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Emmanuel Jacob

Laboratoire de Probabilités et Modèles Aléatoires
Université Pierre et Marie Curie, Paris VI

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Abstract

Consider a Langevin process, that is an integrated Brownian motion, constrained to stay in $[0, \infty)$ by a partially elastic boundary at 0. If the elasticity coefficient of the boundary is greater than or equal to $c_{crit} = \exp(-\sqrt{\pi}/3)$, bounces will not accumulate in a finite time when the process starts from the origin with strictly positive velocity. We will endeavor to show that there exists then a unique entrance law from the boundary with zero velocity, despite the immediate accumulation of bounces. This result of uniqueness is in sharp contrast with the literature on deterministic second order reflection. Our approach uses certain properties of real-valued random walks and a notion of spatial stationarity which may be of independent interest.

Résumé

Considérons un processus de Langevin, c'est-à-dire un mouvement Brownien intégré, contraint à rester dans $[0, \infty)$ par une frontière partiellement élastique en 0. Si le coefficient d'élasticité de la frontière est supérieur ou égal à $c_{crit} = \exp(-\sqrt{\pi}/3)$, les rebonds ne s'accumuleront pas en temps fini si le processus part de l'origine avec une vitesse strictement positive. Nous nous efforcerons de montrer qu'il existe une unique loi d'entrée depuis la frontière avec vitesse nulle, malgré l'accumulation immédiate des rebonds. Ce résultat d'unicité est en fort contraste avec la littérature sur les réflexions au second ordre déterministe. Notre approche utilise certaines propriétés das marches aléatoires à valeurs réelles et une notion de stationarité spatiale, qui pourraient avoir un intérêt propre.

Key words. Langevin process, second order reflexion, renewal theory, stationarity, ladder height processes.

e-mail. emmanuel.jacob@normalesup.org

1 Introduction

Imagine a deterministic particle evolving in $\mathbb{R}_+$, started from 0, submitted to an external force f, and constrained by a partially elastic boundary at the origin. We write $x(t)$ for the position of the particle and we consider the following equations of motion:

$$(SOR) \qquad \begin{cases} x(t) &= \displaystyle\int_0^t \dot{x}(s)\mathrm{d}s \\ \\ \dot{x}(t) &= \dot{x}(0) + \int_0^t f(s)\mathrm{d}s - c\sum_{0<s\leq t} \dot{x}(s-)\mathbf{1}_{x(s)=0}, \end{cases}$$

where the trajectory x is smooth and the velocity $\dot{x}$ is càdlàg.

The coefficient $c > 0$ is the elasticity coefficient of the boundary: after a bounce, the boundary restores a portion c of the incoming speed. The speed $\dot{x}(0) \geq 0$ is the initial condition. We always assume that we start from the origin, that is $x(0) = 0$.

Equations (SOR) describe the so-called second order reflection problem. There exists a large literature on the subject. We mention some names, Bressan in 1960 [7], Percivale in 1985 [14], Schatzman in 1998 [15], or Ballard in 2000 [1]. An important feature is that in the case of an analytic force f, there is existence and uniqueness to the equations (SOR) for any initial condition, but when f is no more analytic (and even if it is $\mathcal{C}^\infty$), uniqueness may fail.

The main difficulty in second order reflection comes from the possibility for bounces to accumulate, in which case the sum in the equation involves infinitely many terms. We distinguish two problems: First bounces may accumulate just before a finite time $t > 0$. Second when we start with zero velocity, bounces may accumulate just after the starting time 0.

In this paper we are interested in Equations (SOR) when the external force f is random and given by a white noise. A fortiori a realization of f will not be analytic and we will not try and work on a fixed realization. The first observation is that outside the boundary, the velocity of the particle behaves like a Brownian motion, hence the particle evolves like a free Langevin process (i.e the integrated Brownian motion). A consequent study about the free Langevin process in general can be found in Lachal [12]. Bect mentioned the reflection and bounds accumulation problems for particles that can be excited by a white noise in his thesis ([2], see part III.4). For the reader interested in the problem of a Langevin process reflected at a totally inelastic boundary, that is $c = 0$, we refer to Bertoin [3, 4] and Jacob [11].

Let us come back to our settings of a white noise and $c > 0$. Then the problem of accumulation of bounces just before a finite time $t > 0$ is easy: We shall see that bounces accumulate if and only if the elasticity coefficient is less than the critical coefficient $c_{crit} = \exp(-\pi/\sqrt{3})$. However the question of starting with zero velocity is more fastidious. We focus on the critical and supercritical cases, the study of the subcritical case being the center of interest of a forthcoming paper. Our main result is that for $c \geq c_{crit}$ the reflected Langevin process starting from the origin with a speed $\varepsilon > 0$ is converging in law, when ε goes to 0, to a non-degenerate process. We observe in this introduction that this result is

easy for the particular case $c = 1$ (perfectly elastic boundary) because a reflected Langevin process can then be constructed from the free Langevin process Y by taking its absolute value $|Y|$. However there is no such construction when the elasticity coefficient is $c \neq 1$. We will also deduce the weak existence and weak uniqueness of solutions to (SOR) with zero initial velocity.

Our method is to focus on the sequence of the velocities of the process after the bounces and start with the crucial observation that their logarithm is a random walk. This enables us to use technics of renewal theory, including results about its associated ladder height process and its overshoot.

We start the preliminaries in Section 2.1 by giving some background on the Langevin process and illustrating the three different regimes (subcritical, supercritical, critical). We then define in Section 2.2 in an abstract context a notion of spatial stationarity, and give a convergence result for spatially stationary processes stated in Lemma 2, which will be proven in the Appendix. We start Section 3 with the statement of our main theorem and important consequences. Section 3.1 uses renewal theory and Lemma 2 to construct a spatially stationary process and reduce the proof of the main theorem to that of Lemma 4. Section 3.2 handles this proof in the supercritical case thanks to an explicit construction[1] of the spatially stationary random walk. However this construction does not hold in the critical case, and Section 3.3 completes then the proof thanks to a disintegration formula[1] for the spatially stationary random walk.

2 Preliminaries

2.1 Background on the Langevin process

We write Y for a free Langevin process, that is $Y_t = \int_0^t \dot{Y}_s ds$ where $\dot{Y}$ is a Brownian motion, and $\dot{Y}_0 = v > 0$ is the starting velocity. Our working space is the space of smooth trajectories starting from zero. We write X for the canonical process, and $\dot{X}$ for its right-derivative, also called velocity process. The latter belongs to $\mathcal{D}(\mathbb{R}_+, \mathbb{R})$, the set of càdlàg paths. In this paper, by a slight abuse of notation, when we define a probability measure P, we also write P for the expectation under this probability measure. When f is a measurable functional and A an event, we even write $P(f, A)$ for the quantity $P(f\mathbf{1}_A)$.

We are ready to describe the law $\mathbb{P}_v^c$ on $\mathcal{D}(\mathbb{R}_+, \mathbb{R})$, defined as the law of the velocity $\dot{X}$ of the process reflected on a boundary with elasticity coefficient $c > 0$, with initial velocity $v > 0$ and killed when bounces accumulate. To this end, let us write $\zeta_0 = 0$ and $\zeta_{n+1} := \inf\{t > \zeta_n : X_t = 0\}$ for the sequence of the successive hitting times of zero (see the figure below for an illustration of the notations). We call an *arch* a part of the path included between two consecutive hitting times of zero. Then, under $\mathbb{P}_v^c$, the reflected process X behaves like Y until the first return time to zero ζ_1, that is the first arch of Y and X have the same law, $(Y_t)_{\zeta_0 \leq t \leq \zeta_1} \overset{d}{=} (X_t)_{\zeta_0 \leq t \leq \zeta_1}$. Then the second arch of the reflected process, $(X_t)_{\zeta_1 \leq t \leq \zeta_2}$, has the same law as the first arch of a Langevin process starting with

[1] These two constructions in particular may be of independent interest.

velocity $\dot{X}_{\zeta_1} := -c\dot{X}_{\zeta_1^-}$. We construct in the same way the sequence of the successive arches of X. We also write V_n^-, and V_n for the speed of the process just before this n-th bounce, and for the speed of the process just after this n-th bounce, respectively, so that we have $V_n = \dot{X}_{\zeta_n} = -c\dot{X}_{\zeta_n^-} = -cV_n^-$. The process is then killed at the time $\zeta_\infty := \sup(\zeta_n) \in]0, \infty]$, that we call the time of accumulation of bounces. This ends the definition of $\mathbb{P}_v^c$. Figure 1 below shows two complete arches and the beginning of a third one.

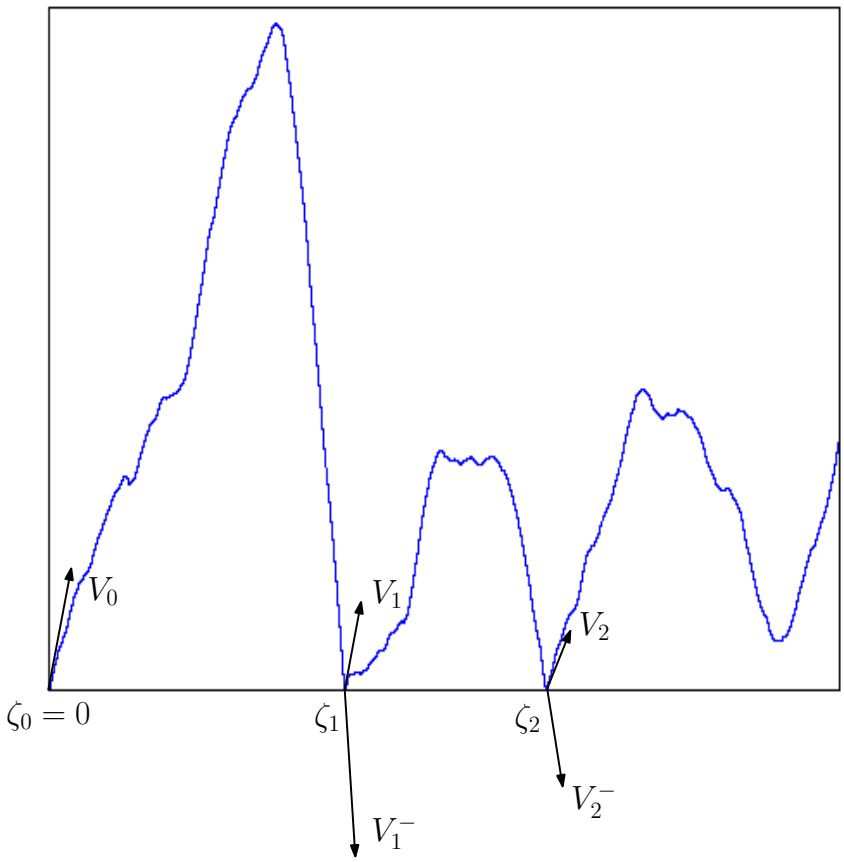

Figure 1: First arches of a reflected Langevin process

We go on with the following crucial lemma:

Lemma 1. *1) Under $\mathbb{P}_v^c$, the sequence $\left(\frac{\zeta_{n+1}-\zeta_n}{V_n^2}, \frac{V_{n+1}}{V_n}\right)_{n\geq 0}$ is i.i.d., of law independent of v and determined by:*

$$\frac{1}{\mathrm{d}s\mathrm{d}v}\mathbb{P}_1^c\left((\zeta_1, V_1/c) \in (\mathrm{d}s, \mathrm{d}v)\right) = \frac{3v}{\pi\sqrt{2}s^2}\exp(-2\frac{v^2-v+1}{s})\int_0^{\frac{4v}{s}} e^{-\frac{3\theta}{2}}\frac{\mathrm{d}\theta}{\sqrt{\pi\theta}}. \qquad (2.1)$$

2) In particular, the sequence $\ln(V_n)$ *is a random walk. The density of its step distribution* $\ln(V_1 - V_0)$ *under* $\mathbb{P}_v^c$ *does not depend of* v *and is given by:*

$$\frac{1}{\mathrm{d}v}\mathbb{P}_1^c(\ln(V_1) \in \mathrm{d}v) = \frac{3}{2\pi}\frac{e^{\frac{5}{2}(v-\ln c)}}{1+e^{3(v-\ln c)}}\mathrm{d}v. \tag{2.2}$$

We also have

$$\mathbb{P}_1^c(\ln V_1) = \frac{\pi}{\sqrt{3}} + \ln c$$

and

$$\ln V_1 \in \mathbf{L}^2(\mathbb{P}_1^c).$$

3) We have, when $t \to \infty$,

$$\mathbb{P}_1^c(\zeta_1 > t) \sim c' t^{-\frac{1}{4}}, \tag{2.3}$$

where c' *is some positive constant.*

Proof. Informally, the first part just comes from the observation that the variable $(\zeta_n - \zeta_{n-1})/(V_{n-1})^2$ (resp. V_n/V_{n-1}) is equal to the duration of the $k-$th arch renormalized to start with speed one (resp. to the absolute value of the speed of the process just before its return time to zero, for this renormalized arch). More precisely:

Given V_n, recall that the process $(X_{(t+\zeta_n)\wedge\zeta_{n+1}})_{t\geq 0}$ is independent of $(X_{t\wedge\zeta_n})_{t\geq 0}$ and has the same law as $(X_{t\wedge\zeta_1})_{t\geq 0}$ under $\mathbb{P}^c_{V_n}$, thus $(\zeta_{n+1}-\zeta_n, V_{n+1}/c)$ is independent of $(\zeta_k, V_k)_{k\leq n}$ has the same law as $(\zeta_1, \frac{1}{c}V_1)$ under $\mathbb{P}^c_{V_n}$ Given V_n, it follows that $((\zeta_{n+1}-\zeta_n)/(V_n)^2, V_{n+1}/V_n)$ is independent of $(\zeta_k, V_k)_{k\leq n}$ and has the same law as $(\zeta_1, \frac{1}{c}V_1)$ under $\mathbb{P}_1^c$. We just have to remove the conditioning to get the i.i.d character of the sequence.

Then all the densities given are already known, and they were first obtained by McKean in [13]. For the convenience of the reader, we still detail a possible calculation of $\mathbb{P}_1^c(\ln V_1)$. It is obtained as a consequence from the following formula, by taking its derivative at 0:

$$\mathbb{P}_1^c\left(\left(\frac{V_1}{c}\right)^x\right) = \frac{1}{2\cos(\frac{x+1}{3}\pi)} \text{ for } x < 1/2. \tag{2.4}$$

Then we should prove this formula for any fixed $x < 1/2$. But from the law $\mathbb{P}_1^c(V_1/c \in \mathrm{d}v) = \frac{3}{2\pi}\frac{v^{\frac{3}{2}}}{1+v^3}\mathrm{d}v$, we get,

$$\begin{aligned}\mathbb{P}_1^c\left(\left(\frac{V_1}{c}\right)^x\right) &= \frac{3}{2\pi}\int_0^\infty \frac{t^{x+3/2}}{1+t^3}\mathrm{d}t \\ &= \frac{1}{2\pi}\int_0^\infty \frac{t^{\frac{x}{3}-\frac{1}{6}}}{1+t}\mathrm{d}t.\end{aligned}$$

As $\cos(\frac{x+1}{3}\pi) = \sin((\frac{x}{3}+\frac{5}{6})\pi)$, in order to get (2.4) we just have to prove

$$\int_0^\infty \frac{t^{y-1}}{1+t}\mathrm{d}t = \frac{\pi}{\sin(\pi y)},$$

where $y = \frac{x}{3} + \frac{5}{6}$ is in $(0,1)$. Now, we have

$$\begin{aligned}\int_0^\infty \frac{t^{y-1}}{1+t}\mathrm{d}t &= \int_0^1 t^y(1-t)^{1-y}\mathrm{d}t \\ &= B(y,1-y) \\ &= \frac{\Gamma(y)\Gamma(1-y)}{\Gamma(1)} \\ &= \frac{\pi}{\sin(\pi y)}.\end{aligned}$$

where B and Γ are the usual Beta and Gamma function, respectively.

Finally the estimation (2.3) is in Goldman [8]. □

From this Lemma we deduce the following important result:

Corollary 1. *The time of accumulation of bounces ζ_∞ is:*

finite $\mathbb{P}_v^c-$almost surely if $c < \exp(-\pi/\sqrt{3})$.
infinite $\mathbb{P}_v^c-$almost surely if $c \geq \exp(-\pi/\sqrt{3})$.

We thus call $c_{crit} := \exp(-\pi/\sqrt{3})$ the critical elasticity coefficient.

Proof. We may express ζ_∞ as the series:

$$\zeta_\infty = \sum_{n=1}^{\infty} \frac{\zeta_n - \zeta_{n-1}}{(V_{n-1})^2}(V_{n-1})^2.$$

For $c < \exp(-\pi/\sqrt{3})$, the law of large numbers tells that the sequence $\frac{1}{k}\ln(V_k)$ is converging to $\ln(c) + \pi/\sqrt{3} < 0$ a.s. On the other hand, it follows from (2.3) that the expectation of $(\ln(\zeta_1))^2$ is finite[2]. Thus, for any fixed $\varepsilon > 0$ there is a.s. only a finite number of k such that $\ln((\zeta_k - \zeta_{k-1})/(V_{k-1})^2)$ is larger than εk. We deduce an a.s. exponential decay for the variables $\zeta_{k+1} - \zeta_k$. *A fortiori* ζ_∞ is a. s. finite.

Take now $c \geq \exp(-\pi/\sqrt{3})$. For $c > \exp(-\pi/\sqrt{3})$, the random walk $\ln V_n$ has a positive drift and is transient. Thus the sequence V_n is diverging to $+\infty$. As $(\zeta_n - \zeta_{n-1})/(V_{n-1})^2$ is independent of V_{n-1} and has a fixed distribution, we deduce that ζ_∞ is infinite. For $c = \exp(-\pi/\sqrt{3})$, the step distribution zero expectation and finite variance, thus the random walk becomes recurrent (from the central limit theorem). This time the sequence V_n is recurrent, but still it is not converging to zero, what is enough to conclude in the same way that ζ_∞ is infinite. □

Figures 2, 3 and 4 below give illustrations of the different regimes, obtained by a basic simulation. We have drawn the reflected Langevin process with three different values of c for the same underlying Brownian motion and the same starting velocity. For $c = 0.1$, that is in the subcritical case, the bounds accumulate (here at about time 7000). For $c = 0.25$, that is in the supercritical case, there are bigger and bigger arches. While in the critical case $c = c_{crit}$ the consequence of the recurrence is that there are both very big arches and very small arches (see the bouncing times ζ_9, ζ_{10}, ζ_{11}), and the bounces do not accumulate.

[2]This result was also stressed by McKean in [13]

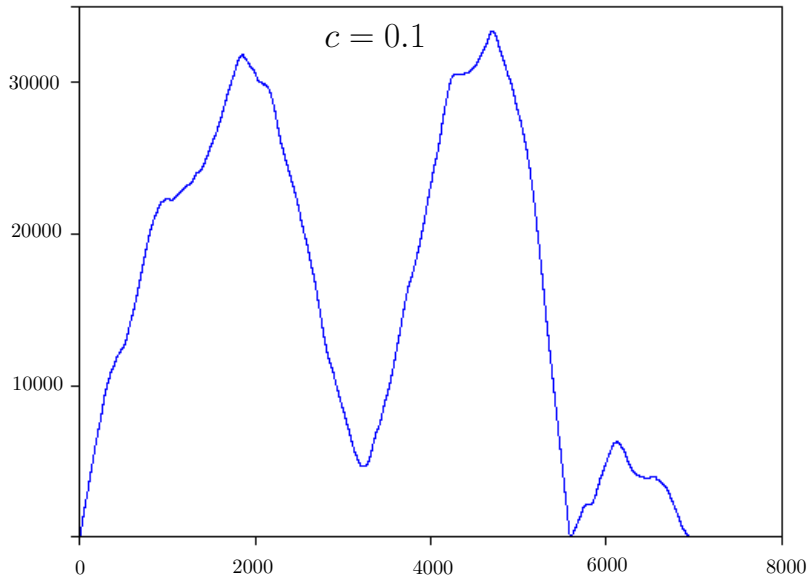

Figure 2: Reflected Langevin process for elasticity coefficient $c = 0.1 < c_{crit}$

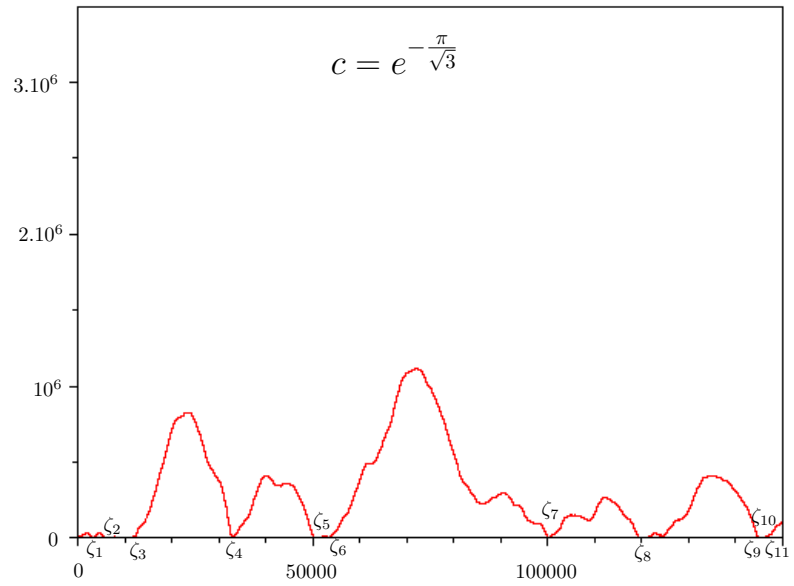

Figure 3: Reflected Langevin process for elasticity coefficient $c = c_{crit} = \exp(-\pi/\sqrt{3})$

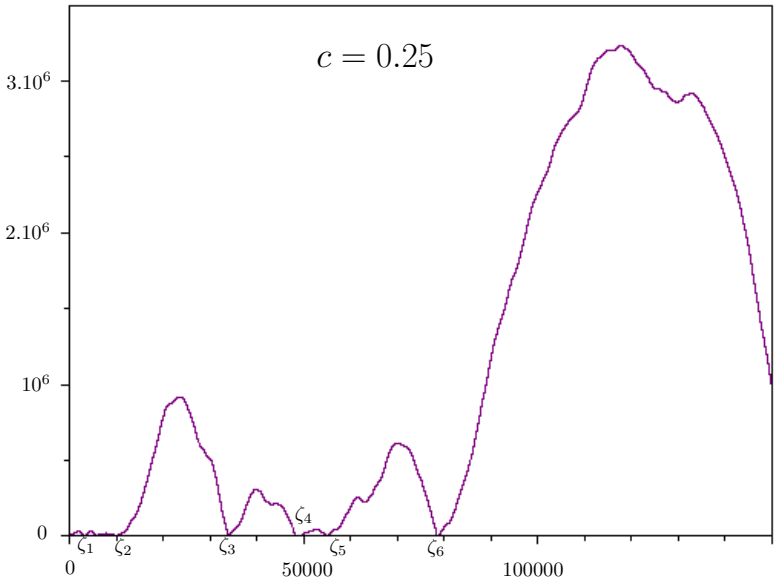

Figure 4: Reflected Langevin process for elasticity coefficient $c = 0.25 > c_{crit}$

2.2 Weak convergence to a spatially stationary process

After this first results on the Langevin process, we give the abstract context for a notion of spatial stationarity and an important lemma that we will need later.

We write Ω for the set of sequences indexed by $\mathbb{Z}$, $\omega = (\omega_n)_{n\in\mathbb{Z}} = (\omega_n^1, \omega_n^2)_{n\in\mathbb{Z}}$, with values in $[-\infty, \infty) \times \mathcal{D}$, where $\mathcal{D}$ is a topological space with an isolated point $\emptyset$. For the moment just consider this space as playing an accessory role that will be understood later.

For any real number x we write T_x for the hitting time of (x, ∞) by the first coordinate, that is

$$T_x = T_x(\omega) = \inf\{n \in \mathbb{Z}, \omega_n^1 > x\}.$$

Under all the measures P that we will consider on Ω we will have

$$\lim_{-\infty} \omega_n^1 = -\infty, \ \limsup_{+\infty} \omega_n^1 = +\infty \qquad \text{P-almost surely,}$$

and as a consequence T_x will have values in $\mathbb{Z}$, P-almost surely. We then define a spatial translation operator Θ on Ω, by:

$$\Theta_x(\omega) := (\omega_{n+T_x}^1 - x, \omega_{n+T_x}^2)_{n\in\mathbb{Z}}. \tag{2.5}$$

This definition immediately yields a notion of spatial stationarity for the probabilities on Ω:

Definition 1. *We say that a probability P on Ω is spatially stationary if $P \circ \Theta_x = P$ for any $x \in \mathbb{R}$.*

We also write

$$\Omega_+ := \left\{\omega \in \Omega : (\omega_n^1, \omega_n^2) = (-\infty, \emptyset) \text{ for all } n < 0\right\},$$

that we shall think of as the sequences indexed by $\mathbb{N}$. We write $\omega^+ \in \Omega_+$ for the projection of $\omega \in \Omega$ defined by:

$$\omega_n^+ = \begin{array}{ll} (-\infty, \emptyset) & \text{if } n < 0 \\ \omega_n & \text{if } n \geq 0 \end{array}$$

If P is a probability on Ω, we write P_+ for the image probability on Ω_+ by this projection. Finally we write $\Rightarrow$ for the weak convergence in the sense of finite dimensional probabilities. The following lemma states a convergence result to spatially stationary probability measure on Ω as a consequence of convergence results on Ω_+:

Lemma 2. *Let $(P_v)_{v>0}$ be a family of probability measures on Ω. We suppose that there is a probability Q on Ω_+ such that:*

$$\forall x \in \mathbb{R}, \quad (P_v \circ \Theta_x)_+ \Rightarrow_{v \to 0} Q.$$

Then there exists a unique spatially stationary probability measure P on Ω such that $P_+ = Q$. Moreover, we have

$$P_v \circ \Theta_x \Rightarrow P.$$

The proof of this technical lemma is based on the Kolmogorov existence theorem, but we postpone it to the appendix, while we explain how we can apply it to our case.

3 Entering with zero velocity

In the following we always consider $c \geq c_{crit}$, that is we shall not treat the subcritical case. We recall that $\mathbb{P}_v^c$ stands for the law of the process reflected with elasticity coefficient c with starting velocity $v > 0$. We don't need to specify that it is killed at ζ_∞, as this time is infinite $\mathbb{P}_v^c$-a.s thanks to Corollary 1

Write $(S_n)_{n\geq 0}$ for the random walk of the logarithm of the (outgoing) velocity at the $n-$th bounce (that is $S_n = \ln(V_n)$), with step distribution given by (2.2). Its corresponding (strictly) ascending ladder height process $(H_n)_{n\geq 0}$ is the random walk with positive jumps defined by $H_0 = S_0$ and $H_k = S_{n_k}$ where $n_0 = 0$ and $n_k = \inf\{n > n_{k-1}, S_n > S_{n_{k-1}}\} \in \mathbb{N}$. From Lemma 1, if $c > c_{crit}$, then the step distribution of $(S_n)_{n\geq 0}$ has strictly positive and finite expectation $\mu := \mathbf{E}_1(S_1 - S_0) = \frac{\pi}{\sqrt{3}} + \ln c$, while if $c = c_{crit}$, then it has zero expectation $\mu = 0$ and finite variance. In both cases, it is known (see Theorem 3.4 in Spitzer [16]) that

the expectation of the step distribution of $(H_n)_{n\geq 0}$, that is $\mu_H := \mathbb{E}(H_1 - H_0)$, belongs to $(0, \infty)$. The probability law

$$m(\mathrm{d}y) := \frac{1}{\mu_H}\mathbf{P}_0(H_1 > y)\mathrm{d}y. \tag{3.1}$$

is known in renewal theory as being the stationary law of the overshoot (see also Part 3.1).

We now state our main theorem and its important corollary:

Theorem 1. *The family of probability measures $(\mathbb{P}^c_v)_{v>0}$ on $\mathcal{D}(\mathbb{R}_+, \mathbb{R})$ has a weak limit when $v \to 0^+$, which we denote by $\mathbb{P}^c_{0^+}$. More precisely, write τ_u for the instant of the first bounce with speed greater than u, that is $\tau_u := \inf\{t > 0, X_t = 0, \dot{X}_t > u\}$. Then the law of $\ln(\dot{X}_{\tau_u}/u)$ is m, and:*

$$(*) \quad \begin{cases} \lim\limits_{u\to 0^+} \tau_u = 0 \qquad \mathbb{P}^c_{0^+}\text{-almost surely} \\ \text{Conditionally on } \dot{X}_{\tau_u} = v, \text{ the process } (\dot{X}_{\tau_u+t})_{t\geq 0} \text{ is independent of } (\dot{X}_s)_{s<\tau_u} \\ \quad \text{and has law } \mathbb{P}^c_v. \end{cases}$$

Corollary 2. *The canonical process X under $\mathbb{P}^c_{0^+}$ is a solution to (SOR), and any solution has this same law. Thus, for $c \geq c_{crit}$, there is weak existence and weak uniqueness of solutions to (SOR) with initial condition $X_0 = \dot{X}_0 = 0$.*

Proof of the corollary. Consider a weaker version of the theorem where the weak convergence holds for the measures on $\mathcal{D}(\mathbb{R}^*_+, \mathbb{R})$ (the time 0 is removed). Actually, both the theorem in its strong version and the corollary follow from this weak version. Indeed, consider the process $(X_t)_{t>0}$ under $\mathbb{P}^c_{0^+}$. It follows from the conditions $(*)$ that $\mathbb{P}^c_{0^+}$- almost surely, the process X is at zero countably many times, thus $\dot{X}$ has countably many jumps and $\ddot{X}$ is well defined. It is clear that $(\ddot{X}_{\tau_u+t})_{t\geq 0}$ is a white noise and hence $\ddot{X}$ is a white noise. It follows that $\dot{X}$ satisfies the equation

$$\dot{X}(t) = \dot{X}(\varepsilon) + \int_\varepsilon^t \ddot{X}(s)\mathrm{d}s - c\sum_{\varepsilon<s\leq t} \dot{X}(s-)\mathbf{1}_{X(s)=0}$$

for any $0 < \varepsilon < t$. It follows that the process $(\dot{X}(t-r) - \dot{X}(t) - c\sum_{t-r<s\leq t} \dot{X}(s-)\mathbf{1}_{X(s)=0})_{0\leq r<t}$ is a Brownian motion, thus it has a finite limit when r goes to t. From this and from $(*)$ which implies $\dot{X}(t)\mathbf{1}_{X(t)=0} \to 0$ when $t \to 0^+$, we deduce $\dot{X}(t) \to 0$ when $t \to 0^+$. Hence we can extend $\mathbb{P}^c_{0^+}$ on $\mathcal{D}(\mathbb{R}_+, \mathbb{R})$ by putting $X(0) = \dot{X}(0) = 0$. The weak convergence of $\mathbb{P}^c_v$ to $\mathbb{P}^c_{0^+}$ clearly holds on $\mathcal{D}(\mathbb{R}_+, \mathbb{R})$.

Taking again the equation above and letting ε go to zero, the term $\dot{X}(\varepsilon)$ is going to zero and the term $\int_\varepsilon^t \ddot{X}(s)\mathrm{d}s$ to $\int_0^t \ddot{X}(s)\mathrm{d}s$. We deduce that the term $\sum_{\varepsilon<s\leq t} \dot{X}(s-)\mathbf{1}_{X(s)=0}$ is converging to the finite term $\sum_{0<s\leq t} \dot{X}(s-)\mathbf{1}_{X(s)=0}$, so that the equations (SOR) are satisfied, with $\ddot{X}$ a white noise.

Consider now any solution X to (SOR) starting from initial velocity 0. If X were not coming back to zero at small times, then there wouldn't be any jumps for $\dot{X}$ at small

times, thus X would behave like a Langevin process. But it is not possible as the Langevin process starting with velocity 0 does come back at 0 at arbitrary small times. The process X necessarily satisfies $(*)$. Now, the process $(\dot{X}_{\tau_u+t})_{t\geq 0}$ is converging in law to $\dot{X}$, thus the law of $\dot{X}$ is an accumulation point of the family $(\mathbb{P}^c_v)_{v>0}$ when $v \to 0$. Thus it must coincide with $\mathbb{P}^c_{0^+}$. □

The rest of the section is devoted to the proof of the theorem in its weak version (see the beginning of the proof of the Corollary). Our method will be to first prove the convergence of the processes translated at the time of the first bounce with speed greater than some fixed u, that is τ_u. A renewal theorem will give us a convergence result of the process $(\dot{X}_{\tau_u+t})_{t\geq 0}$ to a law described simply. Then the lemma will allow to generalize this to the convergence of this family of processes. Then we will still need to prove the simultaneous convergence of the τ_u. All this will appear to be enough to prove the convergence of the non-translated processes.

3.1 Convergence of shifted processes

We recall the notation V_n for the (outgoing) velocity at the n-th bounce and S_n for its logarithm, for $n \geq 0$. We also write $\mathcal{N}_n$ for the translated velocity path starting at the n-th bounce and renormalized so as to start with speed one. That is, $\mathcal{N}_n$ is defined by

$$(\mathcal{N}_n(t))_{t\geq 0} := (V_n^{-1}\dot{X}(\zeta_n + V_n^2 t))_{t\geq 0}.$$

The process $\mathcal{N}_n$ is independent of $(\dot{X}_t)_{0\leq t\leq \zeta_n}$ and has law $\mathbb{P}^c_1$. The knowledge of the process X, or $\dot{X}$, is equivalent to the knowledge of the sequence $(S_n, \mathcal{N}_n)_{n\geq 0}$, or even just $(S_0, \mathcal{N}_0)$. But it is more convenient to first prove convergence results about (translations of) the sequence $(S_n, \mathcal{N}_n)_{n\geq 0}$, then deduce results about X, what we do.

We put down $\mathcal{D} := \mathcal{D}(\mathbb{R}_+, \mathbb{R}) \cup \emptyset$ and we define moreover, for $n < 0$, $(S_n, \mathcal{N}_n) := (-\infty, \emptyset)$, so that the sequence $(S, \mathcal{N}) := (S_n, \mathcal{N}_n)_{n\in\mathbb{Z}}$ lays in Ω_+, in the settings of Section 2.2. We call $\mathbf{P}_v$ its law on Ω_+ (or Ω), under $\mathbb{P}^c_v$. We also use the other notations of Section 2.2 and in particular the spatial translation operator Θ_x defined by (2.5). We now aim at establishing convergence results for the probabilities $\mathbf{P}_v \circ \Theta_x$.

First, observe that under $\mathbf{P}_v$ and for $n \geq 0$, $(S_{n+1}, \mathcal{N}_{n+1})$ is measurable with respect to $(S_n, \mathcal{N}_n)$, and thus $(S, \mathcal{N})$ is entirely determined by $(S_0, \mathcal{N}_0)$, which follows the law $\delta_{\ln v} \otimes \mathbb{P}^c_1$. In other words, there is a deterministic functional F such that $(S_n, \mathcal{N}_n)_{n\geq 0} = F(S_0, \mathcal{N}_0)$, and $\mathbf{P}_v$ is the law on Ω induced by the law $\delta_{\ln v} \otimes \mathbb{P}^c_1$ for $(S_0, \mathcal{N}_0)$. Write now $\mathbf{Q}$ for the law on Ω_+ induced by the law $m \otimes \mathbb{P}^c_1$ for $(S, \mathcal{N})$, where the measure m is the stationary law of the overshoot we introduced earlier, defined by (3.1).

Lemma 3. *For any real number x, we have*

$$(\mathbf{P}_v \circ \Theta_x)_+ \Rightarrow_{v\to 0^+} \mathbf{Q}$$

Proof. It is a direct application of the renewal theory for the random walk with positive jumps H that the law of its overshoot over a level x, that is $H_{T_x(H)} - x$, where the time

$T_x(H)$ is defined by $T_x(H) := \inf\{k, H_k \geq x\}$, is converging, when $x - H_0$ goes to infinity, to m. But it is clear that the random walks $(S_n)_{n\geq 0}$ and $(H_n)_{n\geq 0}$ have the same overshoot, $S_{T_x} - x = H_{T_x(H)} - x$. Under $\mathbf{P}_v$ and when v goes to 0^+, then $x - H_0 = x - \ln v$ goes to $+\infty$ and $S_{T_x} - x$ converges in law to m. In other words, the law of S_0 under $\mathbf{P}_v \circ \Theta_x$ is converging to m.

Now, the usual Markov property and scaling invariance property show that for any x, v, under $\mathbf{P}_v \circ \Theta_x$, $(S_n - S_0, \mathcal{N}_n)_{n\geq 0}$ is independent of S_0 and has the same law as $(S_n, \mathcal{N}_n)_{n\geq 0}$ under $\mathbf{P}_1$. This altogether establishes the convergence of $(\mathbf{P}_v \circ \Theta_x)_+$ to $\mathbf{Q}$. □

Applying Lemma 2, we immediately deduce:

Corollary 3. *For any real number x, we have*

$$\mathbf{P}_v \circ \Theta_x \Rightarrow_{v\to 0^+} \mathbf{P}, \tag{3.2}$$

where $\mathbf{P}$ is the unique spatially stationary probability measure on Ω such that $\mathbf{P}_+ = \mathbf{Q}$.

Remark 1. *Call $\mathbf{P}^1$ the projection of $\mathbf{P}$ on the first coordinate, $\mathbf{Q}^1$ the projection of Q on the first coordinate, and Θ^1 the spatial translation operator induced on the first coordinate (defined by $\Theta^1_x(\omega^1) := (\omega^1_{n+T_x} - x)_{n\in\mathbb{Z}}$). It is immediate that $\mathbf{Q}^1$ is the law of a random walk where S_0 has distribution m, that we have $\mathbf{P}^1_+ = \mathbf{Q}$ and that $\mathbf{P}^1$ is spatially stationary. Similar arguments also show that $\mathbf{P}^1$ is the unique spatially stationary measure such that $\mathbf{P}^1_+ = \mathbf{Q}$. We call it the law of the spatially stationary random walk.*

We now want to deduce the theorem from the corollary. To this end, we have to understand how to reconstruct $\dot{X}$ from $\Theta_x(S, \mathcal{N})$. Write $\alpha_x := \tau_{e^x}$ the instant of the first bounce with speed greater than $\exp(x)$. Observe that α_x can also be expressed as a functional of $\Theta_x(S, \mathcal{N})$ by the following formula:

$$\alpha_x = A(\Theta_x(S, \mathcal{N})), \tag{3.3}$$

where A is defined by

$$A(\omega) = \sum_{n<0} e^{2\omega^1_n} \zeta_1(\omega^2_n), \tag{3.4}$$

with the convention that $\zeta_1(\emptyset) = 0$. Formula (3.4) should not come as a surprise, the sum is just the sum of the duration of the arches with negative indexes. The formula (3.3) holds, the number of non-zero terms be actually finite (as it is the case under $\mathbf{P}_v$) or not (as it will be the case under $\mathbf{P}$).

Now, on the event $\varepsilon > \alpha_x$, the process $(\dot{X}_t)_{t\geq\varepsilon}$ can be expressed as a functional of $(\alpha_x, S_{T_x}, \mathcal{N}_{T_x})$. Accurately, we have

$$\dot{X}_t \mathbf{1}_{t\geq\alpha_x} = e^{S_{T_x}} \mathcal{N}_{T_x}(e^{-2S_{T_x}}(t - \alpha_x)) \mathbf{1}_{t\geq\alpha_x}. \tag{3.5}$$

We work under $\mathbf{P}$. Suppose that we have proven that $\alpha_x < \infty$ almost surely. Then formula (3.5) allows to construct the process $\dot{X}$, and we call $\mathbb{P}^c_{0^+}$ its law on $\mathcal{D}(\mathbb{R}^*_+, \mathbb{R})$. This law

is our candidate for Theorem 1. It is easy to check that under $\mathbb{P}^c_{0^+}$, the law of $\ln(\dot{X}_{\tau_u}/u)$ is m, and that the conditions $(*)$ are satisfied. Then we still should establish the weak convergence on $\mathcal{D}(\mathbb{R}^*_+, \mathbb{R})$, and for this it is enough to prove the convergence of the law of the triplet $(\alpha_x, S_{T_x}, \mathcal{N}_{T_x})$ under $\mathbf{P}_v$ to that under $\mathbf{P}$ (observe that this is not stated directly in Corollary 3). All well considered, all we still need to establish Theorem 1 is the following lemma:

Lemma 4. *1) The time α_x is finite $\mathbf{P}$-almost surely*

2) The law of $(\alpha_x, S_{T_x}, \mathcal{N}_{T_x})$ under $\mathbf{P}_v$ converges to that under $\mathbf{P}$ when $v \to 0^+$.

Thanks to a simple scaling argument we may suppose $x = 0$, what we do. We need to distinguish the supercritical and the critical cases.

3.2 Proof of Lemma 4 in the supercritical case

Throughout this section we suppose that $c > c_{crit}$. We need a more explicit description of $\mathbf{P}$. To this end we will introduce a new measure on Ω, the (temporally) stationary measure Π_λ. If one just considers the projection on the first coordinate S, this will be the measure of the stationary random walk.

Let us begin with this more usual measure. Introduce first P_0 for the law of the random walk $(S_n)_{n\in\mathbb{Z}}$ indexed by $\mathbb{Z}$, where $S_0 = 0$ and $(S_{n+1} - S_n)_{n\in\mathbb{Z}}$ is i.i.d with common law that of the generic step. Then write P_x for the law of $(x + S_n)_{n\in\mathbb{Z}}$ under P_0, and set

$$P_\lambda = \int_{\mathbb{R}} P_x \mathrm{d}x.$$

This σ-finite measure is (temporally) stationary, that is the law of $(S_{k+n})_{n\in\mathbb{Z}}$ under P_λ is P_λ, for any $k \in \mathbb{Z}$. This term "law" has to be understood in a generalized sense, that is in settings where we allow the laws to be not only probability measures but also σ-finite measures. We call this generalized process of law P_λ the (temporally) stationary random walk.

Now start again the same construction, but with adding the second coordinate. We first recall that under $\mathbf{P}_v$ and for $n \geq 0$, $(S_{n+1}, \mathcal{N}_{n+1})$ is measurable with respect to $(S_n, \mathcal{N}_n)$; we have $(S_{n+1}, \mathcal{N}_{n+1}) = F(S_n, \mathcal{N}_n)$, where F is a deterministic functional. For $n \leq 0$, consider Π^n_x for the law of $(S_k, \mathcal{N}_k)_{k\geq n}$, where $\mathcal{N}_n \overset{d}{=} \mathbb{P}^c_1$, $S_n = x - \ln(V_{-n}(\mathcal{N}_n))$ (recall that $V_{-n}(\mathcal{N}_n)$ denotes the velocity of the particle after the $(-n)-$th bounce), and the sequence $(S_k, \mathcal{N}_k)_{k>n}$ is given by $(S_k, \mathcal{N}_k) = F^{k-n}(S_n, \mathcal{N}_n)$.

It should be clear that the laws Π^n_x, $n \leq 0$, are compatible. Kolmogorov existence theorem entails the existence Π_x, the law on Ω under which $(S_k, \mathcal{N}_k)_{k\geq n}$ has law Π^n_x for any $n \leq 0$. Then we just define Π_λ by

$$\Pi_\lambda := \int \Pi_y \mathrm{d}y.$$

Again, it is σ-finite (temporally) stationary measure. Also the law of the first coordinate S under Π_λ is P_λ. The following lemma states a link between Π_λ and $\mathbf{P}$, as well as between P_λ and $\mathbf{P}^1$ (recall Remark 1 after Corollary 3).

Lemma 5. *Suppose $c > c_{crit}$.*
1) We have $P_\lambda(T_0 = 0) = \Pi_\lambda(T_0 = 0) = \mu \in (0, \infty)$.
2) We have $\mathbf{P}^1(\cdot) = P_\lambda(\cdot | T_0 = 0)$ and $\mathbf{P}(\cdot) = \Pi_\lambda(\cdot | T_0 = 0)$.

Proof. We recall that as we are in the supercritical case, the drift $\mu = \mathbf{E}_1(S_1 - S_0) = \frac{\pi}{\sqrt{3}} + \ln c \in$ is strictly positive and finite. We still write $(H_n)_{n\geq 0}$ for the (strictly) ascending ladder height process of the sequence $(S_n)_{n\geq 0}$. Its drift $\mu_H = \mathbf{E}_1(H_1 - H_0)$ is also strictly positive and finite. A result of Woodroofe [17] and Gut [9] states that, for any $y > 0$, we have

$$\frac{1}{\mu_H} P_0(H_1 > y) = \frac{1}{\mu} P_0\left(\inf_{n\geq 1} S_n > y\right). \tag{3.6}$$

The calculation below follows:

$$\begin{aligned}
\Pi_\lambda(T_0 = 0) &= P_\lambda(T_0 = 0) \\
&= \int_0^\infty \mathrm{d}x P_x\left(\sup_{n\leq -1} S_n < 0\right) \\
&= \int_0^\infty \mathrm{d}x P_0\left(\inf_{n\geq 1} S_n > x\right) \\
&= \mu \int_0^\infty \frac{\mathrm{d}x}{\mu_H} P_0(H_1 > x) \\
&= \mu,
\end{aligned}$$

where we used a symmetry property in the third line. As $\mu \in (0, \infty)$ we can condition the infinite measure on the event $\{T_0 = 0\}$ to get the probability measure

$$\Pi_\lambda(\cdot | T_0 = 0) := \frac{1}{\mu} \Pi_\lambda(\cdot \mathbf{1}_{T_0 = 0}).$$

We leave to the reader the simple verification that this measure on Ω is spatially stationary in the sense of Definition 1 and is projected on the measure $\mathbf{Q}$ on Ω_+. Thus it must coincide with $\mathbf{P}$, by Corollary 3. □

We may now prove Lemma 4. For the first part, it is enough to prove that for a fixed y, the sum $A(S, \mathcal{N})$ is finite Π_y-almost surely. Then α_x will be finite Π_y-a.s. , hence also Π_λ-almost surely (by integration) and $\mathbf{P}$-almost surely (by conditioning on a nontrivial event).

Under $\mathbf{P}_{\exp(y)}$, we have seen in Lemma 1 that the sequence of the duration of the arches with positive index, that is with our current notations $(\zeta_1(\mathcal{N}_n))_{n\leq 0}$, is i.i.d, with law that of ζ_1 under $\mathbb{P}_1^c$. Under Π_y, the sequence $(\zeta_1(\mathcal{N}_n))_{n\in\mathbb{Z}}$, with index in $\mathbb{Z}$, is again i.i.d with law that of ζ_1 under $\mathbb{P}_1^c$. Thus for any fixed $\varepsilon > 0$ there is Π_x-a.s. only a finite number of $k > 0$ such that $\ln(\zeta_1(\mathcal{N}_k))$ is bigger than εk. On the other hand, the sequence $(S_{-k-1})_{k\geq 0}$ under Π_x is a simple random walk, with an almost sure linear decay. The sum $A(S, \mathcal{N})$ is thus finite Π_y-a.s, as requested.

For the second part, it is enough to prove the convergence of the Laplace transform $\mathbf{P}_v(e^{-q\alpha_0} f(S_{T_0}, \mathcal{N}_{T_0}))$ to $\mathbf{P}(e^{-q\alpha_0} f(S_{T_0}, \mathcal{N}_{T_0})) = \mathbf{P}(e^{-q\alpha_0} f(S_0, \mathcal{N}_0))$ for any continuous functional $f : \mathbb{R} \times D \to [0, 1]$.

We have

$$\limsup \mathbf{P}_v(e^{-q\alpha_0} f(S_{T_0}, \mathcal{N}_{T_0})) \quad \leq \quad \mathbf{P}_v(e^{-q\alpha_0^l} f(S_{T_0}, \mathcal{N}_{T_0}))$$

for any $l \geq 0$, where we have written $\alpha_0^l = \sum_{T_0 - l < n \leq T_0} e^{2S_n} \zeta_1(\mathcal{N}_n)$ the total duration of the l last arches before index T_0, so that $\alpha_0 = \sup \alpha_0^l$. But we deduce from Corollary 3 that the right hand side converging to $\mathbf{P}(e^{-q\alpha_0^l} f(S_0, N_0))$ when $v \to 0$. Hence, letting l go to infinity, we immediately get by dominated convergence

$$\limsup_{v \to 0+} \mathbf{P}_v(e^{-q\alpha_0} f(S_{T_0}, \mathcal{N}_{T_0})) \leq \mathbf{P}(e^{-q\alpha_0} f(S_0, \mathcal{N}_0)). \tag{3.7}$$

Suppose that for some f we have

$$\liminf_{v \to 0+} \mathbf{P}_v(e^{-q\alpha_0} f(S_{T_0}, \mathcal{N}_{T_0})) < \mathbf{P}(e^{-q\alpha_0} f(S_0, \mathcal{N}_0)).$$

Adding this inequality to the inequality (3.7) stated for the function $1 - f$, we get

$$\liminf_{v \to 0+} \mathbf{P}_v(e^{-q\alpha_0}) < \mathbf{P}(e^{-q\alpha_0}).$$

In particular, for some $v > 0$, we have $\mathbf{P}_v(e^{-q\alpha_0}) < \mathbf{P}(e^{-q\alpha_0})$. We next just have to prove that the variable α_0 under $\mathbf{P}$ *dominates stochastically* the variable α_0 under $\mathbf{P}_v$ to complete the proof by contradiction.

It is immediate that for $0 < u < v$, the variable α_0 under $\mathbb{P}_u^c$ dominates stochastically the variable α_0 under $\mathbb{P}_v^c$ (with a scaling argument for example), and intuitively they should all be dominated by the variable α_0 under $\mathbb{P}_{0+}^c$

Indeed, fix v, fix M, and consider a real $x < \ln v$. We have

$$\begin{aligned}
\mathbf{P}(\alpha_0 \geq M) &\geq \mathbf{P}(\alpha_0^{T_x} \geq M) \\
&\geq \mathbf{P}\left(\sum_{T_x < n \leq T_0} e^{2S_n} \zeta_1(\mathcal{N}_n) \geq M \right) \\
&\geq \int m(\mathrm{d}y) \mathbf{P}_{x+y} \left(\sum_{T_x < n \leq T_0} e^{2S_n} \zeta_1(\mathcal{N}_n) \geq M \right),
\end{aligned}$$

where the last line follows from the relation $(\mathbf{P} \circ \Theta_x)_+ = \mathbf{Q}$ and a rescaling. Hence,

$$\begin{aligned}
\mathbf{P}(\alpha_0 \geq M) &\geq \int_0^{\ln v - x} m(\mathrm{d}y) \mathbf{P}_{x+y} \left(\sum_{T_x < n \leq T_0} e^{2S_n} \zeta_1(\mathcal{N}_n) \geq M \right) \\
&\geq \int_0^{\ln v - x} m(\mathrm{d}y) \mathbf{P}_{x+y}(\alpha_0 \geq M) \\
&\geq m([0, \ln v - x]) \mathbf{P}_v(\alpha_0 \geq M).
\end{aligned}$$

Letting x go to $-\infty$ gives the requested inequality and proves the stochastic domination. The Lemma 4 is proved in the supercritical case.

3.3 Proof of Lemma 4 in the critical case

In the critical case, we certainly can define P_λ and Π_λ as before, but under these measures the time T_0 is almost surely equal to $-\infty$. Lemma 5 thus fails, and so does the previous construction of $\mathbf{P}^1$ and $\mathbf{P}$. The next subsection is devoted to another construction of the law of the spatially stationary random walk $\mathbf{P}^1$. We then will use it to prove again the $\mathbf{P}-$almost sure finiteness of α_0, and Lemma 4 will follow from the same arguments as before. Throughout this section we assume (except in the next two lines) that $c = c_{crit}$.

3.3.1 The spatially stationary random walk in the critical case.

Let us start with an observation in the supercritical case: it follows from Lemma 5 that under $\mathbf{P}^1$, the sequence $(-S_{-n})_{n>0}$ is a random walk conditioned to stay positive, with some initial law. This description stays relevant in the critical case, provided that we precise what we mean by "random walk conditioned to stay positive" (conditioning on a trivial event). This is done in [5] and we recall it here briefly.

Write as usual P_x for the law of the random walk starting from position x. If you write $(D_n)_{n\geq 0}$ for the strictly descending ladder height process (defined in the exact similar way as the strictly ascending ladder height process, and also equal to the opposite of the strictly ascending ladder height process of $\widehat{S} := -S$) , the renewal function h is defined by

$$h(x) := \sum_{k=0}^{\infty} P_x(D_k \geq 0).$$

In particular h is non-decreasing, right-continuous, and we have $h(0) = 1$ and $h(x) = 0$ for $x < 0$. The renewal function is invariant for the random walk killed as it enters the negative half-line, and allows to define the process conditioned on never entering $(-\infty, 0)$ thanks to an usual $h-$transform in the sense of Doob. That is, the law of this process starting from $x > 0$, written $P_x^{\uparrow 0}$, is defined by

$$P_x^{\uparrow 0}(f(S)) = \frac{1}{h(x)} P_x(f(S)h(S_n), \inf_{k\leq n} S_k \geq 0) \tag{3.8}$$

for any $f(S) = f(S_0, ..., S_n)$ functional of the n first steps. For any $a \in \mathbb{R}$ and $x > a$, we also write $P_x^{\uparrow a}$ for the law of the random walk starting from $x > a$ and conditioned on never entering $(-\infty, a)$, defined in the exact same way, by

$$P_x^{\uparrow a}(f(S)) = \frac{1}{h(x-a)} P_x(f(S)h(S_n - a), \inf_{k\leq n} S_k \geq a) \tag{3.9}$$

for any $f(S) = f(S_0, ..., S_n)$ functional of the n first steps. The only other thing we will need to know about h is the following sub-additive inequality, that is a consequence from

a Markov property:

$$h(x+a) - h(x) \leq h(a), \qquad x, a > 0. \tag{3.10}$$

Recall that μ_H is the drift of the strictly ascending ladder height process and write $p(x,y)$ for the transition densities of the random walk. The following proposition gives a desintegration description of the spatially stationary random walk which is very similar to that of the spatially stationary Lévy process introduced by Bertoin and Savov in [6]. The extent of that comparison is as good as it can be given that we are dealing here with random walks, there in the more toughs settings of Lévy processes.

Proposition 1. *The measure* $\nu(\mathrm{d}x\mathrm{d}y) := \frac{1}{\mu_H} p(0, x+y)\mathbf{1}_{x\geq 0, y\geq 0} V(x)\mathrm{d}x\mathrm{d}y$ *is a probability law. The law of* $\mathbf{P}^1$ *is determined by:*
- *Under* $\mathbf{P}^1$, $(-S_{-1}, S_0)$ *has the law* ν.
- *Conditionally on* $-S_{-1} = x$ *and* $S_0 = y$, *the processes* $(-S_{-n-1})_{n\geq 0}$ *and* $(S_n)_{n\geq 0}$ *are independent, the law of* $(-S_{-n-1})_{n\geq 0}$ *is* $P_x^{\uparrow 0}$, *that of* $(S_n)_{n\geq 0}$ *is* P_y.

The measure ν is nothing else than the stationary joint law of the overshoot and the undershoot. The proof of this theorem will last until the end of the subsection. As a preliminary, we introduce a crucial though rather simple lemma.

Lemma 6. *For any* $0 \leq a \leq x$, *we have:*

$$P_x^{\uparrow 0}(\inf_{n\geq 0} S_n \geq a) = \frac{h(x-a)}{h(x)} \tag{3.11}$$

$$P_x^{\uparrow 0}(\cdot | \inf_{n\geq 0} S_n \geq a) = P_x^{\uparrow a}(\cdot). \tag{3.12}$$

Proof. The event $\{\inf_{k\geq 0} S_k \geq a\}$ is the limit of the events $\{\inf_{0\leq k\leq n} S_k \geq a\}$, further

$$\begin{aligned} & P_x^{\uparrow 0}(\inf_{0\leq k\leq n} S_k \geq a) \\ = & \frac{1}{h(x)} P_x(h(S_n), \inf_{0\leq k\leq n} S_k \geq a) \\ = & \frac{1}{h(x)} P_x(h(S_n - a), \inf_{0\leq k\leq n} S_k \geq a) + \frac{1}{h(x)} P_x(h(S_n) - h(S_n - a), \inf_{0\leq k\leq n} S_k \geq a). \end{aligned}$$

The first term of the sum is equal to $\frac{h(x-a)}{h(x)}$ because the function $h(\cdot - a)$ is invariant for the random walk killed when hitting $(-\infty, a)$. The second term is positive and bounded from above by $\frac{h(a)}{h(x)} P_x(\inf_{0\leq k\leq n} S_k \geq a)$, which goes to 0 when n goes to $+\infty$. This proves equation (3.11). Then (3.12) is straightforward: Indeed, for $f(S) = f(S_0, ..., S_n)$ functional of the n first steps, we have:

$$\begin{aligned} P_x^{\uparrow 0}(f(S) | \inf_{k\geq 0} S_k \geq a) &= \frac{1}{P_x^{\uparrow 0}\left(\inf_{k\geq 0} S_k \geq a\right)} P_x^{\uparrow 0}\left(f(S) P_{S_n}^{\uparrow 0}\left(\inf_{k\geq 0} S_k \geq a\right)\right) \\ &= \frac{h(x)}{h(x-a)} \cdot \frac{1}{h(x)} P_x\left(f(S) h(S_n) \frac{h(S_n - a)}{h(S_n)}\right) \\ &= P_x^{\uparrow a}(f(S)). \end{aligned}$$

□

Now, recall that the invariance property of h states that, for any $x \geq 0$, we have

$$h(x) = P_x(h(S_1)\mathbf{1}_{S_1 \geq 0}).$$

Define $\overline{h}$ by $\overline{h}(x) := P_x(h(S_1), S_1 \geq 0)$ for *any real number* x. Thus for $x \geq 0$, $\overline{h}$ and h coincide, but for $x < 0$ they certainly don't. This allows to define the law $P_x^{\uparrow 0}$ of the random walk starting from x and conditioned on never hitting the negative half-line *at times* $n \geq 1$, by the formula:

$$P_x^{\uparrow 0}(f(S)) = \frac{1}{\overline{h}(x)} P_x(f(S)h(S_n), \inf_{1 \leq k \leq n} S_k \geq 0) \tag{3.13}$$

for any functional $f(S) = f(S_0, ..., S_n)$. This is consistent with our previous notations, and, for $y < x$, we have

$$P_x^{\uparrow 0}(\cdot) = P_x^{\uparrow y}(\cdot | \inf_{n \geq 1} S_n \geq 0).$$

We leave to the reader the verification that formula (3.11) can be generalized in

$$P_x^{\uparrow y}(\inf_{n \geq 1} S_n \geq a) = \frac{\overline{h}(x-a)}{\overline{h}(x-y)} \tag{3.14}$$

with the only requirement $y < a$.

The following lemma is a result of straightforward calculations that we leave to the interested reader.

Lemma 7. *Write ν_- (resp. ν_+) for the first (resp. second) marginal of ν. These measures on $\mathbb{R}_+$ are given for $x, y > 0$, by*

$$\begin{aligned} \nu_-(\mathrm{d}x) &= \frac{1}{\mu_H} h(x) P_0(S_1 \geq x) \mathrm{d}x. \\ \nu_+(\mathrm{d}y) &= \frac{1}{\mu_H} \overline{h}(-y) \mathrm{d}y. \end{aligned}$$

Moreover,

$$\begin{aligned} P_{-\nu_-}(S_1 \in \mathrm{d}y | S_1 \geq 0) &= \nu_+(\mathrm{d}y). \\ P^{\uparrow 0}_{-\nu_+(\mathrm{d}y)} &= \nu_-(\mathrm{d}x). \end{aligned}$$

This lemma should make the introduction of the measure ν in the theorem more transparent. It indeed gives us two alternative ways of defining the measure $\mathbf{P}^1$. The first one is to take S_0 distributed according to ν_+ and, conditionally on $S_0 = y$, take $(S_n)_{n \geq 0}$ of law P_y and $(-S_{-n})_{n \geq 0}$ independent and of law $P_{-y}^{\uparrow 0}$ (in the sense defined just before). The second one to take $-S_{-1}$ distributed according to ν_- and, conditionally on $S_{-1} = -x$, take $(S_{n-1})_{n \geq 0}$ of law P_{-x} conditioned on having a first jump no smaller than x, and $(-S_{-n-1})_{n \geq 0}$ independent and of law $P_x^{\uparrow 0}$.

Proof of the proposition. We need to prove three things, the fact that ν is a probability measure (that is, has mass one), the fact that $\mathbf{P}^1$ is spatially stationary and the equality $\mathbf{P}^1_+ = \mathbf{Q}$. We start with the spatial stationarity. Fix $a > 0$. We should prove that $S = (S_n)_{n\in\mathbb{Z}}$ and $R_n := \Theta_a(S) = (S_{T_a+n} - a)_{n\in\mathbb{Z}}$ have the same law under $\mathbf{P}^1$.

Observe that T_a is also equal to the instant of the last passage under level a for the process $(-R_{-n})_{n\geq 0}$. *Suppose that we proved* that $((T_a, -R_{-n})_{0\leq n\leq T_a})$ has the same law a process $(L_a(S), (S_n)_{0\leq n\leq L_a(S)})$ under $P^{\uparrow 0}_{-\nu_+}$. Then, conditionally on $-R_{-T_a} = z$, it is clear that the process $(-R_{-n-T_a})_{n\geq 0} = (a - S_{-n})_{n\geq 0}$ is independent of $(-R_{-n})_{0\leq n\leq T_a}$ and follows the law $P^{\uparrow a}_z$. But we also have that for a process S under $P^{\uparrow 0}_{-\nu_+}$, conditionally on $S_{L_a(S)} = z$, the process $(S_{n+L_a(S)})_{n\geq 0}$ is independent from $(S_n)_{0\leq n\leq L_a(S)}$ and follows the law $P^{\uparrow a}_z$. This altogether proves that the process $(-R_{-n})_{n\geq 0}$ follows the law $P^{\uparrow 0}_{-\nu_+}$. Finally, from a Markov property, it is clear that given $R_0 = y$, the process $(R_n)_{n\geq 0}$ is independent of $(R_n)_{n\leq 0}$ and follows the law P_y, thus the law of $(R_n)_{n\in Z}$ is $\mathbf{P}^1$.

Hence, all we still need to prove is the following duality property that $(T_a, -R_{-n})_{0\leq n\leq T_a})$ has the same law as $(L_a(S), (S_n)_{0\leq n\leq L_a(S)})$ for a process S of law $P^{\uparrow 0}_{-\nu_+}$. This property also finds its analogue in [6], in their Theorem 2. Anyway, fix $n \geq 0$ and $f : \mathbb{R}^{n+1} \to \mathbb{R}$ a positive continuous functional. We should prove the following equality:

$$\mathbf{P}^1(f((-R_{-k})_0 \leq k \leq n)\mathbf{1}_{T_a=n}) = P^{\uparrow 0}_{-\nu_+}(f((S_k)_{0\leq k\leq n})\mathbf{1}_{L_a(S)=n}).$$

The case $n = 0$ is particular and follows from this calculation:

$$\begin{aligned} P^{\uparrow 0}_{-\nu_+}(-S_0 \in \mathrm{d}x, L_a(S) = 0) &= \nu_+(x)P^{\uparrow 0}_{-\nu_+}(\inf_{k\geq 1} S_k \geq a)\mathrm{d}x \\ &= \frac{1}{\mu_H}\overline{h}(-x)\frac{\overline{h}(-a-x)}{\overline{h}(-x)}\mathrm{d}x \\ &= \nu_+(a+x)\mathrm{d}x = \mathbf{P}^1(R_0 \in \mathrm{d}x, T_a = 0). \end{aligned}$$

In the case $n > 0$, we write $\tilde{f}((S_k)_{0\leq k\leq n}) := f((a - S_{n-k})_{0\leq k\leq n})$, the usual duality property for random walks stating

$$P_x(f(S)\mathbf{1}_{a-S_n\in\mathrm{d}y})\mathrm{d}x = P_y(\tilde{f}(S)\mathbf{1}_{a-S_n\in\mathrm{d}x})\mathrm{d}y.$$

We are ready to calculate

$$\begin{aligned} & \mathbf{P}^1(f((-R_{-k})_{0\leq k\leq n})\mathbf{1}_{T_a=n}) \\ = \ & \mathbf{P}^1(\tilde{f}((S_k)_{0\leq k\leq n})\mathbf{1}_{T_a=n}) \\ = \ & \int\int_{(x,y)\in\mathbb{R}_+\times[0,a)} \nu_+(\mathrm{d}y)P_y(\tilde{f}((S_k)_{0\leq k\leq n}), S_n - a \in \mathrm{d}x, \forall 0 \leq i < n, S_i \leq a) \\ = \ & \int\int_{\mathbb{R}_+\times[0,a)} \frac{1}{\mu_H}\overline{h}(-y)\mathrm{d}xP_{-x}(f((S_k)_{0\leq k\leq n}), a - S_n \in \mathrm{d}y, \forall 0 < i \leq n, S_i \geq 0) \\ = \ & \int\int_{\mathbb{R}_+\times[0,a)} \frac{\mathrm{d}x}{\mu_H}P_{-x}(f((S_k)_{0\leq k\leq n})h(S_n), a - S_n \in \mathrm{d}y, \forall 0 < i \leq n, S_i \geq 0)\frac{\overline{h}(-y)}{h(a-y)}. \end{aligned}$$

Using then (3.13) and (3.14), we get

$$\begin{aligned}
& \mathbf{P}^1(f((-R_{-k})_{0\le k\le n})\mathbf{1}_{T_a=n}) \\
= & \int\int_{\mathbb{R}_+\times[0,a)} \nu_+(\mathrm{d}x)P^{\uparrow 0}_{-x}(f((S_k)_{0\le k\le n}), a-S_n\in \mathrm{d}y)P^{\uparrow 0}_{a-y}(\inf_{k\ge 1} S_k\ge a) \\
= & \int_{\mathbb{R}_+} \nu_+(\mathrm{d}x)P^{\uparrow 0}_{-x}(f((S_k)_{0\le k\le n}), S_n<a, \inf_{k>n} S_k\ge a) \\
= & P^{\uparrow 0}_{-\nu_+}(f((S_k)_{0\le k\le n})\mathbf{1}_{L_a(S)=n}).
\end{aligned}$$

The measure $\mathbf{P}^1$ is thus spatially stationary.

Now the two facts that ν has mass one and that $\mathbf{P}^1_+ = \mathbf{Q}$ both follow from the equality

$$\overline{h}(-y) = P_0(H_1\ge y)$$

for $y\ge 0$ (recall that H is the strictly ascending ladder height process). Fix some $y\ge 0$. We already know from (3.14) that $\overline{h}(-y) = P^{\uparrow 0}_0(\inf_{n\ge 0} S_n\ge y)$, thus we should prove

$$P_0(H_1\in \mathrm{d}y) = P^{\uparrow 0}_0(\inf_{n\ge 0} S_n\in \mathrm{d}y). \tag{3.15}$$

This will be a consequence from another duality argument. Write T_{inf} for the instant when S hits its minimum on times $n\ge 1$. Write $\tilde{T}_1 := \inf\{n>0, S_n>S_0\}$ (so that $S_{\tilde{T}_1}=H_1$). Then $(S_k)_{0\le k\le \tilde{T}_1}$ under P_0 and $(S_n)_{0\le k\le T_{inf}}$ under $P^{\uparrow 0}_0$ are in duality. Indeed, fix $n>0$ and $f(S)=f((S_k)_{0\le k\le n})$ a positive continuous functional. Write also $\tilde{f}((S_k)_{0\le k\le n}) := f((S_n-S_{n-k})_{0\le k\le n}$. Then,

$$\begin{aligned}
P^{\uparrow 0}_0(f(S)\mathbf{1}_{T_{inf}=n}) &= P^{\uparrow 0}_0\left(f(S)P^{\uparrow 0}_x(\inf_{n\ge 1} S_n\ge x), \inf_{1\le k\le n} S_k = S_n\right) \\
&= P_0(f(S)\frac{h(x)}{h(x)}, \inf_{1\le k\le n} S_k = S_n) \\
&= P_0(\tilde{f}(S), \sup_{0\le k\le n-1} S_k = 0) \\
&= P_0(\tilde{f}(S)\mathbf{1}_{\tilde{T}_1=n}).
\end{aligned}$$

This duality property implies in particular (3.15). □

3.3.2 Finiteness of α_0 in the critical case.

The only thing we actually need from last subsection is the fact under $\mathbf{P}^1$ (or, equivalently, under $\mathbf{P}$), the sequence $(-S_{-n})_{n\ge 1}$ is a random walk conditioned to stay positive, with some initial law. The paper [10] gives very precise results about the behavior of this random walk conditioned to stay positive, and we deduce in particular the following rough bounds that are sufficient for our purposes:

Lemma 8. *For any $\varepsilon > 0$, we have*

$$n^{-\frac{1}{2}+\varepsilon} S_{-n} \to -\infty \tag{3.16}$$

when $n \to \infty$, $\mathbf{P}$-a.s.

We now work under $\mathbf{P}$ and we recall that α_0 is then given by

$$\alpha_0 = \sum_{n<0} e^{2S_n} \zeta_1(\mathcal{N}_n).$$

We write $L_n := e^{2S_n}\zeta_1(\mathcal{N}_n)$ for the duration of the arch of index n. We need to transfer the results about the behavior of (S_{-n}) to results about the behavior of (L_{-n}). This is made possible by the following lemma:

Lemma 9. *1) Under $\mathbf{P}$ and conditionally on a realization $(S_n)_{n\in\mathbb{Z}} = (s_n)_{n\in\mathbb{Z}}$, the variables $(L_n)_{n\in\mathbb{Z}}$ are mutually independent, and the law of L_n is that of ζ_1 under $\mathbb{P}^c_{\exp(s_n)}(\cdot|V_1 = \exp(s_{n+1}))$.*

2) If $u, v < r$ for some real number r, then

$$\mathbb{P}^c_u(\zeta_1 > tr^2 | V_1 = cv) \leq \frac{16\sqrt{2}}{3\sqrt{\pi}} t^{-\frac{3}{2}}. \tag{3.17}$$

Proof. The result of the first part is easy for $(L_n)_{n\geq 0}$, and we get the result for $(L_n)_{n\in\mathbb{Z}}$ by spatial stationarity.

For the second part, observe that as the law of the couple (ζ_1, V_1) under $\mathbb{P}^c_v$ is known (see Lemma 1 and in particular (2.1) and (2.2)), we get explicitly:

$$\frac{1}{\mathrm{d}s}\mathbb{P}^c_1(\zeta_1 \in \mathrm{d}s | V_1 = cv) = \frac{\sqrt{2}(u^3+v^3)}{s^2 u^{\frac{1}{2}} v^{\frac{1}{2}}} \exp(-2\frac{v^2 - uv + u^2}{s}) \int_0^{\frac{4uv}{s}} e^{-\frac{3\theta}{2}} \frac{\mathrm{d}\theta}{\sqrt{\pi\theta}}.$$

Provided that we take $r > u, v$, we get

$$\begin{aligned} \frac{1}{\mathrm{d}s}\mathbb{P}^c_1(\zeta_1 \in \mathrm{d}s | V_1 = cv) &\leq \frac{\sqrt{4}r^3}{s^2 u^{\frac{1}{2}} v^{\frac{1}{2}}} \int_0^{\frac{4uv}{s}} \frac{\mathrm{d}\theta}{\sqrt{\pi\theta}} \\ &\leq \frac{8\sqrt{2}}{\pi} r^3 s^{-\frac{5}{2}}. \end{aligned}$$

Integrating this inequality between tr^2 and $+\infty$ gives (3.17). □

The $\mathbf{P}$-almost sure finiteness of α_0 follows straightforwardly. Write

$$r_n = e^{s_n} \vee \frac{e^{s_{n+1}}}{c},$$

and, for $n > 0$, write A_n for the event

$$L_{-n} \geq n\ r_{-n}^2.$$

The lemma states that the probability of A_n is bounded above by a constant times $n^{-\frac{3}{2}}$. Hence only a finite number of A_n occur and the $(L_{-n})_{n\geq 0}$ are summable, almost surely. In other words, the time α_0 is finite $\mathbf{P}$-almost surely.

4 Appendix

The appendix is devoted to the proof of the technical lemma 2.

First notice that the uniqueness is immediate. Consider P and P' two probabilities satisfying the conditions of Lemma 2. Then for any real x, we have $(P \circ \Theta_x)_+ = P_+ = Q = (P' \circ \Theta_x)_+$. It follows $P = P'$.

The key point is the construction of the probability P, which will be a consequence of Kolmogorov existence theorem. First, note that we have $\Theta_x \circ \Theta_y = \Theta_{x+y}$ for any x, y reals. Take $x > 0$. On the one hand, from $(P_v \circ \Theta_0)_+ \Rightarrow Q$, we deduce $(P_v \circ \Theta_0)_+ \circ \Theta_x \Rightarrow Q \circ \Theta_x$. On the other hand, we have $\big((P_v \circ \Theta_0)_+ \circ \Theta_x\big)_+ = (P_v \circ \Theta_x)_+ \Rightarrow Q$. Thus the laws $(Q \circ \Theta_x)_+$ and Q are identical for any $x > 0$.

For $x_1 < ... < x_n$ reals, I define first Y^{x_1} as a variable of law Q, then Y^{x_i} by $Y^{x_i} = (\Theta_{x_i - x_1} \circ Y^{x_1})_+$, so that Y_{x_i} has also law Q. I write $Q^{x_1,...,x_n}$ for the law of $(Y^{x_1}, ..., Y^{x_n})$ obtained in that way, law on $\Omega^{x_1,...x_n}$. These laws are compatible. Thus Kolmogorov theorem tells that there exists a law $\overline{Q}$ on $\Omega^{\mathbb{R}}$ such that the finite dimensional marginals of $\overline{Q}$ on say $x_1, ..., x_n$ is equal to $Q^{x_1,...,x_n}$.

Let $(Z^x)_{x \in \mathbb{R}}$ be with law $\overline{Q}$. Then, define Y a random variable on Ω by

$$Y(k) := \lim_{a \to \infty} \Theta_a \circ Z^{-a}(k).$$

This definition requires some explications. First, $\overline{Q}(T_a(Z^{-a}) \geq -k) = \overline{Q}(T_a(Z^0) \geq -k)$ is converging to 1 when a goes to $+\infty$. Thus a.s. for some a we have $T_a(Z^{-a}) \geq -k$ and then $\Theta_a \circ Z^{-a}(k) \neq -\infty$. But for any $x > a$, we have:

$$\begin{aligned} \Theta_a \circ Z^{-a}(k) &= \Theta_a \circ (\Theta_{x-a} \circ Z^{-x})_+(k) \\ &= \Theta_a \circ (\Theta_{x-a} \circ Z^{-x})(k) = \Theta_x \circ Z^{-x}(k), \end{aligned}$$

where we can drop the index $+$ at the second equality because we are on the event $T_a(Z^{-a}) \geq -k$. Thus for each k the family $(\Theta_a \circ Z^{-a}(k))_{a \geq 0}$ is constant as soon as it leaves $-\infty$, and the limit is well-defined.

Observe that the random variable Y satisfies the conditions

$$\lim_{n \to -\infty} Y^1(n) = -\infty, \qquad \limsup_{n \to +\infty} Y^1(n) = +\infty.$$

Its probability law P on Ω not only satisfies $P_+ = Q$, it is also spatially invariant: Indeed, for any x, the variable $\Theta_x(Y)$ has law $P \circ \Theta_x$ and is given by

$$\Theta_x(Y)(k) = \lim_{a \to \infty} \Theta_{x+a} \circ Z^{-a}(k) = \lim_{a \to \infty} \Theta_a \circ Z^{-a-x}(k).$$

But it is obvious that the family $(Z^{a-x})_{a \in \mathbb{R}}$ also has law $\overline{Q}$, hence $\Theta_x(Y)$ has law P.

Finally, we still have to prove $P_v \circ \Theta_x \Rightarrow P$. Take f any positive bounded continuous functional depending on a finite number of variables $\omega_{t_1}, ...\omega_{t_n}$, with $t = t_1 < ... < t_n$, so that $f((\omega_s)_{s \in \mathbb{Z}}) = f((\omega_s)_{s \geq t})$. We suppose without loss of generality $t < 0$. Observe that

under the probability $P_v \circ \Theta_x$ or under P, we have $T_0 = 0$, and the events $T_{-y} \leq t$ and $T_y \circ \Theta_{-y} > -t$ coincide, almost surely. Observe also $Q(T_y \leq -t) \to_{y\to\infty} 0$. Then,

$$\begin{aligned} P_v \circ \Theta_x(f((\omega_s)_{s\geq t})\mathbf{1}_{T_{-y}<t}) &= P_v \circ \Theta_{x-y}(f \circ \Theta_y((\omega_s)_{s\geq t}), T_y > -t) \\ &= (P_v \circ \Theta_{x-y})_+(f \circ \Theta_y((\omega_s)_{s\geq t}), T_y > -t) \\ &\to Q(f \circ \Theta_y((\omega_s)_{s\geq t}), T_y > -t) \\ &= P(f \circ \Theta_y((\omega_s)_{s\geq t}), T_y > -t) \\ &= P(f((\omega_s)_{s\geq t}), T_{-y} < t), \end{aligned}$$

where we get the second line because the functional $\mathbf{1}_{T_y>-t} f \circ \Theta_y((\omega_s)_{s\geq t})$ does not depend on $(\omega_n)_{n<0}$, and obtain the last line by the translation Θ_{-y}. Besides, we have:

$$\begin{aligned} |P_v \circ \Theta_x(f((\omega_s)_{s\geq t})\mathbf{1}_{T_{-y}<t}) - P_v \circ \Theta_x(f((\omega_s)_{s\geq t}))| &\leq \sup(f) P_v \circ \Theta_x(\mathbf{1}_{T_{-y}\geq t}) \\ &= \sup(f) P_v \circ \Theta_{x-y}(\mathbf{1}_{T_y\leq -t}) \\ &\to_{v\to 0^+} \sup(f) Q(\mathbf{1}_{T_y\leq -t}) \to_{y\to\infty} 0, \end{aligned}$$

and in the same way

$$P(f((\omega_s)_{s\geq t}), T_{-y} < -t) \longrightarrow_{y\to\infty} P(f((\omega_s)_{s\geq t})).$$

This is enough to deduce

$$P_v(f((\omega_s)_{s\geq t}) \to P(f((\omega_s)_{s\geq t})).$$

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