k$ -category and let  $X$  be a grading of  $\mathcal{B}$ . The smash product category  $\mathcal{B} \# X$  has set of objects  $\mathcal{B}_0 \times \Gamma_X$ , the morphisms are homogeneous components as follows:*

$${}_{(c,t)}(\mathcal{B} \# X)_{(b,s)} = X^{t^{-1}s} {}_c\mathcal{B}_b.$$

Note that we have slightly changed the notation from the one used in [4] in order to emphasize that the smash product category depends on the grading  $X$  and not just on the corresponding group  $\Gamma_X$ .

Note that the evident functor  $F_X : \mathcal{B} \# X \rightarrow \mathcal{B}$  is a Galois covering.

Let  $\mathcal{B}$  be a connected  $k$ -category with a fixed object  $b_0$ . The category  $\text{Gal}(\mathcal{B}, b_0)$  has as objects the Galois coverings of  $\mathcal{B}$ . A morphism in  $\text{Gal}(\mathcal{B}, b_0)$  from  $F : \mathcal{C} \rightarrow \mathcal{B}$  to  $G : \mathcal{D} \rightarrow \mathcal{B}$  is a morphism of coverings  $(H, J)$ , see Definition 2.5. We will say that  $H$  is a  $J$ -morphism.

The group  $\text{Aut}_1(F_X)$  of a Galois smash product covering can be identified with  $\Gamma_X$ . Since any Galois covering  $F$  of  $\mathcal{B}$  is isomorphic to a smash product Galois

covering by considering the natural grading of  $\mathcal{B}$  by  $\text{Aut}_1 F$ , we consider, as in [6], the full subcategory  $\text{Gal}^\#(\mathcal{B}, b_0)$  whose objects are the smash product Galois coverings provided by connected gradings of  $\mathcal{B}$ . It can be proved that this full subcategory is equivalent to  $\text{Gal}(\mathcal{B}, b_0)$ , see [6]. The fibre functor  $\Phi^\# : \text{Gal}^\#(\mathcal{B}, b_0) \rightarrow \text{Groups}$  given by

$$\Phi^\#(F_X) = F_X^{-1}(b_0) = \Gamma_X$$

is the main ingredient for the definition of the fundamental group, namely

$$\Pi_1(\mathcal{B}, b_0) = \text{Aut}\Phi^\#.$$

Next we will consider in detail this group, and we will prove that an element of  $\Pi_1(\mathcal{B}, b_0)$  is a family of elements in the groups of connected gradings, related through canonical surjective morphisms. We recall an important result obtained by P. Le Meur [9, 10], see also [5].

**Proposition 4.2** *Let  $F$  and  $G$  be Galois coverings of a  $k$ -category  $\mathcal{B}$  and let  $(H, J)$  be a morphism from  $F$  to  $G$  in  $\text{Gal}(\mathcal{B}, b_0)$ . There is a unique surjective group morphism*

$$\lambda_H : \text{Aut}_1 F \longrightarrow \text{Aut}_1 G$$

such that  $Hf = \lambda_H(f)H$ .

**Remark 4.3** *For any smash product Galois covering  $F_X$ , we identify the isomorphic groups  $\text{Aut}_1 F_X$  and  $\Gamma_X$  through left multiplication, that is, by the correspondence  $s : F_X \rightarrow F_X$  with  $s(x) = sx$  for any  $s \in \Gamma_X$ .*

In case of Galois coverings given by smash products the preceding proposition can be reformulated and completed as follows:

**Proposition 4.4** *Let  $X$  and  $Y$  be connected gradings of a  $k$ -category  $\mathcal{B}$ , and let  $F_X$  and  $F_Y$  be the corresponding smash product Galois coverings with groups  $\Gamma_X$  and  $\Gamma_Y$ . Let  $(H, J)$  be a morphism from  $F_X$  to  $F_Y$  in  $\text{Gal}^\#(\mathcal{B}, b_0)$ , where  $H : \mathcal{B}\#X \rightarrow \mathcal{B}\#Y$  is given on objects by  $H = (J, H_\Gamma)$ . Then there exists a unique canonical surjective morphism of groups  $\lambda_H : \Gamma_X \rightarrow \Gamma_Y$  verifying*

$$H_\Gamma(sx) = \lambda_H(s)H_\Gamma(x) \text{ for all } x \in \Gamma_X.$$

Moreover the complete list of  $J$ -morphisms from  $F_X$  to  $F_Y$  is given by  $\{qH\}_{q \in \Gamma_Y}$ , and

$$\lambda_{qH} = q(\lambda_H)q^{-1}.$$

**Proof.** The first part of the proposition is a reformulation of the preceding one. Concerning the second part, note first that  $qH$  is indeed a morphism from  $F_X$  to  $F_Y$  since  $F_Y q = F_Y$ . Moreover we know (see for instance [5]) that  $J$ -morphisms are completely determined by the image of  $b_0$  in its fibre. Since  $\Gamma_Y$  acts transitively and freely on the  $F_Y$ -fibre, we infer that  $\{qH\}_{q \in \Gamma_Y}$  is indeed the complete list without repetitions of  $J$ -morphisms.

Finally we have that

$$(qH)_\Gamma(sx) = q\lambda_H(s)H_\Gamma(x) = q\lambda_H(s)q^{-1}qH_\Gamma(x) = \lambda_{qH}(s)(qH)_\Gamma(x)$$

which proves that  $\lambda_{qH} = q(\lambda_H)q^{-1}$  since  $\lambda_{qH}$  is uniquely determined by last equality.  $\diamond$

**Corollary 4.5** *Under the previous hypothesis,  $H_\Gamma$  is determined by  $\lambda_H$  as follows:*

$$H_\Gamma(s) = \lambda_H(s)H_\Gamma(1).$$

*There is a canonical surjective group morphism  $\mu : \Gamma_X \rightarrow \Gamma_Y$  attached to the fact that there exist morphisms from  $F_X$  to  $F_Y$ . This map is given by*

$$\mu(s) = H_\Gamma(1)^{-1}\lambda_H(s)H_\Gamma(1).$$

*Moreover  $H_\Gamma(s) = H_\Gamma(1)\mu(s)$ .*

**Proof.** The first assertion is clear since  $H_\Gamma(sx) = \lambda_H(s)H_\Gamma(x)$  for all  $x$ . We need to prove that  $\mu$  is independent of the considered morphism. Let  $qH_\Gamma$  be another  $J$ -morphism, then

$$\begin{aligned} (qH_\Gamma(1))^{-1}\lambda_{qH}(s)qH_\Gamma(1) &= H_\Gamma(1)^{-1}q^{-1}q\lambda_H(s)q^{-1}qH_\Gamma(1) = \\ &= H_\Gamma(1)^{-1}\lambda_H(s)H_\Gamma(1) = \mu(s). \end{aligned}$$

The last equality is clear from the previous ones.  $\diamond$

We are now able to describe any automorphism of the fibre functor, namely any element of the intrinsic fundamental group of a  $k$ -category.

Recall that  $\sigma \in \text{Aut}\Phi^\#$  is a family  $\{\sigma_X : \Gamma_X \rightarrow \Gamma_X\}$  where  $X$  is any connected grading of  $\mathcal{B}$ , making commutative the following diagram for any morphism  $H$  in  $\text{Gal}^\#(\mathcal{B}, b_0)$ :

$$\begin{array}{ccc} \Gamma_X & \xrightarrow{\sigma_X} & \Gamma_X \\ H_\Gamma \downarrow & & \downarrow H_\Gamma \\ \Gamma_Y & \xrightarrow{\sigma_Y} & \Gamma_Y \end{array}$$

**Lemma 4.6** *The map  $\sigma_X$  is the right product by an element  $g_X \in \Gamma_X$ .*

**Proof.** In case  $X = Y$ , the vertical arrows in the above diagram can be specialized by any element in  $\text{Aut}_1 F_X = \Gamma_X$ . By Remark 4.3, this vertical morphisms are left product by any  $g \in \Gamma_X$ . We infer  $\sigma_X(g) = \sigma_X(g1) = g\sigma_X(1)$  and we set  $g_X = \sigma_X(1)$ .  $\diamond$

**Proposition 4.7** *The defining elements  $\{g_X\}$  of the above lemma for  $\sigma \in \Pi_1(\mathcal{B}, b_0)$  are the families of group elements verifying  $\mu(g_X) = g_Y$  for each canonical surjective group morphism  $\mu$  obtained in case of existence of a morphism from  $\mathcal{B}\#X$  to  $\mathcal{B}\#Y$ .*

**Proof.** Let  $(H, J)$  be a morphism from  $F_X$  to  $F_Y$  where  $X$  and  $Y$  are connected gradings of  $\mathcal{B}$ , and let  $\mu : \Gamma_X \rightarrow \Gamma_Y$  be the corresponding canonical surjective group morphism. The previous diagram is then:

$$\begin{array}{ccc} \Gamma_X & \xrightarrow{\cdot g_X} & \Gamma_X \\ H_{\Gamma}(1)\mu \downarrow & & \downarrow H_{\Gamma}(1)\mu \\ \Gamma_Y & \xrightarrow{\cdot g_Y} & \Gamma_Y \end{array}$$

Hence

$$H_{\Gamma}(1)\mu(xg_X) = H_{\Gamma}(1)\mu(x)g_Y$$

for any  $x$ . Since  $\mu$  is a group homomorphism we infer  $\mu(g_X) = g_Y$ . Reciprocally a family of elements with the stated property clearly defines an automorphism of the fibre functor.  $\diamond$

## 5 Functoriality

An important tool for proving the functoriality of the intrinsic fundamental group concerns the restriction of a grading to full subcategories.

Let  $\mathcal{B}$  be a  $k$ -category and let  $X$  be a connected grading of  $\mathcal{B}$  by the group  $\Gamma_X$ . Let  $\mathcal{D}$  be a connected full subcategory of  $\mathcal{B}$ . The **restricted grading** by the same group  $\Gamma_X$  is denoted  $X \downarrow_{\mathcal{D}}$ . Note that since  $\mathcal{D}$  is full, each  $k$ -module of morphisms in  $\mathcal{D}$  preserves the same direct sum decomposition of the original grading. Clearly the grading  $X \downarrow_{\mathcal{D}}$  is not connected in general.

Let  $X$  be a non necessarily connected grading of a connected  $k$ -category  $\mathcal{B}$ . Let  ${}_{b_2}(\Gamma_X)_{b_1}$  be the set of **walk's degrees** from  $b_1$  to  $b_2$ , that is, the set of elements in  $\Gamma_X$  which are degrees of homogeneous non-zero walks from  $b_1$  to  $b_2$ . Note that if  $b_1 = b_2$  this set is a subgroup of  $\Gamma_X$  which we denote  $\Gamma_{X, b_1}$ . Let  $s$  be any walk's degree from  $b_1$  to  $b_2$ . Then

$${}_{b_2}(\Gamma_X)_{b_1} = s [\Gamma_{X, b_1}] = [\Gamma_{X, b_2}] s.$$

Recall that by definition, the grading  $X$  is connected if and only if for any objects  $b_1, b_2 \in \mathcal{B}_0$

$${}_{b_2}(\Gamma_X)_{b_1} = \Gamma_X.$$

This is equivalent to  ${}_{b_2}(\Gamma_X)_{b_1} = \Gamma_X$  for some pair of objects, see [6].

Next we will describe the construction of a connected grading out of a non necessarily connected grading of a  $k$ -category  $\mathcal{B}$ . This construction will depend on some choices of non-zero homogeneous walks and it will be used for the restriction

of a connected grading to a full connected subcategory, which is non-connected in general. In this case we will prove that the choices we made are irrelevant with respect to the fundamental group.

**Definition 5.1** Let  $X$  be a non necessarily connected grading of a connected  $k$ -category  $\mathcal{B}$ ,  $(a_b)_{b \in \mathcal{B}_0}$  a family of elements in  $\Gamma_X$ . We define a **conjugated grading**  ${}^a X$  of  $\mathcal{B}$  by the same group  $\Gamma_X$  as follows: we set the  ${}^a X$ -degree of a non-zero homogeneous morphism from  $b$  to  $c$  of  $X$ -degree  $s$  as  $a_c^{-1} s a_b$ , that is,

$$({}^a X)^s {}_c \mathcal{B}_b = X^{a_c s a_b^{-1}} {}_c \mathcal{B}_b.$$

Hence the underlying homogeneous components remain unchanged, and there is no difficulty to prove that  ${}^a X$  is indeed a grading.

We observe that  ${}^a X$  is connected if  $X$  is so.

**Proposition 5.2** Let  $\mathcal{B}$  be a connected  $k$ -category with a non necessarily connected grading  $X$  with group  $\Gamma_X$ . Let  $b_0$  be a fixed object. There exists a (non canonical) connected grading of  $\mathcal{B}$  with group  $\Gamma_{X, b_0}$ .

**Proof.** First we prove that  ${}_b(\Gamma_X)_{b_0}$  is non empty for any object  $b$ . Indeed, since  $\mathcal{B}$  is connected there exists a non-zero walk from  $b_0$  to  $b$ . Each non-zero morphism is a sum of homogeneous morphisms, and at least one of them is non-zero. We replace the morphisms of the walk by one of these non-zero homogeneous morphism, obtaining in this way a non-zero homogeneous walk from  $b_0$  to  $b$ .

Secondly we choose a family  $u = (u_b)_{b \in \mathcal{B}_0}$  of walk's degrees from  $b_0$  to  $b$  for each object  $b$ , with the special choice  $u_{b_0} = 1$ .

Third we consider the conjugated grading  ${}^u X$ , that is, the  ${}^u X$ -degree of a non-zero homogeneous morphism of  $X$ -degree  $s$  from  $b$  to  $c$  is  $u_c^{-1} s u_b$ . Note that this element is also the  $X$ -degree of a non-zero homogeneous closed walk at  $b_0$ , hence it belongs to  $\Gamma_{X, b_0}$ .

Finally we assert that the group of this grading at  $b_0$  is the full  $\Gamma_{X, b_0}$ . Indeed the new degree of each homogeneous non-zero closed walk at  $b_0$  equals its  $X$ -degree since each change of degree of a morphism of the walk compensates with the change of the following one and  $u_{b_0} = 1$ . By definition  $\Gamma_{X, b_0}$  is precisely the set of  $X$ -degrees of non-zero closed walks at  $b_0$ , then  $\Gamma_{{}^u X, b_0} = \Gamma_{X, b_0}$ . We conclude that  ${}^u X$  is a connected grading.  $\diamond$

**Remark 5.3** As quoted before, the connected grading we have constructed depends on the choice  $(u_b)_{b \in \mathcal{B}_0}$ , where  $u_b \in {}_b(\Gamma_X)_{b_0}$ . Nevertheless any other choice  $(u'_b)_{b \in \mathcal{B}_0}$  with  $u'_{b_0} = 1$  is obtained as  $(u_b a_b)_{b \in \mathcal{B}_0}$  where  $a_b \in \Gamma_{X, b_0}$  and  $a_{b_0} = 1$ . Note that the group of the connected gradings is the same for  ${}^u X$  and  ${}^{u'} X$ .

**Theorem 5.4** Let  $\mathcal{B}$  be a connected  $k$ -category with a fixed object  $b_0$  and let  $\mathcal{D}$  be a connected full subcategory containing  $b_0$ . Then there is a canonical group morphism

$$\kappa : \Pi_1(\mathcal{D}, b_0) \longrightarrow \Pi_1(\mathcal{B}, b_0).$$

which makes  $\Pi_1$  a functor.

**Proof.** According to Proposition 4.7, let  $\sigma = (g_Z)$  be an element of  $\Pi_1(\mathcal{D}, b_0)$  where  $Z$  varies over all the connected gradings of  $\mathcal{D}$  and  $g_Z \in \Gamma_Z$ . Recall that  $\mu(g_Z) = g_{Z'}$  for each pair of connected gradings  $Z$  and  $Z'$  such that there is a morphism from  $\mathcal{D}\#Z$  to  $\mathcal{D}\#Z'$ . Let  $X$  be a connected grading of  $\mathcal{B}$ . In order to define  $(\kappa(\sigma))_X$ , we first consider the restricted grading  $X \downarrow_{\mathcal{D}}$  by the same group  $\Gamma_X$ .

This grading is not connected in general. Hence we choose a set of walk's degrees  $u$  from  $b_0$  to each object of  $\mathcal{D}$  in order to construct the associated (non canonical) connected grading  $Y = {}^u(X \downarrow_{\mathcal{D}})$ . Note that the group of this connected grading is  $\Gamma_{X \downarrow_{\mathcal{D}}, b_0}$ . We define

$$\kappa(\sigma)_X = g_Y.$$

In order to prove that  $\kappa$  is well defined we need to check that for another set of walk's degrees  $u' = (u'_b)$  and for the resulting connected grading  $Y' = {}^{u'}(X \downarrow_{\mathcal{D}})$ , we have  $g_Y = g_{Y'}$ . This will be insured by the following Lemma, which shows that there is a morphism between the corresponding Galois smash coverings whose corresponding canonical group map  $\mu$  is the identity (recall that  $\mu(g_Y) = g_{Y'}$ ).

In order to prove that the obtained family of elements is coherent, let  $X$  and  $X'$  be two connected gradings of  $\mathcal{B}$  and let  $(H, J)$  be a morphism of coverings from  $\mathcal{B}\#X \rightarrow \mathcal{B}\#X'$ . Let  $\mu : \Gamma_X \rightarrow \Gamma_{X'}$  be the canonical group map associated to the Galois covering morphism  $\mathcal{B}\#X \rightarrow \mathcal{B}\#X'$ . We know that  $H(b_0, s) = (b_0, H_{\Gamma}(1)\mu(s))$ , hence we can modify  $H$  in order to have  $H(b_0, s) = (b_0, \mu(s))$ . Since  $J$  is the identity on objects of  $\mathcal{B}$ , the full subcategory  $\mathcal{D}$  is preserved by  $J$  and we infer a morphism of Galois coverings  ${}^u(X \downarrow_{\mathcal{D}}) \rightarrow {}^{u'}(X' \downarrow_{\mathcal{D}})$  where  $u = (u_b)$  is a choice of walk's degrees from  $b_0$  to each object  $b$ . The group map  $\mu$  restricts to the corresponding grading groups, since  $\sigma$  is a coherent family of elements for  $\mathcal{D}$  we infer  $\mu(g[{}^u(X \downarrow_{\mathcal{D}})]) = g[{}^{u'}(X' \downarrow_{\mathcal{D}})]$ . Finally note that the map  $\kappa$  clearly preserves composition of inclusions of full subcategories.  $\diamond$

Next we will prove the lemma used in the preceding proof. Observe that the concerned gradings  ${}^u(X \downarrow_{\mathcal{D}})$  and  ${}^{u'}(X \downarrow_{\mathcal{D}})$  are two connected gradings of a linear category which differ in a simple way, since  $u'_b = u_b a_b$ , see Remark 5.3. Note that  $a_{b_0} = 1$ .

**Remark 5.5** *The connected gradings  ${}^u(X \downarrow_{\mathcal{D}})$  and  ${}^{u'}(X \downarrow_{\mathcal{D}})$  are conjugated gradings by the family  $(a_b)_{b \in \mathcal{B}_0}$  with  $a_{b_0} = 1$ .*

**Lemma 5.6** *Let  $(a_b)_{b \in \mathcal{B}_0}$  be a family of elements in  $\Gamma_X$ . There is a covering morphism  $\mathcal{B}\#X \rightarrow \mathcal{B}\#{}^a X$  between conjugated gradings. The corresponding induced canonical group morphism  $\mu : \Gamma_X \rightarrow \Gamma_X$  is conjugation by  $a_{b_0}$ . In particular if  $a_{b_0} = 1$  then  $\mu = 1$ .*

**Proof.** The functor  $H$  is given on objects by  $H(b, s) = (b, sa_b)$  while on morphisms the functor is the identity since

$${}_{(c, ta_c)}(\mathcal{B}\#{}^a X)_{(b, sa_b)} = ({}^a X)^{a_c^{-1} t^{-1} sa_b} {}_c \mathcal{B}_b = X^{t^{-1} s} {}_b \mathcal{B}_c = {}_{(c, t)}(\mathcal{B}\#X)_{(b, s)}.$$

Note that the functor is of the form  $(H, 1)$  and  $H(b_0, s) = (b_0, sa_0) = (b_0, H_\Gamma(s))$ .

In order to compute  $\mu$  we first show that  $\lambda_H = 1$ . Recall that  $\lambda_H : \Gamma_X \rightarrow \Gamma_{a_X}$ , where  $\Gamma_X = \Gamma_{a_X}$ , is a group morphism uniquely determined by the property  $Hs = \lambda_H(s)H$  for any  $s$  in  $\Gamma_X$ . Clearly  $Hs = sH$ , then  $\lambda_H(s) = s$ . The canonical morphism  $\mu$  is given by  $\mu(s) = H_{\Gamma_X}(1)^{-1}\lambda_H(s)H_{\Gamma_X}(1)$ , see Corollary 4.5. Consequently  $\mu(s) = a_{b_0}^{-1}sa_{b_0}$ .  $\diamond$

We end this section by giving a general criterion for  $\kappa$  being injective, and we give a family of cases where the criterion applies.

**Theorem 5.7** *Let  $\mathcal{B}$  be a connected  $k$ -category, let  $b_0$  be a fixed object and let  $\mathcal{D}$  be a connected full subcategory containing  $b_0$ . Assume any connected grading of  $\mathcal{D}$  is a connected component of the restriction of some connected grading of  $\mathcal{B}$ . Then the group morphism*

$$\kappa : \Pi_1(\mathcal{D}, b_0) \longrightarrow \Pi_1(\mathcal{B}, b_0)$$

*is injective.*

**Proof.** Let  $\sigma = (g_Z)$  be a coherent family defining an element in  $\Pi_1(\mathcal{D}, b_0)$ . Assume  $\kappa(\sigma) = 1$ , which means that for any connected grading  $X$  of  $\mathcal{B}$  we have  $\kappa(\sigma)_X = 1$ . Recall that  $\kappa(\sigma)_X = g_{[u(X \downarrow \mathcal{D})]}$ . Consequently those elements are trivial. By hypothesis any connected grading  $Z$  of  $\mathcal{D}$  is of this form, then  $\sigma = 1$ .  $\diamond$

**Definition 5.8** *A subcategory  $\mathcal{D}$  of  $\mathcal{B}$  is said to be **convex** if any morphism of  $\mathcal{D}$  only factors through morphisms in  $\mathcal{D}$ . In case  $\mathcal{D}$  is full, this condition is equivalent to the fact that any composition of an outgoing morphism (with source in  $\mathcal{D}$  and target not in  $\mathcal{D}$ ) and an incoming one (reverse conditions) must be zero.*

The following corollary shows that convex connected full subcategory of  $\mathcal{B}$  is incompressible .

**Corollary 5.9** *Let  $\mathcal{B}$  be a connected  $k$ -category, let  $b_0$  be a fixed object and let  $\mathcal{D}$  be a connected full convex subcategory containing  $b_0$ . Then  $\kappa$  is injective.*

**Proof.** Let  $Z$  be a connected grading of  $\mathcal{D}$ . We extend it to  $\mathcal{B}$  by providing trivial degree to any morphism whose source or target is not in  $\mathcal{D}$ . By hypothesis there is no non-zero morphism of the form  $gf$  where  $f$  has source in  $\mathcal{D}$ ,  $g$  has target in  $\mathcal{D}$ , and the source of  $g$  and the target of  $f$  coincides without being in  $\mathcal{D}$ . We infer that this setting indeed provides a grading. The grading is connected since any element of the group is a walk's degree, already in  $\mathcal{D}$ . The preceding result insures that  $\kappa$  is injective.  $\diamond$



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