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# LAGRANGIAN NAVIER-STOKES DIFFUSIONS ON MANIFOLDS: VARIATIONAL PRINCIPLE AND STABILITY

MARC ARNAUDON AND ANA BELA CRUZEIRO

ABSTRACT.

We prove a variational principle for stochastic Lagrangian Navier-Stokes trajectories on manifolds. We study the behaviour of such trajectories concerning stability as well as rotation between particles; the two-dimensional torus case is described in detail.

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## 1. INTRODUCTION

As discovered by V. I. Arnold ([2]) the motion of an incompressible non viscous fluid can be characterized as a geodesic on a group of diffeomorphisms. This point of view allows in particular to derive properties of the Lagrangian Euler flow, such as stability, through the study of the geometry of the group ([6]).

When the fluid is viscous, namely for the Navier-Stokes equation, one can describe the Lagrangian trajectories as realizations of a stochastic process and interpret the associated drift, solving Navier-Stokes, as an expectation over this process. This intrinsically probabilistic approach we follow here is inspired by [8], [9]. Similar stochastic models are used for example in [4]. Then the trajectories remain, in an appropriate sense, geodesics and are almost sure solutions of a variational principle. This was shown in [3] for the two-dimensional torus.

We prove a variational principle for the Lagrangian Navier-Stokes diffusions in a compact Riemannian manifold. Furthermore we study its stability properties. The behaviour of the trajectories depends on the intensity of the noise as well as on the metric of the underlying manifold. The example of the torus is studied in detail. Finally we describe the evolution in time of the rotation between stochastic Lagrangian particles.

Let  $(M, \mathbf{g})$  be a compact oriented Riemannian manifold without boundary.

Recall that the Itô differential of an  $M$ -valued semimartingale  $Y$  is defined by

$$(1.1) \quad dY_t = P(Y)_t d \left( \int_0^t P(Y)_s^{-1} \circ dY_s \right)_t$$

where

$$(1.2) \quad P(Y)_t : T_{Y_0}M \rightarrow T_{Y_t}M$$

is the parallel transport along  $t \mapsto Y_t$ . Alternatively, in local coordinates,

$$(1.3) \quad dY_t = \left( dY_t^i + \frac{1}{2} \Gamma_{jk}^i(Y_t) dY_t^j \otimes dY_t^k \right) \partial_i$$

where  $\Gamma_{jk}^i$  are the Christoffel symbols of the Levi-Civita connection.

If the semimartingale  $Y_t$  has an absolutely continuous drift, we denote it by  $DY_t dt$ : for every 1-form  $\alpha \in \Gamma(T^*M)$ , the finite variation part of

$$(1.4) \quad \int_0^t \langle \alpha(Y_t), dY_t \rangle$$

is

$$(1.5) \quad \int_0^t \langle \alpha(Y_t), DY_t dt \rangle$$

Let  $G^s$ ,  $s \geq 0$  be the infinite dimensional group of homeomorphisms on  $M$  which belong to  $H^s$ , the Sobolev space of order  $s$ . For  $s > \frac{m}{2} + 1$ ,  $m = \dim M$ ,  $G^s$  is a  $C^\infty$  Hilbert manifold. The volume preserving homeomorphism subgroup will be denoted by  $G_V^s$ :

$$G_V^s = \{g \in G^s, : g_*\mu = \mu\},$$

with  $\mu$  the volume element associated to the Riemannian metric. We denote by  $\mathcal{G}^s$  (resp.  $\mathcal{G}_V^s$ ) the Lie algebra of  $G^s$  (resp.  $G_V^s$ ). See [6] for example.

On  $M$  we consider an incompressible Brownian flow  $g_u(t) \in G_V^0$  with covariance  $a \in \Gamma(TM \odot TM)$  and time dependent drift  $u(t, \cdot) \in \Gamma(TM)$ . We assume that for all  $x \in M$ ,  $a(x, x) = 2\nu \mathbf{g}^{-1}(x)$  for some  $\nu > 0$ . This means that

$$(1.6) \quad dg_u(t)(x) \otimes dg_u(t)(y) = a(g_u(t)(x), g_u(t)(y)) dt,$$

$$(1.7) \quad dg_u(t)(x) \otimes dg_u(t)(x) = 2\nu \mathbf{g}^{-1}(g_u(t)(x)) dt,$$

the drift of  $g_u(t)(x)$  is absolutely continuous and satisfies  $Dg_u(t)(x) = u(t, g_u(t)(x))$ . The generator of this process is

$$L_u = \nu \Delta^h + \frac{\partial}{\partial t} + \partial_u$$

where  $\Delta^h$  is the horizontal Laplacian. The parameter  $\nu$  will be called the speed of the Brownian flow.

If the time is indexed by  $[0, T]$  for some  $T > 0$ , we define the action functional by

$$S(g_u) = \frac{1}{2} \mathbb{E} \left[ \int_0^T \left( \int_M \|Dg_u(t)(x)\|^2 dx \right) dt \right].$$

## 2. THE VARIATIONAL PRINCIPLE

Define

$$(2.1) \quad \mathcal{H} = \{v \in C^1([0, T], \mathcal{G}_V^\infty), v(0, \cdot) = 0, v(T, \cdot) = 0\}$$

Given  $v \in \mathcal{H}$ , consider the following ordinary differential equation

$$(2.2) \quad \begin{aligned} \frac{de_t(v)}{dt} &= \dot{v}(t, e_t(v)) \\ e_0(v) &= e \end{aligned}$$

where  $e$  is the identity of  $G_V^\infty$ . Since  $v$  is divergence free,  $e_t(v)$  is a  $G_V^\infty$ -valued deterministic path.

We denote by  $\mathcal{P}$  the set of continuous  $G_V^0$ -valued semimartingales  $g(t)$  such that  $g(0) = e$ . Then for all  $v \in \mathcal{H}$ , we have  $e_t(v) \circ g_u(t) \in \mathcal{P}$ .

**Definition 2.1.** Let  $J$  be a functional defined on  $\mathcal{P}$  and taking values in  $\mathbb{R}$ . We define its left and right derivatives in the direction of  $h(\cdot) = e_t(v)$ ,  $v \in \mathcal{H}$  at a process  $g \in \mathcal{P}$  respectively, by

$$(2.3) \quad \begin{aligned} (D_L)_h J[g] &= \frac{d}{d\varepsilon} J[e_t(\varepsilon v) \circ g(\cdot)]|_{\varepsilon=0}, \\ (D_R)_h J[g] &= \frac{d}{d\varepsilon} J[g(\cdot) \circ e_t(\varepsilon v)]|_{\varepsilon=0}. \end{aligned}$$

A process  $g \in \mathcal{P}$  will be called a critical point of the functional  $J$  if

$$(2.4) \quad (D_L)_h J[g] = (D_R)_h J[g], \quad \forall h = e_t(v), \quad v \in \mathcal{H}.$$

**Theorem 2.2.** Let  $(t, x) \mapsto u(t, x)$  be a smooth time-dependent divergence-free vector field on  $M$ , defined on  $[0, T] \times M$ . Let  $g_u(t)$  a stochastic Brownian flow with speed  $\nu > 0$  and drift  $u$ . The stochastic process  $g_u(t)$  is a critical point of the energy functional  $S$  if and only if the vector field  $u(t)$  verifies the Navier-Stokes equation

$$(2.5) \quad \frac{\partial u}{\partial t} + \nabla_u u = \nu \square u - \nabla p.$$

For the construction of weak solutions of Navier-Stokes equations on Riemannian manifolds we refer to [7].

*Proof.* Since the functional  $S$  is right invariant, it is enough to consider the left derivative. So we need to compute

$$(2.6) \quad \frac{d}{d\varepsilon}|_{\varepsilon=0} S(e_t(\varepsilon v)(g_u)).$$

We let

$$(2.7) \quad f(\varepsilon) = S(e_t(\varepsilon v)(g_u)).$$

Then

$$(2.8) \quad f(\varepsilon) = \frac{1}{2} \int_M \left( \mathbb{E} \left[ \int_0^T \left( \|De_t(\varepsilon v)(g_u)(t)(x)\|^2 \right) dt \right] \right) dx$$

which yields

$$(2.9) \quad f'(0) = \int_M \left( \mathbb{E} \left[ \int_0^T \left( \langle \nabla_\varepsilon|_{\varepsilon=0} De_t(\varepsilon v)(g_u)(t)(x), u(t, g_u(t)(x)) \rangle \right) dt \right] \right) dx.$$

We need to compute

$$(2.10) \quad \nabla_\varepsilon|_{\varepsilon=0} D e_t(\varepsilon v)(g_u(t)(x)).$$

We have

$$\begin{aligned} \nabla_t \frac{d}{d\varepsilon}|_{\varepsilon=0} e_t(\varepsilon v) &= \nabla_\varepsilon|_{\varepsilon=0} \frac{d e_t(\varepsilon v)}{dt} \\ &= \nabla_\varepsilon|_{\varepsilon=0} \varepsilon \dot{v}(t, e_t(\varepsilon v)) \\ &= \dot{v}(t, e). \end{aligned}$$

Together with  $v(0, \cdot) = 0$ , this implies

$$(2.11) \quad \frac{d}{d\varepsilon}|_{\varepsilon=0} e_t(\varepsilon v)(x) = v(t, x).$$

Consequently

$$(2.12) \quad \frac{d}{d\varepsilon}|_{\varepsilon=0} e_t(\varepsilon v)(g_u(t)(x)) = v(t, g_u(t)(x)).$$

By Itô equation,

$$\begin{aligned} (2.13) \quad & d e_t(\varepsilon v)(g_u(t)(x)) \\ &= \langle d e_t(\varepsilon v)(\cdot), d g_u(t)(x) \rangle + \frac{1}{2} \nabla d e_t(\varepsilon v)(g_u(t)(x)) (d g_u(t)(x) \otimes d g_u(t)(x)) \\ &= \langle d e_t(\varepsilon v)(\cdot), d g_u(t)(x) \rangle + \nu \Delta e_t(\varepsilon v)(g_u(t)(x)) dt. \end{aligned}$$

Here  $\Delta e_t(\varepsilon v)(\cdot)$  denotes the tension field of the map  $e_t(\varepsilon v) : M \rightarrow M$ . This yields

$$\begin{aligned} (2.14) \quad & D e_t(\varepsilon v)(g_u(t)(x)) = \langle d e_t(\varepsilon v)(\cdot), D g_u(t)(x) \rangle + \nu \Delta e_t(\varepsilon v)(g_u(t)(x)) \\ & \quad + \varepsilon \dot{v}(t, e_t(\varepsilon v)(g_u(t)(x))) \\ &= \langle d e_t(\varepsilon v)(\cdot), u(t, g_u(t)(x)) \rangle + \nu \Delta e_t(\varepsilon v)(g_u(t)(x)) \\ & \quad + \varepsilon \dot{v}(t, e_t(\varepsilon v)(g_u(t)(x))). \end{aligned}$$

Differentiating with respect to  $\varepsilon$  at  $\varepsilon = 0$ , we get

$$\begin{aligned} (2.15) \quad & \nabla_\varepsilon|_{\varepsilon=0} D e_t(\varepsilon v)(g_u(t)(x)) \\ &= \langle \nabla_\varepsilon|_{\varepsilon=0} d e_t(\varepsilon v)(\cdot), u(t, g_u(t)(x)) \rangle + \nu \nabla_\varepsilon|_{\varepsilon=0} \Delta e_t(\varepsilon v)(g_u(t)(x)) \\ & \quad + \frac{\partial v}{\partial t}(t, g_u(t)(x)) \\ &= \left\langle \nabla \cdot \frac{d}{d\varepsilon}|_{\varepsilon=0} e_t(\varepsilon v)(\cdot), u(t, g_u(t)(x)) \right\rangle + \nu \square \frac{d}{d\varepsilon}|_{\varepsilon=0} e_t(\varepsilon v)(g_u(t)(x)) \\ & \quad + \frac{\partial v}{\partial t}(t, g_u(t)(x)) \\ &= \langle \nabla \cdot v(t, \cdot), u(t, g_u(t)(x)) \rangle + \nu \square v(t, \cdot)(g_u(t)(x)) + \frac{\partial v}{\partial t}(t, g_u(t)(x)) \\ &= \nabla_{u(t, g_u(t)(x))} v(t, \cdot) + \nu \square v(t, \cdot)(g_u(t)(x)) + \frac{\partial v}{\partial t}(t, g_u(t)(x)). \end{aligned}$$

We used the commutation formula  $\nabla_\varepsilon|_{\varepsilon=0} \Delta = \square \frac{d}{d\varepsilon}$ , where  $\square = dd^* + d^*d$  is the damped Laplacian. Alternatively,

$$(2.16) \quad \square v = \Delta^h v + \text{Ric}^\sharp(v).$$

For a  $TM$ -valued semimartingale  $J_t$  which projects onto the  $M$ -valued semimartingale  $Y_t$ , we denote by  $\mathcal{D}J_t$  the Itô covariant derivative:

$$(2.17) \quad \mathcal{D}J_t = P(Y)_t d(P(Y)_t^{-1} J_t).$$

Then Itô equation yields

$$(2.18) \quad \mathcal{D}u(t, g_u(t)(x)) \simeq \frac{\partial u}{\partial t}(t, g_u(t)(x)) dt + \nabla_{dg_u(t)(x)} u + \nu \Delta^h u(t, g_u(t)(x)) dt$$

and

$$(2.19) \quad \mathcal{D}v(t, g_u(t)(x)) \simeq \frac{\partial v}{\partial t}(t, g_u(t)(x)) dt + \nabla_{dg_u(t)(x)} v + \nu \Delta^h v(t, g_u(t)(x)) dt.$$

where the notation  $\simeq$  means equal up to a martingale:

$$\begin{aligned} & \int_0^\cdot P(g_u(\cdot))_t^{-1} \mathcal{D}u(t, g_u(t)(x)) \\ & - \int_0^\cdot P(g_u(\cdot))_t^{-1} \left( \frac{\partial u}{\partial t}(t, g_u(t)(x)) dt + \nabla_{dg_u(t)(x)} u + \nu \Delta^h u(t, g_u(t)(x)) dt \right) \end{aligned}$$

is a local martingale.

On the other hand, denoting  $u_t = u(t, g_u(t)(x))$  and  $v_t = v(t, g_u(t)(x))$  we have

$$(2.20) \quad \langle u_T, v_T \rangle = \int_0^T \langle \mathcal{D}u_t, v_t \rangle + \int_0^T \langle u_t, \mathcal{D}v_t \rangle + \int_0^T \langle \mathcal{D}u_t, \mathcal{D}v_t \rangle.$$

Let us denote by  $Dv_t$  the drift of  $v_t$  with respect to the damped connection  $\nabla^c$  on  $TM$ , whose geodesics are the Jacobi fields. It is known that,

$$(2.21) \quad \left( Du_t - \nu \text{Ric}^\sharp(u_t) \right) dt \quad \text{is the drift of } \mathcal{D}u_t$$

and

$$(2.22) \quad \left( Dv_t - \nu \text{Ric}^\sharp(v_t) \right) dt \quad \text{is the drift of } \mathcal{D}v_t.$$

As can be seen from (2.15), (2.19) and (2.22), the drift  $Dv_t$  commutes with the derivative with respect to a parameter, so it satisfies

$$(2.23) \quad Dv_t = \nabla_\varepsilon|_{\varepsilon=0} De_t(\varepsilon v)(g_u(t)(x)).$$

Taking the expectation in (2.20) and using (2.23), (2.21) and (2.22), we get by removing the martingale parts

$$(2.24) \quad \begin{aligned} \mathbb{E}[\langle u_T, v_T \rangle] &= \mathbb{E} \left[ \int_0^T \langle \frac{\partial u}{\partial t}(t, g_u(t)(x)) + \nabla_{u_t} u + \nu \Delta^h u(t, g_u(t)(x)), v_t \rangle dt \right] \\ &+ \mathbb{E} \left[ \int_0^T \langle u_t, \nabla_\varepsilon|_{\varepsilon=0} De_t(\varepsilon v)(g_u(t)(x)) - \nu \text{Ric}^\sharp(v_t) \rangle dt \right] \\ &+ \mathbb{E} \left[ 2\nu \int_0^T \text{tr} \langle \nabla \cdot u, \nabla \cdot v \rangle (t, g_u(t)(x)) dt \right]. \end{aligned}$$

Then using the facts that  $v_T = 0$ , together with

$$(2.25) \quad \langle u_t, \text{Ric}^\sharp(v_t) \rangle = \langle \text{Ric}^\sharp(u_t), v_t \rangle$$

and (2.16), we get

$$\begin{aligned}
(2.26) \quad & \mathbb{E} \left[ \int_0^T \langle u_t, \nabla_\varepsilon|_{\varepsilon=0} D e_t(\varepsilon v)(g_u(t)(x)) \rangle dt \right] \\
&= -\mathbb{E} \left[ \int_0^T \left\langle \frac{\partial u}{\partial t}(t, g_u(t)(x)) + \nabla_{u_t} u + \nu \square u(t, g_u(t)(x)), v_t \right\rangle dt \right] \\
&\quad - \mathbb{E} \left[ 2\nu \int_0^T \operatorname{tr} \langle \nabla \cdot u_t, \nabla \cdot v_t \rangle (t, g_u(t)(x)) dt \right].
\end{aligned}$$

Integrating with respect to  $x$  yields

$$\begin{aligned}
(2.27) \quad & f'(0) \\
&= -\mathbb{E} \left[ \int_0^T \left( \int_M \left\langle \left( \left( \frac{\partial}{\partial t} + \nabla_u + \nu \square \right) u \right) (t, g_u(t)(x)), v(t, g_u(t)(x)) \right\rangle dx \right) dt \right] \\
&\quad - \mathbb{E} \left[ 2\nu \int_0^T \left( \int_M \operatorname{tr} \langle \nabla \cdot u, \nabla \cdot v \rangle (t, g_u(t)(x)) dx \right) dt \right].
\end{aligned}$$

Now we use the fact that  $g_u(t)(\cdot)$  is volume preserving:

$$\begin{aligned}
(2.28) \quad & f'(0) \\
&= -\mathbb{E} \left[ \int_0^T \left( \int_M \left\langle \left( \left( \frac{\partial}{\partial t} + \nabla_u + \nu \square \right) u \right) (t, x), v(t, x) \right\rangle dx \right) dt \right] \\
&\quad - \mathbb{E} \left[ 2\nu \int_0^T \left( \int_M \operatorname{tr} \langle \nabla \cdot u, \nabla \cdot v \rangle (t, x) dx \right) dt \right].
\end{aligned}$$

Since  $M$  is compact and orientable, an integration by parts gives

$$(2.29) \quad \int_M \operatorname{tr} \langle \nabla \cdot u, \nabla \cdot v \rangle (t, x) dx = - \int_M \langle \square u, v \rangle (t, x) dx.$$

Replacing in (2.28) we get

$$(2.30) \quad f'(0) = -\mathbb{E} \left[ \int_0^T \left( \int_M \left\langle \left( \left( \frac{\partial}{\partial t} + \nabla_u - \nu \square \right) u \right) (t, x), v(t, x) \right\rangle dx \right) dt \right].$$

The process  $g_u(t)$  is a critical point of the energy functional  $S$  if and only if  $f'(0) = 0$ , which by equation (2.30) is equivalent to

$$(2.31) \quad \left( \frac{\partial}{\partial t} + \nabla_u - \nu \square \right) u = -\nabla p$$

for some function  $p$  on  $[0, T] \times M$ . This achieves the proof.  $\square$

### 3. A MARTINGALE CHARACTERIZATION FOR SOLUTIONS OF NAVIER-STOKES EQUATIONS

In this section, to simplify the equations, we assume the pressure to be constant. The pressure will not be present, in any case, in the weak version of the formulae we derive.

We seek to obtain a formula for the drift of the covariant derivative with respect to a parameter of a family of Navier-Stokes solutions, extending the well-known Jacobi equation.

Consider a family of diffusions  $g^\alpha$ ,  $\alpha \in \mathbb{R}$ , satisfying

$$(3.1) \quad g^\alpha(0) = \varphi(\alpha)$$

where  $\varphi : \mathbb{R} \rightarrow M$  is a smooth path on  $M$ , and solution to the Itô SDE

$$(3.2) \quad dg^\alpha(t) = u(t, g^\alpha(t)) dt + \sigma(g^\alpha(t)) dB_t$$

where  $u$  solves

$$(3.3) \quad \partial_t u + \nabla_u u + \nu \square u = 0,$$

$B_t = (B_t^\ell)_{\ell \geq 0}$  is a family of real Brownian motions,  $\sigma = (\sigma_\ell)_{\ell \geq 0}$ , and for all  $\ell \geq 0$ ,  $\sigma_\ell$  is a vector field on  $M$ . We furthermore assume that

$$(3.4) \quad \sigma \sigma^* = \nu g^{-1}$$

where  $g$  is the Riemannian metric on  $M$ .

We denote by  $u_t^\alpha = Dg^\alpha(t) = u(t, g^\alpha(t))$  the drift of  $g^\alpha$ . We denote by  $\mathcal{D}^c J_t$  the vertical part of the Itô differential (with respect to  $\nabla^c$ ) of a  $TM$ -valued semi-martingale  $J_t$ . It is known that

$$(3.5) \quad \mathcal{D}^c J_t = \mathcal{D} J_t + \frac{1}{2} R(J_t, dX_t) dX_t$$

where  $X_t = \pi(J_t)$  and  $R$  is the curvature tensor. If  $J_t$  has an absolutely continuous drift  $D^c J_t$ , then the finite variation part of  $\mathcal{D}^c J_t$  is  $D^c J_t dt$ .

From the Itô equation

$$(3.6) \quad \mathcal{D}^c u_t^\alpha \simeq \partial_t u(t, g^\alpha(t)) dt + \nabla_{dg^\alpha(t)} u + \nu \square u(t, g^\alpha(t)) dt$$

we deduce that the drift of  $\mathcal{D}^c u_t^\alpha$  is

$$(3.7) \quad D^c u_t^\alpha = \partial_t u(t, g^\alpha(t)) + \nabla_{u_t^\alpha} u + \nu \square u(t, g^\alpha(t)) = 0.$$

From [1] Theorem 4.5, we have formally

$$(3.8) \quad \begin{aligned} & \mathcal{D} \nabla_\alpha u_t^\alpha \\ &= \nabla_\alpha \mathcal{D} u_t^\alpha + R(dg^\alpha(t), \partial_\alpha g^\alpha(t)) u_t^\alpha \\ & \quad + R(dg^\alpha(t), \partial_\alpha g^\alpha(t)) \mathcal{D} u_t^\alpha - \nu d^* R(\partial_\alpha g^\alpha(t)) u_t^\alpha + \frac{1}{2} R(dg^\alpha(t), \mathcal{D} \partial_\alpha g^\alpha(t)) u_t^\alpha. \end{aligned}$$

Using (3.5), we obtain

$$(3.9) \quad \begin{aligned} & \mathcal{D}^c \nabla_\alpha u_t^\alpha \\ &= \nabla_\alpha \mathcal{D}^c u_t^\alpha + R(dg^\alpha(t), \partial_\alpha g^\alpha(t)) u_t^\alpha - \nu \nabla_{\partial_\alpha g^\alpha(t)} \text{Ric}^\sharp(u_t^\alpha) dt \\ & \quad + R(dg^\alpha(t), \partial_\alpha g^\alpha(t)) \mathcal{D}^c u_t^\alpha - \nu d^* R(\partial_\alpha g^\alpha(t)) u_t^\alpha dt + \frac{1}{2} R(dg^\alpha(t), \mathcal{D} \partial_\alpha g^\alpha(t)) u_t^\alpha. \end{aligned}$$

Removing the martingale part we obtain the drift

$$(3.10) \quad \begin{aligned} D^c \nabla_\alpha u_t^\alpha &= \nabla_\alpha D^c u_t^\alpha + R(u_t^\alpha, \partial_\alpha g^\alpha(t)) u_t^\alpha - \nu \nabla_{\partial_\alpha g^\alpha(t)} \text{Ric}^\sharp(u_t^\alpha) \\ & \quad + 2\nu \text{tr} R(\cdot, \partial_\alpha g^\alpha(t)) \nabla \cdot u_t^\alpha - \nu d^* R(\partial_\alpha g^\alpha(t)) u_t^\alpha + \nu \text{tr} R(\sigma(\cdot), \nabla_{\partial_\alpha g^\alpha(t)} \sigma(\cdot)) u_t^\alpha. \end{aligned}$$

Now since  $Du_t^\alpha = 0$  we finally get



**Proposition 3.1.** *The drift of the covariant derivative with respect to  $\alpha$  of the family  $(u_t^\alpha)_{\alpha \in \mathbb{R}}$  of Navier-Stokes solutions is given by*

$$(3.11) \quad \begin{aligned} D^c \nabla_\alpha u_t^\alpha &= R(u_t^\alpha, \partial_\alpha g^\alpha(t)) u_t^\alpha - \nu \nabla_{\partial_\alpha g^\alpha(t)} \text{Ric}^\sharp(u_t^\alpha) \\ &\quad + 2\nu \text{tr} R(\cdot, \partial_\alpha g^\alpha(t)) \nabla \cdot u_t^\alpha - \nu d^* R(\partial_\alpha g^\alpha(t)) u_t^\alpha + \nu \text{tr} R(\sigma(\cdot), \nabla_{\partial_\alpha g^\alpha(t)} \sigma(\cdot)) u_t^\alpha. \end{aligned}$$

This formula extends the well known corresponding (Jacobi) equation for the variation of geodesics.

#### 4. THE TWO-DIMENSIONAL TORUS ENDOWED WITH THE EUCLIDEAN DISTANCE

We study the evolution in time of the  $L^2$  distance between two particles in the two dimensional torus. Notice that, in order to interpret the diffusion processes as a solution of the variational principle described in section 2, there is no canonical choice for the Brownian motion, as far as it corresponds to the same generator. We make here a particular choice.

On the two-dimensional torus  $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z} \times \mathbb{R}/2\pi\mathbb{Z}$  we consider the following vector fields

$$A_k(\theta) = (k_2, -k_1) \cos k.\theta, \quad B_k(\theta) = (k_2, -k_1) \sin k.\theta$$

and the Brownian motion

$$(4.1) \quad dW(t) = \sum_{k \in \mathbb{Z}} \lambda_k \sqrt{\nu} (A_k dx_k + B_k dy_k)$$

where  $x_k, y_k$  are independent copies of real Brownian motions. We assume that  $\sum_k |k|^2 \lambda_k^2 < \infty$ , a necessary and sufficient condition for the Brownian flow to be defined in  $L^2(\mathbb{T})$ . Furthermore we consider  $\lambda_k = \lambda(|k|)$  to be nonzero for a equal number of  $k_1$  and  $k_2$  components. In this case the generator of the process is equal to

$$L_u = C\nu\Delta + \frac{\partial}{\partial t} + \partial_u$$

with  $2C = \sum_k \lambda_k^2$  (c.f.[3] Theorem 2.2). We shall assume  $C$  to be equal to one. Let us take two Lagrangian stochastic trajectories starting from different diffeomorphisms and let us write

$$(4.2) \quad dg_t = (odW(t)) + u(t, g_t)dt, \quad d\tilde{g}_t = (odW(t)) + u(t, \tilde{g}_t)dt$$

with

$$g_0 = \phi, \quad \tilde{g}_0 = \psi, \quad \phi \neq \psi$$

We consider the  $L^2$  distance of the particles defined by

$$\rho^2(\phi, \psi) = \int_{\mathbb{T}} |\phi(\theta) - \psi(\theta)|^2 d\theta.$$

where  $d\theta$  stands for the normalized Lebesgue measure on the torus.

We let  $\rho_t = \rho(g_t, \tilde{g}_t)$  and  $\tau(g, \tilde{g}) = \inf\{t > 0 : \rho_t = 0\}$ .

**Lemma 4.1.** *The stopping time  $\tau(g, \tilde{g})$  is infinite.*

*Proof.* By uniqueness of the solution of the sde for  $\tilde{g}_t$  we can let for all  $t > 0$   $\tilde{g}_t(\theta) = g_t((\phi^{-1} \circ \psi)(\theta))$ . Since  $g_t$ ,  $\varphi$  and  $\psi$  are diffeomorphisms, if  $\varphi(\theta) \neq \psi(\theta)$  then  $g_t(\theta) \neq g_t((\phi^{-1} \circ \psi)(\theta))$ .

Since  $\phi \neq \psi$ , the set  $\{\theta \in \mathbb{T}, \tilde{g}_t(\theta) \neq g_t(\theta)\}$  has positive measure and this implies that  $\rho_t > 0$ , which in turn implies that  $\tau(g, \tilde{g})$  is infinite.  $\square$

Denote by  $L_t(\theta)$  the local time of the process  $|g_t(\theta) - \tilde{g}_t(\theta)|$  when  $(g_t(\theta), \tilde{g}_t(\theta))$  reaches the cutlocus of  $\mathbb{T}$ . By Itô calculus we have

$$\begin{aligned} d\rho_t &= \frac{1}{\rho_t} \sum_k \lambda_k \sqrt{\nu} \langle g_t - \tilde{g}_t, (A_k(g_t) - A_k(\tilde{g}_t)) dx_k(t) + (B_k(g_t) - B_k(\tilde{g}_t)) dy_k(t) \rangle_{\mathbb{T}} \\ &\quad + \frac{1}{\rho_t} \langle g_t - \tilde{g}_t, u(t, g_t) - u(t, \tilde{g}_t) \rangle_{\mathbb{T}} dt - \frac{1}{\rho_t} \int_{\mathbb{T}} |g_t - \tilde{g}_t|(\theta) dL_t(\theta) \\ &\quad + \frac{1}{2\rho_t} \sum_k \lambda_k^2 \nu \left( \|A_k(g_t) - A_k(\tilde{g}_t)\|_{\mathbb{T}}^2 + \|B_k(g_t) - B_k(\tilde{g}_t)\|_{\mathbb{T}}^2 \right) dt \\ &\quad - \frac{1}{2\rho_t^3} \sum_k \lambda_k^2 \nu \left( \langle g_t - \tilde{g}_t, A_k(g_t) - A_k(\tilde{g}_t) \rangle_{\mathbb{T}}^2 + \langle g_t - \tilde{g}_t, B_k(g_t) - B_k(\tilde{g}_t) \rangle_{\mathbb{T}}^2 \right) dt \end{aligned}$$

where  $\langle \cdot, \cdot \rangle_{\mathbb{T}}$  and  $\|\cdot\|_{\mathbb{T}}$  denote, resp., the  $L^2$  inner product and norm. We let

$$(4.3) \quad \delta u(t) = \frac{1}{\rho_t} (u(t, g_t) - u(t, \tilde{g}_t)).$$

We have

$$(4.4) \quad A_k(g_t) - A_k(\tilde{g}_t) = -2 \sin \frac{k \cdot (g_t + \tilde{g}_t)}{2} \sin \left( \frac{k \cdot (g_t - \tilde{g}_t)}{2} \right) k^\perp,$$

$$(4.5) \quad B_k(g_t) - B_k(\tilde{g}_t) = 2 \cos \frac{k \cdot (g_t + \tilde{g}_t)}{2} \sin \left( \frac{k \cdot (g_t - \tilde{g}_t)}{2} \right) k^\perp,$$

where we have noted  $k^\perp = (k_2, -k_1)$ . Then, for  $k \neq 0$  we let

$$(4.6) \quad n_k = \frac{k}{|k|}, \quad \text{and} \quad n_g(t) = \frac{1}{\rho_t} (g_t - \tilde{g}_t).$$

This yields

$$(4.7) \quad A_k(g_t) - A_k(\tilde{g}_t) = -2|k|^2 \rho_t \sin \frac{k \cdot (g_t + \tilde{g}_t)}{2} \frac{\sin \left( \frac{k \cdot (g_t - \tilde{g}_t)}{2} \right)}{|k| \rho_t} n_{k^\perp},$$

$$(4.8) \quad B_k(g_t) - B_k(\tilde{g}_t) = 2|k|^2 \rho_t \left( \cos \frac{k \cdot (g_t + \tilde{g}_t)}{2} \frac{\sin \left( \frac{k \cdot (g_t - \tilde{g}_t)}{2} \right)}{|k| \rho_t} \right) n_{k^\perp}.$$

With these notations we get

$$\begin{aligned}
d\rho_t &= \rho_t \sqrt{\nu} \sum_k \lambda_k |k|^2 \int_{\mathbb{T}} 2 (n_{k^\perp} \cdot n_g(t, \theta)) \frac{\sin}{|k|\rho_t} \left( \frac{k \cdot (g_t(\theta) - \tilde{g}_t(\theta))}{2} \right) \\
&\times \left( -\sin \frac{k \cdot (g_t(\theta) + \tilde{g}_t(\theta))}{2} dx_k(t) + \cos \frac{k \cdot (g_t(\theta) + \tilde{g}_t(\theta))}{2} dy_k(t) \right) d\theta \\
&+ \rho_t \langle n_g(t), \delta u(t) \rangle_{\mathbb{T}} dt - \rho_t \int_{\mathbb{T}} |n_g(t, \theta)| \frac{1}{\rho_t} dL_t(\theta) \\
&+ 2\nu\rho_t \sum_k \lambda_k^2 |k|^4 \left\| \frac{\sin}{|k|\rho_t} \left( \frac{k \cdot (g_t - \tilde{g}_t)}{2} \right) \right\|_{\mathbb{T}}^2 dt \\
&- 2\nu\rho_t \sum_k \lambda_k^2 |k|^4 \\
&\times \left( \int_{\mathbb{T}} (n_{k^\perp} \cdot n_g(t, \theta)) \sin \left( \frac{k \cdot (g_t(\theta) + \tilde{g}_t(\theta))}{2} \right) \frac{\sin}{|k|\rho_t} \left( \frac{k \cdot (g_t(\theta) - \tilde{g}_t(\theta))}{2} \right) d\theta \right)^2 dt \\
&- 2\nu\rho_t \sum_k \lambda_k^2 |k|^4 \\
&\times \left( \int_{\mathbb{T}} (n_{k^\perp} \cdot n_g(t, \theta)) \cos \left( \frac{k \cdot (g_t(\theta) + \tilde{g}_t(\theta))}{2} \right) \frac{\sin}{|k|\rho_t} \left( \frac{k \cdot (g_t(\theta) - \tilde{g}_t(\theta))}{2} \right) d\theta \right)^2 dt.
\end{aligned}$$

And finally:

**Proposition 4.2.** *The Itô equation for the distance  $\rho_t$  between the diffeomorphisms  $g_t$  and  $\tilde{g}_t$  is given by*

$$(4.9) \quad d\rho_t = \rho_t (\sigma_t dz_t + b_t dt + \langle n_g(t), \delta u(t) \rangle_{\mathbb{T}} dt - da_t)$$

where  $z_t$  is a real valued Brownian motion,  $\sigma_t > 0$  is given by

$$\begin{aligned}
(4.10) \quad \sigma_t^2 &= 4\nu \sum_k \lambda_k^2 |k|^4 \\
&\times \left( \int_{\mathbb{T}} (n_{k^\perp} \cdot n_g(t, \theta)) \sin \left( \frac{k \cdot (g_t(\theta) + \tilde{g}_t(\theta))}{2} \right) \frac{\sin}{|k|\rho_t} \left( \frac{k \cdot (g_t(\theta) - \tilde{g}_t(\theta))}{2} \right) d\theta \right)^2 \\
&+ 4\nu \sum_k \lambda_k^2 |k|^4 \\
&\times \left( \int_{\mathbb{T}} (n_{k^\perp} \cdot n_g(t, \theta)) \cos \left( \frac{k \cdot (g_t(\theta) + \tilde{g}_t(\theta))}{2} \right) \frac{\sin}{|k|\rho_t} \left( \frac{k \cdot (g_t(\theta) - \tilde{g}_t(\theta))}{2} \right) d\theta \right)^2,
\end{aligned}$$

the process  $b_t$  satisfies

$$(4.11) \quad b_t + \frac{1}{2}\sigma_t^2 = 2\nu\rho_t \sum_k \lambda_k^2 |k|^4 \left\| \frac{\sin}{|k|\rho_t} \left( \frac{k \cdot (g_t - \tilde{g}_t)}{2} \right) \right\|_{\mathbb{T}}^2 dt$$

and  $a_t$  is defined by

$$(4.12) \quad a_0 = 0, \quad da_t = \int_{\mathbb{T}} |n_g(t, \theta)| \frac{1}{\rho_t} dL_t(\theta).$$

So we have for all  $0 < t_0 < t$ ,

$$(4.13) \quad \rho_t = \rho_{t_0} \exp \left( \int_{t_0}^t \sigma_s dz_s + \int_{t_0}^t \left( b_s - \frac{1}{2} \sigma_s^2 + \langle n_g(s), \delta u(s) \rangle_{\mathbb{T}} \right) ds - (a_t - a_{t_0}) \right).$$

Let

$$(4.14) \quad \delta_k = \delta_k(t, \theta) = \frac{\rho_t (n_g \cdot n_k)}{|g_t(\theta) - \tilde{g}_t(\theta)|}.$$

Notice that

$$\delta_k^2 + \delta_{k^\perp}^2 = 1.$$

**Lemma 4.3.** *We have*

$$(4.15) \quad \sigma_t^2 \leq 4\nu \sum_k \lambda_k^2 |k|^4 \int_{\mathbb{T}} \delta_{k^\perp}^2 \frac{\sin^2}{|k|^2 \rho_t^2} \left( \frac{k \cdot (g_t(\theta) - \tilde{g}_t(\theta))}{2} \right) d\theta$$

and

$$(4.16) \quad b_t \geq 2\nu \sum_k \lambda_k^2 |k|^4 \int_{\mathbb{T}} (n_g \cdot n_k)^2 d\theta \int_{\mathbb{T}} \frac{\sin^2}{|k|^2 \rho_t^2} \left( \frac{k \cdot (g_t(\theta) - \tilde{g}_t(\theta))}{2} \right) d\theta,$$

in particular  $b_t \geq 0$ .

Let  $R > 0$ . Assuming that  $\lambda_k = 0$  for all  $k$  such that  $|k| > R$  then on

$$\left\{ \omega \mid \forall \theta \in \mathbb{T}, |g_t(\theta) - \tilde{g}_t(\theta)| \leq \frac{\pi}{R} \right\}$$

we have

$$b_t - \frac{1}{2} \sigma_t^2 \geq 0.$$

*Proof.* Using Cauchy Schwartz inequality,

$$\begin{aligned} \sigma_t^2 &\leq 4 \sum_k \lambda_k^2 |k|^4 \nu \int_{\mathbb{T}} |(n_g \cdot n_{k^\perp})| \sin^2 \left( \frac{k \cdot (g_t(\theta) + \tilde{g}_t(\theta))}{2} \right) \frac{|\sin|}{|k| \rho_t} \left( \frac{k \cdot (g_t(\theta) - \tilde{g}_t(\theta))}{2} \right) d\theta \\ &\quad \times \int_{\mathbb{T}} |(n_g \cdot n_{k^\perp})| \frac{|\sin|}{|k| \rho_t} \left( \frac{k \cdot (g_t(\theta) - \tilde{g}_t(\theta))}{2} \right) d\theta \\ &+ 4 \sum_k \lambda_k^2 |k|^4 \nu \int_{\mathbb{T}} |(n_g \cdot n_{k^\perp})| \cos^2 \left( \frac{k \cdot (g_t(\theta) + \tilde{g}_t(\theta))}{2} \right) \frac{|\sin|}{|k| \rho_t} \left( \frac{k \cdot (g_t(\theta) - \tilde{g}_t(\theta))}{2} \right) d\theta \\ &\quad \times \int_{\mathbb{T}} |(n_g \cdot n_{k^\perp})| \frac{|\sin|}{|k| \rho_t} \left( \frac{k \cdot (g_t(\theta) - \tilde{g}_t(\theta))}{2} \right) d\theta \\ &= 4\nu \sum_k \lambda_k^2 |k|^4 \left( \int_{\mathbb{T}} |(n_g \cdot n_{k^\perp})| \frac{|\sin|}{|k| \rho_t} \left( \frac{k \cdot (g_t(\theta) - \tilde{g}_t(\theta))}{2} \right) d\theta \right)^2 \\ &= 4\nu \sum_k \lambda_k^2 |k|^4 \left( \int_{\mathbb{T}} \frac{\delta_{k^\perp} |g_t(\theta) - \tilde{g}_t(\theta)|}{\rho_t} \frac{|\sin|}{|k| \rho_t} \left( \frac{k \cdot (g_t(\theta) - \tilde{g}_t(\theta))}{2} \right) d\theta \right)^2 \\ &\leq 4\nu \sum_k \lambda_k^2 |k|^4 \int_{\mathbb{T}} \frac{|g_t(\theta) - \tilde{g}_t(\theta)|^2}{\rho_t^2} d\theta \int_{\mathbb{T}} \delta_{k^\perp}^2 \frac{\sin^2}{|k|^2 \rho_t^2} \left( \frac{k \cdot (g_t(\theta) - \tilde{g}_t(\theta))}{2} \right) d\theta \\ &= 4\nu \sum_k \lambda_k^2 |k|^4 \int_{\mathbb{T}} \delta_{k^\perp}^2 \frac{\sin^2}{|k|^2 \rho_t^2} \left( \frac{k \cdot (g_t(\theta) - \tilde{g}_t(\theta))}{2} \right) d\theta. \end{aligned}$$

On the other hand,

$$b_t + \frac{1}{2}\sigma_t^2 = 2\nu \sum_k \lambda_k^2 |k|^4 \int_{\mathbb{T}} \frac{\sin^2}{|k|^2 \rho_t^2} \left( \frac{k \cdot (g_t(\theta) - \tilde{g}_t(\theta))}{2} \right) d\theta,$$

so that using the bound

$$\begin{aligned} \sigma_t^2 &\leq 4\nu \sum_k \lambda_k^2 |k|^4 \left( \int_{\mathbb{T}} |(n_g \cdot n_{k^\perp})| \frac{|\sin|}{|k| \rho_t} \left( \frac{k \cdot (g_t(\theta) - \tilde{g}_t(\theta))}{2} \right) d\theta \right)^2 \\ &\leq 4\nu \sum_k \lambda_k^2 |k|^4 \int_{\mathbb{T}} (n_g \cdot n_{k^\perp})^2 d\theta \int_{\mathbb{T}} \frac{\sin^2}{|k|^2 \rho_t^2} \left( \frac{k \cdot (g_t(\theta) - \tilde{g}_t(\theta))}{2} \right) d\theta \end{aligned}$$

for  $\sigma_t^2$  yields

$$b_t \geq 2\nu \sum_k \lambda_k^2 |k|^4 \int_{\mathbb{T}} (n_g \cdot n_k)^2 d\theta \int_{\mathbb{T}} \frac{\sin^2}{|k|^2 \rho_t^2} \left( \frac{k \cdot (g_t(\theta) - \tilde{g}_t(\theta))}{2} \right) d\theta$$

where we used

$$\int_{\mathbb{T}} (n_g \cdot n_k)^2 d\theta + \int_{\mathbb{T}} (n_g \cdot n_{k^\perp})^2 d\theta = 1.$$

Since  $\lambda_k$  depends only on  $|k|$ , we have  $\lambda_k = \lambda_{k^\perp}$  for all  $k$ . Then putting together the terms corresponding to  $k$  and  $k^\perp$  we obtain

$$\begin{aligned} b_t + \frac{1}{2}\sigma_t^2 &= \nu \sum_k \lambda_k^2 |k|^4 \\ &\times \int_{\mathbb{T}} \left( \frac{\sin^2}{|k|^2 \rho_t^2} \left( \frac{k \cdot (g_t(\theta) - \tilde{g}_t(\theta))}{2} \right) + \frac{\sin^2}{|k^\perp|^2 \rho_t^2} \left( \frac{k^\perp \cdot (g_t(\theta) - \tilde{g}_t(\theta))}{2} \right) \right) d\theta, \end{aligned}$$

and this yields using the bound for  $\sigma_t^2$  as well as  $\delta_k^2 + \delta_{k^\perp}^2 = 1$

$$\begin{aligned} b_t - \frac{1}{2}\sigma_t^2 &\geq \nu \sum_k \lambda_k^2 |k|^4 \\ &\times \int_{\mathbb{T}} (\delta_k^2 - \delta_{k^\perp}^2) \left( \frac{\sin^2}{|k|^2 \rho_t^2} \left( \frac{k \cdot (g_t(\theta) - \tilde{g}_t(\theta))}{2} \right) - \frac{\sin^2}{|k^\perp|^2 \rho_t^2} \left( \frac{k^\perp \cdot (g_t(\theta) - \tilde{g}_t(\theta))}{2} \right) \right) d\theta \\ &= \nu \sum_k \lambda_k^2 |k|^4 \\ &\times \int_{\mathbb{T}} (\delta_k^2 - \delta_{k^\perp}^2) \left( \frac{\sin^2}{|k|^2 \rho_t^2} \left( \delta_k \frac{|k| |g_t - \tilde{g}_t|(\theta)}{2} \right) - \frac{\sin^2}{|k^\perp|^2 \rho_t^2} \left( \delta_{k^\perp} \frac{|k| |g_t - \tilde{g}_t|(\theta)}{2} \right) \right) d\theta \end{aligned}$$

Assuming that  $\lambda_k = 0$  whenever  $|k| > R$  then on

$$\left\{ \omega \mid \forall \theta \in \mathbb{T}, |g_t(\theta) - \tilde{g}_t(\theta)| \leq \frac{\pi}{R} \right\}$$

the functions inside the integral are nonnegative, consequently

$$b_t - \frac{1}{2}\sigma_t^2 \geq 0.$$

□

Define

$$(4.17) \quad \ell(x) = \frac{\sin x}{x} \text{ for } x \neq 0, \quad \ell(0) = 1.$$

From Lemma 4.3 we easily get the following result.

**Proposition 4.4.** *Let  $R \geq 1$ . Then on*

$$\left\{ \omega | \forall \theta \in \mathbb{T}, |g_t(\theta) - \tilde{g}_t(\theta)| \leq \frac{\pi\sqrt{2}}{R} \right\},$$

letting

$$c_R = \frac{\nu}{8} \ell^2 \left( \frac{\pi}{\sqrt{2}} \right) \sum_{|k| \leq R} \lambda_k^2 |k|^4,$$

we have,

$$(4.18) \quad d\rho_t \geq \rho_t \left( \sigma_t dz_t - \|\delta u(t)\|_{\mathbb{T}} dt - \int_{\mathbb{T}} |n_g(t, \theta)| \frac{1}{\rho_t} dL_t(\theta) + c_R dt \right).$$

Moreover assuming that  $\lambda_k = 0$  whenever  $|k| > R$ , then letting

$$c'_R = \frac{1}{8} \nu \inf_{|v|=1} \sum_{|k| \leq R} \lambda_k^2 |k|^4 \left( (n_k \cdot v)^2 - (n_{k^\perp} \cdot v)^2 \right)^2,$$

on

$$\left\{ \omega | \forall \theta \in \mathbb{T}, |g_t(\theta) - \tilde{g}_t(\theta)| \leq \frac{\pi}{2R} \right\},$$

$$(4.19) \quad d\rho_t \geq \rho_t \left( \sigma_t dz_t + \frac{1}{2} \sigma_t^2 dt - \|\delta u(t)\|_{\mathbb{T}} dt + c'_R dt \right).$$

*Proof.* If  $|g_t(\theta) - \tilde{g}_t(\theta)| \leq \frac{\pi\sqrt{2}}{R}$  then for all  $k$  such that  $|k| \leq R$ ,

$$\ell^2 \left( \frac{k \cdot (g_t(\theta) - \tilde{g}_t(\theta))}{2} \right) \geq \ell^2 \left( \frac{\pi}{\sqrt{2}} \right)$$

and this implies

$$\frac{\sin^2}{|k|^2 \rho_t^2} \left( \frac{k \cdot (g_t(\theta) - \tilde{g}_t(\theta))}{2} \right) \geq \frac{1}{4} \ell^2 \left( \frac{\pi}{\sqrt{2}} \right) (n_k \cdot n_g)^2.$$

So with (4.16) we get

$$\begin{aligned} b_t &\geq \frac{1}{2} \ell^2 \left( \frac{\pi}{\sqrt{2}} \right) \nu \sum_{|k| \leq R} \lambda_k^2 |k|^4 \left( \int_{\mathbb{T}} (n_g \cdot n_k)^2 d\theta \right)^2 \\ &\geq \frac{1}{4} \ell^2 \left( \frac{\pi}{\sqrt{2}} \right) \nu \sum_{|k| \leq R} \lambda_k^2 |k|^4 \left( \left( \int_{\mathbb{T}} (n_g \cdot n_k)^2 d\theta \right)^2 + \left( \int_{\mathbb{T}} (n_g \cdot n_{k^\perp})^2 d\theta \right)^2 \right) \\ &\geq \frac{1}{8} \ell^2 \left( \frac{\pi}{\sqrt{2}} \right) \nu \sum_{|k| \leq R} \lambda_k^2 |k|^4 \end{aligned}$$

(again we used  $\int_{\mathbb{T}} (n_g \cdot n_k)^2 d\theta + \int_{\mathbb{T}} (n_g \cdot n_{k^\perp})^2 d\theta = 1$ ). This establishes (4.18).

Next if  $|g_t(\theta) - \tilde{g}_t(\theta)| \leq \frac{\pi}{2R}$  then from the calculation in the proof of Lemma 4.3

$$\begin{aligned}
b_t - \frac{1}{2}\sigma_t^2 &\geq \nu \sum_{|k| \leq R} \lambda_k^2 |k|^4 \\
&\times \int_{\mathbb{T}} (\delta_k^2 - \delta_{k^\perp}^2) \left( \frac{\sin^2}{|k|^2 \rho_t^2} \left( \delta_k \frac{|k| |g_t - \tilde{g}_t|(\theta)}{2} \right) - \frac{\sin^2}{|k|^2 \rho_t^2} \left( \delta_{k^\perp} \frac{|k| |g_t - \tilde{g}_t|(\theta)}{2} \right) \right) d\theta \\
&\geq \nu \sum_{|k| \leq R} \lambda_k^2 |k|^4 \int_{\mathbb{T}} (\delta_k^2 - \delta_{k^\perp}^2)^2 \frac{|g_t - \tilde{g}_t|^2(\theta)}{8\rho_t^2} d\theta \\
&\geq \int_{\mathbb{T}} \frac{|g_t - \tilde{g}_t|^2(\theta)}{\rho_t^2} c'_R d\theta = c'_R.
\end{aligned}$$

this establishes (4.19).  $\square$

**Theorem 4.5.** *Let  $t > 0$ ,  $R \geq 1$  and*

$$\Omega_t = \left\{ \omega \in \Omega, \forall s \leq t, \forall \theta \in \mathbb{T}, |(g_s(\theta)(\omega) - \tilde{g}_s(\theta)(\omega))| \leq \frac{\pi}{2R} \right\}.$$

*If we assume the initial conditions for the  $L^2$  distance and the  $L^2$  norm of the initial velocity related as  $c = \rho_0 - 2\|u_0\|_{\mathbb{T}} > 0$ , and suppose that  $\int_{\mathbb{T}} u = 0$ , then on  $\Omega_t$ ,*

$$(4.20) \quad \forall s \leq t, \quad \rho_s \geq e^{\int_0^t \sigma_s dz_s + c'_R t} \left( \rho_0 - 2\|u_0\|_{\mathbb{T}} \int_0^t e^{-\int_0^s \sigma_r dz_r - (c'_R + \frac{c}{2})s} ds \right)$$

*as long as the right hand side stays positive.*

*On the other hand if we assume that there exist constants  $c_1, c_2 > 0$  such that for all  $\theta \in \mathbb{T}$  and  $s \in [0, t]$ ,*

$$(4.21) \quad |\nabla u(t, \theta)| \leq c_1 e^{-c_2 t},$$

*then on  $\Omega_t$ ,  $\forall s \leq t$ ,*

$$(4.22) \quad \rho_s \geq \rho_0 \exp \left( \int_0^t \sigma_s dz_s + c'_R t - \frac{c_1}{c_2} (1 - e^{-c_2 t}) \right).$$

*Proof.* Assume that  $\rho_0 - 2\|u_0\|_{\mathbb{T}} > 0$ . From inequality (4.19) we have on  $\Omega_t$ , for  $s \leq t$ ,

$$(4.23) \quad d\rho_s \geq \rho_s \left( \sigma_s dz_s + (c'_R + \frac{1}{2}\sigma_s^2) ds \right) - 2\|u(s, \cdot)\|_{\mathbb{T}} ds.$$

Using the fact that  $u(t, \cdot)$  satisfies Navier-Stokes equation together with Poincaré inequality,

$$\begin{aligned}
\frac{d}{ds} \|u(s, \cdot)\|_{\mathbb{T}}^2 &= -2\nu \|\nabla u(s, \cdot)\|_{\mathbb{T}}^2 \\
&\leq -\nu \|u(s, \cdot)\|_{\mathbb{T}}^2.
\end{aligned}$$

Therefore we have

$$\|u(s, \cdot)\|_{\mathbb{T}} \leq e^{-\frac{\nu}{2}s} \|u_0\|_{\mathbb{T}}.$$

We obtain

$$(4.24) \quad d\rho_s \geq \rho_s \left( \sigma_s dz_s + (c'_R + \frac{1}{2}\sigma_s^2) ds \right) - 2e^{-\frac{\nu}{2}s} \|u_0\|_{\mathbb{T}} ds.$$

From this comparison theorem for solution of sde's yields (4.20).

Now assume (4.21). To prove (4.22) we start with (4.19), and remark that  $\|\delta u(t)\|_{\mathbb{T}} \leq \sup_{\theta \in \mathbb{T}} |\nabla u(t, \theta)|$ . Then with the bound on  $\nabla u(t, \theta)$  we have

$$d\rho_t \geq \rho_t \left( \sigma_t dz_t + \frac{1}{2} \sigma_t^2 dt - c_1 e^{-c_2 t} dt + c'_R dt \right).$$

Integrating the right hand side between  $t_0$  and  $t$  gives the result.  $\square$

**Remark 4.6.** The bound (4.21) is satisfied for instance for solutions  $u(t, \cdot)$  of the form  $e^{-\nu|k|^2 t} A_k$ .

Also notice that, by the expression of the constant  $c'_R$ , the stochastic Lagrangian trajectories for a fluid with a given viscosity constant tend to get apart faster when the higher Fourier modes (and therefore the smaller length scales) are randomly excited.

## 5. THE TWO-DIMENSIONAL TORUS ENDOWED WITH THE EXTRINSIC DISTANCE

It seems difficult to deal with the local time term of Proposition 4.2. To circumvent this problem we propose to endow the torus  $\mathbb{T}$  with a distance  $\rho_{\mathbb{T}}$  equivalent to the one of section 4, but such that  $\rho_{\mathbb{T}}^2$  is smooth on  $\mathbb{T} \times \mathbb{T}$ . Then we will see that when the assumptions of Theorem 4.5 are not fulfilled, then the behaviour of the distance of two diffeomorphisms can be completely different even if their distance is small. So the uniform control of the distance in Theorem 4.5 looks as a necessary condition for an exponential growth of the distance.

The map

$$\begin{aligned} \mathbb{R}/2\pi\mathbb{Z} \times \mathbb{R}/2\pi\mathbb{Z} &\rightarrow [0, 2] \\ (\theta_1, \theta_2) &\mapsto 2 \left| \sin \left( \frac{\theta_2 - \theta_1}{2} \right) \right| \end{aligned}$$

defines a distance on the circle  $\mathbb{R}/2\pi\mathbb{Z}$ : it is the extrinsic distance on the circle embedded in the plane. From this distance we can define the product distance on the torus  $\mathbb{T}$ .

$$\rho_{\mathbb{T}}((\theta_1, \theta_2), (\theta'_1, \theta'_2)) = 2 \left( \sin^2 \left( \frac{\theta'_1 - \theta_1}{2} \right) + \sin^2 \left( \frac{\theta'_2 - \theta_2}{2} \right) \right)^{1/2}.$$

Note

$$\rho_{\mathbb{T}}^2((\theta_1, \theta_2), (\theta'_1, \theta'_2)) = 2(2 - \cos(\theta'_1 - \theta_1) - \cos(\theta'_2 - \theta_2)).$$

The distance  $\rho_{\mathbb{T}}^2$  is smooth on  $\mathbb{T} \times \mathbb{T}$ . Now let  $\phi$  and  $\psi$  be two diffeomorphisms on the torus  $\mathbb{T}$ . We define the distance  $\rho(\phi, \psi)$  with the formula

$$\begin{aligned} \rho^2(\phi, \psi) &= \int_{\mathbb{T}} \rho_{\mathbb{T}}^2(\phi(\theta), \psi(\theta)) d\theta \\ &= 2 \int_{\mathbb{T}} (2 - \cos(\phi^1(\theta) - \psi^1(\theta)) - \cos(\phi^2(\theta) - \psi^2(\theta))) d\theta \\ &= 4 \int_{\mathbb{T}} \left( \sin^2 \left( \frac{\phi^1(\theta) - \psi^1(\theta)}{2} \right) + \sin^2 \left( \frac{\phi^2(\theta) - \psi^2(\theta)}{2} \right) \right) d\theta \end{aligned}$$

Now let

$$\rho_t = \rho(g_t, \tilde{g}_t).$$



From the smoothness of  $\rho_{\mathbb{T}}^2$ , the formula for  $\rho_t$  does not involve a local time. More precisely, letting

$$\begin{aligned}\delta g &= g_t(\theta) - \tilde{g}_t(\theta), \\ \delta \cos k \cdot g &= \cos k \cdot g_t(\theta) - \cos k \cdot \tilde{g}_t(\theta), \\ \delta \sin k \cdot g &= \sin k \cdot g_t(\theta) - \sin k \cdot \tilde{g}_t(\theta), \\ \sin \delta g &= (\sin(\delta g_t)_1(\theta), \sin(\delta g_t)_2(\theta)), \\ \delta u &= (u(t, g_t) - u(t, \tilde{g}_t))\end{aligned}$$

we get from Itô calculus

$$\begin{aligned}d\rho_t &= \rho_t \sum_k \lambda_k \left\langle \frac{\sin \delta g}{\rho_t}, (k_2, -k_1) \left( \frac{\delta \cos k \cdot g}{\rho_t} dx_k + \frac{\delta \sin k \cdot g}{\rho_t} dy_k \right) \right\rangle_{\mathbb{T}} \\ &+ \rho_t \left\langle \frac{\sin \delta g}{\rho_t}, \frac{\delta u}{\rho_t} \right\rangle_{\mathbb{T}} dt \\ &+ \frac{\rho_t}{2} \left( \sum_k \lambda_k^2 \int_{\mathbb{T}} (k_2^2 \cos \delta g_1 + k_1^2 \cos \delta g_2) \frac{(\delta \cos k \cdot g)^2 + (\delta \sin k \cdot g)^2}{\rho_t^2} d\theta \right) dt \\ &- \frac{\rho_t}{2} \sum_k \lambda_k^2 \left( \int_{\mathbb{T}} \left( k_2 \frac{\sin \delta g_1}{\rho_t} - k_1 \frac{\sin \delta g_2}{\rho_t} \right) \frac{\delta \cos k \cdot g}{\rho_t} d\theta \right)^2 dt \\ &- \frac{\rho_t}{2} \sum_k \lambda_k^2 \left( \int_{\mathbb{T}} \left( k_2 \frac{\sin \delta g_1}{\rho_t} - k_1 \frac{\sin \delta g_2}{\rho_t} \right) \frac{\delta \sin k \cdot g}{\rho_t} d\theta \right)^2 dt.\end{aligned}$$

This clearly has the form

$$d\rho_t = \rho_t (\sigma_t dz_t + b_t dt)$$

where  $\sigma_t$  and  $b_t$  are bounded processes and  $z_t$  is a real-valued Brownian motion. However it can happen that the drift is negative even if  $\rho_t$  is small, as the following example shows.

**Example 5.1.** Let  $\alpha > 0$  small and  $\varepsilon > 0$  satisfying  $\varepsilon \ll \alpha$ . Take  $\phi = \text{id}$  and assume that there exist two subsets  $E_1$  and  $E_2$  of  $\mathbb{T}$  such that  $E_1 \subset E_2$ ,  $E_1$  has measure  $\alpha$ ,  $E_2$  has measure  $\alpha + \varepsilon$ ,  $\psi(\theta) = \theta$  for all  $\theta \in \mathbb{T} \setminus E_2$  and  $\psi(\theta) = (\theta_1 + \pi, \theta_2)$  for all  $\theta \in E_1$ . Since  $\varepsilon$  can be as small as we want, we have

$$\rho_0^2 \simeq 4\alpha, \quad (\sin \delta g)_0 \simeq 0, \quad (\delta g_0)_2 \simeq 0, \quad (\delta \sin k \cdot g)_0 \simeq 0,$$

$$\text{on } \mathbb{T} \setminus E_2, \quad (\delta \cos k \cdot g)_0 = 0,$$

$$\text{on } E_1, \quad (\delta \cos k \cdot g)_0 = -2 \text{ if } k_1 \text{ is odd, } (\delta \cos k \cdot g)_0 = 0 \text{ if } k_1 \text{ is even,}$$

so at time  $t = 0$ ,

$$d\rho_t \simeq -\frac{\rho_t}{2} \left( \sum_{k_1 \text{ odd}} \lambda_k^2 k_2^2 \right) dt.$$

To construct a diffeomorphism like  $\psi$ , one can cut an annulus  $E_1$  of width  $\frac{\alpha}{2\pi}$  in  $\mathbb{T}$  and rotate it by  $\pi$ . This yields a one to one map on  $\mathbb{T}$ . Then smoothen it around the boundary of the annulus to get  $\psi$ . The set  $E_2$  can be taken as an annulus of width  $\frac{\alpha + \varepsilon}{2\pi}$  containing  $E_1$ .

6. DISTANCE AND ROTATION PROCESSES OF TWO PARTICLES ON A GENERAL RIEMANNIAN MANIFOLD

6.1. Distance of two particles.

Let  $B_t = (B_t^\ell)_{\ell \geq 0}$  be a family of independent real Brownian motions,  $\sigma = (\sigma_\ell)_{\ell \geq 0}$ , with, for all  $\ell \geq 0$ ,  $\sigma_\ell$  a divergence free vector field on  $M$ . We furthermore assume that

$$(6.1) \quad \sigma(x)\sigma^*(y) = a(x, y).$$

In particular

$$(6.2) \quad \sigma(x)\sigma^*(x) = 2\nu\mathbf{g}^{-1}(x).$$

We let  $\varphi, \psi \in G_V^0$ . In this section we assume that

$$(6.3) \quad dg_t(x) = \sigma(g_t(x)) dB_t + u(t, g_t(x)) dt, \quad g_0 = \varphi$$

and

$$(6.4) \quad d\tilde{g}_t(x) = \sigma(\tilde{g}_t(x)) dB_t + u(t, \tilde{g}_t(x)) dt, \quad \tilde{g}_0 = \psi$$

For simplicity we let  $x_t = g_t(x)$ ,  $y_t = \tilde{g}_t(x)$  and

$$\rho_t(x) = \rho_M(x_t, y_t)$$

For  $x, y \in M$  such that  $y$  does not belong to the cutlocus of  $x$ , we let  $a \mapsto \gamma_a(x, y)$  be the minimal geodesic in time 1 from  $x$  to  $y$  ( $\gamma_0(x, y) = x$ ,  $\gamma_1(x, y) = y$ ). For  $a \in [0, 1]$  we let  $J_a = T\gamma_a$  the tangent map to  $\gamma_a$ . In other words, for  $v \in T_x M$  and  $w \in T_y M$ ,  $J_a(v, w)$  is the value at time  $a$  of the Jacobi field along  $\gamma$ . which takes the values  $v$  at time 0 and  $w$  at time 1.

We first consider the case where  $y_t$  does not belong to the cutlocus of  $x_t$ . We note  $T_a = T_a(t) = \dot{\gamma}_a(x_t, y_t)$  and  $\gamma_a(t) = \gamma_a(x_t, y_t)$ .

Letting  $P(\gamma_a)_t$  be the parallel transport along  $\gamma_a(t)$ , we have for the Itô covariant differential

$$\begin{aligned} \mathcal{D}\dot{\gamma}_a(t) &:= P(\gamma_a)_t d(P(\gamma_a)_t^{-1} \dot{\gamma}_a(t)) \\ &= \nabla_{d\gamma_a(t)} \dot{\gamma}_a + \frac{1}{2} \nabla_{d\gamma_a(t)} \cdot \nabla_{d\gamma_a(t)} \dot{\gamma}_a(t). \end{aligned}$$

On the other hand the Itô differential  $d\gamma_a(t)$  satisfies

$$d\gamma_a(t) = J_a(dx_t, dy_t) + \frac{1}{2} (\nabla_{(dx_t, dy_t)} J_a) (dx_t, dy_t).$$

So we get

$$(6.5) \quad \mathcal{D}\dot{\gamma}_a(t) = \nabla_{J_a(dx_t, dy_t)} \dot{\gamma}_a + \nabla_{\frac{1}{2} (\nabla_{(dx_t, dy_t)} J_a) (dx_t, dy_t)} \dot{\gamma}_a + \frac{1}{2} \nabla_{d\gamma_a(t)} \cdot \nabla_{d\gamma_a(t)} \dot{\gamma}_a(t).$$

Let  $e(t) \in T_{x_t} M$  be the unit vector satisfying  $T_0(t) = \rho_t(x)e(t)$ . For  $\ell \geq 0$  we let  $a \mapsto J_a^\ell(t, x)$  be the Jacobi field such that  $J_0^\ell(t, x) = \sigma_\ell(g_t(x))$ ,  $J_1^\ell(t, x) = \sigma_\ell(\tilde{g}_t(x))$ . Moreover we assume that  $\nabla_{J_0^\ell(t, x)} J_0^\ell(t, x) = 0$  and  $\nabla_{J_1^\ell(t, x)} J_1^\ell(t, x) = 0$ .

With these notations, equation (6.5) rewrites as

$$\begin{aligned} \mathcal{D}T_a &= \nabla_{J_a(dx_t, dy_t)} T_a + \frac{1}{2} \sum_{\ell \geq 0} \nabla_{\nabla_{J_a^\ell} J_a^\ell} T_a dt + \frac{1}{2} \sum_{\ell \geq 0} \nabla_{J_a^\ell} \cdot \nabla_{J_a^\ell} T_a dt \\ &= \dot{J}_a(dx_t, dy_t) + \frac{1}{2} \sum_{\ell \geq 0} \nabla_{J_a^\ell} \nabla_{J_a^\ell} T_a dt. \end{aligned}$$

We have

$$\begin{aligned} d\rho_t(x) &= d \left( \left( \int_0^1 \langle T_a(t), T_a(t) \rangle da \right)^{1/2} \right) \\ &= \frac{1}{2\rho_t(x)} \left( 2 \int_0^1 \langle \mathcal{D}T_a(t), T_a(t) \rangle da + \int_0^1 \langle \mathcal{D}T_a(t), \mathcal{D}T_a(t) \rangle da \right) \\ &\quad - \frac{1}{8\rho_t(x)^3} d(\|T_0\|^2) \cdot d(\|T_0\|^2) \\ &= \sum_{\ell \geq 0} \left\langle \dot{J}_0^\ell(t, x), e_t(x) \right\rangle dB_t^\ell + \left\langle \dot{J}_0(u(t, g_t(x)), u(t, \tilde{g}_t(x))), e_t(x) \right\rangle \\ &\quad + \frac{1}{2\rho_t(x)} \left( \int_0^1 \sum_{\ell \geq 0} \langle \nabla_{J_a^\ell} \nabla_{J_a^\ell} T_a, T_a \rangle da dt + \sum_{\ell \geq 0} \int_0^1 \|j_a^\ell\|^2 da \right) \\ &\quad - \frac{1}{2\rho_t(x)} \sum_{\ell \geq 0} \langle \dot{J}_0^\ell(t, x), e_t(x) \rangle^2. \end{aligned}$$

Note

$$\begin{aligned} \int_0^1 \langle \nabla_{J_a^\ell} \nabla_{J_a^\ell} T_a, T_a \rangle da &= \int_0^1 \langle \nabla_{J_a^\ell} \nabla_{T_a} J_a^\ell, T_a \rangle da \\ &= \int_0^1 \langle \nabla_{T_a} \nabla_{J_a^\ell} J_a^\ell, T_a \rangle da - \int_0^1 \langle R(T_a, J_a^\ell) J_a^\ell, T_a \rangle da \\ &= \int_0^1 T_a \langle \nabla_{J_a^\ell} J_a^\ell, T_a \rangle da - \int_0^1 \langle R(T_a, J_a^\ell) J_a^\ell, T_a \rangle da \\ &= [\langle \nabla_{J_a^\ell} J_a^\ell, T_a \rangle]_0^1 - \int_0^1 \langle R(T_a, J_a^\ell) J_a^\ell, T_a \rangle da \\ &= - \int_0^1 \langle R(T_a, J_a^\ell) J_a^\ell, T_a \rangle da \end{aligned}$$

using the fact that  $\nabla_{J_a^\ell} J_a^\ell = 0$  for  $a = 0, 1$ . So finally,

$$\begin{aligned} d\rho_t(x) &= \sum_{\ell \geq 0} \left\langle \dot{J}_0^\ell(t, x), e_t(x) \right\rangle dB_t^\ell \\ &\quad + \left\langle \dot{J}_0(u(t, g_t(x)), u(t, \tilde{g}_t(x))), e_t(x) \right\rangle \\ &\quad + \frac{1}{2\rho_t(x)} \left( \int_0^1 \sum_{\ell \geq 0} \left( \|J_a^{\ell, N}\|^2 - \langle R(T_a(t, x), J_a^{\ell, N}(t, x)) J_a^{\ell, N}(t, x), T_a(t, x) \rangle \right) da \right) dt \end{aligned}$$

with  $J_a^{\ell, N}(t, x)$  the part of  $J_a^\ell(t, x)$  normal to  $T_a$ .

Removing the assumption that  $y_t$  does not belong to the cutlocus of  $x_t$ , it is well known (see [5] for a similar argument) that the formula becomes

$$\begin{aligned} d\rho_t(x) &= \sum_{\ell \geq 0} \left\langle \dot{J}_0^\ell(t, x), e_t(x) \right\rangle dB_t^\ell \\ &\quad + \left\langle \dot{J}_0(u(t, g_t(x)), u(t, \tilde{g}_t(x))), e_t(x) \right\rangle - dL_t(x) \\ &\quad + \frac{1}{2\rho_t(x)} \left( \int_0^1 \sum_{\ell \geq 0} \left( \|J_a^{\ell, N}\|^2 - \langle R(T_a(t, x), J_a^{\ell, N}(t, x))J_a^{\ell, N}(t, x), T_a(t, x) \rangle \right) da \right) dt \end{aligned}$$

where  $-L_t(x)$  is the local time of  $\rho_t(x)$  when  $(g_t(x), \tilde{g}_t(x))$  visits the cutlocus. Then letting

$$\rho_t = \rho(g_t, \tilde{g}_t) = \left( \int_M \rho_t^2(x) dx \right)^{1/2},$$

we get

$$\begin{aligned} d\rho_t &= \frac{1}{\rho_t} \sum_{\ell \geq 0} \left( \int_M \rho_t(x) \left\langle \dot{J}_0^\ell(t, x), e_t(x) \right\rangle dx \right) dB_t^\ell \\ &\quad + \frac{1}{\rho_t} \int_M \rho_t(x) \left\langle \dot{J}_0(u(g_t(x)), u(\tilde{g}_t(x))), e_t(x) \right\rangle dx dt - \frac{1}{\rho_t} \int_M \rho_t(x) L_t(x) dx \\ &\quad + \frac{1}{2\rho_t} \left( \int_M \sum_{\ell \geq 0} \left( \int_0^1 \left( \|J_a^{\ell, N}\|^2 - \langle R(T_a(t, x), J_a^{\ell, N}(t, x))J_a^{\ell, N}(t, x), T_a(t, x) \rangle \right) da \right) dx \right) dt \\ &\quad + \frac{1}{2\rho_t} \int_M \sum_{\ell \geq 0} \left\langle \dot{J}_0^\ell(t, x), e_t(x) \right\rangle^2 dx dt \\ &\quad - \frac{1}{2\rho_t^3} \sum_{\ell \geq 0} \left( \int_M \rho_t(x) \left\langle \dot{J}_0^\ell(t, x), e_t(x) \right\rangle dx \right)^2 dt. \end{aligned}$$

For a vector  $w \in T_{g_t(x)}M$ , we let  $w^T$  the part of  $w$  tangential to  $T_0(t, x)$ . Letting

$$\cos \left( \dot{J}_0^{\ell, T}(t, \cdot), T_0(t, \cdot) \right) = \frac{\int_M \left\langle \dot{J}_0^{\ell, T}(t, x), T_0(t, x) \right\rangle dx}{\rho_t \left( \int_M \left\| \dot{J}_0^{\ell, T}(t, x) \right\|^2 dx \right)^{1/2}}$$

(observe  $\rho_t^2 = \int_M \|T_0(t, x)\|^2 dx$ ), we finally proved

**Proposition 6.1.** *The Itô differential of the distance  $\rho_t$  between  $g_t$  and  $\tilde{g}_t$  is given by*

$$\begin{aligned} d\rho_t &= \frac{1}{\rho_t} \sum_{\ell \geq 0} \left( \int_M \rho_t(x) (P_{\tilde{g}_t(x), g_t(x)}(\sigma_\ell^T(\tilde{g}_t(x))) - \sigma_\ell^T(g_t(x))) dx \right) dB_t^\ell \\ &\quad + \frac{1}{\rho_t} \int_M \rho_t(x) (P_{\tilde{g}_t(x), g_t(x)}(u^T(\tilde{g}_t(x))) - u^T(g_t(x))) dx dt - \frac{1}{\rho_t} \int_M \rho_t(x) dL_t(x) dx \\ &\quad + \frac{1}{2\rho_t} \left( \int_M \sum_{\ell \geq 0} \left( \int_0^1 (\|J_a^{\ell, N}\|^2 - \langle R(T_a(t, x), J_a^{\ell, N}(t, x))J_a^{\ell, N}(t, x), T_a(t, x) \rangle) da \right) dx \right) dt \\ &\quad + \frac{1}{2\rho_t} \sum_{\ell \geq 0} \left( 1 - \cos^2 \left( j_0^{\ell, T}(t, \cdot), T_0(t, \cdot) \right) \right) \int_M \|j_0^{\ell, T}(t, x)\|^2 dx dt. \end{aligned}$$

In the case of manifolds with negative curvature we may observe a similar phenomena to the one of the torus with the Euclidean distance treated in Section 4: as long as the  $L^\infty$  norm stays sufficiently small to avoid the cut-locus of the manifold, the  $L^2$  mean distance between the stochastic particles tends to increase exponentially fast.

## 6.2. The rotation process.

In the following we would like to study the rotation of two particles  $g_t(x)$  and  $\tilde{g}_t(x)$  when they are in a close distance one to another. Recall that we have noted  $x_t = g_t(x)$ ,  $y_t = \tilde{g}_t(x)$ . We always assume that the distance from  $x_t$  to  $y_t$  is small: we are interested in the behaviour of  $e(t)$  as  $\rho_t(x)$  goes to 0. We let

$$(6.6) \quad d_m x(t)^N = \sigma(x_t) dB_t - \langle \sigma(x_t) dB_t, e(t) \rangle e(t)$$

and

$$(6.7) \quad d_m y(t)^N = \sigma(y_t) dB_t - \langle \sigma(y_t) dB_t, P_{x_t, y_t} e(t) \rangle P_{x_t, y_t} e(t)$$

where  $P_{x_t, \gamma_a(t)}$  denotes the parallel transport along  $\gamma_a$ .

From Itô formula we have

$$(6.8) \quad \mathcal{D}T_0 = \rho_t(x) \mathcal{D}e(t) + d\rho_t(x) e(t) + d\rho_t(x) \mathcal{D}e(t)$$

and this yields

$$\begin{aligned}
 \mathcal{D}e(t) &= \frac{1}{\rho_t(x)} \mathcal{D}T_0 - \frac{1}{\rho_t(x)} d\rho_t(x) e(t) - \frac{1}{2} \frac{1}{\rho_t(x)} d\rho_t(x) \mathcal{D}e(t) \\
 &= \frac{1}{\rho_t(x)} \dot{J}_0(d_m x(t)^N, d_m y(t)^N) \\
 &\quad + \frac{1}{\rho_t(x)} \dot{J}_0(u(t, x_t), u(t, y_t)) dt + \frac{1}{2\rho_t(x)} \sum_{\ell \geq 0} \nabla_{J_0^\ell} \nabla_{J_0^\ell} T_0 dt \\
 &\quad - \frac{1}{\rho_t(x)} \langle P_{y_t, x_t}(u(t, y_t)) - u(t, x_t), e(t) \rangle e(t) \\
 &\quad - \frac{1}{2\rho_t(x)^2} \left( \int_0^1 \sum_{\ell \geq 0} (\|\nabla_{T_a} J_a^\ell\|^2 - R(T_a, J_a^\ell) J_a^\ell, T_a) da \right) e(t) \\
 &\quad - \frac{1}{2} \frac{1}{\rho_t(x)} d\rho_t(x) \mathcal{D}e(t) \\
 &= \frac{1}{\rho_t(x)} \dot{J}_0(d_m x(t)^N, d_m y(t)^N) + \frac{1}{\rho_t(x)} \dot{J}_0(u^N(t, x_t), u^N(t, y_t)) \\
 &\quad + \frac{1}{2\rho_t(x)} \sum_{\ell \geq 0} \nabla_{J_0^\ell} \nabla_{J_0^\ell} T_0 dt \\
 &\quad - \frac{1}{2\rho_t(x)^2} \left( \int_0^1 \sum_{\ell \geq 0} (\|\nabla_{T_a} J_a^\ell\|^2 - R(T_a, J_a^\ell) J_a^\ell, T_a) da \right) e(t)
 \end{aligned}$$

where we used the fact that  $d\rho_t(x) \mathcal{D}e(t) = 0$ , and where  $u^N$  denotes the part of  $u$  which is normal to the geodesic  $\gamma_a$ . Now as before

$$\nabla_{J_0^\ell} \nabla_{J_0^\ell} T_0 = \nabla_{T_0} \nabla_{J_0^\ell} J_0^\ell - R(T_0, J_0^\ell) J_0^\ell.$$

Finally we get

**Lemma 6.2.**

$$\begin{aligned}
 \mathcal{D}e(t) &= \frac{1}{\rho_t(x)} \dot{J}_0(d_m x(t)^N, d_m y(t)^N) + \frac{1}{\rho_t(x)} \dot{J}_0(u^N(t, x_t), u^N(t, y_t)) \\
 &\quad + \frac{1}{2\rho_t(x)} \sum_{\ell \geq 0} \nabla_{T_0} \nabla_{J_0^\ell} J_0^\ell - R(T_0, J_0^\ell) J_0^\ell dt \\
 &\quad - \frac{1}{2\rho_t(x)^2} \left( \int_0^1 \sum_{\ell \geq 0} (\|\nabla_{T_a} J_a^\ell\|^2 - R(T_a, J_a^\ell) J_a^\ell, T_a) da \right) e(t).
 \end{aligned}$$

From now on we assume that  $M = \mathbb{T}$  the two dimensional torus.

In this situation the curvature tensor vanishes and we have the formulas

$$J_a(v, w) = v + a(w - v), \quad \dot{J}_a(v, w) = w - v.$$

We immediately get

$$\begin{aligned}
 de(t) = \mathcal{D}e(t) &= \frac{1}{\rho_t(x)} (d_m y(t)^N - d_m x(t)^N) + \frac{1}{\rho_t(x)} ((u^N(t, y_t) - u^N(t, x_t)) dt \\
 &\quad - \frac{1}{2\rho_t(x)^2} \sum_{\ell \geq 0} \|\sigma_\ell(y_t) - \sigma_\ell(x_t)\|^2 dt e(t)
 \end{aligned}$$

where we used the fact that  $\nabla_{T_0} \nabla_{J^\ell} J^\ell = 0$ , as a consequence of  $\nabla_{J_0^\ell} J_0^\ell = 0$ ,  $\nabla_{J_1^\ell} J_1^\ell = 0$ , and  $R \equiv 0$ .

Let us specialize again to the case where the vector fields are given by

$$A_k(\theta) = (k_2, -k_1) \cos k \cdot \theta, \quad B_k(\theta) = (k_2, -k_1) \sin k \cdot \theta$$

and the Brownian motion

$$(6.9) \quad dW(t) = \sum_{k \in \mathbb{Z}} \lambda_k \sqrt{\nu} (A_k dx_k + B_k dy_k)$$

where  $x_k, y_k$  are independent copies of real Brownian motions. As in section 4 we assume that  $\sum_k |k|^2 \lambda_k^2 < \infty$  and we consider  $\lambda_k = \lambda(|k|)$  to be nonzero for a equal number of  $k_1$  and  $k_2$  components. Again we write

$$(6.10) \quad dg_t = (odW(t)) + u(t, g_t) dt, \quad d\tilde{g}_t = (odW(t)) + u(t, \tilde{g}_t) dt$$

with

$$g_0 = \phi, \quad \tilde{g}_0 = \psi, \quad \phi \neq \psi.$$

Changing the notation to  $g_t = g_t(\theta) = x_t$ ,  $\tilde{g}_t = \tilde{g}_t(\theta) = y_t$ , we get

$$\begin{aligned} de(t) &= \frac{1}{\rho_t(\theta)} \sum_{|k| \neq 0} \lambda_k \sqrt{\nu} (\cos k \cdot \tilde{g}_t - \cos k \cdot g_t) k^{\perp, N} dx_k \\ &+ \frac{1}{\rho_t(\theta)} \sum_{|k| \neq 0} \lambda_k \sqrt{\nu} (\sin k \cdot \tilde{g}_t - \sin k \cdot g_t) k^{\perp, N} dy_k \\ &+ \frac{1}{\rho_t(\theta)} ((u^N(t, \tilde{g}_t) - u^N(t, g_t)) dt \\ &- \frac{1}{2\rho_t^2(\theta)} \sum_{|k| \neq 0} \lambda_k^2 \nu |k^{\perp, N}|^2 ((\cos k \cdot \tilde{g}_t - \cos k \cdot g_t)^2 + (\sin k \cdot \tilde{g}_t - \sin k \cdot g_t)^2) e(t) dt \\ &= \frac{1}{\rho_t(\theta)} \sum_{|k| \neq 0} \lambda_k \sqrt{\nu} k^{\perp, N} \left( 2 \sin \frac{k \cdot (\tilde{g}_t - g_t)}{2} \right) dz_k \\ &+ \frac{1}{\rho_t(\theta)} ((u^N(t, \tilde{g}_t) - u^N(t, g_t)) dt \\ &- \frac{2}{\rho_t^2(\theta)} \sum_{|k| \neq 0} \lambda_k^2 \nu |k^{\perp, N}|^2 \sin^2 \left( \frac{k \cdot (\tilde{g}_t - g_t)}{2} \right) e(t) dt \end{aligned}$$

where  $z_k$  is the Brownian motion defined by

$$dz_k = -\sin \frac{k \cdot (\tilde{g}_t + g_t)}{2} dx_k + \cos \frac{k \cdot (\tilde{g}_t + g_t)}{2} dy_k.$$

Noting  $|k^{\perp, N}|^2 = |k|^2(n_k \cdot e(t))^2$ , we obtain

$$\begin{aligned}
 (6.11) \quad de(t) &= \frac{1}{\rho_t(\theta)} \sum_{|k| \neq 0} |k| \lambda_k \sqrt{\nu}(n_k \cdot e(t)) e'(t) \left( 2 \sin \frac{k \cdot (\tilde{g}_t - g_t)}{2} \right) dz_k \\
 &+ \frac{1}{\rho_t(\theta)} \left( (u^N(t, \tilde{g}_t) - u^N(t, g_t)) dt \right. \\
 &\left. - \frac{2}{\rho_t^2(\theta)} \sum_{|k| \neq 0} |k|^2 \lambda_k^2 \nu(n_k \cdot e(t))^2 \sin^2 \frac{k \cdot (\tilde{g}_t - g_t)}{2} e(t) dt \right)
 \end{aligned}$$

where  $e'(t)$  is a unit vector in  $\mathbb{T}$  orthonormal to  $e(t)$ . Now for every  $K > 0$ , if  $\rho_t(\theta) \leq \frac{\pi}{2K}$  then for all  $k$  such that  $|k| \leq K$ ,

$$\frac{\sin^2 \frac{k \cdot (\tilde{g}_t - g_t)}{2}}{|k|^2 \rho_t^2(\theta) (n_k \cdot e(t))^2} \geq \frac{1}{\pi^2}.$$

Now using  $|k| = |k^\perp|$  and  $(n_k \cdot e(t))^2 + (n_{k^\perp} \cdot e(t))^2 = 1$ , we get

$$(6.12) \quad \frac{2}{\rho_t^2(\theta)} \sum_{|k| \neq 0} |k|^2 \lambda_k^2 \nu(n_k \cdot e(t))^2 \sin^2 \frac{k \cdot (\tilde{g}_t - g_t)}{2} \geq \frac{\nu}{2\pi^2} \sum_{0 < |k| < K} \lambda_k^2 |k|^4.$$

Observe that the term in the left is the second part of the drift in equation (6.11) as well as the derivative of the quadratic variation of  $e(t)$ . This yields the following result.

**Proposition 6.3.** *Identifying  $T\mathbb{T}$  with  $\mathbb{C}$ , we have  $e(t) = e^{iX_t}$  where  $X_t$  is a real-valued semimartingale with quadratic variation*

$$(6.13) \quad d[X, X]_t = \frac{4}{\rho_t^2(\theta)} \sum_{|k| \neq 0} |k|^2 \lambda_k^2 \nu(n_k \cdot e(t))^2 \sin^2 \frac{k \cdot (\tilde{g}_t - g_t)}{2} dt$$

and drift

$$(6.14) \quad \int_0^t \frac{1}{\rho_s(\theta)} \langle u(s, \tilde{g}_s) - u(s, g_s), ie(s) \rangle ds.$$

We have for all  $K > 0$ , on  $\left\{ \rho_t(\theta) \leq \frac{\pi}{2K} \right\}$ ,

$$(6.15) \quad d[X, X]_t \geq \frac{\nu}{\pi^2} \sum_{0 < |k| < K} \lambda_k^2 |k|^4.$$

If  $\sum_{|k| \neq 0} \lambda_k^2 |k|^4 = +\infty$ , then as  $\tilde{g}_t(\theta)$  gets closer and closer to  $g_t(\theta)$ , the rotation  $e(t)$

becomes more and more irregular in the sense that the derivative of the quadratic variation of  $X_t$  tends to infinity.

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