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The number of absorbed individuals in branching Brownian motion with a barrier

PASCAL MAILLARD*

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Summary. We study supercritical branching Brownian motion on the real line starting at the origin and with constant drift c . At the point $x > 0$, we add an absorbing barrier, i.e. individuals touching the barrier are instantly killed without producing offspring. It is known that there is a critical drift c_0 , such that this process becomes extinct almost surely if and only if $c \geq c_0$. In this case, if Z_x denotes the number of individuals absorbed at the barrier, we give an equivalent for $P(Z_x = n)$ as n goes to infinity. In the case of a b -ary offspring distribution, this answers a conjecture by David Aldous about the total progeny of the process, originally stated for the branching random walk.

Keywords. Branching Brownian motion with absorption, FKPP equation, travelling wave, singularity analysis, Galton–Watson process.

MSC2010. Primary: 60J80. Secondary: 34M35.

1 Introduction

We define branching Brownian motion as follows. Starting with an initial individual sitting at the origin of the real line, this individual moves according to a 1-dimensional Brownian motion with drift c until an independent exponentially distributed time with rate β . At that moment it dies and produces L (identical) offspring, L being a random variable in the non-negative integers with $P(L = 1) = 0$. Starting from the position at which its parent has died, each child repeats this process, all independently of one another and of their parent. For a rigorous definition, see for example [8].

We assume that $m = E[L] - 1 \in (0, \infty)$, which means that the process is supercritical. At position $x > 0$, we add an *absorbing barrier*, i.e. individuals hitting the barrier are instantly killed without producing offspring. Under some moment conditions, this process becomes extinct almost surely if and only if the drift $c \geq c_0 = \sqrt{2\beta m}$. A conjecture of David Aldous ([3]), originally stated for the branching random walk, says that the number N_x of individuals that have lived during the lifetime of the process satisfies $E[N_x] < \infty$ and $E[N_x \log N_x] = \infty$ in the critical speed area ($c = c_0$), and $P(N_x > n) \sim Kn^{-\gamma}$ in the subcritical speed area ($c > c_0$), with some $K > 0, \gamma > 1$. For the branching random walk, the conjecture of the critical case was proven by Addario-Berry and Broutin [1] for general reproduction laws

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satisfying a mild integrability assumption. Aidékon [2] refined the results for constant L by showing that there are positive constants ρ, C_1, C_2 , such that for every x , we have

$$\frac{C_1 x e^{\rho x}}{n(\log n)^2} \leq P(N_x > n) \leq \frac{C_2 x e^{\rho x}}{n(\log n)^2} \quad \text{for large } n.$$

Assuming L constant has the advantage that N_x is directly related to the number Z_x of individuals *absorbed at the barrier* by $N_x - 1 = (Z_x - 1)(L/(L - 1))$, hence it is possible to study N_x through Z_x .

In this sense, Neveu [18] had already proven the critical case conjecture for branching Brownian motion since he showed that the process $Z = (Z_x)_{x \geq 0}$ is actually a continuous-time Galton–Watson process of finite expectation, but not belonging to the $L \log L$ class in the critical case. In this paper, we propose a refinement of this result:

Theorem 1.1. *Assume $c = c_0$. Assume that $E[L(\log L)^{2+\varepsilon}] < \infty$ for some $\varepsilon > 0$. For $n \geq 2$, let $q_n = \lim_{x \rightarrow 0} \frac{1}{x} P(Z_x = n)$. Then for each $x \geq 0$ we have*

$$\sum_{k=n}^{\infty} q_k \sim \frac{c_0}{n(\log n)^2} \quad \text{and} \quad P(Z_x > n) \sim \frac{c_0^2 x e^{c_0 x}}{n(\log n)^2}, \quad \text{as } n \rightarrow \infty.$$

The heavy tail of Z_x suggests that its generating function is amenable to singularity analysis in the sense of [10]. This is in fact the case in both the critical and subcritical cases if we impose a stronger condition upon the offspring distribution and leads to the next theorem.

Theorem 1.2. *Define $f(s) = E[s^L]$ the generating function of the offspring distribution. Assume that 1 is a regular point of f , i.e. that f can be analytically extended in 1. Let δ be the span of $L - 1$, i.e. the greatest positive integer, such that $L - 1$ is concentrated on $\delta\mathbb{Z}$. Let $(q_n)_{n \geq 2}$ be defined as in Theorem 1.1. Let $\lambda_c \leq \bar{\lambda}_c$ be the roots of the equation $\lambda^2 - 2c\lambda + c_0^2$ and denote by $d = \frac{\bar{\lambda}_c}{\lambda_c}$ the ratio of the two roots.*

– In the critical speed area ($c = c_0$) we have for all $x \geq 0$, as $n \rightarrow \infty$,

$$q_{\delta n + 1} \sim \frac{c_0}{\delta n^2 (\log n)^2} \quad \text{and} \quad P(Z_x = \delta n + 1) \sim \frac{c_0^2 x e^{c_0 x}}{\delta n^2 (\log n)^2}.$$

– In the subcritical speed area ($c > c_0$) there exists a constant $K = K(c, \beta, f) > 0$, such that for all $x \geq 0$, as $n \rightarrow \infty$,

$$q_{\delta n + 1} \sim \frac{K}{n^{d+1}} \quad \text{and} \quad P(Z_x = \delta n + 1) \sim \frac{e^{\bar{\lambda}_c x} - e^{\lambda_c x}}{\bar{\lambda}_c - \lambda_c} \frac{K}{n^{d+1}}.$$

Furthermore, $q_{\delta n + k} = 0$ for all $n \in \mathbb{N}_0 = \{0, 1, 2, \dots\}$ and $k \in \{2, \dots, \delta\}$.

The important condition in Theorem 1.2 is that 1 be a regular point of f . For example, the theorem does not apply if L has a power-law tail. Pringsheim’s theorem (see e.g. [5], Band II, p. 289) entails that this condition is in fact equivalent to $f(s) = \sum P(L = n) s^n$ having a radius of convergence greater than one.

The content of the paper is organised as follows: In Section 2 we derive some preliminary results by probabilistic means. Section 3 is devoted to the proof of Theorem 1.1, which draws on a Tauberian theorem and known asymptotics of travelling wave solutions to the FKPP equation. In Section 4 we review results about complex differential equations, singularity analysis of generating functions and continuous-time Galton–Watson processes. Those are needed for the proof of Theorem 1.2, which is done in Section 5.

2 First results by probabilistic methods

The goal of this section is to prove

Proposition 2.1. *Assume $c > c_0$ and $E[L^2] < \infty$. Let $\lambda_c < \overline{\lambda}_c$ be the roots of $\lambda^2 - 2c\lambda + c_0^2$ and denote by $d = \overline{\lambda}_c/\lambda_c$ their ratio. Then there exists a constant $C = C(x, c, \beta, L) > 0$, such that*

$$P(Z_x > n) \geq \frac{C}{n^d} \quad \text{for large } n.$$

This result is needed to assure that the constant K in Theorem 1.2 is non-zero. It is independent from Section 3 and in particular from Theorem 1.1. Its proof is entirely probabilistic and follows closely [2].

2.1 Notation and preliminary remarks

Our notation borrows from [16]. An *individual* is an element in the space of Ulam–Harris labels

$$U = \bigcup_{n \in \mathbb{N}_0} \mathbb{N}^n,$$

where $\mathbb{N} = \{1, 2, 3, \dots\}$ and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. This space is endowed with the ordering relations \preceq and \prec defined by

$$u \preceq v \iff \exists w \in U : v = uw \quad \text{and} \quad u \prec v \iff u \preceq v \text{ and } u \neq v.$$

The space of Galton–Watson trees is the space of subsets $\mathfrak{t} \subset U$, such that $\emptyset \in \mathfrak{t}$, $v \in \mathfrak{t}$ if $v \prec u$ and $u \in \mathfrak{t}$ and for every u there is a number $L_u \in \mathbb{N}_0$, such that for all $j \in \mathbb{N}$, $uj \in \mathfrak{t}$ if and only if $j \leq L_u$. Thus, L_u is the number of children of the individual u .

Branching Brownian motion is defined on the filtered probability space $(\mathcal{T}, \mathcal{F}, (\mathcal{F}_t), P)$. Here, \mathcal{T} is the space of Galton–Watson trees with each individual $u \in \mathfrak{t}$ having a mark $(\zeta_u, X_u) \in \mathbb{R}^+ \times C(\mathbb{R}^+, \mathbb{R})$, where ζ_u is the life length and $X_u(s)$ the position of u at the time t (respectively, the position of its ancestor that was alive at time t , or the cemetery symbol Δ if u dies before the time t). The sigma-field \mathcal{F}_t contains all the information up to time t , and $\mathcal{F} = \bigcup_{t \geq 0} \mathcal{F}_t$.

Let $y, c \in \mathbb{R}$, $\beta \in \mathbb{R}^+$ and L be some random variable with values in \mathbb{N}_0 . $P = P^{y, c, \beta, L}$ is the unique probability measure, such that, starting with a single individual at the point y ,

- each individual moves according to a Brownian motion with drift c until an independent time ζ_u following an exponential distribution with parameter β .
- At the time ζ_u the individual dies and leaves L_u offspring at the position where it has died, with L_u being an independent copy of L .
- Each child of u repeats this process, all independently of one another and of the past of the process.

A common technique in branching processes since [17] is to select a genealogical line of descent from the ancestor \emptyset , called the *spine* and written as $\xi = (\xi_0 = \emptyset, \xi_1, \xi_2, \dots)$, where ξ_{n+1} is a child of ξ_n for each $n \in \mathbb{N}_0$. This gives the space $\tilde{\mathcal{T}}$ of marked trees with spine and the sigma-fields $\tilde{\mathcal{F}}$ and $\tilde{\mathcal{F}}_t$.

Assume from now on that $m = E[L] - 1 \in (0, \infty)$. Let N_t be the set of those individuals alive at time t . Note that every $\tilde{\mathcal{F}}_t$ -measurable function $f(T, \xi)$, $T \in \mathcal{T}$, admits a

representation

$$f(T, \xi) = \sum_{u \in N_t} f_u(T) \mathbb{1}_{u \in \xi},$$

where f_u are \mathcal{F}_t -measurable functions for each $u \in U$. We can therefore define a measure \tilde{P} on $(\tilde{\mathcal{T}}, \tilde{\mathcal{F}}, (\tilde{\mathcal{F}}_t))$ by

$$\tilde{P}_{|\tilde{\mathcal{F}}_t}(f(T, \xi)) = e^{-\beta mt} P_{|\mathcal{F}_t} \left(\sum_{u \in N_t} f_u(T) \right).$$

It is known ([16]) that this definition is sound and that \tilde{P} is actually a probability measure with the following properties:

- Under \tilde{P} , the individuals on the spine move according to a Brownian motion with drift c and die at an accelerated rate $(m+1)\beta$, independent of the motion.
- When an individual on the spine dies, it leaves a number of offspring at the point where it has died, this number following the size-biased distribution of L . In other words, let \tilde{L} be a random variable whose law $\mathcal{L}(\tilde{L})$ has the density $L/(m+1)$ with respect to $\mathcal{L}(L)$. Then the number of offspring is an independent copy of \tilde{L} .
- Amongst those offspring, the next individual on the spine is chosen uniformly. This individual repeats the behaviour of its parent.
- The other offspring initiate branching Brownian motions according to the law P .

2.2 Branching Brownian motion with two barriers

Let $a, b \in \mathbb{R}$ such that $y \in (a, b)$. Let $\tau = \tau_{a,b}$ be the (random) set of those individuals whose paths enter $(-\infty, a] \cup [b, \infty)$ and all of whose ancestors' paths have stayed inside (a, b) . For $u \in \tau$ we denote by $\tau(u)$ the first exit time from (a, b) by u 's path, i.e.

$$\tau(u) = \inf\{t \geq 0 : X_u(t) \notin (a, b)\} = \min\{t \geq 0 : X_u(t) \in \{a, b\}\},$$

and set $\tau(u) = \infty$ for $u \notin \tau$. τ is an (optional) stopping line in the sense of [8].

Lemma 2.2. *Assume $|c| > c_0 = \sqrt{2\beta m}$ and define $\rho = \sqrt{c^2 - c_0^2}$. Let $Z_{a,b}$ denote the number of individuals leaving the interval (a, b) at the point a , i.e.*

$$Z_{a,b} = \sum_{u \in \tau} \mathbb{1}_{X_u(\tau)=a},$$

where $X_u(\tau) = X_u(\tau(u))$. Then

$$E^y[Z_{a,b}] = e^{c(a-y)} \frac{\sinh((b-y)\rho)}{\sinh((b-a)\rho)}.$$

If, furthermore, $\sigma = E[L(L-1)] < \infty$, then

$$E^y[Z_{a,b}^2] = \frac{2\beta\sigma e^{c(a-y)}}{\rho \sinh^3((b-a)\rho)} \left[\sinh((b-y)\rho) \int_a^y e^{c(a-r)} \sinh^2((b-r)\rho) \sinh((r-a)\rho) dr \right. \\ \left. + \sinh((y-a)\rho) \int_y^b e^{c(a-r)} \sinh^3((b-r)\rho) dr \right] + E^y[Z_{a,b}].$$

Proof. Define $\xi(\tau) = \xi_i$ if $\xi_i \in \tau$ and write $\tau(\xi) = \tau(\xi(\tau))$ and $X_\xi(\tau) = X_{\xi(\tau)}(\tau)$. Then

$$E^y[Z_{a,b}] = \tilde{P}^y(e^{\beta m \tau(\xi)}, X_\xi(\tau) = a).$$

Let W^y be the law of standard Brownian motion started at y and $(B_t)_{t \geq 0}$ the canonical process. Let $T = T_{a,b}$ be the first exit time from (a, b) of B_t . By Girsanov's theorem, the above quantity is then equal to

$$W^y[e^{c(B_T - y) - \frac{1}{2}(c^2 - c_0^2)T}, B_T = a].$$

Evaluating this expression ([7], p. 212, 3.0.5) gives the first equality.

For $u \in U$, let Θ_u be the operator that maps a tree in \mathcal{T} to its subtree rooted in u . Then note that for each $u \in \tau$ we have

$$Z_{a,b} = 1 + \sum_{v \prec u} \sum_{\substack{w \in L_v \\ w \neq u}} Z_{a,b} \circ \Theta_w,$$

hence

$$E^y[Z_{a,b}^2] = E^y \left[\sum_{u \in \tau} \mathbb{1}_{X_u(\tau) = a} Z_{a,b} \right] = E^y[Z_{a,b}] + \tilde{P}^y \left[e^{\beta m \tau(\xi)} \sum_{v \prec \xi(\tau)} \sum_{\substack{w \in L_v \\ w \neq \xi(\tau)}} Z_{a,b} \circ \Theta_w, X_\xi(\tau) = a \right].$$

Conditioning on the path of the spine and the fission times on the spine and using the strong branching property, the second term equals

$$\tilde{P}^y \left[e^{\beta m \tau(\xi)} \sum_{v \prec \xi(\tau)} E[L_v - 1] E^{X_v(\zeta_v^-)}[Z_{a,b}], X_\xi(\tau) = a \right],$$

where ζ_v denotes the time of death of the individual v . But since under \tilde{P} the fission times form a Poisson process of intensity $\beta(m+1)$ and the reproduction on the spine follows the size-biased law of L , both being independent from the motion, this equals

$$\begin{aligned} W^y \left[e^{c(B_T - y) - \frac{1}{2}\rho^2 T} \int_0^T \beta \sigma E^{B_t}[Z_{a,b}] dt, B_T = a \right] \\ = \beta \sigma e^{c(a-y)} \int_a^b E^r[Z_{a,b}] W^y \left[e^{-\frac{1}{2}\rho^2 T} L_T^r, B_T = a \right] dr, \end{aligned}$$

where L_T^r is the local time of (B_t) at the time T and the point r . The last expression can be evaluated explicitly ([7], p. 215, 3.3.8) and gives the desired equality. \square

Corollary 2.3. *Under the assumptions of Lemma 2.2, for each $b > 0$ there are $C_b^{(1)}, C_b^{(2)} > 0$, such that as $a \rightarrow -\infty$,*

$$\begin{aligned} E^0[Z_{a,b}] &\sim C_b^{(1)} e^{(c+\rho)a}, \quad \text{and} \\ E^0[Z_{a,b}^2] &\sim C_b^{(2)} \begin{cases} e^{(c+\rho)a}, & \text{if } c > c_0 \\ e^{2(c+\rho)a}, & \text{if } c < -c_0 \end{cases}. \end{aligned}$$

2.3 Proof of Proposition 2.1

We now assume $y = 0$ and $E[L^2] < \infty$. Let $x > 0$ and let $\tau = \tau_x$ be the stopping line of those individuals hitting the point x for the first time. We set $Z_x = |\tau_x|$.

Proof of Proposition 2.1. Let $a < 0$ and $n \in \mathbb{N}$. Then, by the strong branching property,

$$P^0(Z_x > n) \geq P^0(Z_x > n \mid Z_{a,x} \geq 1)P^0(Z_{a,x} \geq 1) \geq P^a(Z_x > n)P^0(Z_{a,x} \geq 1).$$

If P_-^0 denotes the law of branching Brownian motion started at the point 0 with drift $-c$, then

$$P^a(Z_x > n) = P_-^0(Z_{a-x} > n) \geq P_-^0(Z_{a-x,1} > n).$$

In order to bound this quantity, we choose $a = a_n$ in such a way that $n = \frac{1}{2}E_-^0[Z_{a_n-x,1}]$. By Corollary 2.3 and the Paley–Zygmund inequality, there is then a constant $C_1 > 0$, such that

$$P_-^0(Z_{a_n-x,1} > n) \geq \frac{1}{4} \frac{E_-^0[Z_{a_n-x,1}]^2}{E_-^0[Z_{a_n-x,1}^2]} \geq C_1 \quad \text{for large } n.$$

Furthermore, by Corollary 2.3, $a_n = -(1/\lambda_c) \log n + O(1)$, which entails that there exists $C_2 > 0$, such that

$$P^0(Z_{a_n,x} \geq 1) \geq \frac{E^0[Z_{a_n,x}]^2}{E^0[Z_{a_n,x}^2]} \geq \frac{C_2}{n^d} \quad \text{for large } n,$$

again by the Paley–Zygmund inequality. This proves the proposition with $C = C_1 C_2$. \square

3 Tail asymptotics in the critical case

In Section 3.1 we review the known relation between the process $(Z_x)_{x \geq 0}$ and a differential equation called the FKPP equation. It will be of frequent use during the rest of the paper. Section 3.2 contains the proof of Theorem 1.1 and is independent from Sections 2, 4 and 5.

3.1 The FKPP equation

As was already observed by Neveu ([18]), the translational invariance of Brownian motion and the strong branching property immediately imply that $Z = (Z_x)_{x \geq 0}$ is a homogeneous Galton–Watson process. There is therefore an infinitesimal generating function

$$a(s) = \alpha \left(\sum_{n=0}^{\infty} p_n s^n - s \right), \quad \alpha > 0, \quad p_1 = 0, \quad (3.1)$$

associated to it ([4], p. 106, [12]). Its probabilistic interpretation is

$$\alpha = \lim_{x \rightarrow 0} \frac{1}{x} P(Z_x \neq 1) \quad \text{and} \quad p_n = \lim_{x \rightarrow 0} P(Z_x = n \mid Z_x \neq 1),$$

hence $q_n = \alpha p_n$. Note that with no further conditions on c and L , $\sum_n p_n$ need not necessarily be 1.

Neveu showed that $a(s)$ is intimately linked with the so-called *travelling wave* solutions of a reaction-diffusion equation called the Fisher–Kolmogorov–Petrovskii–Piskounov (FKPP) equation. Although he treated dyadic branching Brownian motion only (i.e. $L = 2$ almost surely), it is trivial to extend his result to general offspring distributions.

Fact 3.1. Assume $c \geq c_0$. Define $f(s) = E[s^L]$ and let q be the unique fixed point of f in $[0, 1)$. There exists $t_0 \in \mathbb{R} \cup \{-\infty\}$ and a strictly decreasing smooth function $\psi : (t_0, \infty) \rightarrow (q, 1)$ with $\psi(t_0+) = 1$ and $\psi(\infty) = q$, such that $a = \psi' \circ \psi^{-1}$ on $(q, 1)$, $F_x(s) = E[s^{Z_x}] = \psi(\psi^{-1}(s) + x)$ and ψ satisfies the following differential equation on (t_0, ∞) :

$$\frac{1}{2}\psi'' - c\psi' = \beta(\psi - f \circ \psi). \quad (3.2)$$

Remark 3.2. In this paper, we are only interested in the case where $Z_x < \infty$ almost surely for every $x \geq 0$. This corresponds to $t_0 = -\infty$ in Fact 3.1. It is known (e.g. [16]) that a sufficient condition for this to happen is

$$c > c_0 \text{ and } E[L \log L] < \infty, \text{ or } c = c_0 \text{ and } E[L(\log L)^{2+\varepsilon}] < \infty \text{ for some } \varepsilon > 0. \quad (C)$$

In this case, it is easy to show (see the proof of Lemma 5.1), that $a'(1) \in \{\lambda_c, \bar{\lambda}_c\}$. In particular, $a'(1) = c_0$, if $c = c_0$.

3.2 Proof of Theorem 1.1

We start with the following Abelian-type lemma:

Lemma 3.3. Let X be a random variable concentrated on \mathbb{N}_0 and let $\varphi(s) = E[s^X]$ be its generating function. Assume that $E[X(\log X)^\gamma] < \infty$ for some $\gamma > 0$. Then

$$\varphi'(1) - \varphi'(1-s) = O((\log \frac{1}{s})^{-\gamma}), \quad \text{as } s \rightarrow 0.$$

Proof. Let $s_0 > 0$ be such that the function $s \mapsto s(\log \frac{1}{s})^\gamma$ is increasing on $[0, s_0]$. Let $s \in (0, s_0)$. Then, with $p_k = P(X = k)$,

$$(\varphi'(1) - \varphi'(1-s))(\log \frac{1}{s})^\gamma = \sum_{k=1}^{\infty} k p_k (1 - (1-s)^{k-1}) (\log \frac{1}{s})^\gamma.$$

If $k > s^{-1}$, then $(1 - (1-s)^{k-1})(\log \frac{1}{s})^\gamma < (\log k)^\gamma$. If $\lceil s_0^{-1} \rceil \leq k < s^{-1}$, then $s(\log \frac{1}{s})^\gamma < \frac{1}{k}(\log k)^\gamma$ and thus $(1 - (1-s)^{k-1})(\log \frac{1}{s})^\gamma < k s (\log \frac{1}{s})^\gamma \leq (\log k)^\gamma$. Furthermore,

$$\sum_{k=1}^{\lceil s_0^{-1} \rceil} k p_k (1 - (1-s)^{k-1}) (\log \frac{1}{s})^\gamma < (s(\log \frac{1}{s})^\gamma) \sum_{k=1}^{\lceil s_0^{-1} \rceil} k^2 p_k < C,$$

for some $C > 0$. Collecting these results, we have

$$(\varphi'(1) - \varphi'(1-s))(\log \frac{1}{s})^\gamma < C + \sum_{k=\lceil s_0^{-1} \rceil}^{\infty} p_k k (\log k)^\gamma \leq C + E[X(\log X)^\gamma] < \infty,$$

for all $s < s_0$, which proves the lemma. \square

Proof of Theorem 1.1. We have $c = c_0$ by hypothesis. Let ψ be the travelling wave from (3.2). By Remark 3.2, the hypothesis $E[L(\log L)^{2+\varepsilon}] < \infty$ implies that ψ is defined on \mathbb{R} . Let $\phi(x) = 1 - \psi(-x)$, such that $a(1-s) = \phi'(\phi^{-1}(s))$ and $u(1-s, x) = 1 - \phi(\phi^{-1}(s) - x)$. It is known from [16] that there exists $K \in (0, \infty)$, such that $\phi(x) \sim Kx e^{-c_0 x}$ as $x \rightarrow \infty$. Since $a'(1) = c_0$ by Remark 3.2, this entails that $\phi'(x) \sim -c_0 Kx e^{-c_0 x}$.

Let $\varphi_1 = \phi'$, $\varphi_2 = \phi$ and $g(s) = 2\beta((m+1)s - (1 - f(1-s)))$. By Lemma 3.3, $g(s) = o(s(\log \frac{1}{s})^{-2})$. Equation (3.2) now implies

$$\frac{d}{dx} \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} = M \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} + \begin{pmatrix} g(\varphi_2(x)) \\ 0 \end{pmatrix}, \quad \text{with } M = \begin{pmatrix} -2c_0 & -c_0^2 \\ 1 & 0 \end{pmatrix}. \quad (3.3)$$

The Jordan decomposition of M is given by

$$J = A^{-1}MA = \begin{pmatrix} -c_0 & 1 \\ 0 & -c_0 \end{pmatrix}, \quad A = \begin{pmatrix} -c_0 & 1 - c_0 \\ 1 & 1 \end{pmatrix}. \quad (3.4)$$

Setting $\begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} = A \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}$, we get with $\xi = \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}$:

$$\xi' = J\xi + O(g(\xi_1 + \xi_2)),$$

which, in integrated form, becomes

$$\xi(x) = C_0 e^{xJ} + \int_{x_0}^x e^{(x-y)J} O(g(\xi_1(y) + \xi_2(y))) dy,$$

for some constant vector C_0 and some $x_0 \in \mathbb{R}$. But $\xi_i(x) = O(xe^{-c_0x})$, $i = 1, 2$, hence $g(\xi_1(x) + \xi_2(x)) = o(e^{-c_0x}/x)$. It follows that $\varphi_i(x) = C_{i,1}xe^{-c_0x} + C_{i,2}e^{-c_0x} + o(e^{-c_0x}/x)$, $i = 1, 2$, and by translating ϕ appropriately along the real line, we can choose it in such a way that for some constant C we have

$$\phi(x) = Cxe^{-c_0x} + o(e^{-c_0x}/x), \quad \phi'(x) = -c_0Cxe^{-c_0x} + Ce^{-c_0x} + o(e^{-c_0x}/x).$$

With these equations and the fact that $f'(1) - f(1-s) = o((\log \frac{1}{s})^{-2})$ by hypothesis and Lemma 3.3, one can show by elementary but tedious calculus that

$$a''(1-s) \sim \frac{c_0}{s(\log \frac{1}{s})^2} \quad \text{and} \quad F_x''(1-s) \sim \frac{c_0^2 xe^{-c_0x}}{s(\log \frac{1}{s})^2}, \quad \text{as } s \downarrow 0.$$

By standard Tauberian theorems ([9], XIII.5, Theorem 2), the first equivalence implies that

$$U(n) = \sum_{k=1}^n k^2 \alpha p_k \sim c_0 \frac{n}{(\log n)^2} \quad \text{as } n \rightarrow \infty.$$

By integration by parts, this entails that

$$\sum_{k=n}^{\infty} \alpha p_k = \int_{n-}^{\infty} x^{-2} U(dx) \sim c_0 \left(2 \int_n^{\infty} \frac{1}{x^2 (\log x)^2} dx - \frac{1}{n(\log n)^2} \right).$$

But the last integral is equivalent to $1/(n(\log n)^2)$ (see e.g. [9], VIII.9, Theorem 1), which proves the first part of the theorem. The second part is proven analogously. \square

4 Preliminaries for the proof of Theorem 1.2

In the light of Proposition 2.1, one may suggest that under suitable conditions on L one may extend the proof of Theorem 1.1 to the subcritical case $c > c_0$ and prove that $P(Z_x > n) \sim C'/(n^d)$ for some constant C' . In order to apply the Tauberian theorems, one would then have to establish asymptotics for the $(\lfloor d \rfloor + 1)$ -th derivatives of $a(s)$ and $F_x(s)$ as $s \rightarrow 1$. In trying to do this, one quickly sees that the known asymptotics for the travelling wave ($1 - \psi(x) \sim \text{const} \times e^{-\lambda_c x}$ as $x \rightarrow -\infty$, see [16]) are not precise enough for this method to work. However, instead of relying on Tauberian theorems, one can analyse the behaviour of the *holomorphic* function $a(s)$ near its singular point 1. This method is widely used in combinatorics at least since the seminal paper by Flajolet and Odlyzko [10] and is the basis for our proof of Theorem 1.2. Not only does it work in both the critical and subcritical cases, it even yields asymptotics for the density instead of the tail only. But before turning to the details, we give an overview of the several steps of our proof.

1. From (3.2), we derive a first order complex differential equation for $a(s)$ valid in the unit disk of the complex plane.
2. Using results from Chapters 3 and 4 in [6], we analyse this equation locally around the point 1 of the complex plane to establish asymptotics for $a(s)$ when s is near 1.
3. With the help of an equation valid for general continuous-time Galton–Watson processes, we transfer these asymptotics to asymptotics for the generating functions $F_x(s) = E[s^{Z_x}]$.
4. Finally, we apply results from [10] to turn this analysis into asymptotics for the reproduction probabilities and the density of Z_x .

In the rest of this section, we will define our notation for the complex analytic part of the proof and review some necessary general complex analytic results.

4.1 Notation

In the course of the paper, we will work in the spaces \mathbb{C} and \mathbb{C}^2 , endowed with the Euclidean topology. An open connected set is called a *region*, a simply connected region containing a point z_0 is also called a *neighbourhood* of z_0 . The closure of a set D is denoted by \overline{D} , its border by ∂D . The ball of radius r around z_0 is denoted by $\mathbb{D}(z_0, r) = \{z \in \mathbb{C} : |z - z_0| < r\}$, its closure and border by $\overline{\mathbb{D}}(z_0, r)$ and $\partial\mathbb{D}(z_0, r)$, respectively. We further use the abbreviation $\mathbb{D} = \mathbb{D}(0, 1)$ for the unit disk. For $0 < \varphi < \pi$, $r > 0$ and $x \in \mathbb{R}$, we define

$$\begin{aligned} G(\varphi, r) &= \{z \in \mathbb{D}(1, r) \setminus \{1\} : |\arg(1 - z)| < \pi - \varphi\}, & S_+(\varphi, x) &= [x, \infty) \times (-\varphi, \varphi), \\ \Delta(\varphi, r) &= \{z \in \mathbb{D}(0, 1 + r) \setminus \{1\} : |\arg(1 - z)| < \pi - \varphi\}, & S_-(\varphi, x) &= (-\infty, x] \times (-\varphi, \varphi). \end{aligned}$$

Let G be a region in \mathbb{C} , $z_0 \in \partial G$ and f and g analytic functions in G with $g(z) \neq 0$ for all $z \in G$. We write

$$\begin{aligned} f(z) \in o(g(z)) &\iff \lim_{z \rightarrow z_0} \frac{|f(z)|}{|g(z)|} = 0, \\ f(z) \in O(g(z)) &\iff \exists K \in \mathbb{C} : f(z) = Kg(z) + o(g(z)), \\ f(z) \sim g(z) &\iff f(z) = g(z) + o(g(z)). \end{aligned}$$

4.2 Complex differential equations

In this section, we review some basics about complex differential equations. We start with the fundamental existence and uniqueness theorem, which can be found for example in [6], [13] or [15].

Fact 4.1. *Let G be a region in \mathbb{C}^2 and (w_0, z_0) a point in G . Let $f(w, z) : G \rightarrow \mathbb{C}$ be analytic in G , i.e. f is continuous and both partial derivatives exist and are continuous. Then there exists a unique function $w(z)$ analytic in a neighbourhood U of z_0 , such that*

1. $w(z_0) = w_0$,
2. $(w(z), z) \in G$ for all $z \in U$ and
3. $w'(z) = f(w(z), z)$ for all $z \in U$.

In other words, the differential equation $w' = f(w, z)$ with boundary condition $w(z_0) = w_0$ has exactly one solution $w(z)$ which is analytic in z_0 .

A consequence is the following standard result.

Fact 4.2. *Let H be a region in \mathbb{C} and $w(z)$ analytic in H . Let G be a region in \mathbb{C}^2 , such that $(w(z), z) \in G$ for each $z \in H$ and suppose that there exists an analytic function $f(w, z) : G \rightarrow \mathbb{C}$, such that $w'(z) = f(w(z), z)$ for each $z \in H$. Let $z_0 \in \partial H$. Suppose that $w(z)$ is continuous in z_0 and that $(w(z_0), z_0) \in G$. Then z_0 is a regular point of $w(z)$, i.e. $w(z)$ admits an analytic extension in z_0 .*

4.3 Singularity analysis

We summarise results about the singularity analysis of generating functions. The basic references are [10] and [11], Chapter VI. The results are of two types: those that establish an equivalent for the coefficients of functions that are explicitly known, and those that estimate the coefficients of functions which are dominated by another function. We start with the results of the first type:

Fact 4.3. *Let $d \in (1, \infty) \setminus \mathbb{N}$, $k \in \mathbb{N}$, $\gamma \in \mathbb{R} \setminus \{0\}$, $\delta \in \mathbb{R}$ and the functions f_1, f_2 defined by*

$$f_1(z) = (1 - z)^d \quad \text{and} \quad f_2(z) = (1 - z)^k \left(\log \frac{1}{1 - z} \right)^\gamma \left(\log \log \frac{1}{1 - z} \right)^\delta,$$

for $z \in \mathbb{C} \setminus [1, +\infty)$. Let $(p_n^{(i)})$ be the coefficients of the Taylor expansion of f_i around the origin, $i = 1, 2$. Then $(p_n^{(i)})$ satisfy the following equivalence relations as $n \rightarrow \infty$:

$$p_n^{(1)} \sim \frac{K_1}{n^{d+1}} \quad \text{and} \quad p_n^{(2)} \sim \frac{K_2 (\log n)^{\gamma-1} (\log \log n)^\delta}{n^{k+1}},$$

for some non-zero constants $K_1 = K_1(d), K_2 = K_2(k, \gamma, \delta)$. We have $K_2(1, -1, 0) = 1$.

Proof. For f_1 , this is Proposition 1 from [10]. For f_2 this is remark 3 at the end of chapter 3 in the same paper. Note that the additional factors $\frac{1}{z}$ do not change the nature of the singularities, since $\frac{1}{z}$ is analytic in 1. For the last statement, note that $\frac{d}{ds} \frac{1}{\Gamma(-s)}(1) = 1$, since Γ has a pole of order 1 at -1 with residue -1 . \square

The results of the second type are contained in the next theorem. It is identical to Corollary 4 in [10]. Note that a potential difficulty here is that it requires analytical extension outside the unit disk.

Fact 4.4. Let $0 < \varphi < \pi/2$, $r > 0$ and $f(z)$ be analytic in $\Delta(\varphi, r)$. Assume that as $z \rightarrow 1$,

$$f(z) = o\left((1-z)^\alpha L\left(\frac{1}{1-z}\right)\right) \quad \text{where } L(u) = (\log u)^\gamma (\log \log u)^\delta$$

for some real constants α, γ, δ . Then the coefficients (p_n) of the Taylor expansion of f around 0 satisfy

$$p_n = o\left(\frac{L(n)}{n^{\alpha+1}}\right) \quad \text{as } n \rightarrow \infty.$$

4.4 An equation for continuous-time Galton–Watson processes

The following proposition establishes a relation between the infinitesimal generating function of a Galton–Watson process and its generating function at time t . We will use it to transfer asymptotics for $a(s)$ to the generating functions $F_x(s) = E[s^{Z_x}]$. For real s , the formulae stated in the proposition are well known, but we will need to use them for complex s , which is why we have to include some (complicated) hypotheses to be sure that the functions and integrals appearing in the formulae are well defined.

Proposition 4.5. Let $(Y_t)_{t \geq 0}$ be a continuous-time Galton–Watson process starting at 1. Let $a(s)$ be its infinitesimal generating function and $F_t(s) = E[s^{Y_t}]$. Let $a'(1) = \lambda \in (0, \infty)$. Suppose that $a(s)$ has an analytic extension into some simply connected region D_a . Let $Z_a = \{s \in D_a : a(s) = 0\}$. Suppose further that $F_t(s)$ has an analytic extension into some simply connected region D_F . Let there be a simply connected region $G \subset D_a \setminus Z_a$ and a simply connected region $D \subset G \cap D_F$ with $F_t(D) \subset G$ and $D \cap \mathbb{D} \neq \emptyset$. Then the following equations hold for all $s \in D$:

$$\int_s^{F_t(s)} \frac{1}{a(r)} dr = t, \tag{4.1}$$

and

$$1 - F_t(s) = e^{\lambda t}(1-s) \exp\left(\int_{F_t(s)}^s f^*(r) dr\right), \tag{4.2}$$

where $f^*(s)$ is defined for all $s \in D_a \setminus Z_a$ as

$$f^*(s) = \frac{a'(1)}{a(s)} + \frac{1}{1-s}. \tag{4.3}$$

Proof. We note first that $\frac{1}{a(s)}$ and f^* are analytic in the simply connected region G , hence the integrals are univalent analytic functions in D . Equation (4.1) now follows readily from Kolmogorov’s backward equation for $s \in \mathbb{D} \cap D$ (see [4], p. 106). The law of the permanence of functional equations (see for example [5], Band I, p. 203) then implies that (4.1) holds for every $s \in D$. This proves the first equation. Since G is simply connected and does not contain the point $1 \in Z_a$, the primitive $\int \frac{ds}{1-s} = -\log(1-s)$ is univalent on G , which immediately gives the second equation. \square

Corollary 4.6. If 1 is a regular point of $a(s)$, then it is a regular point for $F_t(s)$ for every $t \geq 0$.

Proof. Define $G = \{s \in \mathbb{D} : \operatorname{Re} s > q\}$. Then $G \cap Z_a = \emptyset$, since q is the only zero of a in \mathbb{D} (every probability generating function g with $g'(1) > 1$ has exactly one fixed point q in \mathbb{D} ; this can easily be seen by applying Schwarz's lemma to $\tau^{-1} \circ g \circ \tau$, where τ is the Möbius transform that maps 0 onto q). Let $s_1 \in (q, 1)$ be such that $F_t(s) \in G$ for every $s \in H = \{s \in \mathbb{D} : \operatorname{Re} s > s_1\}$. We can then apply Proposition 4.5 to conclude that (4.2) holds for every $s \in H$.

Since $a(s)$ is analytic in a neighbourhood U of 1 by hypothesis, it is easy to show that f^* is analytic in U as well. Thus, f^* has a primitive F^* in $H \cup U$. We define the function $g(s) = (1-s)\exp(F^*(s))$ on $H \cup U$. Since $g'(1) = -\exp(F^*(1)) \neq 0$, there exists an inverse g^{-1} of g in a neighbourhood U_1 of $g(1) = 0$. Let $U_2 \subset U$ be a neighbourhood of 1, such that $e^{\lambda t}g(s) \in U_1$ for every $s \in U_2$. Define the analytic function $\tilde{F}_t(s) = g^{-1}(e^{\lambda t}g(s))$ for $s \in U_2$. Then by (4.2), we have $F_t(s) = \tilde{F}_t(s)$ for every $s \in H \cap U_2$, hence \tilde{F}_t is an analytic extension of F_t in 1. \square

Corollary 4.7. *Suppose that $a(s)$ has an analytic extension into $G(\varphi_0, r_0)$ for some $0 < \varphi_0 < \pi$ and $r_0 > 0$. Then for every $\varphi_0 < \varphi < \pi$ there exists $r > 0$, such that $F_t(s)$ can be extended into $G(\varphi, r)$, mapping $G(\varphi, r)$ into $G(\varphi_0, r_0)$.*

Proof. Suppose w.l.o.g. that $a(s) \neq 0$ in $G(\varphi_0, r_0)$. Let $A(s)$ be a primitive of $(a(s))^{-1}$ on $G(\varphi_0, r_0)$. Since $a'(0) = \lambda > 0$, it is easy to show that there is $r_1 \leq r_0$, such that $A(s)$ is injective on $G(\varphi_0, r_1)$. Fix $\varphi \in (\varphi_0, \pi)$ and define $\varphi_1 = (\varphi_0 + \varphi)/2$. Since $A(s) \sim \lambda^{-1} \log(1-s)$ as $s \rightarrow 1$, there exists $R \in \mathbb{R}$, such that $A(G(\varphi_0, r_1))$ covers the strip $S = S_-(R, \pi - \varphi_1)$, hence A^{-1} exists on S . Furthermore, there is an $r > 0$, such that $A(s) + t \in S$ for every $s \in G(\varphi, r)$. We can thus define the function $\tilde{F}_t(s) = A^{-1}(t + A(s))$ on $G(\varphi, r)$. By Proposition 4.5, $\tilde{F}_t(s) = F_t(s)$ on $G(\varphi, r) \cap \mathbb{D}$, hence \tilde{F}_t is an analytic extension of F_t , mapping $G(\varphi, r)$ into $G(\varphi_0, r_0)$ by definition. \square

5 Proof of Theorem 1.2

We turn back to the branching Brownian motion and to our Galton–Watson process $Z = (Z_x)_{x \geq 0}$ of the number of individuals absorbed at the point x . Throughout this section, we place ourselves under the hypotheses of Theorem 1.2, i.e. we assume that $c \geq c_0 = \sqrt{2\beta m}$ and that the radius of convergence of $f(s) = E[s^L]$ is greater than 1. The equation $\lambda^2 - 2c\lambda + c_0^2 = 0$ then has the solutions $\lambda_c = c - \sqrt{c^2 - c_0^2}$ and $\bar{\lambda}_c = c + \sqrt{c^2 - c_0^2}$, hence $\lambda_c = \bar{\lambda}_c = c_0$ if $c = c_0$ and $\lambda_c < c_0 < \bar{\lambda}_c$ otherwise. The ratio $d = \bar{\lambda}_c/\lambda_c$ is therefore greater than or equal to one, according to whether $c > c_0$ or $c = c_0$, respectively.

We recall from section 3.1 that there is an infinitesimal generating function $a(s)$ associated to Z , which will be analysed thoroughly throughout this section. As a first step, we derive a first-order complex differential equation for $a(s)$:

Lemma 5.1. *On $\bar{\mathbb{D}}$, we have:*

$$a'(s)a(s) = 2ca(s) + 2\beta(s - f(s)). \quad (5.1)$$

Proof. On $(q, 1)$, we have by Fact 3.1

$$a' = \frac{\psi'' \circ \psi^{-1}}{\psi' \circ \psi^{-1}} = 2c + 2\beta \frac{\psi \circ \psi^{-1} - f \circ \psi \circ \psi^{-1}}{\psi' \circ \psi^{-1}} = 2c + 2\beta \frac{\operatorname{Id} - f}{a},$$

whence (5.1) follows. This implies that the equation holds on \mathbb{D} as well and if $a(s)$ and $a'(s)$ converge for each $s \in \partial\mathbb{D}$, Abel's limit theorem entails that the equation is satisfied on $\overline{\mathbb{D}}$. Since the coefficients of $a + \alpha s$ and $a' + \alpha$ are real and non-negative, it is enough to verify the convergence in 1. But since condition (C) in Remark 3.2 is verified by hypothesis, $a(1) = 0$ and since the coefficients of a are positive,

$$a'(1) = \lim_{s \uparrow 1} a'(s) = 2c + 2\beta \lim_{s \uparrow 1} \frac{s - f(s)}{a(s)} = 2c - \frac{2\beta m}{a'(1)},$$

which implies $a'(1) < \infty$. \square

The next lemma assures that we can ignore certain degenerate cases appearing in the course of the analysis of (5.1). It is the analytic interpretation of the probabilistic results in Section 2.

Lemma 5.2. *1 is a singular point of $a(s)$. If $c = c_0$, then $a''(1) = +\infty$.*

Proof. If $c = c_0$, the second assertion follows from Theorem 1.1 or from Neveu's result that (Z_x) does not belong to the L log L class (see the remark before Theorem 1.1). This implies that the radius of convergence of the power series $a(s)$ is 1, and Pringsheim's theorem now implies that 1 is a singular point of $a(s)$. If $c > c_0$, Proposition 2.1 implies that $E[s^{Z_x}] = \infty$ for every $s > 1$, whence 1 is a singular point of the generating function $F_x(s)$ by Pringsheim's theorem. By Corollary 4.6, it follows that 1 is a singular point of $a(s)$ as well. \square

The next theorem is the core of the proof of Theorem 1.2.

Theorem 5.3. *Let λ_c , $\overline{\lambda_c}$ and d be defined as in Theorem 1.2. Then for every angle $\varphi \in (0, \pi)$ there exists $r > 0$, such that $a(s)$ possesses an analytical extension (denoted by $a(s)$ as well) into $G(\varphi, r)$. Moreover, as $s \rightarrow 1$ in $G(\varphi, r)$, the following holds.*

– If $d = 1$, then

$$a(1-s) = -c_0 s + c_0 \frac{s}{\log \frac{1}{s}} + c_0 s \frac{\log \log \frac{1}{s}}{(\log \frac{1}{s})^2} + O\left(\frac{s}{(\log \frac{1}{s})^2}\right). \quad (5.2)$$

– If $d > 1$, then there is a $K = K(c, \beta, f) \in \mathbb{C} \setminus \{0\}$ and a polynomial $h(s) = \sum_{n=2}^{\lfloor d \rfloor} c_n s^n$, such that

$$\text{if } d \notin \mathbb{N}: \quad a(1-s) = -\lambda_c s + h(s) + K s^d + o(s^d), \quad (5.3)$$

$$\text{if } d \in \mathbb{N}: \quad a(1-s) = -\lambda_c s + h(s) + K s^d \log s + o(s^d). \quad (5.4)$$

Before turning to the proof, we explain its basic ideas in the case $c > c_0$. Starting point is the differential equation (5.1). We will see that the behaviour of $a(s)$ around the point 1 will be closely related to the behaviour of the solutions to an equation of the form

$$\frac{db}{ds} = d \cdot \frac{b + \text{terms in } b \text{ and } s \text{ of order } \geq 2}{s + \text{terms in } b \text{ and } s \text{ of order } \geq 2},$$

around the point 0. If we remove the higher order terms from this equation, the family of solutions is given by $b(s) = K s^d$ for $K \in \mathbb{C}$, hence we can hope that our solution behaves

asymptotically as Ks^d for some $K \in \mathbb{C}$. This is actually the case if $d \notin \mathbb{N}$, but if d is a natural number, it will behave like a solution to

$$\frac{db}{ds} = d \cdot \frac{b + Cs^d}{s},$$

where C is some non-zero constant. But the family of solutions to this equation is given by $b(s) = Cs^d(\log s + K)$ for $K \in \mathbb{C}$, which is where the additional logarithmic factor comes into play.

At first sight, this dichotomy might seem strange, but it becomes evident if one remembers that we expect the coefficients of $a(s)$ (i.e. the infinitesimal reproduction probabilities) to behave like $1/n^{d+1}$. In light of Fact 4.3, a logarithmic factor must therefore appear if d is a natural number, otherwise $a(s)$ would be analytic in 1, in which case its coefficients would decrease at least exponentially.

Proof of Theorem 5.3. During this proof, we need to successively shrink the radius r to ensure analyticity of the functions we are studying. These radii will be denoted by $r_1 \geq r_2 \geq r_3 \geq \dots$, in order of appearance.

We set $b(s) = a(1 - s)$. Then

$$-b'(s)b(s) = 2cb(s) + 2\beta(1 - s - f(1 - s)) \quad \text{on } \overline{\mathbb{D}}(1, 1).$$

Since f is regular in 1 by hypothesis, $f(1) = 1$ and $f'(1) = E[L] = m + 1$, there exists $0 < r_1 < 1 - q$ and a function h_1 analytic on $\mathbb{D}(0, r_1)$ with $h(0) = h'(0) = 0$, such that $f(1 - s) = 1 - (m + 1)s + h_1(s)$ for $s \in \mathbb{D}(0, r_1)$.

As a first step, we analyse the above equation for real non-negative s . Since $b(s) \neq 0$ on $(0, r_1)$, we can divide both sides by $b(s)$ to obtain

$$\frac{db}{ds} = \frac{-2cb - c_0^2s + 2\beta h_1(s)}{b} \quad \text{on } (0, r_1). \quad (5.5)$$

We introduce the parameter $dt = \frac{ds}{b}$, living on $[t_0, \infty)$ for some $t_0 \in \mathbb{R}$, to obtain the system

$$\frac{db}{dt} = -2cb - c_0^2s + 2\beta h_1(s), \quad \frac{ds}{dt} = b,$$

which, in matrix form, becomes

$$\frac{d}{dt} \begin{pmatrix} b \\ s \end{pmatrix} = M \begin{pmatrix} b \\ s \end{pmatrix} + \begin{pmatrix} 2\beta h_1 \\ 0 \end{pmatrix}, \quad M = \begin{pmatrix} -2c & -c_0^2 \\ 1 & 0 \end{pmatrix}. \quad (5.6)$$

Note that this is nothing else than (3.3). Thus, the Jordan decomposition of M is given by (3.4), if $c = c_0$, and

$$A^{-1}MA = \begin{pmatrix} -\overline{\lambda}_c & 0 \\ 0 & -\lambda_c \end{pmatrix}, \quad A = \begin{pmatrix} -\overline{\lambda}_c & -\lambda_c \\ 1 & 1 \end{pmatrix}, \quad \text{if } c > c_0. \quad (5.7)$$

Setting $\begin{pmatrix} b \\ s \end{pmatrix} = A \begin{pmatrix} B \\ S \end{pmatrix}$ transforms (5.6) into

$$\frac{dB}{dt} = -\overline{\lambda}_c B + P(B, S), \quad \frac{dS}{dt} = -\lambda_c S + Q(B, S), \quad \text{if } c > c_0 \text{ and} \quad (5.8)$$

$$\frac{dB}{dt} = -c_0 B + S + D(B, S), \quad \frac{dS}{dt} = -c_0 S + E(B, S), \quad \text{if } c = c_0, \quad (5.9)$$

where P , Q , D and E are of the form

$$P(w, z) = \sum' p_{ij} w^i z^j, \quad Q(w, z) = \sum' q_{ij} w^i z^j, \quad \text{etc.}, \quad \text{with } \sum' = \sum_{i, j \in \mathbb{N}_0, i+j \geq 2} \quad (5.10)$$

and all are analytic in a neighbourhood $(\mathbb{D}(0, r_2))^2$ of the point $(0, 0)$.

In Chapter 4 of [6], Bieberbach now tells us how to get rid of the non-linear terms. Let us first treat the subcritical case ($c > c_0$). Then $\overline{\lambda}_c > \lambda_c$. He proves in this case that it is possible to find $u(w, z)$ and $v(w, z)$, analytic in $(\mathbb{D}(0, r_3))^2$ and of the form

$$u(w, z) = w + \sum' u_{ij} w^i z^j, \quad v(w, z) = z + \sum' v_{ij} w^i z^j, \quad (5.11)$$

such that, when passing to $\mathbf{b} = u(B, S)$ and $\mathbf{s} = v(B, S)$, the equation (5.8) becomes, if $d \notin \mathbb{N}$:

$$\frac{d\mathbf{b}}{dt} = -\overline{\lambda}_c \mathbf{b}, \quad \frac{d\mathbf{s}}{dt} = -\lambda_c \mathbf{s}, \quad (5.12)$$

and if $d = n \in \mathbb{N} \setminus \{1\}$: either (5.12) or

$$\frac{d\mathbf{b}}{dt} = -\overline{\lambda}_c \mathbf{b} + C \mathbf{s}^n, \quad \frac{d\mathbf{s}}{dt} = -\lambda_c \mathbf{s}, \quad C \in \mathbb{C} \setminus \{0\}. \quad (5.13)$$

The general solution of (5.12) is

$$\mathbf{b}(t) = \text{const} \times e^{-\overline{\lambda}_c t}, \quad \mathbf{s}(t) = \text{const} \times e^{-\lambda_c t}, \quad (5.14)$$

and the solutions of (5.13) are

$$\mathbf{b}(t) = CC' t e^{-\overline{\lambda}_c t} + \text{const} \times e^{-\overline{\lambda}_c t}, \quad \mathbf{s}(t) = C' e^{-\lambda_c t}, \quad C' \neq 0, \quad (5.15)$$

$$\text{or } \mathbf{b}(t) = \text{const} \times e^{-\overline{\lambda}_c t}, \quad \mathbf{s}(t) = 0, \quad (5.16)$$

where const denotes any complex number. Thus, \mathbf{b} and \mathbf{s} must be of this form. They therefore possess a unique analytical extension onto \mathbb{C} , defined by (5.14), (5.15) or (5.16) for all $t \in \mathbb{C}$.

Let us now work the way back to $b(s)$. First of all, we note that by (5.11) and the implicit function theorem (see for example [14], Theorem 2.1.2), there are u^* , v^* , such that

$$\begin{pmatrix} B \\ S \end{pmatrix} = \begin{pmatrix} u^*(\mathbf{b}, \mathbf{s}) \\ v^*(\mathbf{b}, \mathbf{s}) \end{pmatrix} = \begin{pmatrix} \mathbf{b} + \sum' u_{ij}^* \mathbf{b}^i \mathbf{s}^j \\ \mathbf{s} + \sum' v_{ij}^* \mathbf{b}^i \mathbf{s}^j \end{pmatrix}, \quad (5.17)$$

and u^* , v^* are analytic on $(\mathbb{D}(0, r_4))^2$. Furthermore, we have $b = -\lambda_c s + (\lambda_c - \overline{\lambda}_c) B$, so we have to define B in dependence on s . In the degenerate cases where $\mathbf{b} = 0$ or $\mathbf{s} = 0$ it is easy to see that $B(s)$ is analytic in 0 and therefore also $b(s)$, which is impossible because of Lemma 5.2. We can therefore assume $\mathbf{b} \neq 0$ and $\mathbf{s} \neq 0$.

If $d \notin \mathbb{N}$, (5.14) and (5.17) now imply as $\text{Re } t \rightarrow \infty$, with $n = \lfloor d \rfloor$:

$$s = c e^{-\lambda_c t} + c_2 e^{-2\lambda_c t} + \dots + c_n e^{-n\lambda_c t} + c_d e^{-d\lambda_c t} + o(e^{-d\lambda_c t}), \quad (5.18)$$

$$\text{and } s' = -(c/\lambda_c) e^{-\lambda_c t} + O(e^{-\min(d, 2)\lambda_c t}), \quad (5.19)$$

where c and c_d are nonzero (remember that $\overline{\lambda}_c = d\lambda_c$). Fix $0 < \varphi < \pi$. From (5.18) and (5.19) it is easy to show that there is a $t_1 \in \mathbb{R}$, such that $s(t)$ maps $S_+(t_1, (\pi - \varphi/2)/\lambda_c)$

injectively onto $G = \{s \in \mathbb{D}(0, r) : |\arg s| < \pi - \varphi\}$. Taking into account that, because of (5.14) and (5.17),

$$B = c'_2 e^{-2\lambda ct} + \dots + c'_n e^{-n\lambda ct} + c'_d e^{-d\lambda ct} + o(e^{-d\lambda ct}) \quad \text{as } \operatorname{Re} t \rightarrow \infty, \quad (5.20)$$

with c'_d nonzero, it follows that

$$B(s) = c''_2 s^2 + \dots + c''_n s^n + \frac{c'_d}{c^d} s^d + o(s^d) \quad \text{as } s \rightarrow 0, \quad (5.21)$$

which proves (5.3).

In the case $d = n \in \mathbb{N} \setminus \{1\}$, assume that \mathfrak{b} and \mathfrak{s} are of the form (5.14), which entails $\mathfrak{b} = \mathfrak{s}^n$. Again, this implies that $b(s)$ is analytic in a neighbourhood of 0, contradicting Lemma 5.2.

Therefore, \mathfrak{b} and \mathfrak{s} must be of the form (5.15). Similar arguments as above then lead to

$$B = c''_2 s^2 + \dots + c''_n s^n - K s^n \log s + o(s^n) \quad \text{as } \operatorname{Re} t \rightarrow \infty,$$

for some non-zero constant K , proving (5.4).

We turn to the case $d = 1$. In what follows, we denote by $\varepsilon_1, \varepsilon_2, \dots$ some positive constants that are as small as necessary. We have $B = -b + (1 - c_0)s$, which is a strictly increasing function of s on $(0, \varepsilon_1)$, since $\frac{db}{ds}(0) = -c_0$. But s is a decreasing function of t for large t , thus B admits an inverse $t = t(B)$ on $(0, \varepsilon_2)$. Setting $\tilde{B} = -c_0 B$, (5.9) then entails

$$\frac{dS}{d\tilde{B}} = \frac{S + \tilde{E}(\tilde{B}, S)}{\tilde{B} + S + \tilde{D}(\tilde{B}, S)} \quad \text{on } \left(-\frac{\varepsilon_2}{c_0}, 0\right), \quad (5.22)$$

where \tilde{E} and \tilde{D} are of the form (5.10). Now it is possible to show (see Lemma 5.5 below) that there exists $h(z) = \sum_{k \geq 2} h_k z^k$ analytic in 0, such that $\tilde{S} = S - h(\tilde{B})$ verifies

$$\frac{d\tilde{S}}{d\tilde{B}} = \frac{\tilde{S}}{\tilde{B} + \tilde{S} + F(\tilde{B}, \tilde{S})} \quad \text{on } (-\varepsilon_3, 0), \quad (5.23)$$

where F is of the form (5.10). Now, $\tilde{S} = b + c_0 s + O(s^2)$, hence $\frac{d\tilde{S}}{ds}(0) = 0$. But by Lemma 5.2, $\frac{d^2 b}{ds^2}(0) = +\infty$, hence \tilde{S} is a strictly increasing function of s on $(0, \varepsilon_4)$. It therefore has an inverse $t = t(\tilde{S})$ on $(0, \varepsilon_5)$ and

$$\frac{d\tilde{B}}{d\tilde{S}} = \frac{\tilde{B} + \tilde{S} + F(\tilde{B}, \tilde{S})}{\tilde{S}} \quad \text{on } (0, \varepsilon_5). \quad (5.24)$$

According to [6], §4.6, there exists now an analytic function

$$u(w, z) = w + \sum_{i, j \geq 0, i+2j \geq 2} u_{i, 2j} w^i z^j, \quad (5.25)$$

such that $\mathfrak{b} = u(\tilde{B}, \tilde{S})$ and $\mathfrak{s} = \tilde{S}$ satisfy the equation

$$\frac{d\mathfrak{b}}{d\mathfrak{s}} = \frac{\mathfrak{b} + \mathfrak{s}}{\mathfrak{s}} \quad \text{on } (0, \varepsilon_6), \quad (5.26)$$

whose solution verifies $\mathfrak{b}(\mathfrak{s}) = \mathfrak{s}(\log \mathfrak{s} + c)$ for some $c \in \mathbb{C}$ and can thus be extended onto the whole slit plane $\mathbb{C} \setminus (-\infty, 0]$. Introducing the parameter $r = \log \mathfrak{s} \in \mathbb{R} \times (-\pi, \pi)$ gives $\mathfrak{b} = re^r + ce^r$ and $\mathfrak{s} = e^r$.

As above, the implicit function theorem entails that there exists an analytic function $v(w, z)$ of the form (5.25), such that $\tilde{B} = v(\mathfrak{b}, \mathfrak{s})$. We get, as $\operatorname{Re} r \rightarrow -\infty$,

$$s = -\frac{1}{c_0} \tilde{B} + h(\tilde{B}) + \mathfrak{s} = -\frac{1}{c_0} re^r + O(e^r),$$

$$\text{and } s'(r) = -\frac{1}{c_0} re^r + O(e^r).$$

Again, we can show that for every $0 < \varphi < \pi$ there exists $r_1 \in \mathbb{R}$, such that the map $s(r)$ is injective on the strip $S_-(r_1, \pi - \varphi/2)$ and covers the region $G = \{s \in \mathbb{D}(0, \rho) : |\arg s| < \pi - \varphi\}$ for some $\rho > 0$. Consequently, the inverse $r = r(s)$ exists on G . Elementary calculus shows that

$$r = \log(c_0 s) - \log \log \frac{1}{s} + \frac{\log \log \frac{1}{s}}{\log \frac{1}{s}} + O\left(\frac{1}{\log \frac{1}{s}}\right) \quad \text{as } s \rightarrow 0.$$

Since $b = -c_0 s + S$ and $S = e^r + O(r^2 e^{2r})$ as $\operatorname{Re} r \rightarrow -\infty$, this immediately implies (5.2). \square

Remark 5.4. The reason why we cannot explicit the constant K in Theorem 5.3 is that we are analysing (5.1) only *locally* around the point 1. Since the solution of (5.1) with boundary conditions $a(q) = a(1) = 0$ is unique (this follows from the uniqueness of the travelling wave solutions to the FKPP equation), a *global* analysis of this equation should be able to exhibit the value of K . But it is probably easier to refine the probabilistic arguments of Section 2, which already give a lower bound that can be easily made explicit.

The following lemma was needed in the proof of Theorem 5.3.

Lemma 5.5. *Let $w(z)$ satisfy the following equation on $(0, \varepsilon)$ or $(-\varepsilon, 0)$ for some $\varepsilon > 0$:*

$$\frac{dw}{dz} = \frac{w + P(w, z)}{z + w + Q(w, z)}, \quad w(0) = 0, \quad (5.27)$$

with P, Q of the form (5.10). Then there exists $h(z) = \sum_{k \geq 2} h_k z^k$ analytic in 0, such that $\tilde{w} = w - h(z)$ satisfies

$$\frac{d\tilde{w}}{dz} = \frac{\tilde{w}}{z + \tilde{w} + P^*(\tilde{w}, z)}, \quad \tilde{w}(0) = 0, \quad (5.28)$$

with P^ of the form (5.10).*

Proof. We first show that (5.27) has an analytic solution $w_0(z)$ with $w_0(z) = w'_0(z) = 0$. We choose the *ansatz* $w_0 = z \cdot w_1$ and get

$$w'_1 = \frac{-w_1^2 + zP_1(w_1, z) - zw_1Q_1(w_1, z)}{z(1 + w_1 + zQ_1(w_1, z))}, \quad (5.29)$$

where $P_1(w, z) = P(zw, z)/z^2$ and $Q_1(w, z) = Q(zw, z)/z^2$ are analytic in $(0, 0)$. Writing $1/(1 + w_1 + zQ_1(w_1, z))$ as a power series in w_1 and z and multiplying the numerator in (5.29) by it, we get

$$w'_1 = \frac{-w_1^2 + zR(w_1, z)}{z}, \quad \text{with } R(w, z) = \sum_{i, j \geq 0} r_{ij} w^i z^j. \quad (5.30)$$

Setting further $w_1 = z(w_2 + r_{00})$ with $w_2(0) = 0$ gives

$$w_2' = \frac{-w_2 + zR_1(w_2, z)}{z}, \quad R_1(w, z) \text{ analytic in } (0, 0). \quad (5.31)$$

We can now continue as in [6], p. 64 to conclude that the coefficients of w_2 can be determined from (5.31) and form a convergent power series in a neighbourhood of $(0, 0)$. We then continue as in [6], p. 65 by showing that $\tilde{w} = w - w_0$ satisfies (5.28). \square

The asymptotics established in Theorem 5.3 for the infinitesimal generating function can now be readily transferred to the generating functions $F_x(s)$.

Corollary 5.6. *Let λ_c , $\overline{\lambda}_c$ and d be defined as in Theorem 1.2. Set $F_x(s) = E[s^{Z_x}]$ for $s \in \mathbb{D}$. Then for every $x \geq 0$ and angle $0 < \varphi < \pi$ there exists $r > 0$, such that F_x possesses an analytical extension (denoted by F_x as well) into $G(\varphi, r)$. Furthermore, for every $x \geq 0$ the following holds as $s \rightarrow 1$ in $G(\varphi, r)$.*

– If $d = 1$, then

$$F_x(1 - s) = 1 - c_0s + c_0^2xe^{c_0x} \left(\frac{s}{\log \frac{1}{s}} + \frac{s \log \log \frac{1}{s}}{(\log \frac{1}{s})^2} \right) + O\left(\frac{s}{(\log \frac{1}{s})^2} \right). \quad (5.32)$$

– If $d > 1$, then there is a polynomial $h_x(s) = \sum_{n=2}^{\lfloor d \rfloor} c_n s^n$, such that

$$\text{if } d \notin \mathbb{N}: \quad F_x(1 - s) = 1 - \lambda_c s + h_x(s) + K_x ds^d + o(s^d), \quad (5.33)$$

$$\text{if } d \in \mathbb{N}: \quad F_x(1 - s) = 1 - \lambda_c s + h_x(s) + K_x s^d \log s + o(s^d), \quad (5.34)$$

where $K_x = K(e^{\overline{\lambda}_c x} - e^{\lambda_c x})/(\overline{\lambda}_c - \lambda_c)$, with K being the constant from Theorem 5.3.

Proof. Let $0 < \varphi_0 < \varphi$ and let r_0 be such that $a(s)$ can be analytically extended onto $G(\varphi_0, r_0)$. Such an r_0 exists by the virtue of Theorem 5.3. Corollary 4.7 now implies that there exists $r > 0$, such that $F_x(s)$ can be analytically extended onto $G(\varphi, r)$ and maps $G(\varphi, r)$ into $G(\varphi_0, r_0)$. Hence, the functions

$$w(s) = 1 - F_x(1 - s) \quad \text{and} \quad I(s) = \int_s^{w(s)} f^*(1 - r) dr,$$

where $f^*(s)$ is defined as in (4.3), are univalent analytic functions in

$$D = 1 - G(\varphi, r) = \{s \in \mathbb{D}(0, r) \setminus \{0\} : |\arg s| < \pi - \varphi\}.$$

In what follows, we always assume that $s \in D$.

We treat the case $d > 1$ first. By Theorem 5.3, $a(1 - s) = -\lambda_c s + s^2 h_0(s) + \sigma(s) + o(s^d)$, where $h_0(s)$ is a polynomial in s and $\sigma(s) = Ks^d$ or $\sigma(s) = Ks^d \log s$, according to whether $d \notin \mathbb{N}$ or $d \in \mathbb{N}$, respectively. It follows that for some polynomial $h_1(s)$ with $h_1(0) = 0$,

$$I(s) = [h_1(r)]_s^{w(s)} - \int_s^{w(s)} \frac{\sigma(r)}{\lambda_c r^2} dr + o(s^{d-1}). \quad (5.35)$$

By Proposition 4.5, we further have

$$w(s) = se^{\lambda_c x} \exp(I(s)) = se^{\lambda_c x} (1 + I(s) + O(I(s)^2)) \quad \text{as } s \rightarrow 0. \quad (5.36)$$

Repeated application of (5.36) shows that $w(s) = P(s) + o(s^{\lceil d-1 \rceil})$, where P is a polynomial of degree $\lceil d-1 \rceil$ with $P(0) = 0$. Straightforward calculus now shows that for some $C \in \mathbb{C}$,

$$\int_s^{w(s)} \frac{\sigma(r)}{\lambda_c r^2} dr = \frac{K_x}{e^{\lambda_c x}} \frac{\sigma(s)}{s} + C s^{\lceil d-1 \rceil} + o(s^{d-1}), \quad (5.37)$$

yielding (5.33) and (5.34).

In the critical case $d = 1$, Theorem 5.3 tells us that

$$f^*(1-s) = -\frac{1}{s} \left(\frac{1}{\log \frac{1}{s}} + \frac{\log \log \frac{1}{s}}{(\log \frac{1}{s})^2} + O\left(\frac{1}{(\log \frac{1}{s})^2}\right) \right). \quad (5.38)$$

Write $\lambda = \lambda_c = c_0$. For our first approximation of $w(s)$, we note that

$$I(s) \sim -\int_s^{se^{\lambda x}} \frac{1}{r \log \frac{1}{r}} dr \sim -\frac{\lambda x}{\log \frac{1}{s}} \quad \text{as } s \rightarrow 0,$$

hence, by the virtue of (5.36),

$$w(s) = se^{\lambda x} \left(1 - \frac{\lambda x}{\log \frac{1}{s}} + o\left(\frac{1}{\log \frac{1}{s}}\right) \right). \quad (5.39)$$

To obtain a finer approximation, we decompose $I(s)$ into

$$I(s) = \int_s^{se^{\lambda x}} f^*(1-r) dr + \int_{se^{\lambda x}}^{w(s)} f^*(1-r) dr =: I_1(s) + I_2(s).$$

We then have

$$I_1(s) = -\frac{\lambda x}{\log \frac{1}{s}} - \lambda x \frac{\log \log \frac{1}{s}}{(\log \frac{1}{s})^2} + O\left(\frac{1}{(\log \frac{1}{s})^2}\right),$$

and, because of (5.39),

$$-I_2(s) \sim \int_{se^{\lambda x}}^{se^{\lambda x}(1+\frac{\lambda x}{\log s})} \frac{1}{r \log \frac{1}{r}} dr \sim \frac{\lambda x}{(\log \frac{1}{s})^2}.$$

This finishes the proof. \square

Proof of Theorem 1.2. Most of the work has been done in Theorem 5.3 and Corollary 5.6. In order to apply the methods from singularity analysis reviewed in Section 4.3, it remains to check the necessary analyticity conditions and to treat the case where the law of L has a span of 2 or more.

We first turn to the infinitesimal probabilities. It is clear that if $(p_k)_{k \in \mathbb{N}_0}$ represents the infinitesimal reproduction law of (Z_x) , the span of $(p_{k+1})_{k \in \mathbb{N}_0}$ is the same as the span δ of $L-1$, since the process starts with one individual and the number of individuals increases by $l-1$ when an individual gives birth to l children.

Let us first treat the case $\delta = 1$. Denote $h(s) = \sum_{k \in \mathbb{N}_0} p_k s^k$. Let $s_0 \in \partial \mathbb{D} \setminus \{1\}$. Since $\delta = 1$, we have $|h(s_0)| < 1 = |s_0|$, hence $a_0 := a(s_0) \neq 0$. We want to show that $a(s)$ can be analytically extended in s_0 . For this purpose, write the differential equation (5.1) in the form

$$a' = \frac{2ca + 2\beta(s - f(s))}{a} =: g(a, s). \quad (5.40)$$

Since the radius of convergence of f is greater than 1 by hypothesis, g is analytic in (a_0, s_0) . Fact 4.2 then shows that s_0 is a regular point of a . Since $s_0 \in \partial\mathbb{D} \setminus \{1\}$ was chosen arbitrarily, this means that every point of $\partial\mathbb{D} \setminus \{1\}$ is a regular point of a .

Now let $0 < \varphi < \pi/2$ and $r > 0$ from Theorem 5.3, i.e. a can be analytically extended onto $G(\varphi, r)$ and satisfies one of the asymptotic relations (5.2), (5.3) or (5.4), according to whether $d = 1$, $d \notin \mathbb{N}$ or $d \in \mathbb{N} \setminus \{1\}$, respectively. Since every point of $\partial\mathbb{D} \setminus \{1\}$ is regular, it follows by elementary arguments that a can actually be extended onto $\Delta(\varphi, r_0)$ for some $r_0 > 0$. Facts 4.3 and 4.4 together with (5.2), (5.3) and (5.4) immediately prove the first half of Theorem 1.2 (note our definition of $O(\cdot)$ in Section 4.1).

If $\delta \geq 2$, then $p_{k+\delta n} = 0$ for all $k \in \{2, \dots, n\}$ and $a(s)/s$ is δ -periodic. We can then show as above that a has exactly δ singularities on $\partial\mathbb{D}$ located on $1, e^{i2\pi/\delta}, \dots, e^{i2(\delta-1)\pi/\delta}$. If we set

$$h(s) = \alpha \left(\sum_n p_{1+\delta n} s^n - 1 \right),$$

then $a(s) = s \cdot h(s^\delta)$, and so the only singularity on $\partial\mathbb{D}$ of h is the point 1. Furthermore, if $q(s)$ denotes a version of $\sqrt[\delta]{s}$ which is analytic in a neighbourhood U of 1 and which satisfies $q(1) = 1$, then $h(s) = a(q(s))/q(s)$ near 1. But since $q'(1) = 1/\delta$, we have

$$h(1-s) = a \left(1 - \left(\frac{1}{\delta} s + c_2 s^2 + c_3 s^3 + \dots \right) \right) (1 + c'_1 s + c'_2 s + \dots),$$

for some constants c_n, c'_n , and so equations (5.2), (5.3) and (5.4) transfer to h with the coefficient of the main singular term divided by δ^d . We can therefore use Facts 4.3 and 4.4 for h to conclude.

The proof of the statements for the probabilities $P(Z_x = n)$ is completely analogous, drawing on Corollary 5.6 instead. The only thing that remains to show is that for each $x \geq 0$, F_x is analytic in every point $s_0 \in \partial\mathbb{D}$ (we assume $\delta = 1$, the case $\delta \geq 2$ is handled as above). To this end, we deduce from Kolmogorov's forward and backward equations (see e.g. [4], p. 106 or [12], p. 99) the equation

$$F'_x(s) = \frac{a(F_x(s))}{a(s)} \quad \text{on } \mathbb{D}.$$

Now, since $a(s)$ is analytic and non-zero in s_0 , the function $f(w, s) = a(w)/a(s)$ is analytic in $(a(s_0), s_0)$, hence we can apply Fact 4.2 to conclude that F_x is analytic in s_0 as well. This finishes the proof of the theorem. \square

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References

- [1] ADDARIO-BERRY, L., AND BROUTIN, N. Total progeny in killed branching random walk. <http://arxiv.org/abs/0908.1083v1>, 2009.

- [2] AÏDÉKON, E. Tail asymptotics for the total progeny of the critical killed branching random walk. <http://arxiv.org/abs/0911.0877v1>, 2009.
- [3] ALDOUS, D. Power laws and killed branching random walk. Website, available at <http://www.stat.berkeley.edu/~aldous/Research/OP/brw.html>.
- [4] ATHREYA, K. B., AND NEY, P. E. *Branching processes*. Springer-Verlag, New York, 1972. Die Grundlehren der mathematischen Wissenschaften, Band 196.
- [5] BIEBERBACH, L. *Lehrbuch der Funktionentheorie. I. Elemente der Funktionentheorie. II. Moderne Funktionentheorie*. Chelsea Publishing Company, New York, 1945.
- [6] BIEBERBACH, L. *Theorie der gewöhnlichen Differentialgleichungen auf funktionentheoretischer Grundlage dargestellt*. Zweite umgearbeitete und erweiterte Auflage. Die Grundlehren der Mathematischen Wissenschaften, Band 66. Springer - Verlag, Berlin, 1965.
- [7] BORODIN, A. N., AND SALMINEN, P. *Handbook of Brownian motion—facts and formulae*, second ed. Probability and its Applications. Birkhäuser Verlag, Basel, 2002.
- [8] CHAUVIN, B. Product martingales and stopping lines for branching Brownian motion. *Ann. Probab.* 19, 3 (1991), 1195–1205.
- [9] FELLER, W. *An introduction to probability theory and its applications. Vol II. 2nd ed.* Wiley Series in Probability and Mathematical Statistics. New York etc.: John Wiley and Sons, Inc. XXIV, 669 p. , 1971.
- [10] FLAJOLET, P., AND ODLYZKO, A. Singularity analysis of generating functions. *SIAM J. Discrete Math.* 3, 2 (1990), 216–240.
- [11] FLAJOLET, P., AND SEDGEWICK, R. *Analytic combinatorics*. Cambridge University Press, Cambridge, 2009.
- [12] HARRIS, T. E. *The theory of branching processes*. Die Grundlehren der Mathematischen Wissenschaften, Bd. 119. Springer-Verlag, Berlin, 1963.
- [13] HILLE, E. *Ordinary differential equations in the complex domain*. Wiley-Interscience [John Wiley & Sons], New York, 1976. Pure and Applied Mathematics.
- [14] HÖRMANDER, L. *An introduction to complex analysis in several variables*, revised ed. North-Holland Publishing Co., Amsterdam, 1973. North-Holland Mathematical Library, Vol. 7.
- [15] INCE, E. L. *Ordinary differential equations*. Dover Publications, New York, 1944.
- [16] KYPRIANOU, A. E. Travelling wave solutions to the K-P-P equation: alternatives to Simon Harris’ probabilistic analysis. *Ann. Inst. H. Poincaré Probab. Statist.* 40, 1 (2004), 53–72.
- [17] LYONS, R., PEMANTLE, R., AND PERES, Y. Conceptual proofs of $L \log L$ criteria for mean behavior of branching processes. *Ann. Probab.* 23, 3 (1995), 1125–1138.
- [18] NEVEU, J. Multiplicative martingales for spatial branching processes. Stochastic processes, Semin., Princeton/New Jersey 1987, Prog. Probab. Stat. 15, 223–242., 1988.