



**HAL**  
open science

# Wreath products with the integers, proper actions and Hilbert space compression

Yves Stalder, Alain Valette

► **To cite this version:**

Yves Stalder, Alain Valette. Wreath products with the integers, proper actions and Hilbert space compression. *Geometriae Dedicata*, 2007, 124, pp.199-211. 10.1007/s10711-006-9119-3 . hal-00472504

**HAL Id: hal-00472504**

**<https://hal.science/hal-00472504>**

Submitted on 20 Dec 2019

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# Wreath products with the integers, proper actions and Hilbert space compression

Yves STALDER\* and Alain VALETTE

October 13, 2006

*A la mémoire de Michel Matthey*

## Abstract

We prove that the properties of acting metrically properly on some space with walls or some  $CAT(0)$  cube complex are closed by taking the wreath product with  $\mathbb{Z}$ . We also give a lower bound for the (equivariant) Hilbert space compression of  $H \wr \mathbb{Z}$  in terms of the (equivariant) Hilbert space compression of  $H$ .

## Introduction

A *space with walls*, as defined by Haglund and Paulin [HP98], is a pair  $(X, \mathcal{W})$  where  $X$  is a set and  $\mathcal{W}$  is a set of partitions of  $X$  (called *walls*) into two classes, submitted to the condition that any two points of  $X$  are separated by finitely many walls.

The main examples of spaces with walls are given by  $CAT(0)$  cube complexes (see [BH99]), i.e. metric polyhedral complexes in which each  $k$ -cell is isomorphic to the Euclidean cube  $[-1/2, 1/2]^k$ , and the gluing maps are isometries. Indeed, it is a result of Sageev [Sag95] that hyperplanes in a  $CAT(0)$  cube complex endow the set of vertices with a structure of space with walls (see [ChN05] and [Nic04] for more on the relation between spaces with walls and  $CAT(0)$  cube complexes).

Our first result is the following:

---

\*The first-named author gratefully acknowledges supports given by the Swiss FNRS (grant 20-109130) and the Université de Neuchâtel, where he was employed during the research presented in this article

**Theorem 0.1** *Suppose that a group  $H$  acts metrically properly either on some space with walls, or on some  $CAT(0)$  cube complex. Then so does the restricted wreath product  $H \wr \mathbb{Z} := (\bigoplus_{\mathbb{Z}} H) \rtimes \mathbb{Z}$ .*

Guentner-Kaminker defined the *Hilbert space compression* and the *equivariant Hilbert space compression* for any unbounded metric space (endowed with a group action in the latter case) [GK04]. Since we will deal with uniformly discrete<sup>1</sup> spaces, the following definitions are equivalent to theirs.

Let  $(X, d)$  be a uniformly discrete metric space. We define the *Hilbert space compression* of  $X$  as the supremum of the numbers  $\alpha \in [0, 1]$  such that there exists a Hilbert space  $\mathcal{H}$ , positive constants  $C_1, C_2$  and a map  $f : X \rightarrow \mathcal{H}$  with

$$C_1 \cdot d(x, y)^\alpha \leq \|f(x) - f(y)\| \leq C_2 \cdot d(x, y) \quad \forall x, y \in X .$$

It is denoted by  $R(X, d)$  and it is a quasi-isometry invariant of  $(X, d)$ . If  $H$  is a group acting on  $(X, d)$  by isometries, the *equivariant Hilbert space compression* of  $X$  is the supremum of the numbers  $\alpha \in [0, 1]$  such that there exists a Hilbert space  $\mathcal{H}$  endowed with an action of  $H$  by affine isometries, positive constants  $C_1, C_2$  and a  $H$ -equivariant map  $f : X \rightarrow \mathcal{H}$  with

$$C_1 \cdot d(x, y)^\alpha \leq \|f(x) - f(y)\| \leq C_2 \cdot d(x, y) \quad \forall x, y \in X .$$

It is denoted by  $R_H(X, d)$ . One has trivially  $R_H(X, d) \leq R(X, d)$ .

We may view a group  $H$  as a metric space thanks to the word length associated with some (not necessarily finite) generating subset  $S$ . We denote then by  $R(H, S)$  the Hilbert space compression and by  $R_H(H, S)$  the equivariant Hilbert space compression. In case  $H$  is finitely generated, note that, up to bilipschitz equivalence, the word metric does not depend on the finite generating set, so that the compressions do not depend on the choice of a finite generating set. In this case we write  $R(H)$  and  $R_H(H)$  for the corresponding compressions. We also use these shorter notations in the general case if there is no ambiguity about the generating set. It is a remarkable observation of Gromov (see [CTV, Proposition 4.4] for a proof) that  $R(H) = R_H(H)$  for  $H$  finitely generated and amenable.

The first examples of finitely generated groups whose Hilbert space compression is different from 0 and 1 appeared recently in [AGS]: Thompson's

---

<sup>1</sup>That is, there exists a constant  $\delta > 0$  such that  $d(x, y) \geq \delta$  whenever  $x \neq y$ .

group  $F$  and the wreath product  $\mathbb{Z} \wr \mathbb{Z}$  (see Section 4 for more on this). Our next Theorem allows in particular to construct more examples.

Given a generating set  $S$  for  $H$ , if  $\Gamma = H \wr \mathbb{Z}$ , we always take  $\Sigma = S \cup \{s\}$  as generating set for  $\Gamma$ , where  $s$  is the positive generator of  $\mathbb{Z}$ .

**Theorem 0.2** *Let  $H$  be a group, with generating set  $S$  and let  $\Gamma = H \wr \mathbb{Z}$ . The non equivariant and equivariant Hilbert space compressions satisfy:*

$$R(H, S) \geq R(\Gamma, \Sigma) \geq \frac{R(H, S)}{R(H, S) + 1} ;$$

$$R_H(H, S) \geq R_\Gamma(\Gamma, \Sigma) \geq \max \left\{ R_H(H, S) - \frac{1}{2}, \frac{R_H(H, S)}{2R_H(H, S) + 1} \right\} .$$

In order to select the best bound, we mention that one has  $t - 1/2 \geq t/(2t + 1)$  if and only if  $t \geq (1 + \sqrt{5})/4 \cong 0.809\dots$  (for  $t \in [0, 1]$ ). The above-mentioned observation by Gromov gives immediately a stronger estimate for the equivariant compression.

**Corollary 0.3** *Let  $H$  be a finitely generated and amenable group and let  $\Gamma = H \wr \mathbb{Z}$ . The equivariant Hilbert space compression satisfies:*

$$R_H(H) \geq R_\Gamma(\Gamma) \geq \frac{R_H(H)}{R_H(H) + 1} .$$

The proofs of Theorems 0.1 and 0.2 rest on a similar idea: we express  $H \wr \mathbb{Z}$  as an HNN-extension in two different ways, which provide two different actions of  $H \wr \mathbb{Z}$  on a tree. In Theorem 0.1 we use the product of these two trees, while in Theorem 0.2 we appeal to the affine actions naturally associated with each of these trees (see section 7.4.1 in [CCJ<sup>+</sup>01]).

## 1 Preliminaries: wreath products and trees

Let  $\Lambda$  be a group,  $H$  a subgroup and  $\vartheta : H \rightarrow \Lambda$  an injective homomorphism. The *HNN-extension* with *basis*  $\Lambda$  and *stable letter*  $t$  relatively to  $H$  and  $\vartheta$  is defined by  $HNN(\Lambda, H, \vartheta) = \langle \Lambda, t \mid t^{-1}ht = \vartheta(h) \forall h \in H \rangle$ .

Our definition of graphs and trees are those of [Ser77]. Given an HNN-extension  $\Gamma = HNN(\Lambda, H, \vartheta)$ , the associated *Bass-Serre tree* is defined by

$$V(T) = \Gamma/\Lambda ; E(T) = \Gamma/H \sqcup \Gamma/\vartheta(H) ; \overline{\gamma H} = \gamma t \vartheta(H) ; \overline{\gamma \vartheta(H)} = \gamma t^{-1} H ;$$

$$(\gamma H)^- = \gamma \Lambda ; (\gamma H)^+ = \gamma t \Lambda ; (\gamma \vartheta(H))^- = \gamma \Lambda ; (\gamma \vartheta(H))^+ = \gamma t^{-1} \Lambda$$

where, given an edge  $e$ , its origin is denoted by  $e^-$  and its terminal vertex by  $e^+$ . It is a tree [Ser77, Theorem 12]. We turn  $T$  to an oriented tree by setting  $Ar_+(T) = \Gamma/H$ ,  $(\gamma H)^- = \gamma \Lambda$ ,  $(\gamma H)^+ = \gamma t \Lambda$  and the  $\Gamma$ -action on  $T$  preserves this orientation. Moreover we remark that the oriented tree is bi-regular: for each vertex of  $T$  the outgoing edges are in bijection with  $\Lambda/H$  and the incoming edges are in bijection with  $\Lambda/\vartheta(H)$ .

We turn to wreath products. Let  $G, H$  be groups. We set

$$\Lambda = H^{(G)} = \bigoplus_{g \in G} H = \{ \lambda : G \rightarrow H \text{ with finite support} \} .$$

The group  $G$  acts on  $\Lambda$  by automorphisms:  $(g \cdot \lambda)(x) = \lambda(g^{-1}x)$ . The *restricted wreath product*  $H \wr G$  is the semi-direct product  $\Lambda \rtimes G$ , with respect to the above action. The group  $H$  embeds in  $H \wr G$  and will be identified with its copy indexed by  $1_G \in G$ . It is easy to see that, given generating sets of  $G$  and  $H$ , their union generates  $H \wr G$ .

In case  $G = \mathbb{Z}$ , one may express  $H \wr \mathbb{Z}$  as an HNN-extension in two ways (we denote by  $s$  the positive generator of  $\mathbb{Z}$  in  $H \wr \mathbb{Z}$  and by  $t_+, t_-$  the stable letters of the HNN-extensions)<sup>2</sup>:

1. Set  $\Lambda_+ = \bigoplus_{n \geq 0} H$  and  $\vartheta_+ : \Lambda_+ \rightarrow \Lambda_+$  given by  $\vartheta_+(\lambda)_0 = 1_H$  and  $\vartheta_+(\lambda)_n = \lambda_{n-1}$  for  $n \geq 1$ . One has  $HNN(\Lambda_+, \Lambda_+, \vartheta_+) = H \wr \mathbb{Z}$  and the isomorphism is given by  $\lambda \mapsto \lambda$  and  $t_+ \mapsto s^{-1}$ ;
2. Set  $\Lambda_- = \bigoplus_{n \leq 0} H$  and  $\vartheta_- : \Lambda_- \rightarrow \Lambda_-$  given by  $\vartheta_-(\lambda)_0 = 1_H$  and  $\vartheta_-(\lambda)_n = \lambda_{n+1}$  for  $n \leq -1$ . One has  $HNN(\Lambda_-, \Lambda_-, \vartheta_-) = H \wr \mathbb{Z}$  and the isomorphism is given by  $\lambda \mapsto \lambda$  and  $t_- \mapsto s$ ;

Given a wreath product  $H \wr \mathbb{Z}$ , we will denote by  $T_+$ , respectively  $T_-$ , the Bass-Serre tree associated to the first, respectively second, HNN-extension above. We take as base points (when necessary) the vertices  $\Lambda_+$  and  $\Lambda_-$ .

We collect now some observations about the  $H \wr \mathbb{Z}$ -actions on  $T_+$  and  $T_-$  which will be relevant in the next sections. Set  $\Gamma = H \wr \mathbb{Z}$  and  $\gamma = \lambda s^n \in \Gamma$ . If  $\lambda$  is nontrivial, we set  $m = \min\{k \in \mathbb{Z} : \lambda_k \neq 1_H\}$  and  $M = \max\{k \in \mathbb{Z} : \lambda_k \neq 1_H\}$ . The following lemma is straightforward.

<sup>2</sup>There is a third way of expressing  $H \wr \mathbb{Z}$  as an HNN-extension: set  $\vartheta : \Lambda \rightarrow \Lambda$ ;  $\vartheta(\lambda)_n = \lambda_{n-1}$ . One has  $HNN(\Lambda, \Lambda, \vartheta) = H \wr \mathbb{Z} = \Lambda \rtimes_{\vartheta} \mathbb{Z}$  and the isomorphism is given by  $\lambda \mapsto \lambda$  and  $t \mapsto s^{-1}$ . Nevertheless, this expression will be useless in this article.

**Lemma 1.1** *If  $\lambda = 1$ , one has  $d_{T_+}(\Lambda_+, \gamma\Lambda_+) = |n| = d_{T_-}(\Lambda_-, \gamma\Lambda_-)$ .*

**Lemma 1.2** *If  $\lambda \neq 1$ , the distances  $d_{T_\pm}(\Lambda_\pm, \gamma\Lambda_\pm)$  are given by formulas:*

$$d_{T_+}(\Lambda_+, \gamma\Lambda_+) = \begin{cases} |n| & \text{if } m \geq n \text{ or } m \geq 0 \\ n - 2m & \text{if } m < n \text{ and } m < 0 \end{cases} ;$$

$$d_{T_-}(\Lambda_-, \gamma\Lambda_-) = \begin{cases} |n| & \text{if } M \leq n \text{ or } M \leq 0 \\ 2M - n & \text{if } M > n \text{ and } M > 0 \end{cases} .$$

*In particular, the inequalities  $d_{T_+}(\Lambda_+, \gamma\Lambda_+) \geq -m$ ,  $d_{T_-}(\Lambda_-, \gamma\Lambda_-) \geq M$ ,  $d_{T_+}(\Lambda_+, \gamma\Lambda_+) \geq |n|$  and  $d_{T_-}(\Lambda_-, \gamma\Lambda_-) \geq |n|$  hold.*

PROOF. We prove the second equality, leaving the first one, which is very similar, to the reader. We remark that  $\gamma = \lambda s^n = \lambda t_-^n$  and that, for any  $k \in \mathbb{Z}$ , the stabilizer of the vertex  $t_-^k \Lambda_-$  satisfies

$$\text{Stab}(t_-^k \Lambda_-) = t_-^k \Lambda_- t_-^{-k} = s^k \Lambda_- s^{-k} = \bigoplus_{i \leq k} H ,$$

so that we obtain

$$\lambda \in \text{Stab}(t_-^k \Lambda_-) \iff k \geq M . \quad (1.3)$$

Consider now the infinite geodesic  $c$ , whose set of vertices is  $\{t_-^k \Lambda_- : k \in \mathbb{Z}\}$ . By equation (1.3), half of it is fixed by  $\lambda$ , while the remainder is not (see Figure 1). To draw points  $\Lambda_-$ ,  $t_-^n \Lambda_-$  and  $\gamma\Lambda_- = \lambda t_-^n \Lambda_-$  in Figure 1 (dis-

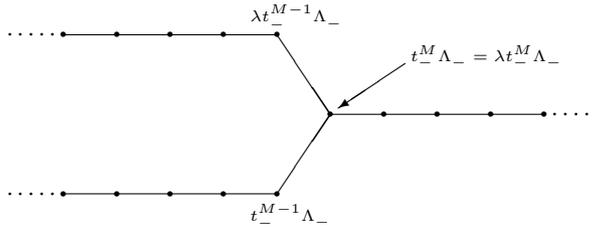


Figure 1: The geodesics  $c$  and  $\lambda \cdot c$ .

tinguish cases  $M > \max(0, n)$ ,  $M \leq 0$  and  $M \leq n$ ) will now convince the reader that the result holds.  $\square$

Let us now state a formula computing the length of an element of  $H \wr \mathbb{Z}$ , which is a direct consequence of [Par92, Theorem 1.2]. Note that, even if the theorem was stated for finitely generated groups, it also applies in our case.

**Proposition 1.4** *Keep the above notations. Let  $\gamma = \lambda s^n \in \Gamma = H \wr \mathbb{Z}$ . In case  $\lambda = 1$ , one has  $|\gamma| = |n|$ , while in case  $\lambda \neq 1$ , the length of  $\gamma$  satisfies:*

$$|\gamma| = L_{\mathbb{Z}}(\gamma) + \sum_{i \in \mathbb{Z}} |\lambda_i| ,$$

where  $L_{\mathbb{Z}}(\gamma)$  denotes the length of the shortest path starting from 0, ending at  $n$  and passing through  $m$  and  $M$  in the (canonical) Cayley graph of  $\mathbb{Z}$ .

The length  $L_{\mathbb{Z}}(\gamma)$  appearing in Proposition 1.4 can be estimated as follows:

**Proposition 1.5** *Let  $\gamma \in \Gamma = H \wr \mathbb{Z}$ . The following inequalities hold:*

$$d_{T_{\pm}}(\Lambda_{\pm}, \gamma \Lambda_{\pm}) \leq L_{\mathbb{Z}}(\gamma) \leq d_{T_+}(\Lambda_+, \gamma \Lambda_+) + d_{T_-}(\Lambda_-, \gamma \Lambda_-)$$

PROOF. If  $\gamma = s^n$ , the result is obvious. We suppose now  $\gamma = \lambda s^n$  with  $\lambda \neq 1$ . The proof is then a consideration of eight cases which are listed in the following table:

$n$	$m$	$M$	$d_{T_+}$	$d_{T_-}$	$L_{\mathbb{Z}}$	$d_{T_+} + d_{T_-}$
$\geq 0$	$\geq 0$	$> n$	$n$	$2M - n$	$2M - n$	$2M$
$\geq 0$	$\geq 0$	$\leq n$	$n$	$n$	$n$	$2n$
$\geq 0$	$< 0$	$> n$	$n - 2m$	$2M - n$	$2M - 2m - n$	$2M - 2m$
$\geq 0$	$< 0$	$\leq n$	$n - 2m$	$n$	$n - 2m$	$2n - 2m$
$< 0$	$< n$	$\leq 0$	$n - 2m$	$-n$	$n - 2m$	$-2m$
$< 0$	$\geq n$	$\leq 0$	$-n$	$-n$	$-n$	$-2n$
$< 0$	$< n$	$> 0$	$n - 2m$	$2M - n$	$2M - 2m + n$	$2M - 2m$
$< 0$	$\geq n$	$> 0$	$-n$	$2M - n$	$2M - n$	$2M - 2n$

The values of  $d_{T_{\pm}}(\Lambda_{\pm}, \gamma \Lambda_{\pm})$  come from Lemma 1.2; those of  $L_{\mathbb{Z}}(\gamma)$  are easy to compute. We now observe that the result is true in the eight cases.  $\square$

Combining Propositions 1.4 and 1.5, one obtains immediately:

**Corollary 1.6** *Let  $\gamma = \lambda s^n \in \Gamma = H \wr \mathbb{Z}$ . The following inequalities hold:*

$$d_{T_{\pm}}(\Lambda_{\pm}, \gamma \Lambda_{\pm}) + \sum_{i \in \mathbb{Z}} |\lambda_i| \leq |\gamma| \leq d_{T_+}(\Lambda_+, \gamma \Lambda_+) + d_{T_-}(\Lambda_-, \gamma \Lambda_-) + \sum_{i \in \mathbb{Z}} |\lambda_i|$$

## 2 Metrically proper actions

Let us consider a group  $G$ , acting by isometries on a metric space  $X$ .

**Definition 2.1** *The action is metrically proper if, whenever  $B$  is a bounded subset of  $X$ , the set  $\{g \in G : g \cdot B \cap B \neq \emptyset\}$  is finite.*

From now on, we shall write “proper” instead of “metrically proper”. Let us recall that the action is proper if and only if the following property holds, for some  $z \in X$ :

$$\text{for any } R > 0, \text{ the set } \{g \in G : d(z, g \cdot z) \leq R\} \text{ is finite.} \quad (Prop_z)$$

Let now  $\mathcal{F} = (X_i, d_i, b_i)_{i \in I}$  be a family of pointed metric spaces (where  $d_i$  is the metric on  $X_i$  and  $b_i \in X_i$  is a base-point), and let  $p \geq 1$ . We call  $\ell^p$ -product of the family the space

$$\ell^p(\mathcal{F}) = \left\{ x \in \prod_{i \in I} X_i : \sum_{i \in I} d_i(b_i, x_i)^p < +\infty \right\} .$$

It is a metric space with metric  $\delta(x, y) = (\sum_{i \in I} d_i(x_i, y_i)^p)^{1/p}$ . We set  $(b_i)_{i \in I}$  as base point. Consider now the case  $(X_i, b_i) = (X, b)$  for all  $i \in I$ . One has:

$$\ell^p(I; X, b) := \ell^p(\mathcal{F}) = \left\{ \phi : I \rightarrow X : \sum_{i \in I} d(b, \phi_i)^p < +\infty \right\} .$$

If a group  $H$  acts by isometries on  $X$ , the group  $H \wr G$  acts by isometries on  $\ell^p(G; X, b)$  in the following way:

$$\begin{cases} (\lambda \cdot \phi)_g &= \lambda_g \cdot \phi_g & \text{for } \lambda \in \bigoplus_{g \in G} H ; \\ (g \cdot \phi)_{g'} &= \phi_{g^{-1}g'} & \text{for } g \in G . \end{cases} \quad (2.2)$$

Given  $G$  infinite, observe that, even if the action of  $H$  is proper, the action of  $H \wr G$  on  $\ell^p(G; X, b)$  is *not*. Indeed,  $G$  has a global fixed point in  $\ell^p(G; X, b)$ .

Theorem 0.1 will follow from the following statement.

**Proposition 2.3** *Let  $H$  be a group acting properly on a metric space  $X$ ,  $b \in X$  and  $p \geq 1$ . Then, the action of  $\Gamma = H \wr \mathbb{Z}$  on  $T_+ \times T_- \times \ell^p(\mathbb{Z}; X, b)$ , where the product is endowed with the  $\ell^p$  metric, is proper.*

PROOF. We are going to prove property  $(Prop_z)$  for  $z = (\Lambda_+, \Lambda_-, (b)_{i \in \mathbb{Z}})$ . Thus let  $R > 0$  and  $A = \{\gamma \in \Gamma : d(z, \gamma \cdot z) \leq R\}$ . Take  $\gamma = \lambda s^n \in A$ . We have  $d_{T_+}(\Lambda_+, \gamma \Lambda_+) \leq R$ ,  $d_{T_-}(\Lambda_-, \gamma \Lambda_-) \leq R$  and  $\sum_{i \in \mathbb{Z}} d(b, \lambda_i \cdot b)^p \leq R^p$ .

By lemmas 1.1 and 1.2, one has  $M \leq R$ ,  $m \geq -R$  (if  $M$  and  $m$  are defined) and  $|n| \leq R$ . Set  $B = \{h \in H : d(b, h \cdot b) \leq R\}$ . It is a finite set since the  $H$ -action is proper.

Hence, one has  $|n| \leq R$ ,  $\lambda_i = 1_H$  for  $|i| > R$  and  $\lambda_i \in B$  for  $|i| \leq R$ . This leaves finitely many choices for  $\gamma$ , and proves thus that  $A$  is finite.  $\square$

**Remark 2.4** The space  $T_+ \times T_- \times \ell^p(\mathbb{Z}; X, b)$  is canonically isometric to the product  $\ell^p(\mathcal{F})$  with  $I = \{+, -\} \cup \mathbb{Z}$  and  $\mathcal{F}$  given by  $\mathcal{F}(+) = (T_+, \Lambda_+)$ ,  $\mathcal{F}(-) = (T_-, \Lambda_-)$  and  $\mathcal{F}(i) = (X, b)$  for  $i \in \mathbb{Z}$ .

PROOF OF OF THEOREM 0.1. We recall first that a tree is a CAT(0) cube complex, hence a space with walls.

It is shown in [CMV04, Section 5] that a  $\ell^1$ -product of spaces with (measured) walls carries the same structure. Hence the conclusion for spaces with walls by proposition 2.3.

Given a CAT(0) cube complex  $Y$ , we denote by  $Y^{(k)}$  the set of  $k$ -cells in  $Y$ . Take now a family  $\mathcal{F} = (X_i, b_i)_{i \in I}$  of CAT(0) cube complexes with  $b_i \in X_i^{(0)}$  and set  $\mathcal{F}^{(0)} = (X_i^{(0)}, b_i)_{i \in I}$ . We are going to construct a subspace  $X$  of  $\ell^2(\mathcal{F})$  which is a CAT(0) cube complex.

We define first  $X^{(0)} = \ell^2(\mathcal{F}^{(0)})$ . Since the distance between two distinct vertices is at least 1, one has

$$X^{(0)} = \bigoplus_{i \in I} (X_i^{(0)}, b_i) := \left\{ v \in \prod_{i \in I} X_i^{(0)} : \{i \in I : v_i \neq b_i\} \text{ is finite} \right\}.$$

For  $k \geq 1$ , we define then the set of  $k$ -cells as

$$X^{(k)} = \left\{ c \in \prod_{i \in I} (X_i^{(0)} \cup \dots \cup X_i^{(k)}) : \sum_{i \in I} \dim(c_i) = k \text{ and } \{i \in I : c_i \neq b_i\} \text{ is finite} \right\}.$$

It is clear that every  $k$ -cell, as a subset of  $\ell^2(\mathcal{F})$ , is isometric to  $[-1/2, 1/2]^k$ . If  $c \in X^{(k)}$ , the faces of  $c$  are the  $(k-1)$ -cells  $c'$  such that  $c'_j$  is a face of  $c_j$  for some  $j$  and  $c'_i = c_i$  for  $i \neq j$ . The gluing maps are isometric. Finally, the space  $\ell^2(\mathcal{F})$  inherits the CAT(0) property, so that  $X$  is a CAT(0) cube complex.

Suppose now that  $H$  acts on a CAT(0) cube complex  $Y$  and take  $v_0$  a vertex of  $Y$ . We consider the family  $\mathcal{F}$  given by  $I = \{+, -\} \cup \mathbb{Z}$ ,  $\mathcal{F}(+) = (T_+, \Lambda_+)$ ,  $\mathcal{F}(-) = (T_-, \Lambda_-)$  and  $\mathcal{F}(i) = (Y, v_0)$  for  $i \in \mathbb{Z}$ . The action of  $H \wr \mathbb{Z}$  on  $\ell^2(\mathcal{F})$

is proper by proposition 2.3 and the CAT(0) cube complex  $X$  constructed as above is an invariant subset, so that it is endowed with a proper action of  $H \wr \mathbb{Z}$  too.  $\square$

**Remark 2.5** The same techniques can be used, if  $H$  acts properly on some Hilbert space  $\mathcal{H}$ , to prove that  $H \wr \mathbb{Z}$  acts properly on the Hilbert direct sum  $\ell^2(E(T_+)) \oplus \ell^2(E(T_-)) \oplus \bigoplus_{i \in \mathbb{Z}} \mathcal{H}$ . Hence, we recover the known fact that Haagerup property is preserved by taking wreath products with  $\mathbb{Z}$  [CCJ<sup>+</sup>01, Proposition 6.1.1 and Example 6.1.6]. The interest of our technique is that we obtain an *explicit* proper action of  $H \wr \mathbb{Z}$ , knowing a proper action of  $H$ .

**Remark 2.6** It is known [CMV04, Theorem 1], that a discrete group satisfies the Haagerup property if and only if it acts properly on some space with measured walls. According to Remark 2.5, whenever  $H$  acts properly on a space with measured walls, the same holds for  $H \wr \mathbb{Z}$ . Again, our techniques give an explicit action, as Theorem 0.1 is also valid for spaces with measured walls.

### 3 Hilbert space compression: Theorem 0.2

We recall that a map  $f : X \rightarrow Y$  between metric spaces is *Lipschitz* if there exists  $M > 0$  such that  $d_Y(f(x), f(y)) \leq M \cdot d_X(x, y)$  for all  $x, y \in X$ . Given a Lipschitz map  $f : X \rightarrow \mathcal{H}$  of a metric space  $X$  into a Hilbert space  $\mathcal{H}$ , we set<sup>3</sup>  $R_f$  to be the supremum of the numbers  $\alpha \in [0, 1]$  such that there exists  $D > 0$  with  $D \cdot d_X(x, y)^\alpha \leq \|f(x) - f(y)\|$  for all  $x, y \in X$ .

Given a generating set  $S$  of a group  $H$ , we recall our convention to take  $\Sigma = S \cup \{s\}$  as generating set for  $H \wr \mathbb{Z}$ , where  $s$  is the positive generator of  $\mathbb{Z}$ . In order to simplify notations, we do not mention explicitly  $S$  and  $\Sigma$ , which we fix throughout this section.

The key result to prove Theorem 0.2 is the following:

**Proposition 3.1** *Let  $H$  be a group (with a generating set  $S$ ) and  $\Gamma = H \wr \mathbb{Z}$ . Suppose that maps  $f : H \rightarrow \mathcal{H}$  and  $f_\pm : V(T_\pm) \rightarrow \mathcal{H}_\pm$  are Lipschitz with  $R_{f_+} = R_{f_-} > 0$  and  $R_f > 0$ . Consider the map*

$$\sigma : \Gamma \rightarrow \mathcal{H}' := \mathcal{H}_+ \oplus \mathcal{H}_- \oplus \bigoplus_{i \in \mathbb{Z}} \mathcal{H} ,$$

---

<sup>3</sup>It does not coincide with the *asymptotic compression* of  $f$  defined in [GK04].

where, given  $\gamma = \lambda s^n \in H \wr \mathbb{Z}$ , we set  $\sigma(\gamma)_\pm = f_\pm(\gamma\Lambda_\pm)$  and  $\sigma(\gamma)_i = f(\lambda_i)$  for  $i \in \mathbb{Z}$ . Then  $R_\sigma \geq R_f \cdot R_{f_\pm} / (R_f + R_{f_\pm})$  and  $R_\sigma \geq \min\{R_{f_\pm}, R_f - \frac{1}{2}\}$ .

Moreover, if  $f$  is  $H$ -equivariant and  $f_\pm$  are  $\Gamma$ -equivariant (with respect to some actions by affine isometries),  $\sigma$  is  $\Gamma$ -equivariant with respect to some action of  $\Gamma$  on  $\mathcal{H}'$  by affine isometries.

PROOF. We show first that  $\sigma$  is Lipschitz (the reader could remark that it is trivial if  $H$  is a finitely generated group; however, this case is also covered by the proof below). Let us take  $K, K_+, K_- > 0$  such that

$$\begin{aligned} \|f(h_1) - f(h_2)\| &\leq K \cdot |h_1^{-1}h_2| \quad \forall h_1, h_2 \in H ; \\ \|f_\pm(u) - f_\pm(v)\| &\leq K_\pm \cdot d_{T_\pm}(u, v) \quad \forall u, v \in V(T_\pm) . \end{aligned}$$

Let  $x, y \in \Gamma$ . We set  $\gamma = x^{-1}y$  and write  $x = \xi s^p$ ,  $y = \eta s^q$ ,  $\gamma = \lambda s^n$  in  $H \wr \mathbb{Z} = \Lambda \rtimes \mathbb{Z}$  (so that  $n = q - p$  and  $\lambda_i = \xi_{i-p}^{-1}\eta_{i-p}$ ). One has then

$$\left( \sum_{i \in \mathbb{Z}} \|\sigma(x)_i - \sigma(y)_i\|^2 \right)^{\frac{1}{2}} \leq \sum_{i \in \mathbb{Z}} \|f(\xi_i) - f(\eta_i)\| \leq \sum_{j \in \mathbb{Z}} K \cdot |\lambda_j| \leq K \cdot |\gamma| .$$

Corollary 1.6 implies:

$$\|\sigma(x)_\pm - \sigma(y)_\pm\| \leq K_\pm \cdot d_{T_\pm}(x\Lambda_\pm, y\Lambda_\pm) = K_\pm \cdot d_{T_\pm}(\Lambda_\pm, \gamma\Lambda_\pm) \leq K_\pm \cdot |\gamma| .$$

Thus, we get finally  $\|\sigma(x) - \sigma(y)\| \leq (K_+ + K_- + K) \cdot |x^{-1}y|$ , which proves that  $\sigma$  is Lipschitz, as desired.

We now turn to the estimation of  $R_\sigma$ . Fix any  $\alpha, \beta$  such that  $0 < \alpha < R_f$  and  $0 < \beta < R_{f_\pm}$ . There exists constants  $C, C_+, C_- > 0$  such that:

$$\begin{aligned} \|f(h_1) - f(h_2)\| &\geq C \cdot |h_1^{-1}h_2|^\alpha \quad \forall h_1, h_2 \in H ; \\ \|f_\pm(u) - f_\pm(v)\| &\geq C_\pm \cdot d_{T_\pm}(u, v)^\beta \quad \forall u, v \in V(T_\pm) . \end{aligned}$$

We notice first that  $\sigma$  is injective. More precisely, for any  $x, y \in \Gamma$ , one has

$$x \neq y \implies \|\sigma(x) - \sigma(y)\| \geq \min\{C, C_+, C_-\} . \quad (3.2)$$

Indeed, we express  $x = \xi s^p$  and  $y = \eta s^q$  as above. If  $p \neq q$ , we obtain  $\|\sigma(x)_\pm - \sigma(y)_\pm\| \geq C_\pm \cdot d_{T_\pm}(x\Lambda_\pm, y\Lambda_\pm)^\beta \geq C_\pm$  and if  $\xi_i \neq \eta_i$  for some  $i$ , we obtain  $\|\sigma(x)_i - \sigma(y)_i\| \geq C \cdot |\xi_i^{-1}\eta_i|^\alpha \geq C$ .

Let us take  $x, y$  and  $\gamma$  as above. According to Corollary 1.6, one (at least) of the following cases occurs. We treat them separately. As the case  $x = y$  is trivial, we assume  $x \neq y$ , that is  $|\gamma| \geq 1$ , in what follows.

(a) **Case**  $d_{T_+}(\Lambda_+, \gamma\Lambda_+) \geq \frac{1}{3}|\gamma|$ : We obtain

$$\begin{aligned} \|\sigma(x) - \sigma(y)\| &\geq \|\sigma(x)_+ - \sigma(y)_+\| \geq C_+ \cdot d_{T_+}(x\Lambda_+, y\Lambda_+)^\beta \\ &= C_+ \cdot d_{T_+}(\Lambda_+, \gamma\Lambda_+)^\beta \geq \frac{C_+}{3^\beta} |x^{-1}y|^\beta. \end{aligned}$$

(b) **Case**  $d_{T_-}(\Lambda_-, \gamma\Lambda_-) \geq \frac{1}{3}|\gamma|$ : We obtain the same way

$$\|\sigma(x) - \sigma(y)\| \geq \|\sigma(x)_- - \sigma(y)_-\| \geq \frac{C_-}{3^\beta} |x^{-1}y|^\beta.$$

(c) **Case**  $\sum_{i \in \mathbb{Z}} |\lambda_i| \geq \frac{1}{3}|\gamma|$ : We establish two independent estimates.

First, for all  $i \in \mathbb{Z}$ , one has  $\|\sigma(x)_i - \sigma(y)_i\| = \|f(\xi_i) - f(\eta_i)\| \geq C|\xi_i^{-1}\eta_i|^\alpha = C|\lambda_{i+p}|^\alpha$ . Recall that  $m = \min\{k \in \mathbb{Z} : \lambda_k \neq 1_H\}$  and  $M = \max\{k \in \mathbb{Z} : \lambda_k \neq 1_H\}$ . Using the Cauchy-Schwarz inequality for the third step below and  $\alpha \leq 1$  for the fourth one, we obtain

$$\begin{aligned} \|\sigma(x) - \sigma(y)\| &\geq \left( \sum_{i \in \mathbb{Z}} \|\sigma(x)_i - \sigma(y)_i\|^2 \right)^{\frac{1}{2}} \geq C \left( \sum_{j \in \mathbb{Z}} |\lambda_j|^{2\alpha} \right)^{\frac{1}{2}} \\ &\geq \frac{C}{\sqrt{M-m+1}} \sum_{j=m}^M |\lambda_j|^\alpha \geq \frac{C}{\sqrt{M-m+1}} \left( \sum_{j=m}^M |\lambda_j| \right)^\alpha \end{aligned}$$

By Proposition 1.4, one has  $|\gamma| \geq M - m + 1$ , so that we obtain

$$\|\sigma(x) - \sigma(y)\| \geq \frac{C}{\sqrt{|\gamma|}} \left( \frac{1}{3} \cdot |\gamma| \right)^\alpha \geq \frac{C}{3^\alpha} \cdot |\gamma|^{\alpha - \frac{1}{2}}. \quad (*)$$

This is our first estimate for case (c).

Second, we fix any  $\zeta \in ]0, 1[$ . Then, either there exists  $k \in \mathbb{Z}$  such that  $|\lambda_k| \geq (\frac{1}{3} \cdot |\gamma|)^\zeta$ , or one has  $M - m + 1 \geq (\frac{1}{3} \cdot |\gamma|)^{1-\zeta}$ . We distinguish the two subcases:

- if there exists  $k \in \mathbb{Z}$  such that  $|\lambda_k| \geq (\frac{1}{3} \cdot |\gamma|)^\zeta$ , we have

$$\begin{aligned} \|\sigma(x) - \sigma(y)\| &\geq \|\sigma(x)_{k-p} - \sigma(y)_{k-p}\| \geq C \cdot |\xi_{k-p}^{-1}\eta_{k-p}|^\alpha \\ &= C \cdot |\lambda_k|^\alpha \geq \frac{C}{3^{\alpha\zeta}} |\gamma|^{\alpha\zeta}; \end{aligned}$$

- in case  $M - m + 1 \geq (\frac{1}{3} \cdot |\gamma|)^{1-\zeta}$ , having  $L_{\mathbb{Z}}(\gamma) \geq M - m$  by definition, Proposition 1.5 gives

$$d_{T_+}(\Lambda_+, \gamma\Lambda_+) + d_{T_-}(\Lambda_-, \gamma\Lambda_-) \geq L_{\mathbb{Z}}(\gamma) \geq \left( \frac{1}{3} \cdot |\gamma| \right)^{1-\zeta} - 1.$$

Thus,  $\exists j \in \{+, -\}$  such that  $d_{T_j}(\Lambda_j, \gamma\Lambda_j) \geq \frac{1}{2}(\frac{1}{3} \cdot |\gamma|)^{1-\zeta} - \frac{1}{2}$ . For  $|\gamma| \geq 4$ , there exists  $K > 0$  such that  $d_{T_j}(\Lambda_j, \gamma\Lambda_j) \geq K \cdot |\gamma|^{1-\zeta}$ , so that  $\|\sigma(x) - \sigma(y)\| \geq C_j K^\beta \cdot |x^{-1}y|^{\beta(1-\zeta)}$  as in cases (a)-(b). Otherwise, for  $|\gamma| \leq 3$ , equation (3.2) gives

$$\|\sigma(x) - \sigma(y)\| \geq (\min\{C, C_+, C_-\}) \cdot 3^{-\beta(1-\zeta)} \cdot |\gamma|^{\beta(1-\zeta)}.$$

Hence, there exists  $C'_\zeta > 0$  with  $\|\sigma(x) - \sigma(y)\| \geq C'_\zeta \cdot |x^{-1}y|^{\beta(1-\zeta)}$ .

Consequently, setting  $m_\zeta = \min\{\alpha\zeta, \beta(1-\zeta)\}$ , it comes

$$\|\sigma(x) - \sigma(y)\| \geq \min\left\{\frac{C}{3\alpha\zeta}, C'_\zeta\right\} \cdot |\gamma|^{m_\zeta}. \quad (**_\zeta)$$

The largest value for  $m_\zeta$  is obtained for  $\alpha\zeta = \beta(1-\zeta)$ , that is  $\zeta = \frac{\beta}{\alpha+\beta}$ . It gives  $m_\zeta = \frac{\alpha\beta}{\alpha+\beta}$ . This is our second estimate for case (c).

As one has  $\beta > \alpha\beta/(\alpha+\beta)$ , combination of cases (a)-(c) gives

$$\begin{aligned} \|\sigma(x) - \sigma(y)\| &\geq C'' \cdot |x^{-1}y|^{\frac{\alpha\beta}{\alpha+\beta}} && \forall x, y \in \Gamma \\ \|\sigma(x) - \sigma(y)\| &\geq C'' \cdot |x^{-1}y|^{\min\{\beta, \alpha-\frac{1}{2}\}} && \forall x, y \in \Gamma \end{aligned}$$

for some  $C'' > 0$ . Hence, we get  $R_\sigma \geq \alpha\beta/(\alpha+\beta)$  and  $R_\sigma \geq \min\{\beta, \alpha-\frac{1}{2}\}$  for all  $\alpha, \beta$  satisfying  $0 < \alpha < R_f$  and  $0 < \beta < R_{f_\pm}$ . This implies immediately  $R_\sigma \geq R_f \cdot R_{f_\pm}/(R_f + R_{f_\pm})$  and  $R_\sigma \geq \min\{R_{f_\pm}, R_f - \frac{1}{2}\}$ .

To conclude the proof of Proposition 3.1, we pass now to the last statement. We thus suppose that  $f$  is  $H$ -equivariant and  $f_\pm$  are  $\Gamma$ -equivariant (with respect to some actions by affine isometries). To establish the  $\Gamma$ -equivariance of  $\sigma$ , we only have to define a  $\Gamma$ -action (by affine isometries) on  $\bigoplus_{i \in \mathbb{Z}} \mathcal{H}$  and check the  $\Gamma$ -equivariance with respect to it.

The  $\Gamma$ -action on  $\bigoplus_{i \in \mathbb{Z}} \mathcal{H} = \ell^2(\mathbb{Z}, \mathcal{H}, 0)$  is defined by equation (2.2). To check the equivariance, we set  $\gamma = \lambda s^n$  and  $g = \mu s^p$  with  $\lambda, \mu \in \Lambda$  and  $n, p \in \mathbb{Z}$ . We have  $(\gamma \cdot \sigma(g))_i = \lambda_i \cdot f(\mu_{i-n})$  and  $\sigma(\gamma g)_i = f(\lambda_i \mu_{i-n})$  and we get  $(\gamma \cdot \sigma(g))_i = \sigma(\gamma g)_i$  for all  $i$  by  $H$ -equivariance of  $f$ .  $\square$

Theorem 0.2 will be obtained by applying Proposition 3.1 with good embeddings of the trees  $T_\pm$ . We explain now how to embed a tree in a Hilbert space with high values of the constant “ $R_f$ ”. First, the following result can be obtained by a straightforward adaptation of [GK04, Proposition 4.2].

**Proposition 3.3** *Let  $T = (V, E)$  be a tree. Then  $R(V) = 1$ .*

More precisely, if we denote by  $E_G$  the set of geometric (or unoriented) edges of  $T$  and if we fix a base vertex  $v_0$ , then for any  $\varepsilon \in ]0, 1/2[$  we may consider the map

$$f_\varepsilon : V \longrightarrow \ell^2(E_G) ; x \longmapsto \sum_{k=1}^{d(v_0, x)} k^\varepsilon \delta_{e_k(x)} ,$$

where the  $e_k(x)$ 's are the consecutive edges on the unique geodesic from  $x$  to  $v_0$  and  $\delta_e$  is the Dirac mass at  $e$ . It is a Lipschitz map with  $R_{f_\varepsilon} \geq 1/2 + \varepsilon$ . We refer to the proof of [GK04, Proposition 4.2] for this fact.

To prove the ‘‘equivariant’’ part of Theorem 0.2, we need some explicit equivariant embeddings into Hilbert spaces. Let  $T = (V, E)$  be a tree. We recall from Section 7.4.1 in [CCJ<sup>+</sup>01] how to embed equivariantly  $T$  in a Hilbert space. We recall that we denote by  $e \mapsto \bar{e}$  the ‘‘orientation-reversing’’ involution on  $E$ , and we endow  $\ell^2(E)$  with the scalar product:

$$\langle \xi | \eta \rangle = \frac{1}{2} \sum_{e \in E} \xi(e) \overline{\eta(\bar{e})} .$$

Define a map  $c : V \times V \rightarrow \ell^2(E) : (x, y) \mapsto c(x, y)$  with

$$c(x, y) = \sum_{e \in (x \rightarrow y)} \delta_e - \delta_{\bar{e}}$$

where  $\delta_e$  is the Dirac mass at  $e$  and the summation is taken over coherently oriented edges in the oriented geodesic from  $x$  to  $y$ . The map  $c$  satisfies, for every  $x, y, z \in V$ :

$$c(x, y) + c(y, z) = c(x, z); \tag{3.4}$$

$$\|c(x, y)\|^2 = d(x, y). \tag{3.5}$$

Moreover if a group  $G$  acts on  $T$ , then for every  $g \in G$ :

$$c(gx, gy) = \pi(g)c(x, y) \tag{3.6}$$

where  $\pi$  is the permutation representation of  $G$  on  $\ell^2(E)$ .

Fix now a base-vertex  $v_0 \in V$ . Define a map

$$\iota_{v_0} : V \rightarrow \ell^2(E) : v \mapsto c(v_0, v)$$

and, for  $g \in G$ , an affine isometry  $\alpha_{v_0}(g)$  of  $\ell^2(E)$ :

$$\alpha_{v_0}(g)\xi = \pi(g)\xi + c(v_0, gv_0).$$

Using equations (3.4) – (3.6) above, the following lemma is immediate.

- Lemma 3.7** 1. For all  $g, h \in G : \alpha_{v_0}(gh) = \alpha_{v_0}(g)\alpha_{v_0}(h)$ , so that  $\alpha_{v_0}$  defines an affine isometric action of  $G$  on  $\ell^2(E)$ ;
2. the map  $\iota_{v_0}$  is  $G$ -equivariant with respect to the action  $\alpha_{v_0}$  on  $\ell^2(E)$ ;
3. one has  $\|\iota_{v_0}(x) - \iota_{v_0}(y)\| = \sqrt{d(x, y)}$  for all  $x, y \in V$ , so that  $R_{\iota_{v_0}} = 1/2$ .

It is an immediate consequence that  $R_G(V) \geq 1/2$ .

PROOF OF THEOREM 0.2. The inequalities  $R(H) \geq R(\Gamma)$  and  $R_H(H) \geq R_\Gamma(\Gamma)$  are trivial.

One has  $R(V(T_\pm)) = 1$  by Proposition 3.3, so that Proposition 3.1 gives  $R(\Gamma) \geq R(V(T_\pm)) \cdot R(H)/(R(V(T_\pm)) + R(H)) = R(H)/(R(H) + 1)$ .

Finally, one has  $R_\Gamma(V(T_\pm)) \geq 1/2$  by Lemma 3.7, so that we obtain

$$R_\Gamma(\Gamma) \geq \frac{R_\Gamma(V(T_\pm)) \cdot R_H(H)}{R_\Gamma(V(T_\pm)) + R_H(H)} \geq \frac{R_H(H)}{2R_H(H) + 1}$$

$$R_\Gamma(\Gamma) \geq \min \left\{ R_\Gamma(V(T_\pm)), R_H(H) - \frac{1}{2} \right\} = R_H(H) - \frac{1}{2}$$

by Proposition 3.1. □

## 4 Hilbert space compression: examples

We begin this section with known results about the compression of groups of the form  $H \wr \mathbb{Z}$ . Let us first state a generalization of [AGS, Theorem 3.9] which gives upper bounds for many of them.

**Proposition 4.1** *Let  $G$  be a finitely generated group with growth function satisfying  $\kappa(n) \asymp n^k$  for some  $k > 0$  and let  $H$  be a group. We assume the generating set of  $H$  chosen such that the word metric is unbounded. Then, the Hilbert space compression of  $\Gamma = H \wr G$  satisfies*

$$R(\Gamma, \Sigma) \leq \frac{1 + k/2}{1 + k},$$

where  $\Sigma$  is the union of the generating sets of  $G$  and  $H$ . In particular, with  $G = \mathbb{Z}$ , we get  $R(H \wr \mathbb{Z}) \leq 3/4$ .

The proof is a straightforward adaptation of [AGS, Theorem 3.9]<sup>4</sup>.

**Remark 4.2** If  $H$  is finitely generated, the hypothesis “the word metric is unbounded” means exactly that  $H$  is infinite.

Lower bounds on compression were found by Tessera [Tes, Corollary 14]. In particular:

**Proposition 4.3** *Let  $H$  be a finitely generated group. If  $H$  has polynomial growth, one has  $R(H \wr \mathbb{Z}) \geq 2/3$ .*

Together, Propositions 4.1 and 4.3 give immediately:

**Corollary 4.4** *If  $H$  is an infinite group with polynomial growth, then one has  $R(H \wr \mathbb{Z}) \in [2/3, 3/4]$ .*

In a similar spirit, Proposition 4.1 and our Theorem 0.2 imply immediately:

**Corollary 4.5** *Let  $H$  be an infinite, finitely generated group.*

- a) *If  $R(H) = 1$ , then  $R(H \wr \mathbb{Z}) \in [1/2, 3/4]$ .*
- b) *If  $R(H) = R_H(H) = 1/2$ , then  $R(H \wr \mathbb{Z}) \in [1/3, 1/2]$  and  $R_{H \wr \mathbb{Z}}(H \wr \mathbb{Z}) \in [1/4, 1/2]$  (in particular, if  $R_{H \wr \mathbb{Z}}(H \wr \mathbb{Z}) < 1/3$ , then  $H$  is non-amenable).*

□

The interest of part (a) in Corollary 4.5 stems from the fact that numerous groups satisfy  $R(H) = 1$ : among amenable groups, we mention polycyclic groups and lamplighter groups  $F \wr \mathbb{Z}$  with  $F$  finite [Tes, Theorem 1]; among (usually) non-amenable groups, we cite hyperbolic groups [BS, Theorem 4.2], groups acting properly co-compactly on finite-dimensional  $CAT(0)$  cube complexes [CaN05], co-compact lattices in connected Lie groups, irreducible lattices in higher rank semi-simple Lie groups [Tes, Theorem 2].

Our excuse for isolating (b) in Corollary 4.5 is a remarkable result by Arzhantseva, Guba and Sapir [AGS, Theorem 1.8]: for Thompson’s group  $F$ , one has  $R(F) = R_F(F) = 1/2$ .

---

<sup>4</sup>The only point to notice is that, since the word length is unbounded on  $H$ , for every  $b$  in the ball of radius  $n$  in  $G$ , we can find an element  $h_b \in H$  such that  $|h_b| = 2n + 1 - |b|$ . In [AGS], the hypothesis  $H = \mathbb{Z}$  is used only to have an explicit formula for  $h_b$ .

**Acknowledgements.** The authors thank Yves de Cornulier for his relevant remarks on a previous version and the referee for a very careful reading and for remarks which improved the presentation of the paper.

## References

- [AGS] Goulnara N. Arzhantseva, Victor Guba, and Mark Sapir. Metrics on diagram groups and uniform embeddings in a Hilbert space. Preprint arXiv:math.GR/0411605.
- [BH99] Martin R. Bridson and André Haefliger. *Metric spaces of non-positive curvature*, volume 319 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 1999.
- [BS] Nikolay Brodskiy and Dmitriy Sonkin. Compression of uniform embeddings into Hilbert space. arXiv:math.GR/0509108, preprint, 2005.
- [CCJ<sup>+</sup>01] Pierre-Alain Cherix, Michael Cowling, Paul Jolissaint, Pierre Julg, and Alain Valette. *Groups with the Haagerup property*, volume 197 of *Progress in Mathematics*. Birkhäuser Verlag, Basel, 2001. Gromov’s a-T-menability.
- [CMV04] Pierre-Alain Cherix, Florian Martin, and Alain Valette. Spaces with measured walls, the Haagerup property and property (T). *Ergodic Theory Dynam. Systems*, 24(6):1895–1908, 2004.
- [CaN05] Sarah Campbell and Graham A. Niblo. Hilbert space compression and exactness of discrete groups. *J. Funct. Anal.*, 222(2):292–305, 2005.
- [ChN05] Indira Chatterji and Graham Niblo. From wall spaces to CAT(0) cube complexes. *Internat. J. Algebra Comput.*, 15(5&6):875–885, 2005.
- [CTV] Yves de Cornulier, Romain Tessera, and Alain Valette. Isometric group actions on Hilbert spaces: growth of cocycles. arXiv:math.GR/0509527, preprint, 2005.

- [GK04] Erik Guentner and Jerome Kaminker. Exactness and uniform embeddability of discrete groups. *J. London Math. Soc. (2)*, 70(3):703–718, 2004.
- [HP98] Frédéric Haglund and Frédéric Paulin. Simplicité de groupes d’automorphismes d’espaces à courbure négative. In *The Epstein birthday schrift*, volume 1 of *Geom. Topol. Monogr.*, pages 181–248 (electronic). Geom. Topol. Publ., Coventry, 1998.
- [Nic04] Bogdan Nica. Cubulating spaces with walls. *Algebr. Geom. Topol.*, 4:297–309 (electronic), 2004.
- [Par92] Walter Parry. Growth series of some wreath products. *Trans. Amer. Math. Soc.*, 331(2):751–759, 1992.
- [Sag95] Michah Sageev. Ends of group pairs and non-positively curved cube complexes. *Proc. London Math. Soc. (3)*, 71(3):585–617, 1995.
- [Ser77] Jean-Pierre Serre. *Arbres, amalgames,  $SL_2$* . Société Mathématique de France, Paris, 1977. Avec un sommaire anglais, Rédigé avec la collaboration de Hyman Bass, Astérisque, No. 46.
- [Tes] Romain Tessera. Asymptotic isoperimetry on groups and uniform embeddings into Banach spaces. arXiv:math.GR/0603138, preprint, 2006.

Authors addresses:

Y.S., Laboratoire de Mathématiques, Université Blaise Pascal, Campus universitaire des Cézaux, F-63177 Aubière Cedex, FRANCE

yves.stalder@math.univ-bpclermont.fr

A.V., Institut de Mathématiques, Université de Neuchâtel, Rue Emile Argand 11, Case postale 158, CH-2009 Neuchâtel, SWITZERLAND

alain.valette@unine.ch