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Abstract: Consider a Langevin process, that is an integrated Brownian motion, constrained to stay in $[0, \infty)$ by a partially elastic boundary at 0. If the elasticity coefficient of the boundary is greater than or equal to $c_{crit} = \exp(-\sqrt{\pi}/3)$, bounces will not accumulate in a finite time when the process starts from the origin with strictly positive velocity. We will show that there exists then a unique entrance law from the boundary with zero velocity, despite the immediate accumulation of bounces. This result of uniqueness is in sharp contrast with the literature on deterministic second order reflection. Our approach uses certain properties of real-valued random walks and a notion of spatial stationarity which may be of independent interest.

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1. Introduction

Imagine a deterministic particle evolving in $\mathbb{R}_+$, star-ted from 0, submitted to an external force f, and constrained by a partially elastic boundary at the origin. We write $x(t)$ for the position of the particle and we consider the following equations of motion:

$$(SOR) \qquad \begin{cases} x(t) &= x(0) + \int_0^t \dot{x}(s)\mathrm{d}s \\ \dot{x}(t) &= \dot{x}(0) + \int_0^t f(s)\mathrm{d}s - (1+c)\sum_{0<s\leq t} \dot{x}(s-)\mathbb{1}_{x(s)=0}, \end{cases}$$

where the velocity $\dot{x}$ is càdlàg. The first equation states that x is continuous and has a right-derivative, $\dot{x}$.

The coefficient $c > 0$ is the elasticity coefficient of the boundary: after a bounce, the boundary restores a portion c of the incoming speed. The couple $(x(0), \dot{x}(0))$ is called the starting condition, while $x(0)$ is the starting position and $\dot{x}(0)$ the starting velocity.

Equations (SOR) describe the so-called second order reflection problem. There is a large literature on the subject. To mention some names, Bressan in 1960 [7], Percivale in 1985 [17], Schatzman in 1998 [18], or Ballard in 2000 [1]. An important feature is that in the case of an analytic force f, there is existence and uniqueness to the equations (SOR) for any initial condition, but when f is not analytic (even if it is $\mathcal{C}^\infty$), uniqueness may fail.

The main difficulty in second order reflection comes from the possibility for bounces to accumulate, in which case the sum in the equation involves infinitely many terms. We distinguish two problems: First, bounces may accumulate just before a finite time $t > 0$. Second, when the particle starts from zero position with zero velocity (initial condition $(0, 0)$), bounces may accumulate just after the starting time 0.

In this paper we are interested in Equations (SOR) when the external force f is random and given by a white noise. A realization of f will *a fortiori* not be analytic and we will not try to work on a fixed realization. The first observation is that outside the boundary, the velocity of the particle behaves like a Brownian motion, hence the particle evolves like a free Langevin process (i.e the integrated Brownian motion). A consequent study about the free Langevin process in general can be found in Lachal [15]. Bect mentioned the reflection and bounds accumulation problems for particles that can be excited by a white noise in his thesis ([2], see part III.4). For the reader interested in the problem of a Langevin process reflected at a totally inelastic boundary, that is $c = 0$, we refer to Bertoin [3, 4] and Jacob [13].

Let us return to consider (SOR) for f a white noise and $c > 0$. Then the problem of accumulation of bounces just before a finite time $t > 0$ is simple enough: We shall see that bounces accumulate if and only if the elasticity coefficient is less than the critical coefficient $c_{crit} = \exp(-\pi/\sqrt{3})$. However the question of starting with zero position and zero velocity is more fastidious. We focus on the critical and supercritical cases, the study of the subcritical case being the center of interest of a forthcoming paper.

Our main result is the following. For $c \geq c_{crit}$, the reflected Langevin process starting from the origin with a speed $v > 0$ converges in law, when v goes to 0, to a non-degenerate process. Moreover, the law of the process yields the unique law of a solution to the equations (SOR) with white noise forcing and initial condition $(0, 0)$.

We observe in this introduction that these results are fairly simple for the particular case $c = 1$ (perfectly elastic boundary) because a reflected Langevin process can then be constructed from the free Langevin process Y by taking its absolute value $|Y|$. However there is no such construction when the elasticity coefficient is $c \neq 1$.

Our method is to focus on the velocities of the process at the bouncing times. A crucial observation is that the sequence of their logarithms is a random walk. This enables us to use technics of renewal theory for random walks, including results about its associated ladder height process and the law of its overshoot.

Next section is devoted to the preliminaries. In Subsection 2.2, we give some background on the Langevin process and we characterize the phase transition at $c = c_{crit}$. In Subsection 2.3, we define a notion of spatial stationarity, in an abstract context. We obtain a convergence result for spatially stationary processes, stated as Lemma 2 and proved in the Appendix. Then, Section 3 starts with the statement of our main theorem and its main consequences. Section 3.1 uses renewal theory and Lemma 2 to construct a spatially stationary process and reduce the proof of the main theorem to that of Lemma 5. Section 3.2 handles this proof in the supercritical case, thanks to an explicit construction[1] of the spatially stationary random walk. However this construction does not hold in the critical case, and Section 3.3 completes then the proof, thanks to a disintegration formula[1] for the spatially stationary random walk.

2. Preliminaries

2.1. Notations

The (free) Kolmogorov process $(Y, \dot{Y})$ with starting position x and starting velocity v — sometimes, we will also say with starting condition (x, v) — is defined by

$$\begin{cases} Y_t &= x + \int_0^t \dot{Y}_s \mathrm{d}s \\ \dot{Y}_t &= v + B_t, \end{cases}$$

where B is a standard Brownian motion. Its first coordinate Y is called the (free) Langevin process. Before writing the second order reflection equations for the Langevin process, we introduce $D = (\{0\} \times \mathbb{R}_+^*) \cup (\mathbb{R}_+^* \times \mathbb{R})$ and $D^0 := D \cup \{(0,0)\}$. Our working space is $\mathcal{C}$, the space of càdlàg trajectories $(x, \dot{x}) : [0, \infty) \to D^0$, which satisfy

$$x(t) = x(0) + \int_0^t \dot{x}(s) \mathrm{d}s.$$

This space is endowed with the $\sigma-$algebra generated by the coordinate maps and with the topology induced by the following injection:

$$\begin{array}{ccc} \mathcal{C} & \to & \mathbb{R}_+ \times \mathcal{D}(\mathbb{R}_+) \\ (x, \dot{x}) & \mapsto & \big(x(0), \dot{x}\big), \end{array}$$

where $\mathcal{D}(\mathbb{R}_+)$ is the space of càdlàg trajectories on $\mathbb{R}_+$, equipped with Skorohod topology. We denote by $(X, \dot{X})$ the canonical process and by $(\mathcal{F}_t, t \geq 0)$ its natural filtration, satisfying the usual conditions of right continuity and completeness. Besides, by a slight abuse of notation, when we define a probability measure P, we also write P for the expectation under this probability measure. When f is a measurable functional and A an event, we also write $P(f, A)$ for the quantity $P(f \mathbb{1}_A)$.

[1] These two constructions in particular may be of independent interest.

For any $(x, v) \in D^0$, the second order reflection of the Langevin process starting from position x and velocity v leads to the following equation:

$$(SOR) \qquad \begin{cases} X_t &= x + \int_0^t \dot{X}_s \mathrm{d}s \\ \\ \dot{X}_t &= v + B_t - (1+c) \sum_{0<s\leq t} \dot{X}_{s-} \mathbb{1}_{X_s=0}, \end{cases}$$

where B is a Brownian motion. Problems of existence and uniqueness of second order reflection equations can only arise around the point $(0,0)$. For any $(x,v) \in D$, we write $\mathbb{P}^c_{x,v}$ for the solution to equations (SOR), *killed at its first hitting time of* $(0,0)$. This process is a well-defined strong Markov process, and will be called the killed reflected Langevin process, or more concisely the reflected Langevin process. We will almost exclusively consider the case when the starting position is 0, and write $\mathbb{P}^c_v$ for $\mathbb{P}^c_{0,v}$ (with $v > 0$).

Let us write $\zeta_0 = 0$ and $\zeta_{n+1} := \inf\{t > \zeta_n : X_t = 0\}$ for the sequence of successive hitting times of zero (see Figure 1 below for an illustration of the notations). We call an *arch* a part of the path included between two consecutive hitting times of zero. Then, under $\mathbb{P}^c_v$, the killed reflected process X behaves like Y until the first return time to zero ζ_1, that is the first arch of Y and X have the same law, $(Y_t)_{\zeta_0 \leq t \leq \zeta_1} \overset{d}{=} (X_t)_{\zeta_0 \leq t \leq \zeta_1}$. Then the second arch of the killed reflected process, $(X_t)_{\zeta_1 \leq t \leq \zeta_2}$, has the same law as the first arch of a Langevin process starting with velocity $\dot{X}_{\zeta_1} := -c\dot{X}_{\zeta_1^-}$. We construct in the same way the sequence of successive arches of X. We also write V_n^-, and V_n for the speed of the process just before this n-th bounce, and for the speed of the process just after this n-th bounce, respectively, so that we have $V_n = \dot{X}_{\zeta_n} = -c\dot{X}_{\zeta_n^-} = -cV_n^-$. Please note that the event that for some n, we have $V_n = 0$, has probability 0. We call time of accumulation of bounces the time $\zeta_\infty := \sup(\zeta_n) \in]0, \infty]$. It coincides almost surely with the hitting time of $(0,0)$. Figure 1 below shows two complete arches and the beginning of a third one.

In the particular case $c = 1$, the killed reflected Langevin process has the same law as the absolute value of the free Langevin process. Then, the sequence $(\zeta_n, V_n)_{n\geq 0}$ coincides with the sequence of the successive passage times to zero and absolute value of the speed of the process at this times, for the free Langevin process. This sequence has been studied by McKean [16]. He shows that it is a homogeneous Markov chain with explicit transition probabilities. Lachal furthers this study in [14] by giving explicit formulas for the law of (ζ_n, V_n) for a fixed n.

Remark. *Wong also studies in [20, 21] the passage times to zero for a certain stationary process, which is obtained from the Langevin process by an exponential change of scale in both time and space. The passage times to zero of this stationary process are closely related to a certain stationary random walk that we will introduce later on. However, this process shall not be confused with the "stationary Langevin process" introduced in [13]. The two processes do not seem to be directly related.*

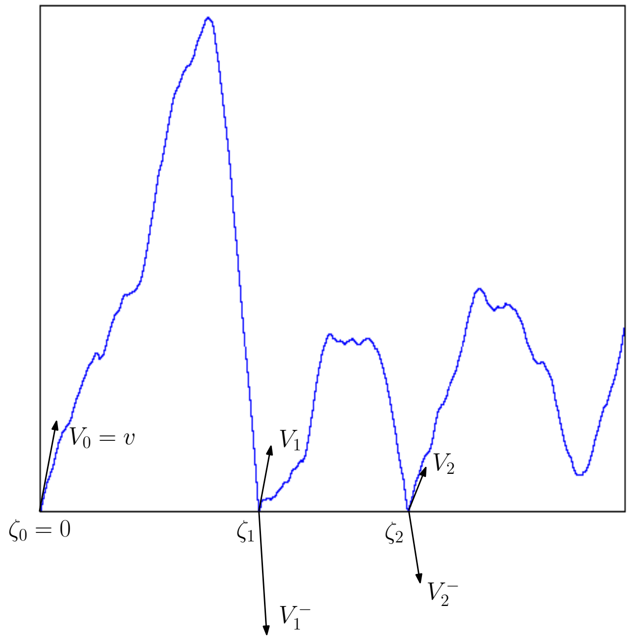

FIG 1. *First arches of a killed reflected Langevin process*

In the next subsection, we present essentially the same results as those of McKean, but stated in the general case $c \neq 1$. The infiniteness of ζ_∞ is then no more guaranteed.

2.2. The sequence $(\zeta_n, V_n)_{n\geq 0}$

Lemma 1. *1. The law of* $(\zeta_1, V_1/c)$ *under* $\mathbb{P}_1^c$ *is given by*

$$\frac{1}{\mathrm{d}s\mathrm{d}v}\mathbb{P}_1^c\left((\zeta_1, V_1/c) \in (\mathrm{d}s, \mathrm{d}v)\right) = \frac{3v}{\pi\sqrt{2}s^2}\exp(-2\frac{v^2-v+1}{s})\int_0^{\frac{4v}{s}} e^{-\frac{3\theta}{2}}\frac{\mathrm{d}\theta}{\sqrt{\pi\theta}}. \tag{2.1}$$

2. *Under* $\mathbb{P}_v^c$*, the sequence* $\left(\frac{\zeta_{n+1}-\zeta_n}{V_n^2}, \frac{V_{n+1}}{V_n}\right)_{n\geq 0}$ *is i.i.d. The common law of its marginals, also independent of* v*, is that of* (ζ_1, V_1) *under* $\mathbb{P}_1^c$*.*
3. *In particular, the sequence* $\ln(V_n)$ *is a random walk. The density of its step distribution* $\ln(V_1/V_0)$ *under* $\mathbb{P}_v^c$ *does not depend on* v *and is given by:*

$$\frac{1}{\mathrm{d}v}\mathbb{P}_1^c(\ln(V_1) \in \mathrm{d}v) = \frac{3}{2\pi}\frac{e^{\frac{5}{2}(v-\ln c)}}{1+e^{3(v-\ln c)}}\mathrm{d}v. \tag{2.2}$$

In particular $\ln(V_1)$ *has finite variance and expectation*

$$\mathbb{P}_1^c(\ln V_1) = \frac{\pi}{\sqrt{3}} + \ln c.$$

4. We have, when $t \to \infty$,

$$\mathbb{P}_1^c(\zeta_1 > t) \sim c' t^{-\frac{1}{4}}, \tag{2.3}$$

where $c' = 3\Gamma(1/4)/(2^{3/4}\pi^{3/2})$.

Proof. The three first points are essentially results given by McKean [16] or direct consequences of these. The last point is similar to a result of Goldman for the law of the process with zero starting velocity and nonzero starting position [9], and follows from (2.1) by standard integral calculus.

For the convenience of the reader, we explain the second point. It follows from the observation that the variable $(\zeta_n - \zeta_{n-1})/(V_{n-1})^2$ (resp. V_n/V_{n-1}) is equal to the duration of the $n-$th arch renormalized to start with speed one (resp. to the absolute value of the speed of the process just before its return time to zero, for this renormalized arch). More precisely:

Recall that, conditionally on $V_n = v$, the process $(X_{(t+\zeta_n)\wedge\zeta_{n+1}})_{t\geq 0}$ is independent of $(X_{t\wedge\zeta_n})_{t\geq 0}$ and has the same law as $(X_{t\wedge\zeta_1})_{t\geq 0}$ under $\mathbb{P}_v^c$, thus $(\zeta_{n+1} - \zeta_n, V_{n+1}/c)$ is independent of $(\zeta_k, V_k)_{k\leq n}$ has the same law as $(\zeta_1, \frac{1}{c}V_1)$ under $\mathbb{P}_v^c$. It follows that the variable $((\zeta_{n+1} - \zeta_n)/(V_n)^2, V_{n+1}/V_n)$ is independent of $(\zeta_k, V_k)_{k\leq n}$ and has the same law as (ζ_1, V_1) under $\mathbb{P}_1^c$ (conditionally on $V_n = v$, but this conditioning can simply be removed). The statement follows. □

From this Lemma we deduce the following important result:

Corollary 1. *The time of accumulation of bounces* ζ_∞ *is:*

finite $\mathbb{P}_v^c-$*almost surely if* $c < \exp(-\pi/\sqrt{3})$,
infinite $\mathbb{P}_v^c-$*almost surely if* $c \geq \exp(-\pi/\sqrt{3})$.

We thus call $c_{crit} := \exp(-\pi/\sqrt{3})$ *the critical elasticity coefficient. We call the case* $c > c_{crit}$ *the supercritical regime, the case* $c < c_{crit}$ *the subcritical regime, the case* $c = c_{crit}$ *the critical regime.*

Proof. We may express ζ_∞ as the series:

$$\zeta_\infty = \sum_{n=1}^{\infty} \frac{\zeta_n - \zeta_{n-1}}{(V_{n-1})^2}(V_{n-1})^2.$$

For $c < \exp(-\pi/\sqrt{3})$, the law of large numbers tells that the sequence $\frac{1}{k}\ln(V_k)$ converges to $\ln(c) + \pi/\sqrt{3} < 0$ a.s. On the other hand, it follows from (2.3) that the expectation of $(\ln(\zeta_1))^2$ is finite[2]. Thus, for any fixed $\varepsilon > 0$ there are a.s. only a finite number of k such that $\ln((\zeta_k - \zeta_{k-1})/(V_{k-1})^2)$ is larger than εk.

[2]This result was also stressed by McKean in [16]

We deduce an a. s. exponential decay for the variables $\zeta_{k+1} - \zeta_k$. *A fortiori* ζ_∞ is a. s. finite.

Take now $c \geq \exp(-\pi/\sqrt{3})$. For $c > \exp(-\pi/\sqrt{3})$, the random walk $\ln V_n$ has a positive drift and is transient. Thus the sequence V_n is diverging to $+\infty$. As $(\zeta_n - \zeta_{n-1})/(V_{n-1})^2$ is independent of V_{n-1} and has a fixed distribution, we deduce that ζ_∞ is infinite. For $c = \exp(-\pi/\sqrt{3})$, the step distribution has zero expectation and finite variance, thus the random walk is recurrent (from the central limit theorem). Then the sequence V_n is recurrent, but it is still not converging to zero, which is enough to conclude in the same way that ζ_∞ is infinite. □

From now, we suppose $c \geq c_{crit}$. Then, for any $(x, v) \in D$, we have $\zeta_\infty = \infty$ almost surely under $\mathbb{P}^c_{x,v}$. The process is thus defined on $\mathbb{R}_+$ (it is not killed), and we simply call it the reflected Langevin process. It is also the unique solution (in the strong sense) of equations (SOR) with starting condition (x, v). A natural question is to ask wether we can define the reflected Langevin process starting from condition $(0, 0)$. The purpose of this work will be to answer positively this question.

Note that the particular case $c = 1$ is trivial, just consider the absolute value of the free Langevin process with initial condition $(0, 0)$. For this process, it is natural to try to describe the instants of bounces and the velocity of the process at these instants. One way to do this, adopted by McKean [16] and Lachal [14], is to define two sequences, the first one corresponding to the successive bounds happening after time 1, the second one to the successive bounds happening before time 1, counted backwardly. This is barely different from considering a single sequence, indexed by $\mathbb{Z}$, where one puts arbitrarily the 0 at the last bounce happening before time 1.

In a fairly similar manner, in the general case $c \neq 0$, we will get to consider a sequence indexed by $\mathbb{Z}$, and whose 0 will be put at the first bounce for which the speed is greater than 1.

2.3. Weak convergence to a spatially stationary process

After these first results on the Langevin process, we give the abstract context for a notion of spatial stationarity and an important lemma that we will need later.

We write Ω for the set of sequences indexed by $\mathbb{Z}$, $\omega = (\omega_n)_{n\in\mathbb{Z}} = (\omega^1_n, \omega^2_n)_{n\in\mathbb{Z}}$, with values in $[-\infty, \infty) \times \mathcal{C}^0$, where $\mathcal{C}^0$ is a topological space with an isolated point $\emptyset$. For now, just consider this space as playing an accessory role that will be clarified later. The set Ω is endowed with the usual product topology.

For any real number x we write T_x for the hitting time of (x, ∞) by the first coordinate, that is

$$T_x = T_x(\omega) = \inf\{n \in \mathbb{Z}, \omega^1_n > x\}.$$

Under all the measures P that we will consider on Ω we will have

$$\lim_{-\infty} \omega_n^1 = -\infty, \ \limsup_{+\infty} \omega_n^1 = +\infty \qquad \text{P-almost surely,}$$

and as a consequence T_x will have values in $\mathbb{Z}$, P-almost surely. We then define a spatial translation operator Θ on Ω, by:

$$\Theta_x(\omega) := (\omega^1_{n+T_x} - x, \omega^2_{n+T_x})_{n\in\mathbb{Z}}. \tag{2.4}$$

This definition immediately yields a notion of spatial stationarity for the probabilities on Ω:

Definition 1. *We say that a probability P on Ω is spatially stationary if $P \circ \Theta_x = P$ for any $x \in \mathbb{R}$.*

We also write

$$\Omega_+ := \left\{\omega \in \Omega : (\omega_n^1, \omega_n^2) = (-\infty, \emptyset) \text{ for all } n < 0\right\},$$

that we shall think of as the sequences indexed by $\mathbb{N}$. We write $\omega^+ \in \Omega_+$ for the projection of $\omega \in \Omega$ defined by:

$$\omega_n^+ = \begin{cases} (-\infty, \emptyset) & \text{if } n < 0, \\ \omega_n & \text{if } n \geq 0. \end{cases}$$

If P is a probability on Ω, we write P_+ for the image probability on Ω_+ by this projection. Finally we write $\Rightarrow$ for the weak convergence on the topological space Ω. The following lemma states a convergence result to a spatially stationary probability measure on Ω, as a consequence of convergence results on Ω_+:

Lemma 2. *Let $(P_v)_{v>0}$ be a family of probability measures on Ω. We suppose that there is a probability Q on Ω_+ such that:*

$$\forall x \in \mathbb{R}, \quad (P_v \circ \Theta_x)_+ \Rightarrow_{v\to 0} Q.$$

Then there exists a unique spatially stationary probability measure P on Ω such that $P_+ = Q$. Moreover, we have

$$P_v \circ \Theta_x \Rightarrow P.$$

The proof of this technical lemma is based on the Kolmogorov existence theorem. We postpone it to the appendix.

3. Entering with zero velocity

Recall that we are in the critical or supercritical regime, $c \geq c_{crit}$. Write $(S_n)_{n\geq 0}$ for the sequence of the logarithm of the (outgoing) velocity at the successive bounces, defined by $S_n = \ln(V_n)$. From Lemma 1, under $\mathbb{P}^c_{x,v}$, it is a random walk with step distribution given by (2.2) and drift

$$\mu := \mathbb{P}^c_1(S_1 - S_0) = \frac{\pi}{\sqrt{3}} + \ln c = \ln(c/c_{crit}).$$

In the supercritical case $c > c_{crit}$ the drift is strictly positive, while in the critical case $c = c_{crit}$ the step distribution has zero drift and finite variance.

We introduce the (strictly) ascending ladder height process $(H_n)_{n\geq 0}$ associated to the random walk $(S_n)_{n\geq 0}$, that is the random walk with positive jumps defined by $H_0 = S_0$ and $H_k = S_{n_k}$, where $n_0 = 0$ and $n_k = \inf\{n > n_{k-1}, S_n > S_{n_{k-1}}\} \in \mathbb{N}$. In both cases (positive drift, or null drift and finite variance), it is known (see Theorem 3.4 in Spitzer [19]) that the expectation of the step distribution of $(H_n)_{n\geq 0}$, that is $\mu_H := \mathbb{P}_1^c(H_1 - H_0)$, belongs to $(0, \infty)$. The probability law

$$m(\mathrm{d}y) := \frac{1}{\mu_H}\mathbb{P}_1^c(H_1 - H_0 > y)\mathrm{d}y. \tag{3.1}$$

is known in renewal theory as the stationary law of the overshoot (see also Part 3.1).

We now state our main theorem and its important corollary. The theorem is a convergence result for the probability laws $(\mathbb{P}_v^c)_{v>0}$ when $v \to 0^+$, while the corollary states the weak existence and uniqueness of solutions to (SOR) equations with initial condition $X_0 = \dot{X}_0 = 0$.

Theorem 1. *The family of probability measures $(\mathbb{P}_v^c)_{v>0}$ on $\mathcal{C}$ has a weak limit when $v \to 0^+$, which we denote by $\mathbb{P}_{0+}^c$. More precisely, write τ_u for the instant of the first bounce with speed greater than u, that is $\tau_u := \inf\{t > 0, X_t = 0, \dot{X}_t > u\}$. Then the law $\mathbb{P}_{0+}^c$ satisfies the following conditions:*

$$(*) \quad \begin{cases} \lim\limits_{u\to 0^+} \tau_u = 0. \\ \text{For any } u, v > 0, \text{ and conditionally on } \dot{X}_{\tau_u} = v, \text{ the process} \\ (X_{\tau_u+t}, \dot{X}_{\tau_u+t})_{t\geq 0} \text{ is independent of } (X_s, \dot{X}_s)_{s<\tau_u} \text{ and has law } \mathbb{P}_v^c. \end{cases}$$

$(**)$ *For any $u > 0$, the law of $\ln(\dot{X}_{\tau_u}/u)$ is m.*

Corollary 2. • *Consider $(X, \dot{X})$ a process of law $\mathbb{P}_{0+}^c$. Then the jumps of $\dot{X}$ on any finite interval are summable and the process B defined by*

$$B_t = \dot{X}_t + (1 + c) \sum_{0<s\leq t} \dot{X}_{s-}\mathbb{1}_{X_s=0}$$

is a Brownian motion. As a consequence the triplet $(X, \dot{X}, B)$ is a solution to (SOR) with initial condition $(0, 0)$.
• *For any solution $(X, \dot{X}, B)$ to (SOR) with initial condition $(0, 0)$, the law of $(X, \dot{X})$ is $\mathbb{P}_{0+}^c$.*

Let us introduce a slightly larger working space,

$$\mathcal{C}^* := \{(x_t, \dot{x}_t)_{t>0}, \forall\varepsilon > 0, (x_{\varepsilon+t}, \dot{x}_{\varepsilon+t})_{t\geq 0} \in \mathcal{C}\}.$$

We mention that $\mathcal{C}$ can be seen as a subspace of $\mathcal{C}^*$, by removing time 0 from the trajectories. This inclusion is strict: an element of $\mathcal{C}^*$ is a trajectory (indexed by $\mathbb{R}_+^*$) which does not necessarily have a limit at 0+. The theorem and its corollary will actually both be a consequence from the following lemma, which can be seen as a weak version of Theorem 1, and whose proof is reported to later.

Lemma 3. *There exists a law $\mathbb{P}^{c*}_{0+}$ on $\mathcal{C}^*$ such that:*
• *We have $\tau_u > 0$ for any $u > 0$, $\mathbb{P}^{c*}_{0+}$-almost surely.*
• *conditions $(*)$ and $(**)$ are satisfied*
• *For any $u > 0$, the joint law of τ_u and $(X_{\tau_u+t}, \dot{X}_{\tau_u+t})_{t\geq 0}$ under $\mathbb{P}^c_v$ converges weakly, when v goes to 0, to that under $\mathbb{P}^{c*}_{0+}$.*

Note that the convergence stated in this lemma holds on $\mathcal{C}$.

Proof of Theorem 1 and Corollary 2. Consider, under $\mathbb{P}^{c*}_{0+}$, the canonical process $(X_t, \dot{X}_t)_{t>0}$. From conditions $(*)$ and the Markov property, we deduce that $(X_t, \dot{X}_t)_{t>0}$ is a strong Markov process with values in D and transitions that of the reflected Langevin process.

It follows that for any $r > 0$, there exists a Brownian motion B^r independant of $\mathcal{F}_r$ and such that, for $t \geq r$,

$$\begin{cases} X_t &= X_r + \int_r^t \dot{X}_s \mathrm{d}s \\ \dot{X}_t &= \dot{X}_r + B^r_{t-r} - (1+c)\sum_{r<s\leq t} \dot{X}_{s-} \mathbb{1}_{X_s=0}. \end{cases}$$

The Brownian motions B^r are linked by $B^r_{t-r} = B^q_{t-q} - B^q_{r-q}$ for $q \leq r \leq t$. We introduce $M_s = B^{1-s}_s$, $0 \leq s < 1$. For any $t < 1$, we have

$$(M_s)_{0\leq s\leq t} = (B^{1-t}_t - B^{1-t}_{t-s})_{0\leq s\leq t}.$$

Therefore $(M_s)_{0\leq s\leq t}$ is a Brownian motion. It follows that $(M_s)_{0\leq s<1}$ is a Brownian motion. Write M_1 for its limit when s tends to 1. Now, define the process B by

$$B_s = \begin{cases} M_1 - M_{1-s} & , \ 0 \leq s < 1. \\ M_1 + B^1_{s-1} & , \ 1 \leq s. \end{cases}$$

It is easy to check that B is a Brownian motion and satisfies $B_t - B_r = B^r_{t-r}$ for $t \geq r$. Hence, for $t \geq r$,

$$\begin{cases} X_t &= X_r + \int_r^t \dot{X}_s \mathrm{d}s \\ \dot{X}_t &= \dot{X}_r + B_t - B_r - (1+c)\sum_{r<s\leq t} \dot{X}_{s-} \mathbb{1}_{X_s=0}. \end{cases} \tag{3.2}$$

The increments of $\dot{X}$ are equal to the sum of two terms, on the one side the increments of B, and on the other side, the jumps, which are happening at the bouncing times. Besides, conditions $(*)$ imply $\dot{X}_t \mathbb{1}_{X_t=0} \underset{t\to 0}{\to} 0$. That is, the value of $\dot{X}$ at a bouncing time is going to 0 when this time goes to 0. It follows $\dot{X}_t \underset{t\to 0}{\to} 0$. Therefore we also have $X_t \to 0$. Consequently, by setting $X_0 = \dot{X}_0 = 0$, we define a process in $\mathcal{C}$. We call its law $\mathbb{P}^c_{0+}$. Now, take again

System (3.2) and let r go to 0. First, we obtain that the sum of the jumps happening just after the initial time (or in a finite time interval) is finite. Then we deduce that under $\mathbb{P}^c_{0+}$, $(X, \dot{X}, B)$ is a solution to (SOR) with starting condition $(0, 0)$.

In summary, we defined a law $\mathbb{P}^c_{0+}$ on $\mathcal{C}$ satisfying conditions $(*)$ and $(**)$, and thus $\tau_u > 0$ and $\tau_u \to 0$ almost surely. Besides, the joint law of τ_u and $(X_{\tau_u+t}, \dot{X}_{\tau_u+t})_{t\geq 0}$ under $\mathbb{P}^c_v$ converges weakly to that under $\mathbb{P}^c_{0+}$. In order to deduce the convergence of $\mathbb{P}^c_v$ to $\mathbb{P}^c_{0+}$, we just need to control what happens on $[0, \tau_u[$. More precisely, it is enough to control the velocity $\dot{X}$. Let us call M_u the sup of $\dot{X}_t$ on $[0, \tau_u[$. It will be enough to prove that when u is small, the variable M_u is small with high probability, uniformly on v small, in the following sense:

$$\forall \varepsilon > 0, \forall \delta > 0, \exists u_0, v_0 > 0, \forall 0 < u \leq u_0, \forall 0 < v \leq v_0, \quad \mathbb{P}^c_v(M_u \geq \delta) \leq \varepsilon.$$

Start from the basic observation $M_u \leq u + \sup_{s,t\in[0,\tau_u[} |B_t - B_s|$, where B is the underlying Brownian motion. It follows

$$\begin{aligned} \mathbb{P}^c_v(M_u \geq u + \delta) &\leq \mathbb{P}^c_v(\tau_u \geq \eta) + \mathbb{P}^c_v\left(\sup_{s,t\in[0,\eta)} |B_t - B_s| \geq \delta\right) \\ &\leq \mathbb{P}^c_v(\tau_u \geq \eta) + \varepsilon, \end{aligned}$$

for a well-chosen $\eta > 0$, independent of v. Now, by writing the right side in the form $\mathbb{P}^c_v(\tau_u \geq \eta) - \mathbb{P}^c_{0+}(\tau_u \geq \eta) + \mathbb{P}^c_{0+}(\tau_u \geq \eta) + \varepsilon$, and using $\tau_u \underset{u\to 0}{\to} 0$, $\mathbb{P}^c_{0+}$−a.s., we get that the following inequality

$$\mathbb{P}^c_v(M_u \geq u + \delta) \quad \leq \quad \mathbb{P}^c_v(\tau_u \geq \eta) - \mathbb{P}^c_{0+}(\tau_u \geq \eta) + 2\varepsilon$$

is satisfied for u small enough. Choose u_0, smaller than δ, such that the inequality is satisfied. Then, from the convergence of the law of τ_{u_0} under $\mathbb{P}^c_v$ to that under $\mathbb{P}^c_{0+}$, we get that for v smaller than some $v_0 > 0$, we have

$$\mathbb{P}^c_v(M_{u_0} \geq 2\delta) \leq 3\varepsilon.$$

Now it is clear that the inequality stays satisfied for $u < u_0$, which ends the proof. The law $\mathbb{P}^c_v$ converges weakly to $\mathbb{P}^c_{0+}$, and Theorem 1 is proved.

Finally, we should prove the uniqueness in Corollary 2. Consider any solution $(X, \dot{X}, B)$ to (SOR) with starting condition $(0, 0)$. If X were not coming back to zero at small times, then there wouldn't be any jumps for $\dot{X}$ at small times, thus X would behave like a Langevin process. But this is not possible as the Langevin process starting from zero with zero velocity does come back at zero at arbitrary small times. As a consequence, the process $(X, \dot{X})$ necessarily satisfies condition $(*)$. Now, the process $(X_{\tau_u+t}, \dot{X}_{\tau_u+t})_{t\geq 0}$ converges in law to $(X, \dot{X})$, thus the law of $(X, \dot{X})$ is an accumulation point of the family $(\mathbb{P}^c_v)_{v>0}$ when $v \to 0$. It must coincide with $\mathbb{P}^c_{0+}$. □

The rest of the section is devoted to the proof of Lemma 3. It can be sketched as follows. First, using renewal theory, we get, for any fixed $u > 0$, the convergence of the law of the process $(\dot{X}_{\tau_u+t})_{t\geq 0}$ to a law that can be described in a simple way. Then Lemma 2 allows, in a certain sense, to include negative times in this convergence result. The last step will be to prove that τ_u converges in law to a finite valued random variable.

3.1. Convergence of shifted processes

We recall the notation V_n for the (outgoing) velocity at the n-th bounce and S_n for its logarithm, for $n \geq 0$. We also write $\mathcal{N}_n$ for the translated velocity path starting at the n-th bounce and renormalized so as to start with speed one. That is, $\mathcal{N}_n$ is defined by

$$(\mathcal{N}_n(t))_{t\geq 0} := (V_n^{-1}\dot{X}(\zeta_n + V_n^2 t))_{t\geq 0}. \tag{3.3}$$

The process $\mathcal{N}_n$ is independent of $(\dot{X}_t)_{0\leq t\leq \zeta_n}$ and has law $\mathbb{P}_1^c$. The knowledge of the process X, or $\dot{X}$, is equivalent to the knowledge of the sequence $(S_n, \mathcal{N}_n)_{n\geq 0}$, or even just $(S_0, \mathcal{N}_0)$. But it is more convenient to first prove convergence results about (translations of) the sequence $(S_n, \mathcal{N}_n)_{n\geq 0}$, then deduce results about X, which we do.

We work with $\mathcal{C}^0 := \mathcal{C} \cup \emptyset$ and we define moreover, for $n < 0$, $(S_n, \mathcal{N}_n) := (-\infty, \emptyset)$, so that the sequence $(S, \mathcal{N}) := (S_n, \mathcal{N}_n)_{n\in\mathbb{Z}}$ lays in Ω_+, in the settings of Section 2.3. We call $\mathbf{P}_v$ its law on Ω_+ (or Ω), under $\mathbb{P}_v^c$. We also use the other notations of Section 2.3, such as $T_x(S) = \inf\{n, S_n \geq x\}$, which we will simply write T_x, or the spatial translation operator Θ_x, defined by (2.4). We now aim at establishing convergence results for the probabilities $\mathbf{P}_v \circ \Theta_x$.

First, observe that under $\mathbf{P}_v$ and for $n \geq 0$, $(S_{n+1}, \mathcal{N}_{n+1})$ is measurable with respect to $(S_n, \mathcal{N}_n)$, and thus $(S, \mathcal{N})$ is entirely determined by $(S_0, \mathcal{N}_0)$, which follows the law $\delta_{\ln v} \otimes \mathbb{P}_1^c$. In other words, there is a deterministic functional G such that $(S_n, \mathcal{N}_n)_{n\geq 0} = G(S_0, \mathcal{N}_0)$, and $\mathbf{P}_v$ is the law on Ω induced by the law $\delta_{\ln v} \otimes \mathbb{P}_1^c$ for $(S_0, \mathcal{N}_0)$. Write now $\mathbf{Q}$ for the law on Ω_+ induced by the law $m \otimes \mathbb{P}_1^c$ for $(S, \mathcal{N})$, where the measure m is the stationary law of the overshoot we introduced earlier, defined by (3.1).

Lemma 4. *For any real number x, we have*

$$(\mathbf{P}_v \circ \Theta_x)_+ \Rightarrow_{v\to 0^+} \mathbf{Q}$$

Proof. Consider the ascending ladder height process H defined at the beginning of Section 3. It is a random walk with positive jumps and finite expectation. It is nonarithmetic in the sense that its jumping law is not included in $d\mathbb{Z}$ for any $d > 0$ (nonarithmeticity is obvious for laws with densities). Renewal theory for random walks with positive jumps (see for example [11], p.62, or [8], p.355) gives the following result: the law of the overshoot over a level x, that is $H_{T_x(H)} - x$, converges to m when $x - H_0$ goes to infinity. This result is transmitted directly to the random walk $(S_n)_{n\geq 0}$, simply because it has the same overshoot: $S_{T_x} - x =$

$H_{T_x(H)} - x$. Under $\mathbf{P}_v$, we have $x - H_0 = x - \ln v \to_{v\to 0+} +\infty$. Hence, when v goes to 0+, the law of the variable $S_{T_x} - x$ under $\mathbf{P}_v$, or, equivalently, that of S_0 under $\mathbf{P}_v \circ \Theta_x$, converges to m.

Now, the usual Markov and scaling invariance properties show that for any x, v, under $\mathbf{P}_v \circ \Theta_x$, $(S_n - S_0, \mathcal{N}_n)_{n\geq 0}$ is independent of S_0 and has the same law as $(S_n, \mathcal{N}_n)_{n\geq 0}$ under $\mathbf{P}_1$. This altogether establishes the convergence of $(\mathbf{P}_v \circ \Theta_x)_+$ to $\mathbf{Q}$. □

Applying Lemma 2, we immediately deduce:

Corollary 3. *For any real number x, we have*

$$\mathbf{P}_v \circ \Theta_x \Rightarrow_{v\to 0^+} \mathbf{P}, \tag{3.4}$$

where $\mathbf{P}$ is the unique spatially stationary probability measure on Ω such that $\mathbf{P}_+ = \mathbf{Q}$.

Remark 1. *Call $\mathbf{P}^1$, resp. $\mathbf{Q}^1$, the projection of $\mathbf{P}$, resp. $\mathbf{Q}$, on the first coordinate. Call Θ^1 the spatial translation operator induced on the first coordinate (defined by $\Theta^1_x(\omega^1) := (\omega^1_{n+T_x} - x)_{n\in\mathbb{Z}}$). Then $\mathbf{Q}^1$ is the law of the random walk with starting position distributed according to m. Moreover, we have $\mathbf{P}^1_+ = \mathbf{Q}^1$, and $\mathbf{P}^1$ is spatially stationary. Similar arguments show that $\mathbf{P}^1$ is the unique spatially stationary measure such that $\mathbf{P}^1_+ = \mathbf{Q}^1$. We call it the law of the spatially stationary random walk.*

We now want to deduce Lemma 3 from Corollary 3. To this end, we have to understand how to reconstruct $\dot{X}$ from $\Theta_x(S, \mathcal{N})$. We start by working under $\mathbf{P}_v$, for some $v > 0$. We introduce an important variable, $\alpha_x := \tau_{e^x}$, the instant of the first bounce with speed greater than $\exp(x)$ for the process $(X, \dot{X})$.

Observe that the definition (3.3) of $\mathcal{N}_n$ induces that the length of the first arch of $\mathcal{N}_n$, that is $\zeta_1(\mathcal{N}_n)$, is equal to V_n^{-2} times the length of the $(1+n)$-th arch of $\dot{X}$. Writing α_x as the sum of the length of arches happening before time α_x, we get $\alpha_x = \sum_{n<T_x} V_n^{-2} \zeta_1(\mathcal{N}_n)$. We may also express α_x as a functional of $\Theta_x(S, \mathcal{N})$ by setting

$$\alpha_x = e^{2x} A(\Theta_x(S, \mathcal{N})), \tag{3.5}$$

where A is defined by

$$A(\omega) = \sum_{n<0} e^{2\omega^1_n} \zeta_1(\omega^2_n), \tag{3.6}$$

with the convention $\zeta_1(\emptyset) = 0$. Now, the process $(X_t, \dot{X}_t)_{t\geq\alpha_x}$ is given as the following functional of $\Theta_x(S, \mathcal{N})$:

$$\begin{cases} \dot{X}_t &= e^{S_{T_x}} \mathcal{N}_{T_x}(e^{-2S_{T_x}}(t - \alpha_x)) \\ X_t &= \int_{\alpha_x}^t \dot{X}_u \mathrm{d}u \end{cases}, \quad t \geq \alpha_x.$$

Now, let us work under $\mathbf{P}$. It is natural to keep the definition of α_x given by Formula (3.5). Please note however that the sum defining α_x now contains an infinite number of nonzero terms.

Lemma 5. *1)* $\mathbf{P}$*-almost surely, the time* α_x *is finite for any* $x > 0$*, and* α_x *goes to 0 when* x *goes to* $-\infty$*,*

2) The law of $(\alpha_x, S_{T_x}, \mathcal{N}_{T_x})$ *under* $\mathbf{P}_v$ *converges to that under* $\mathbf{P}$ *when* $v \to 0^+$*.*

The proof of Lemma 5 is postponed to the next subsections. Taking Lemma 5 for granted, we may proceed to the proof of Lemma 3.

Proof of Lemma 3. The first part of Lemma 5 enables us to define a process $(X_t, \dot{X}_t)_{t>0}$ on $\mathcal{C}^*$ by

$$\begin{cases} \dot{X}_t &= e^{S_{T_x}} \mathcal{N}_{T_x}(e^{-2S_{T_x}}(t-\alpha_x)) \\ X_t &= \int_{\alpha_x}^t \dot{X}_u \mathrm{d}u \end{cases} \quad \text{, for any } t, x \text{ such that } t \geq \alpha_x.$$

This construction is coherent. We call $\mathbb{P}^{c*}_{0+}$ its law on $\mathcal{C}^*$.

Under $\mathbb{P}^{c*}_{0+}$, the instant $\tau_u := \alpha_{\ln(u)}$ is the instant of the first bounce with speed greater than u. It is positive and converges a.s. to 0 when u goes to 0. Besides, the law of $S_{T_{\ln u}} - \ln u$ is equal to m, because by spatial stationarity, $\mathbf{P} \circ \Theta_{\ln u} = \mathbf{P}$. Now, take $x = \ln u$ and $t \geq \alpha_x = \tau_u$ in the formula above. It follows that under $\mathbb{P}^{c*}_{0+}$, the law of $\ln(\dot{X}_{\tau_u}/u)$ is m, and that conditionally on $\dot{X}_{\tau_u} = v$, the process $(X_{\tau_u+t}, \dot{X}_{\tau_u+t})_{t\geq 0}$ has law $\mathbb{P}^c_v$. We leave to the reader the verification that it is also independent of $(X_s, \dot{X}_s)_{0<s<\tau_u}$. Hence the law $\mathbb{P}^{c*}_{0+}$ satisfies conditions $(*)$ and $(**)$.

The second part of the lemma proves that for any fixed $u > 0$, the joint law of τ_u and $(X_{\tau_u+t}, \dot{X}_{\tau_u+t})_{t\geq 0}$ under $\mathbb{P}^c_v$ converges to that under $\mathbb{P}^{c*}_{0+}$, as laws on $\mathcal{C}$. □

Finally, all we have to do is to prove Lemma 5. By scaling, it suffices to show that α_0 is finite $\mathbf{P}-$a.s. to prove the first part. We also can suppose $x = 0$ for the second part. Finally, note that under $\mathbf{P}$, we have almost surely $T_0 = 0$ and hence $\alpha_0 = A(\Theta_0(S, \mathcal{N})) = A(S, \mathcal{N})$.

This proof will be based on a more explicit description of the spatially stationary measures $\mathbf{P}$ and $\mathbf{P}^1$. We must distinguish between critical and supercritical cases.

3.2. Proof of Lemma 5 in the supercritical case

Throughout this section we suppose that $c > c_{crit}$. Therefore the drift $\mu = \mathbf{P}_1(S_1 - S_0) = \frac{\pi}{\sqrt{3}} + \ln c$ is strictly positive. We propose a construction of $\mathbf{P}$ based on the introduction of a temporally stationary measure on Ω. If one just considers the first coordinate, this is a construction of the law of the spatially stationary random walk $\mathbf{P}^1$, using the temporally stationary random walk.

First, let us define this temporally stationary random walk. Introduce P_0, law of the random walk $(S_n)_{n\in\mathbb{Z}}$ indexed by $\mathbb{Z}$, where $S_0 = 0$ and $(S_{n+1} - S_n)_{n\in\mathbb{Z}}$

is i.i.d with common law that of the generic step. Then write P_x for the law of $(x+S_n)_{n\in\mathbb{Z}}$ under P_0, and set

$$P_\lambda = \int_{\mathbb{R}} P_x \mathrm{d}x.$$

This σ-finite measure is (temporally) stationary, in the sense that for any $k\in\mathbb{Z}$, the sequences $(S_n)_{n\in\mathbb{Z}}$ and $(S_{k+n})_{n\in\mathbb{Z}}$ have the same law under P_λ. This term "law" has to be understood in a generalized sense, that is in settings where we allow the laws to be not only probability measures but more generally σ-finite measures. We call this generalized process of law P_λ the (temporally) stationary random walk.

Now start again the same construction, but with adding the second coordinate. We first recall that under $\mathbf{P}_v$ and for $n\ge 0$, $(S_{n+1},\mathcal{N}_{n+1})$ is measurable with respect to $(S_n,\mathcal{N}_n)$; we have $(S_{n+1},\mathcal{N}_{n+1}) = F(S_n,\mathcal{N}_n)$, where F is a deterministic functional. For $n\le 0$, consider Π_x^n for the law of $(S_k,\mathcal{N}_k)_{k\ge n}$, where $\mathcal{N}_n \overset{d}{=} \mathbb{P}_1^c$, $S_n = x-\ln(V_{-n}(\mathcal{N}_n))$ (recall that $V_{-n}(\mathcal{N}_n)$ denotes the velocity of the particle after the $(-n)-$th bounce), and the sequence $(S_k,\mathcal{N}_k)_{k>n}$ is given by $(S_k,\mathcal{N}_k) = F^{k-n}(S_n,\mathcal{N}_n)$.

It should be clear that the laws Π_x^n, $n\le 0$, are compatible. Kolmogorov's existence theorem entails the existence of Π_x, the law on Ω under which $(S_k,\mathcal{N}_k)_{k\ge n}$ has law Π_x^n for any $n\le 0$. Then we just define Π_λ by

$$\Pi_\lambda := \int \Pi_y \mathrm{d}y.$$

Again, this is a σ-finite (temporally) stationary measure. Besides, the law of the first coordinate S under Π_λ is P_λ.

Now, consider the event $\{T_x = n\}$, for $x\in\mathbb{R}$ and $n\in\mathbb{Z}$. It should be clear that its measure under P_λ is independent of x and n. The following lemma gives its value and states a link between Π_λ and $\mathbf{P}$, as well as between P_λ and $\mathbf{P}^1$ (recall Remark 1 after Corollary 3 for the introduction of the law of the spatially stationary random walk, $\mathbf{P}^1$).

Lemma 6. *Suppose $c > c_{crit}$.*
1) We have $P_\lambda(T_0=0) = \Pi_\lambda(T_0=0) = \mu \in (0,\infty)$.
2) We have $\mathbf{P}^1(\cdot) = P_\lambda(\cdot|T_0=0)$ and $\mathbf{P}(\cdot) = \Pi_\lambda(\cdot|T_0=0)$.

Proof. Recall that $\mu = \mathbf{P}_1(S_1-S_0) = \frac{\pi}{\sqrt{3}} + \ln c$ is strictly positive and finite. We still write $(H_n)_{n\ge 0}$ for the (strictly) ascending ladder height process of the sequence $(S_n)_{n\ge 0}$. Its drift $\mu_H = \mathbf{P}_1(H_1-H_0)$ is also strictly positive and finite. A result of Woodroofe [22] and Gut [10] states that, for any $y>0$, we have

$$\frac{1}{\mu_H}P_0(H_1>y) = \frac{1}{\mu}P_0\left(\inf_{n\ge 1} S_n > y\right). \tag{3.7}$$

The calculation below follows:

$$\begin{aligned}\Pi_\lambda(T_0=0) &= P_\lambda(T_0=0)\\ &= \int_0^\infty \mathrm{d}x P_x\left(\sup_{n\le -1} S_n<0\right)\\ &= \int_0^\infty \mathrm{d}x P_0\left(\inf_{n\ge 1} S_n>x\right)\\ &= \mu\int_0^\infty \frac{\mathrm{d}x}{\mu_H}P_0(H_1>x)\\ &= \mu,\end{aligned}$$

where we used a symmetry property in the third line. As $\mu\in(0,\infty)$ we can condition the infinite measure on the event $\{T_0=0\}$ to get the probability measure

$$\Pi_\lambda(\cdot|T_0=0):=\frac{1}{\mu}\Pi_\lambda(\cdot\mathbb{1}_{T_0=0}).$$

We leave to the reader the simple verification that this measure on Ω is spatially stationary in the sense of Definition 1 and is projected on the measure $\mathbf{Q}$ on Ω_+. Thus it must coincide with $\mathbf{P}$, by Corollary 3. □

We may now prove the first part of Lemma 5.

Proof of Lemma 5.1). Recall that we need to prove the $\mathbf{P}$-a.s. finiteness of the sum $A(S,\mathcal{N})$.

We start by proving that it is finite Π_x-almost surely, for a fixed x. Under Π_x, the sequence $(\zeta_1(\mathcal{N}_k))_{k\in\mathbb{Z}}$ is i.i.d with law that of ζ_1 under $\mathbb{P}_1^c$. Using the Borel-Cantelli lemma and estimate (2.3), we get that there are Π_x-a.s. only a finite number of $k>0$ such that $\zeta_1(\mathcal{N}_{-k})$ is bigger than $\exp(\sqrt{k})$. On the other hand, the sequence $(S_{-k})_{k\ge 0}$ under Π_x is a simple random walk, with an almost sure linear decay. Hence, the sum $A(S,\mathcal{N})$ is finite Π_x-a.s. It follows that it is also finite Π_λ-almost surely (by integration) and $\mathbf{P}$-almost surely (by conditioning on a nontrivial event). □

For Lemma 5.2), we need to prove the weak convergence of the law of $(\alpha_0,S_{T_0},\mathcal{N}_{T_0})$ under $\mathbf{P}_v$ to that under $\mathbf{P}$, when $v\to 0^+$. We start by introducing another notation,

$$\alpha_{x,y}:=\alpha_y-\alpha_x=\sum_{T_x\le n<T_y} V_n^{-2}\zeta_1(\mathcal{N}_n)\qquad,\text{ for } x<y.$$

It is clear that under $\mathbf{P}$, as well as under $\mathbf{P}_v$, we have almost surely $\alpha_x \underset{x\to-\infty}{\longrightarrow} 0$ and $\alpha_{x,y}\underset{x\to-\infty}{\longrightarrow}\alpha_y$. We also have a uniform convergence result: the law of the time α_x under $\mathbf{P}_v$ converges in probability to 0 when x goes to $-\infty$, *uniformly on* v, in the following sense:

$$\forall\varepsilon>0,\forall\eta>0,\exists x_0,\forall x\le x_0,\forall v>0,\quad \mathbf{P}_v(\alpha_x\ge\varepsilon)\le\eta. \tag{3.8}$$

Indeed, for any given $\varepsilon > 0$ and $\eta > 0$, we may choose y_0 such that $m([0, y_0]) \geq 1 - \eta$. Now, take $v > 0$. If $v > \exp(x)$, then $\alpha_x = 0$, and there is nothing to prove. We suppose $v \leq \exp(x)$. From a scaling property, for any $y \geq 0$, we have

$$\begin{aligned} \mathbf{P}_v(\alpha_x \geq \varepsilon) &= \mathbf{P}_{ve^y}(\alpha_{x+y} \geq \varepsilon e^{2y}) \\ &\leq \mathbf{P}_{ve^y}(\alpha_{x+y} \geq \varepsilon). \end{aligned}$$

Besides, under $\mathbf{P}_{ve^y}$, we have $T_{\ln v} = 0$ and thus $\alpha_{x+y} = \alpha_{\ln v, x+y}$. Hence, we have

$$\begin{aligned} \mathbf{P}_v(\alpha_x \geq \varepsilon) &\leq \int_{\mathbb{R}_+} m(\mathrm{d}y)\mathbf{P}_{ve^y}(\alpha_{\ln v, x+y} \geq \varepsilon) \\ &\leq \eta + \int_{[0,y_0]} m(\mathrm{d}y)\mathbf{P}_{ve^y}(\alpha_{\ln v, x+y_0} \geq \varepsilon) \\ &\leq \eta + \int_{\mathbb{R}_+} m(\mathrm{d}y)\mathbf{P}_{ve^y}(\alpha_{\ln v, x+y_0} \geq \varepsilon) \\ &\leq \eta + \mathbf{P}(\alpha_{\ln v, x+y_0} \geq \varepsilon) \\ &\leq \eta + \mathbf{P}(\alpha_{x+y_0} \geq \varepsilon), \end{aligned}$$

where the next to last line is a disintegration formula for $\mathbf{P}$ at time $T_{\ln v}$ (recall that the law of $S_{T_{\ln v}} - \ln v$ under $\mathbf{P}$ is m). Now, for x small enough, and uniformly on v, we get $\mathbf{P}_v(\alpha_x \geq \varepsilon) \leq 2\eta$. The uniform convergence result is proved.

We are ready to tackle the proof of Lemma 5.2).

Proof of Lemma 5.2). It is enough to prove the convergence of the expectation $\mathbf{P}_v(f(S_{T_0}, \mathcal{N}_{T_0}), \alpha_0 \geq a)$ to $\mathbf{P}(f(S_{T_0}, \mathcal{N}_{T_0}), \alpha_0 \geq a)$ for any continuous functional $f : \mathbb{R} \times \mathcal{C} \to [0, 1]$ and any $a > 0$.

But Corollary 3 induces the convergence of the law of $(\alpha_{x,0}, S_{T_0}, \mathcal{N}_{T_0})$ under $\mathbf{P}_v$ to that under $\mathbf{P}$. It follows that $\mathbf{P}_v(f(S_{T_0}, \mathcal{N}_{T_0}), \alpha_{x,0} \geq a)$ goes to $\mathbf{P}(f(S_{T_0}, \mathcal{N}_{T_0}), \alpha_{x,0} \geq a)$ when v goes to 0. This term in turn converges to $\mathbf{P}(f(S_{T_0}, \mathcal{N}_{T_0}), \alpha_0 \geq a)$ when x goes to $-\infty$. As $\alpha_0 \geq \alpha_{x,0}$ for any x, it follows

$$\liminf_{v \to 0} \mathbf{P}_v(f(S_{T_0}, \mathcal{N}_{T_0}), \alpha_0 \geq a) \geq \mathbf{P}(f(S_{T_0}, \mathcal{N}_{T_0}), \alpha_0 \geq a). \tag{3.9}$$

On the other hand, for any $\eta > 0$, choose $\varepsilon > 0$ such that $\mathbf{P}(\alpha_0 \in [a - \varepsilon, a[) \leq \eta$, and then choose x, given by the uniform convergence (3.8), such that for any $v > 0$, $\mathbf{P}_v(\alpha_x \geq \varepsilon) \leq \eta$. Then, considering the inequality

$$\begin{aligned} &\mathbf{P}_v(f(S_{T_0}, \mathcal{N}_{T_0}), \alpha_0 \geq a) \\ \leq\ &\mathbf{P}_v(f(S_{T_0}, \mathcal{N}_{T_0}), \alpha_{x,0} \geq a - \varepsilon) + \mathbf{P}_v(f(S_{T_0}, \mathcal{N}_{T_0}), \alpha_x \geq \varepsilon) \end{aligned}$$

and taking the lim sup, we get

$$\begin{aligned} \limsup_{v \to 0} \mathbf{P}_v(f(S_{T_0}, \mathcal{N}_{T_0}), \alpha_0 \geq a) &\leq \mathbf{P}(f(S_{T_0}, \mathcal{N}_{T_0}), \alpha_{x,0} \geq a - \varepsilon) + \eta \\ &\leq \mathbf{P}(f(S_{T_0}, \mathcal{N}_{T_0}), \alpha_0 \geq a) + 2\eta. \end{aligned}$$

This together with (3.9) gives the desired result. $\square$

We finish this subsection with a corollary of Lemma 6.

Corollary 4. *Under $\mathbf{P}^1$, conditionally on $S_0 = x \geq 0$, the sequence $(-S_{-n})_{n\geq 0}$ has the law of the random walk starting from $-x$ and conditioned to stay positive at times $n \geq 1$.*

Proof. Under P_λ and conditionally on $S_0 = x$, the sequence $(-S_{-n})_{n\geq 0}$ has the law of the random walk starting from $-x$. The event $\{T_0 = 0\}$, which is also equal to the event $\{S_0 > 0, \forall n < 0, S_n < 0\}$, has a positive and finite probability when $x \geq 0$. The expression of $\mathbf{P}^1$ given in Lemma 6 directly implies the corollary. □

3.3. Proof of Lemma 5 in the critical case

In the critical case, we certainly can define P_λ and Π_λ as before, but under these measures the time T_0 is almost surely equal to $-\infty$. Lemma 6 thus fails, and so does the previous construction of $\mathbf{P}^1$ and $\mathbf{P}$.

However, an analogue of Corollary 4 will stay true and induce another construction of the law of the spatially stationary random walk $\mathbf{P}^1$. We will then use it to prove again the $\mathbf{P}-$almost sure finiteness of α_0, and Lemma 5 will follow from the same arguments as before. Throughout this subsection we assume $c = c_{crit}$.

3.3.1. The spatially stationary random walk in the critical case.

In order to formulate the analogue of Corollary 4, we need to define the "random walk conditioned to stay positive" for a random walk with null drift, for which the event of staying positive for all positive times has probability 0. This is done in [5]. We recall it here briefly.

Write as usual P_x for the law of the random walk starting from position x. If you write $(D_n)_{n\geq 0}$ for the strictly descending ladder height process (defined in the exact similar way as the strictly ascending ladder height process, and also equal to the opposite of the strictly ascending ladder height process of $\widehat{S} := -S$) , the renewal function h is defined by

$$h(x) := \sum_{k=0}^{\infty} P_x(D_k \geq 0).$$

In particular h is non-decreasing, right-continuous, and we have $h(0) = 1$ and $h(x) = 0$ for $x < 0$. The renewal function is invariant for the random walk killed as it enters the negative half-line. It enables us to define the process conditioned on never entering $(-\infty, 0)$, thanks to a usual $h-$transform, in the sense of Doob. That is, the law of this process starting from $x > 0$, written $P_x^{\uparrow 0}$, is defined by

$$P_x^{\uparrow 0}(f(S)) = \frac{1}{h(x)} P_x\Big(f(S)h(S_n), \inf_{k\leq n} S_k \geq 0\Big) \tag{3.10}$$

for any $f(S) = f(S_0, ..., S_n)$ functional of the first n steps. For any $a \in \mathbb{R}$ and $x > a$, we also write $P_x^{\uparrow a}$ for the law of the random walk starting from $x > a$ and conditioned on never entering $(-\infty, a)$, defined in the exact same way, by

$$P_x^{\uparrow a}(f(S)) = \frac{1}{h(x-a)} P_x\Big(f(S)h(S_n - a), \inf_{k \le n} S_k \ge a\Big) \qquad (3.11)$$

for any $f(S) = f(S_0, ..., S_n)$ functional of the first n steps. The only other thing we will need to know about h is the following sub-additive inequality, which is a consequence of a Markov property:

$$h(x+a) - h(x) \le h(a), \qquad x, a > 0. \qquad (3.12)$$

Recall that μ_H is the drift of the strictly ascending ladder height process and write $p(x, y)$ for the transition densities of the random walk. The following proposition gives a disintegration description of the spatially stationary random walk, which is very similar to that of the spatially stationary Lévy process introduced by Bertoin and Savov in [6].

Proposition 1. *The measure*

$$\nu(\mathrm{d}x\mathrm{d}y) := \frac{1}{\mu_H} p(0, x+y) \mathbb{1}_{x \ge 0, y \ge 0} h(x) \mathrm{d}x\mathrm{d}y$$

is a probability law. The law of $\mathbf{P}^1$ is determined by:
- *Under $\mathbf{P}^1$, $(-S_{-1}, S_0)$ has the law ν.*
- *Conditionally on $-S_{-1} = x$ and $S_0 = y$, the processes $(-S_{-n-1})_{n \ge 0}$ and $(S_n)_{n \ge 0}$ are independent, the law of $(-S_{-n-1})_{n \ge 0}$ is $P_x^{\uparrow 0}$, that of $(S_n)_{n \ge 0}$ is P_y.*

The measure ν is nothing else than the stationary joint law of the overshoot and the undershoot. The proof of this proposition will last until the end of the subsection. As a preliminary, we introduce a crucial though rather simple lemma.

Lemma 7. *For any $0 \le a \le x$, we have:*

$$P_x^{\uparrow 0}\Big(\inf_{n \ge 0} S_n \ge a\Big) = \frac{h(x-a)}{h(x)} \qquad (3.13)$$

$$P_x^{\uparrow 0}\Big(\cdot \mid \inf_{n \ge 0} S_n \ge a\Big) = P_x^{\uparrow a}(\cdot). \qquad (3.14)$$

Proof. By expressing the event $\{\inf_{k \ge 0} S_k \ge a\}$ as the limit of the events $\{\inf_{0 \le k \le n} S_k \ge a\}$, we get

$$\begin{aligned} & P_x^{\uparrow 0}\Big(\inf_{0 \le k \le n} S_k \ge a\Big) \\ = & \frac{1}{h(x)} P_x\Big(h(S_n), \inf_{0 \le k \le n} S_k \ge a\Big) \\ = & \frac{1}{h(x)} P_x\Big(h(S_n - a), \inf_{0 \le k \le n} S_k \ge a\Big) \\ & + \frac{1}{h(x)} P_x\Big(h(S_n) - h(S_n - a), \inf_{0 \le k \le n} S_k \ge a\Big). \end{aligned}$$

The first term of the sum is equal to $\frac{h(x-a)}{h(x)}$ because the function $h(\cdot - a)$ is invariant for the random walk killed when hitting $(-\infty, a)$. The second term is positive and bounded from above by $\frac{h(a)}{h(x)}P_x(\inf_{0\le k\le n} S_k \ge a)$, which goes to 0 when n goes to $+\infty$. This proves equation (3.13). Then (3.14) is straightforward: Indeed, for $f(S) = f(S_0, ..., S_n)$ functional of the first n steps, we have:

$$\begin{aligned}
& P_x^{\uparrow 0}\Big(f(S) | \inf_{k\ge 0} S_k \ge a\Big) \\
= & \frac{1}{P_x^{\uparrow 0}\Big(\inf_{k\ge 0} S_k \ge a\Big)} P_x^{\uparrow 0}\left(f(S) P_{S_n}^{\uparrow 0}\Big(\inf_{k\ge 0} S_k \ge a\Big), \inf_{0\le k\le n} S_k \ge a\right) \\
= & \frac{h(x)}{h(x-a)} \cdot \frac{1}{h(x)} P_x\left(f(S) h(S_n) \frac{h(S_n - a)}{h(S_n)}, \inf_{0\le k\le n} S_k \ge a\right) \\
= & P_x^{\uparrow a}(f(S)).
\end{aligned}$$

□

Now, recall that the invariance property of h yields that, for any $x \ge 0$, we have

$$h(x) = P_x(h(S_1)\mathbb{1}_{S_1\ge 0}).$$

Define $\overline{h}$ by $\overline{h}(x) := P_x(h(S_1), S_1 \ge 0)$ for *any real number* x. Thus for $x \ge 0$, $\overline{h}$ and h coincide, but for $x < 0$ they certainly don't. This enables us to define, for any $x, a \in \mathbb{R}$, the law $P_x^{\uparrow a}$ of the random walk starting from x and conditioned on never entering $(-\infty, a)$ *at times* $n \ge 1$, by the formula:

$$P_x^{\uparrow a}(f(S)) = \frac{1}{\overline{h}(x-a)} P_x\Big(f(S) h(S_n - a), \inf_{1\le k\le n} S_k \ge a\Big) \tag{3.15}$$

for any functional $f(S) = f(S_0, ..., S_n)$. This definition is of course consistent with our previous notations. The following generalization of Lemma 7 and its corollary are consequences of straightforward calculations, that we leave to the interested reader

Lemma 8. *For any $y \le a$, any $x \in \mathbb{R}$, we have*

$$P_x^{\uparrow y}(\inf_{n\ge 1} S_n \ge a) = \frac{\overline{h}(x-a)}{\overline{h}(x-y)} \tag{3.16}$$

$$P_x^{\uparrow y}(\cdot | \inf_{n\ge 1} S_n \ge a) = P_x^{\uparrow a}(\cdot) \tag{3.17}$$

Corollary 5. *Write ν_- (resp. ν_+) for the first (resp. second) marginal of ν. These measures on $\mathbb{R}_+$ are given for $x, y > 0$, by*

$$\begin{aligned}
\nu_-(\mathrm{d}x) &= \frac{1}{\mu_H} h(x) P_0(S_1 \ge x)\mathrm{d}x. \\
\nu_+(\mathrm{d}y) &= \frac{1}{\mu_H} \overline{h}(-y)\mathrm{d}y.
\end{aligned}$$

Moreover,

$$\begin{aligned} P_{-\nu_-}(S_1 \in \mathrm{d}y | S_1 \geq 0) &= \nu_+(\mathrm{d}y) \\ P^{\uparrow 0}_{-\nu_+}(\mathrm{d}x) &= \nu_-(\mathrm{d}x), \end{aligned}$$

where we have written $P_{-\nu_-}(...)$ *for* $\int P_{-x}(...)\nu_-(\mathrm{d}x)$*, as well as* $P_{-\nu_+}(...)$ *for* $\int P_{-x}(...)\nu_+(\mathrm{d}x)$.

This corollary should make the introduction of the measure ν in the proposition more transparent. Indeed, it gives us two alternative ways of defining the measure $\mathbf{P}^1$. First, take S_0 distributed according to ν_+ and, conditionally on $S_0 = y$, take $(S_n)_{n\geq 0}$ of law P_y and $(-S_{-n})_{n\geq 0}$ independent and of law $P^{\uparrow 0}_{-y}$ (in the sense defined just before). Second, take $-S_{-1}$ distributed according to ν_- and, conditionally on $S_{-1} = -x$, take $(S_{n-1})_{n\geq 0}$ of law P_{-x} conditioned on having a first jump no smaller than x, and $(-S_{-n-1})_{n\geq 0}$ independent and of law $P^{\uparrow 0}_{x}$.

Proof of the proposition. We need to prove three things, the fact that ν is a probability measure (that is, has mass one), the fact that $\mathbf{P}^1$ is spatially stationary, and the equality $\mathbf{P}^1_+ = \mathbf{Q}$. We start with the spatial stationarity. Fix $a > 0$. We should prove that $S = (S_n)_{n\in\mathbb{Z}}$ and $R := \Theta_a(S) = (S_{T_a+n} - a)_{n\in\mathbb{Z}}$ have the same law under $\mathbf{P}^1$.

We introduce the notation L_a for the instant of the last passage under level a for the process S. Besides, observe that T_a is also equal to the instant of the last passage under level a for the process $(-R_{-n})_{n\geq 0}$. *Suppose that we proved* that $((T_a, -R_{-n})_{0\leq n\leq T_a})$ has the same law as the process $(L_a, (S_n)_{0\leq n\leq L_a})$ under $P^{\uparrow 0}_{-\nu_+}$. Then, conditionally on $-R_{-T_a} = z$, it is clear that the process $(-R_{-n-T_a})_{n\geq 0} = (a - S_{-n})_{n\geq 0}$ is independent of $(-R_{-n})_{0\leq n\leq T_a}$ and follows the law $P^{\uparrow a}_z$. Besides, for a process S under $P^{\uparrow 0}_{-\nu_+}$, conditionally on $S_{L_a} = z$, the process $(S_{n+L_a})_{n\geq 0}$ is independent from $(S_n)_{0\leq n\leq L_a}$ and follows the law $P^{\uparrow a}_z$. This altogether proves that the process $(-R_{-n})_{n\geq 0}$ follows the law $P^{\uparrow 0}_{-\nu_+}$. Finally, from a Markov property, it is clear that given $R_0 = y$, the process $(R_n)_{n\geq 0}$ is independent of $(R_n)_{n\leq 0}$ and follows the law P_y, thus the law of $(R_n)_{n\in\mathbb{Z}}$ is $\mathbf{P}^1$.

Therefore, the only thing we still need to prove is the following duality property[3]: the variable $(T_a, (-R_{-n})_{0\leq n\leq T_a})$ has the same law as the variable $(L_a, (S_n)_{0\leq n\leq L_a})$ for a process S of law $P^{\uparrow 0}_{-\nu_+}$. Fix $n \geq 0$ and $f : \mathbb{R}^{n+1} \to \mathbb{R}$ a positive continuous functional. We should prove the following equality:

$$\mathbf{P}^1(f((-R_{-k})_{0\leq k\leq n})\mathbb{1}_{T_a=n}) = P^{\uparrow 0}_{-\nu_+}(f((S_k)_{0\leq k\leq n})\mathbb{1}_{L_a=n}).$$

[3]This property also finds its analogue in [6], in their Theorem 2.

The case $n = 0$ is particular and follows from this calculation:

$$\begin{aligned} P^{\uparrow 0}_{-\nu_+}(-S_0 \in \mathrm{d}x, L_a = 0) &= P^{\uparrow 0}_{-x}(\inf_{k\geq 1} S_k \geq a)\nu_+(\mathrm{d}x) \\ &= \frac{1}{\mu_H}\overline{h}(-x)\frac{\overline{h}(-a-x)}{\overline{h}(-x)}\mathrm{d}x \\ &= \nu_+(a+\mathrm{d}x) = \mathbf{P}^1(R_0 \in \mathrm{d}x, T_a = 0). \end{aligned}$$

In the case $n > 0$, we write $\tilde{f}((S_k)_{0\leq k\leq n}) := f((a - S_{n-k})_{0\leq k\leq n})$, the usual duality property for random walks stating

$$P_x(f(S)\mathbb{1}_{a-S_n\in\mathrm{d}y})\mathrm{d}x = P_y(\tilde{f}(S)\mathbb{1}_{a-S_n\in\mathrm{d}x})\mathrm{d}y.$$

We are ready to calculate

$$\begin{aligned} \mathbf{P}^1(f((-R_{-k})_{0\leq k\leq n})\mathbb{1}_{T_a=n}) &= \mathbf{P}^1(f((-R_{-k})_{0\leq k\leq n})\mathbb{1}_{T_a=n}) \\ &= \int\int_{\mathbb{R}_+\times[0,a)} \nu_2(\mathrm{d}x\otimes\mathrm{d}y), \end{aligned}$$

where $\nu_2(\mathrm{d}x\otimes\mathrm{d}y)$ is equal to

$$\begin{aligned} &\nu_+(\mathrm{d}y)P_y(\tilde{f}((S_k)_{0\leq k\leq n}), S_n - a \in \mathrm{d}x, \forall 0\leq i<n, S_i \leq a) \\ = &\frac{1}{\mu_H}\overline{h}(-y)\mathrm{d}xP_{-x}(f((S_k)_{0\leq k\leq n}), a - S_n \in \mathrm{d}y, \forall 0 < i \leq n, S_i \geq 0) \\ = &\frac{\overline{h}(-y)\mathrm{d}x}{h(a-y)\mu_H}P_{-x}\Big(f((S_k)_{0\leq k\leq n})h(S_n), a - S_n \in \mathrm{d}y, \forall 0 < i \leq n, S_i \geq 0\Big). \end{aligned}$$

Using then (3.15) and (3.16), it follows

$$\begin{aligned} &\mathbf{P}^1(f((-R_{-k})_{0\leq k\leq n})\mathbb{1}_{T_a=n}) \\ = &\int\int_{\mathbb{R}_+\times[0,a)} \nu_+(\mathrm{d}x)P^{\uparrow 0}_{-x}\big(f((S_k)_{0\leq k\leq n}), a - S_n \in \mathrm{d}y\big)P^{\uparrow 0}_{a-y}\Big(\inf_{k\geq 1} S_k \geq a\Big) \\ = &\int_{\mathbb{R}_+} \nu_+(\mathrm{d}x)P^{\uparrow 0}_{-x}\Big(f((S_k)_{0\leq k\leq n}), S_n < a, \inf_{k>n} S_k \geq a\Big) \\ = &P^{\uparrow 0}_{-\nu_+}\big(f((S_k)_{0\leq k\leq n})\mathbb{1}_{L_a=n}\big). \end{aligned}$$

The measure $\mathbf{P}^1$ is thus spatially stationary.

Now the two facts that ν has mass one and that $\mathbf{P}^1_+ = \mathbf{Q}^1$ both follow from the equality

$$\overline{h}(-y) = P_0(H_1 \geq y)$$

for $y \geq 0$ (recall that H is the strictly ascending ladder height process). Fix some $y \geq 0$. We already know from (3.16) that $\overline{h}(-y) = P^{\uparrow 0}_0(\inf_{n\geq 0} S_n \geq y)$, thus we should prove

$$P_0(H_1 \in \mathrm{d}y) = P^{\uparrow 0}_0(\inf_{n\geq 0} S_n \in \mathrm{d}y). \tag{3.18}$$

This will be a consequence from another duality argument. Write T_{inf} for the instant when S hits its minimum on times $n \geq 1$. Write $\tilde{T}_1 := \inf\{n > 0, S_n > S_0\}$ (so that $S_{\tilde{T}_1} = H_1$). Then $(S_k)_{0\leq k\leq \tilde{T}_1}$ under P_0 and $(S_k)_{0\leq k\leq T_{inf}}$ under $P_0^{\uparrow 0}$ are in duality. Indeed, fix $n > 0$ and $f(S) = f((S_k)_{0\leq k\leq n})$ a positive continuous functional. Write also $\tilde{f}((S_k)_{0\leq k\leq n}) := f((S_n - S_{n-k})_{0\leq k\leq n})$. Then,

$$\begin{aligned}
& P_0^{\uparrow 0}(f(S)\mathbb{1}_{\{T_{inf}=n\}}) \\
= \; & P_0^{\uparrow 0}\Big(f(S), \inf_{1\leq k\leq n-1} S_k > S_n, \inf_{k\geq n+1} S_k \geq S_n\Big) \\
= \; & P_0^{\uparrow 0}\left(f(S)P_x^{\uparrow 0}\Big(\inf_{k\geq 1} S_k \geq x\Big)\Big|_{x=S_n}, \inf_{1\leq k\leq n-1} S_k > S_n\right) \\
= \; & P_0^{\uparrow 0}\left(\frac{f(S)}{h(S_n)}, \inf_{1\leq k\leq n-1} S_k > S_n\right) \\
= \; & P_0\Big(f(S), \inf_{1\leq k\leq n-1} S_k > S_n \geq 0\Big) \\
= \; & P_0\Big(\tilde{f}(S), \sup_{1\leq k\leq n-1} S_k < 0, S_n \geq 0\Big) \\
= \; & P_0\Big(\tilde{f}(S), \sup_{1\leq k\leq n-1} S_k < 0, S_n > 0\Big) \\
= \; & P_0\big(\tilde{f}(S)\mathbb{1}_{\{\tilde{T}_1=n\}}\big).
\end{aligned}$$

This duality property implies in particular (3.18). □

3.3.2. Finiteness of α_0 in the critical case.

The only thing we actually need from the last subsection is the fact that under $\mathbf{P}^1$ (or, equivalently, under $\mathbf{P}$), the sequence $(-S_{-n})_{n\geq 1}$ is a random walk conditioned to stay positive, with some initial law. The paper [12] gives very precise results about the behavior of this random walk conditioned to stay positive, and we deduce in particular the following rough bounds that are sufficient for our purposes:

Lemma 9. *For any $\varepsilon > 0$, we have*

$$n^{-\frac{1}{2}+\varepsilon} S_{-n} \to -\infty \tag{3.19}$$

when $n \to \infty$, $\mathbf{P}$-a.s.

We now work under $\mathbf{P}$ and we recall that α_0 is then given by

$$\alpha_0 = \sum_{n<0} e^{2S_n}\zeta_1(\mathcal{N}_n).$$

We write $L_n := e^{2S_n}\zeta_1(\mathcal{N}_n)$ for the duration of the arch of index n. We need to transfer the results about the behavior of (S_{-n}) to results about the behavior of (L_{-n}). This is made possible by the following lemma:

Lemma 10. *1) Under* $\mathbf{P}$ *and conditionally on a realization* $(S_n)_{n\in\mathbb{Z}} = (s_n)_{n\in\mathbb{Z}}$*, the variables* $(L_n)_{n\in\mathbb{Z}}$ *are mutually independent, and the law of* L_n *is that of* ζ_1 *under* $\mathbb{P}^c_{\exp(s_n)}(\cdot|V_1 = \exp(s_{n+1}))$*.*

2) If $u, v \leq a$ *for some real number* a*, then*

$$\mathbb{P}^c_u(\zeta_1 > ta^2 | V_1 = cv) \leq \frac{16\sqrt{2}}{3\sqrt{\pi}} t^{-\frac{3}{2}}. \tag{3.20}$$

Proof. The result of the first part is easy for $(L_n)_{n\geq 0}$, and we get the result for $(L_n)_{n\in\mathbb{Z}}$ by spatial stationarity.

For the second part, recall that the law of the couple (ζ_1, V_1) under $\mathbb{P}^c_u$ is known (see Lemma 1, Formulas (2.1) and (2.2)). We obtain, explicitly:

$$\begin{aligned}\frac{1}{\mathrm{d}s}\mathbb{P}^c_u\,(\zeta_1 \in \mathrm{d}s | V_1 = cv) \\ &= \frac{\sqrt{2}(u^3+v^3)}{s^2 u^{\frac{1}{2}} v^{\frac{1}{2}}} \exp\Big(-2\frac{v^2 - uv + u^2}{s}\Big) \int_0^{\frac{4uv}{s}} e^{-\frac{3\theta}{2}} \frac{\mathrm{d}\theta}{\sqrt{\pi\theta}}.\end{aligned}$$

Provided that we take $u, v \leq a$ we get

$$\begin{aligned}\frac{1}{\mathrm{d}s}\mathbb{P}^c_u\,(\zeta_1 \in \mathrm{d}s | V_1 = cv) &\leq \frac{2\sqrt{2}a^3}{s^2 u^{\frac{1}{2}} v^{\frac{1}{2}}} \int_0^{\frac{4uv}{s}} \frac{\mathrm{d}\theta}{\sqrt{\pi\theta}} \\ &\leq \frac{8\sqrt{2}}{\sqrt{\pi}} a^3 s^{-\frac{5}{2}}.\end{aligned}$$

Integrating this inequality between ta^2 and $+\infty$ gives (3.20). □

The $\mathbf{P}$-almost sure finiteness of α_0 follows straightforwardly. Write

$$A_n = e^{S_n} \vee \frac{e^{S_{n+1}}}{c},$$

and, for $n > 0$, write E_n for the event

$$L_{-n} \geq n\ A^2_{-n}.$$

The lemma states that the probability of E_n is bounded above by a constant times $n^{-\frac{3}{2}}$. Hence only a finite number of E_n occur, almost surely. This together with (3.19) gives that the $(L_{-n})_{n\geq 0}$ are summable, almost surely. This shows the $\mathbf{P}$-almost sure finiteness of $A(S, \mathcal{N})$ and concludes the proof.

Appendix A: Proof of Lemma 2

The uniqueness stated in the lemma is immediate. Indeed, consider P and P' two probabilities satisfying the conditions of Lemma 2. Then for any real x, we have $(P \circ \Theta_x)_+ = P_+ = Q = (P' \circ \Theta_x)_+$. It follows $P = P'$.

Now, the key point is the construction of the probability P, which will be a consequence of Kolmogorov's existence theorem. First, note that we have

$\Theta_x \circ \Theta_y = \Theta_{x+y}$ for any x, y real numbers. Take $x > 0$. On the one hand, from $(P_v \circ \Theta_0)_+ \Rightarrow Q$, we deduce $(P_v \circ \Theta_0)_+ \circ \Theta_x \Rightarrow Q \circ \Theta_x$. On the other hand, we have $\big((P_v \circ \Theta_0)_+ \circ \Theta_x\big)_+ = (P_v \circ \Theta_x)_+ \Rightarrow Q$. Thus the laws $(Q \circ \Theta_x)_+$ and Q are identical for any $x > 0$.

For $x_1 < ... < x_n$ real numbers, we define first Y^{x_1} as a variable of law Q on Ω, then Y^{x_i} by $Y^{x_i} = (\Theta_{x_i - x_1}(Y^{x_1}))_+$, so that Y_{x_i} also has law Q. We write $Q^{x_1,\dots,x_n}$ for the law of $(Y^{x_1}, ..., Y^{x_n})$ obtained in that way, on $\Omega^{x_1,\dots x_n}$. These laws are compatible. Thus Kolmogorov's theorem tells that there exists a law $\overline{Q}$ on $\Omega^{\mathbb{R}}$ such that the finite dimensional marginals of $\overline{Q}$ on say $x_1, ..., x_n$ is equal to $Q^{x_1,\dots,x_n}$.

Let $(Z^x)_{x\in\mathbb{R}}$ be with law $\overline{Q}$. Define a random variable $Y = (Y(k))_{k\in\mathbb{Z}}$ on Ω by setting

$$Y(k) := \lim_{a\to+\infty} \Theta_a(Z^{-a})(k).$$

This definition requires some explanation. We start by noticing that the probability $\overline{Q}(T_a(Z^{-a}) \geq -k) = \overline{Q}(T_a(Z^0) \geq -k)$ converges to 1 when a goes to $+\infty$. Thus, a.s., for some a we have $T_a(Z^{-a}) \geq -k$ and then $\Theta_a(Z^{-a})(k) \neq -\infty$. But for any $x > a$, we have:

$$\begin{aligned}\Theta_a(Z^{-a})(k) &= \Theta_a((\Theta_{x-a}(Z^{-x}))_+)(k)\\ &= \Theta_a \circ \Theta_{x-a}(Z^{-x})(k) = \Theta_x(Z^{-x})(k),\end{aligned}$$

where we can drop the index $+$ at the second equality because we are on the event $T_a(Z^{-a}) \geq -k$. Thus for each k the family $(\Theta_a(Z^{-a})(k))_{a\geq 0}$ is constant as soon as it leaves $-\infty$, and the limit is well-defined.

Observe that the random variable Y satisfies the conditions

$$\lim_{k\to-\infty} Y^1(k) = -\infty, \qquad \limsup_{k\to+\infty} Y^1(k) = +\infty.$$

Its probability law P on Ω not only satisfies $P_+ = Q$, it is also spatially invariant: Indeed, for any x, the variable $\Theta_x(Y)$ has law $P \circ \Theta_x$ and is given by

$$\Theta_x(Y)(k) = \lim_{a\to+\infty} \Theta_{x+a}(Z^{-a})(k) = \lim_{a\to+\infty} \Theta_a(Z^{-a-x})(k).$$

But it is obvious that the family $(Z^{a-x})_{a\in\mathbb{R}}$ also has law $\overline{Q}$, hence $\Theta_x(Y)$ has law P.

Finally, we still have to prove $P_v \circ \Theta_x \Rightarrow P$. Take f any positive bounded continuous functional depending on a finite number of variables $\omega_{t_1}, ...\omega_{t_n}$, with $t = t_1 < ... < t_n$, so that $f((\omega_s)_{s\in\mathbb{Z}}) = f((\omega_s)_{s\geq t})$. We suppose without loss of generality $t < 0$. Observe that under the probability $P_v \circ \Theta_x$ or under P, we have $T_0 = 0$, and the events $T_{-y} \leq t$ and $T_y \circ \Theta_{-y} > -t$ coincide, almost surely.

Observe also $Q(T_y \leq -t) \to_{y\to\infty} 0$. Then,

$$\begin{aligned} P_v \circ \Theta_x(f((\omega_s)_{s\geq t})\mathbb{1}_{T_{-y}<t}) &= P_v \circ \Theta_{x-y}(f\circ\Theta_y((\omega_s)_{s\geq t}), T_y > -t) \\ &= (P_v \circ \Theta_{x-y})_+(f\circ\Theta_y((\omega_s)_{s\geq t}), T_y > -t) \\ &\underset{v\to 0^+}{\to} Q(f\circ\Theta_y((\omega_s)_{s\geq t}), T_y > -t) \\ &= P(f\circ\Theta_y((\omega_s)_{s\geq t}), T_y > -t) \\ &= P(f((\omega_s)_{s\geq t}), T_{-y} < t), \end{aligned}$$

where we get the second line because the functional $\mathbb{1}_{T_y>-t} f \circ \Theta_y((\omega_s)_{s\geq t})$ does not depend on $(\omega_n)_{n<0}$, and where we obtain the last line thanks to the translation Θ_{-y}. Besides, we have:

$$\begin{aligned} & |P_v \circ \Theta_x(f((\omega_s)_{s\geq t})\mathbb{1}_{T_{-y}<t}) - P_v \circ \Theta_x(f((\omega_s)_{s\geq t}))| \\ \leq\ & (\sup f).\ P_v \circ \Theta_x(\mathbb{1}_{T_{-y}\geq t}) \\ =\ & (\sup f).\ P_v \circ \Theta_{x-y}(\mathbb{1}_{T_y\leq -t}) \\ \to_{v\to 0^+}\ & (\sup f).\ Q(\mathbb{1}_{T_y\leq -t}) \to_{y\to\infty} 0, \end{aligned}$$

and in the same way

$$P(f((\omega_s)_{s\geq t}), T_{-y} < -t) \longrightarrow_{y\to\infty} P(f((\omega_s)_{s\geq t})).$$

This is enough to deduce

$$P_v \circ \Theta_x(f((\omega_s)_{s\geq t}) \to P(f((\omega_s)_{s\geq t})).$$

The law $P_v \circ \Theta_x$ does converge weakly to P.

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