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Luc Guyot* and Yves Stalder**

Abstract. We give a parametrization by m-adic integers of the limits of Baumslag–Solitar groups (marked with a canonical set of generators). It is shown to be continuous and injective on the invertible m-adic integers. We show that all such limits are extensions of a free group by a lamplighter group and all but possibly one are not finitely presented. Finally, we give presentations related to natural actions on trees.

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Introduction

The set $\mathcal{G}_k$ of marked groups on k generators (see Section 1 for definitions) has a natural topology, which turns it into a metrizable, compact and totally disconnected space. This topology, introduced explicitly by Grigorchuk [Gri84], corresponds to an earlier and more general construction by Chabauty [Cha50]. Let us mention that Grigorchuk and Żuk also defined spaces of marked graphs [GŻ99].

The Grigorchuk topology was introduced in the context of growth of finitely generated groups.[1] Grigorchuk constructed a set of 3-generated groups in $\mathcal{G}_4$ that is homeomorphic to the Cantor set and he fruitfully studied the neighborhood of *the first* (*intermediate growth*) *Grigorchuk group* inside this set [Gri84]. The Grigorchuk topology has various connections with problems in group theory. Let us cite the following examples:

- Stepin used the Cantor set constructed by Grigorchuk to deduce the existence of uncountably many amenable but non-elementary amenable groups without

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[1] Note that a very similar topology was used by Gromov when he discussed further its celebrated polynomial growth theorem [Gro81], Final remarks, p. 71.

growth estimates [Ste86].

- Shalom proved that every finitely generated Kazhdan group is a quotient of some finitely *presented* Kazhdan group by showing that Kazhdan's property (T) defines an open subset of the space of marked groups [Sha00].
- Champetier showed that the quotient of the space of marked groups on k generators by the group isomorphism relation is not a standard Borel space. He also proved that the closure of non-elementary hyperbolic groups contains a G_δ dense subset consisting of infinite torsion groups [Cha00].
- Champetier and Guirardel characterized *limit groups* of Sela (this class coincides with fully residually free groups and with groups having the same universal theory as a free group) as limits of free groups. They also related the Grigorchuk topology to the universal theory of groups and ultraproducts [CG05].
- Nekrashevych constructed a minimal Cantor set in $\mathscr{G}_3$ [Nek07] and proved that it contains a group with non-uniform exponential growth locally isomorphic to the iterated monodromy group of $z^2 + i$ [Nek].

We are interested in the closure of Baumslag–Solitar groups (with their standard marking) and its elements, which we study for their own right. Let us recall that Baumslag–Solitar groups are defined by

$$\mathrm{BS}(m,n) = \langle a, b \mid ab^m a^{-1} = b^n \rangle \quad \text{for } m, n \in \mathbb{Z} \setminus \{0\}.$$

These groups have been introduced in [BS62], in order to give the first examples of non-Hopfian one-relator groups. Since then they received much attention and served as test cases for several problems in combinatorial and geometric group theory. They bear further pathologies in various fields such as topology, geometry and algebraic geometry over groups. Like $\mathrm{BS}(2,3)$, many of them

- do not embed into the fundamental group of any sufficiently large, irreducible, compact and connected 3-manifold [JS79],
- have no proper action on a CAT(0) cube complex [Hag],
- are not equationally Noetherian [Bau99].

Let us recall that they were classified up to group isomorphism by Moldavanskiĭ [Mol91] and up to quasi-isometry by Farb and Mosher [FM98] and Whyte [Why01].

This paper is a continuation of [Sta06], where the second-named author characterized convergent sequences among Baumslag–Solitar groups (marked by generators a and b), provided that the two parameters m and n are coprime. Given any non-zero integer m, let us recall that the topological ring $\mathbb{Z}_m$ of m-adic integers is the projective limit of $\mathbb{Z}/m^h\mathbb{Z}$ for $h \in \mathbb{N}^*$ (see Section 1.1 or [HR63], Chapter II, §10, for more details). Given any ξ in $\mathbb{Z}_m$, using results of [Sta06], we may define (see Definition 1.7):

$$\overline{\mathrm{BS}}(m,\xi) = \lim_{n\to\infty} \mathrm{BS}(m,\xi_n)$$

where $(\xi_n)_n$ is a sequence of rational integers such that ξ_n tends to ξ in $\mathbb{Z}_m$ and $|\xi_n|$ tends to infinity as n goes to infinity. This describes a parametrization $\overline{\mathrm{BS}}_m\colon \mathbb{Z}_m \to \mathscr{G}_2$ given by $\xi \mapsto \overline{\mathrm{BS}}(m,\xi)$. Our first results, proven in Section 2 are the following:

Theorem 1 (Theorem 2.1 and Corollary 2.2). *Let $m \in \mathbb{Z}\setminus\{0\}$ and let $\xi,\eta \in \mathbb{Z}_m$. The equality of marked groups $\overline{\mathrm{BS}}(m,\xi) = \overline{\mathrm{BS}}(m,\eta)$ holds if and only if there is some $d \in \mathbb{Z}\setminus\{0\}$ such that $\gcd(\xi,m) = \gcd(\eta,m) = d$ and the images of ξ/d and η/d in $\mathbb{Z}_{m/d}$ are equal. In particular, the map $\overline{\mathrm{BS}}_m$ is injective on the set of invertible m-adic integers.*

Theorem 2 (Corollary 2.11). *The map $\overline{\mathrm{BS}}_m$ is continuous.*

These theorems allow us to describe (in Corollary 2.13) the set of groups $\overline{\mathrm{BS}}(m,\xi)$, with ξ invertible, as a boundary of a set Baumslag–Solitar groups (marked by the generators a and b) in $\mathscr{G}_2$.

It is well-known that a Baumslag–Solitar group acts on its Bass–Serre tree by automorphisms and on $\mathbb{Q}$ by affine transformations. The first action is faithful whereas the second is not in general. Section 3 is devoted to actions and structure of the groups $\overline{\mathrm{BS}}(m,\xi)$. The groups $\overline{\mathrm{BS}}(m,\xi)$ mainly act by automorphisms on a tree which is in some sense a "limit" of Bass–Serre trees (Theorem 3.5). These considerations imply

Theorem 3 (Theorem 3.11). *For any $m \in \mathbb{Z}\setminus\{0\}$ and $\xi \in \mathbb{Z}_m$, there exists an exact sequence $1 \to \mathbb{F} \to \overline{\mathrm{BS}}(m,\xi) \to \mathbb{Z} \wr \mathbb{Z} \to 1$, where $\mathbb{F}$ is a free group.*

Note that some properties obviously hold for the groups $\overline{\mathrm{BS}}(m,\xi)$, since these properties define closed sets in the space of marked groups: the groups $\overline{\mathrm{BS}}(m,\xi)$ are torsion free and centerless, their subgroup generated by a and bab^{-1} is a non-abelian free group for $|m| \geq 2$, and they are non-Kazhdan. We now present some consequences of our Theorem 3, which are not obvious in the former sense.

Corollary 4 (Corollary 3.13). *The limits $\overline{\mathrm{BS}}(m,\xi)$*

(1) *have the Haagerup property;*

(2) *are residually solvable.*

In the last section of the paper, we discuss presentations of the limits $\overline{\mathrm{BS}}(m,\xi)$. We prove in particular

Theorem 5 (Theorem 4.1). *For any $m \in \mathbb{Z}\setminus\{0\}$ and $\xi \in \mathbb{Z}_m \setminus m\mathbb{Z}_m$, the group $\overline{\mathrm{BS}}(m,\xi)$ is not finitely presented.*

We then exhibit (in Theorem 4.4) presentations of the limits by again using their actions by tree automorphisms.

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1. Definitions and preliminaries

Nota Bene. We denote by $\mathbb{N}$ the set $\{0, 1, 2, \dots\}$ of non-negative integers and by $\mathbb{N}^*$ (respectively $\mathbb{Z}^*$) the sets of positive (respectively non-zero integers). If A is any ring, then $A^\times$ is the set of invertible elements of A. For instance, $\mathbb{Z}^\times = \{-1, 1\}$ whereas $\mathbb{Z}^* = \mathbb{Z} \setminus \{0\}$.

We refer to $\mathbb{Z}$ as the ring of *rational integers* to avoid confusion with the ring of m-adic integers defined in the next section.

1.1. The ring of m-adic integers. Let $m \in \mathbb{Z}^*$. Recall that the ring of m-adic integers $\mathbb{Z}_m$ is the projective limit (in the category of topological rings) of the system

$$\cdots \to \mathbb{Z}/m^h\mathbb{Z} \to \mathbb{Z}/m^{h-1}\mathbb{Z} \to \cdots \to \mathbb{Z}/m^2\mathbb{Z} \to \mathbb{Z}/m\mathbb{Z},$$

where the arrows are the canonical (surjective) homomorphisms. This shows that $\mathbb{Z}_m$ is compact. This topology is compatible with the ultrametric distance given, for $\xi \neq \eta$, by

$$d_m(\xi, \eta) = |m|^{-\max\{k \in \mathbb{N} \mid \xi - \eta \in m^k \mathbb{Z}_m\}}.$$

We collect now (without proofs) some easy facts about m-adic integers which are useful in the following sections. Detailed proofs were given in the second-named author's Ph.D. thesis [Sta05], Appendix C.[2] Notice that $\mathbb{Z}_m$ is the zero ring if $|m| = 1$.

Proposition 1.1. *Let $m \in \mathbb{Z}$ such that $|m| \geqslant 2$ and let $m = \pm p_1^{k_1} \dots p_\ell^{k_\ell}$ be its decomposition in prime factors.*

(a) *One has an isomorphism of topological rings $\mathbb{Z}_m \cong \mathbb{Z}_{p_1} \oplus \cdots \oplus \mathbb{Z}_{p_\ell}$. In particular, if m is not a power of a prime number, the ring $\mathbb{Z}_m$ has zero divisors.*

(b) *The group of invertible elements of $\mathbb{Z}_m$ is given by*

$$\mathbb{Z}_m^\times = \mathbb{Z}_m \setminus (p_1\mathbb{Z}_m \cup \cdots \cup p_\ell\mathbb{Z}_m).$$

(c) *Any ideal of $\mathbb{Z}_m$ is principal. More precisely, any non-zero ideal of $\mathbb{Z}_m$ can be written as $p_1^{i_1} \dots p_\ell^{i_\ell}\mathbb{Z}_m$ with $i_1, \dots i_\ell \in \mathbb{N}$.*

(d) *For any $i_1, \dots i_\ell \in \mathbb{N}$, one has $\mathbb{Z} \cap p_1^{i_1} \dots p_\ell^{i_\ell}\mathbb{Z}_m = p_1^{i_1} \dots p_\ell^{i_\ell}\mathbb{Z}$.*

[2] Note that the second part of statement (a) is false there.

Definition 1.2. Let $m \in \mathbb{Z}$ such that $|m| \geqslant 2$ and let $p_1, \dots, p_\ell$ be its prime factors. If E is a subset of $\mathbb{Z}_m$ containing a non-zero integer, the *greatest common divisor* (gcd) of the elements of E is the (unique) number $p_1^{i_1} \dots p_\ell^{i_\ell}$ (with $i_1, \dots, i_\ell \in \mathbb{N}$) such that the ideal generated by E is $p_1^{i_1} \dots p_\ell^{i_\ell} \mathbb{Z}_m$.

If $|m| = 1$, we set by convention $\gcd(\mathbb{Z}_m) = 1$.

Lemma 1.3. *Let $m \in \mathbb{Z}^*$ and let m' be a divisor of m. Let us write $m' = \pm p_1^{j_1} \dots p_\ell^{j_\ell}$ and $m = \pm p_1^{k_1} \dots p_\ell^{k_\ell}$ their decomposition in prime factors ($j_s \leqslant k_s$ for all $s = 1, \dots, \ell$). Let $\pi : \mathbb{Z}_m \to \mathbb{Z}_{m'}$ be the ring morphism induced by projections $\mathbb{Z}/m^h\mathbb{Z} \to \mathbb{Z}/(m')^h\mathbb{Z}$ (for $h \geqslant 1$). Then the following hold:*

(a) *One has $\pi(n) = n$ for any integer n.*

(b) *For any $d = \pm p_1^{i_1} \dots p_\ell^{i_\ell}$ with $i_1, \dots, i_\ell \in \mathbb{N}$, one has $\pi^{-1}(d\mathbb{Z}_{m'}) = d\mathbb{Z}_m$.*

The ideal $d\mathbb{Z}_m$ is both open and closed in $\mathbb{Z}_m$. More generally we have

Lemma 1.4. *Let $m \in \mathbb{Z}^*$ and let m' be an integer whose prime divisors divide m.*

There exists h which depends only on m' and such that for any ξ, η in $\mathbb{Z}_m$, the inequality $d_m(\xi, \eta) < |m|^{-h}$ implies $\gcd(\xi, m') = \gcd(\eta, m')$. In particular, the set of m-adic integers ξ such that $\gcd(\xi, m') = d$ is both open and closed in $\mathbb{Z}_m$.

Proof. Consider h such that m' divides m^h. If $d_m(\xi, \eta) < |m|^{-h}$, one has $\xi = k + m^h\mu$ and $\eta = k + m^h\nu$ with $k \in \mathbb{Z}$ and $\mu, \nu \in \mathbb{Z}_m$ by Proposition 1.1 (d). Hence $\gcd(\xi, m') = \gcd(k, m') = \gcd(\eta, m')$. □

1.2. Marked groups and their topology. Introductory expositions of these topics can be found in [Cha00] or [CG05]. We only recall some basics and what we need in the following sections.

The free group on k generators will be denoted by $\mathbb{F}_k$, or $\mathbb{F}_S$ (with $S = (s_1, \dots, s_k)$) if we want to specify the names of (canonical) generating elements. A *marked group on k generators* is a pair (G, S) where G is a group and $S = (s_1, \dots, s_k) \in G^k$ is a family which generates G. A marked group (G, S) is endowed with a canonical epimorphism $\phi : \mathbb{F}_S \to G$, which induces an isomorphism of marked groups between $\mathbb{F}_S/\ker\phi$ and G. Hence a class of marked groups can always be represented by a quotient of $\mathbb{F}_S$. In particular if a group is given by a presentation, this defines a marking on it. The non-trivial elements of $\mathcal{R} := \ker\phi$ are called *relations* of (G, S). Given $w \in \mathbb{F}_k$ we will often write "$w = 1$ in G" or "$w =_G 1$" to say that the image of w in G is trivial.

Let $w = x_1^{\varepsilon_1} \dots x_n^{\varepsilon_n}$ be a reduced word in $\mathbb{F}_S$ (with $x_i \in S$ and $\varepsilon_i \in \{\pm 1\}$). The integer n is called the *length* of w and denoted $|w|$. If (G, S) is a marked group on k

generators and $g \in G$, the *length* of g is

$$\begin{aligned} |g|_G &:= \min\{n \mid g = s_1 \dots s_n \text{ with } s_i \in S \sqcup S^{-1}\} \\ &= \min\{|w| \mid w \in \mathbb{F}_S,\ \phi(w) = g\}. \end{aligned}$$

Let $\mathcal{G}_k$ be the set of marked groups on k generators (up to marked isomorphism). Let us recall that the topology on $\mathcal{G}_k$ comes from the following ultrametric distance: for $(G_1, S_1) \neq (G_2, S_2) \in \mathcal{G}_k$ we set $d((G_1, S_1), (G_2, S_2)) := e^{-\lambda}$, where λ is the length of a shortest element of $\mathbb{F}_k$ which vanishes in one group and not in the other one. But what one has to keep in mind is the following characterization of convergent sequences.

Lemma 1.5 ([Sta06], Proposition 1). *Let $(G_n)_n$ be a sequence of marked groups in $\mathcal{G}_k$. The sequence $(G_n)_{n\geq 0}$ converges if and only if for any $w \in \mathbb{F}_k$ we have either $w = 1$ in G_n for n large enough, or $w \neq 1$ in G_n for n large enough.*

Observe that the latter condition characterizes exactly Cauchy sequences. Sequences converging to a finitely presented group enjoy a very special and useful property:[3]

Lemma 1.6 ([CG05], Lemma 2.3). *If a sequence $(G_n)_n$ in $\mathcal{G}_k$ converges to a marked group $G \in \mathcal{G}_k$ which is given by a finite presentation, then, for n large enough, G_n is a marked quotient of G.*

We refer to the given references for proofs.

1.3. Notation and conventions. We define a family of limits of Baumslag–Solitar groups in the following way.

Definition 1.7. For $m \in \mathbb{Z}^*$ and $\xi \in \mathbb{Z}_m$, one defines a marked group on two generators $\overline{\mathrm{BS}}(m, \xi)$ by the formula

$$\overline{\mathrm{BS}}(m, \xi) = \lim_{n\to\infty} \mathrm{BS}(m, \xi_n),$$

where $(\xi_n)_n$ is any sequence of rational integers such that ξ_n tends to ξ in $\mathbb{Z}_m$ and $|\xi_n|$ tends to infinity as n goes to infinity.

Notice that $\overline{\mathrm{BS}}(m, \xi)$ is well defined for any $\xi \in \mathbb{Z}_m$ by Theorem 6 of [Sta06]. Note also that for any $n \in \mathbb{Z}^*$, one has $\overline{\mathrm{BS}}(m, n) \neq \mathrm{BS}(m, n)$. Indeed, the word $ab^m a^{-1} b^{-n}$ represents the identity element in $\mathrm{BS}(m, n)$, but not in $\overline{\mathrm{BS}}(m, n)$.

[3] This property is already used in [Gri84], Proof of Theorem 6.2.

When considered as marked groups, the free group $\mathbb{F}_2 = \mathbb{F}(a,b)$, the Baumslag–Solitar groups, and the groups $\overline{\mathrm{BS}}(m,\xi)$ are all (unless stated otherwise) marked by the pair (a,b).

Another group which plays an important role in this article is:

$$\mathbb{Z} \wr \mathbb{Z} = \mathbb{Z} \ltimes_t \mathbb{Z}[t,t^{-1}] \cong \mathbb{Z} \ltimes_s \bigoplus_{\mathbb{Z}} \mathbb{Z},$$

where the generator of the first copy of $\mathbb{Z}$ acts on $\mathbb{Z}[t,t^{-1}]$ by multiplication by t or, equivalently, on $\bigoplus_{\mathbb{Z}} \mathbb{Z}$ by shifting the indices. This group is assumed (unless specified otherwise) to be marked by the generating pair consisting of elements $(1,0)$ and $(0,t^0)$.

The last groups we introduce here are $\Gamma(m,n) = \mathbb{Z} \ltimes_{\frac{n}{m}} \mathbb{Z}\big[\frac{\gcd(m,n)}{\mathrm{lcm}(m,n)}\big]$ $(m,n \in \mathbb{Z}^*)$ where the generator of the first copy of $\mathbb{Z}$ acts on $\mathbb{Z}\big[\frac{\gcd(m,n)}{\mathrm{lcm}(m,n)}\big]$ by multiplication by $\frac{n}{m}$. This group is assumed (unless specified otherwise) to be marked by the generating pair consisting of elements $(1,0)$ and $(0,1)$. The latter elements are the images of $(1,0)$ and $(0,t^0)$ by the homomorphism $\mathbb{Z} \wr \mathbb{Z} \to \Gamma(m,n)$ given by the evaluation $t = \frac{n}{m}$; they are also the images of the elements a and b of $\mathrm{BS}(m,n)$ by the homomorphism defined by $a \mapsto (1,0)$ and $b \mapsto (0,1)$. Observe that the group $\mathbb{Z} \ltimes \mathbb{Z}\big[\frac{\gcd(m,n)}{\mathrm{lcm}(m,n)}\big]$ acts affinely on $\mathbb{Q}$ (or $\mathbb{R}$) by $(1,0)\cdot x = \frac{n}{m}x$ and $(0,y)\cdot x = x + y$.

We introduce the homomorphism $\sigma_a \colon \mathbb{F}_2 \to \mathbb{Z}$ defined by $\sigma_a(a) = 1$ and $\sigma_a(b) = 0$. It factors through all groups $\mathrm{BS}(m,n)$, $\overline{\mathrm{BS}}(m,\xi)$, $\mathbb{Z} \wr \mathbb{Z}$ and $\Gamma(m,n)$. The induced morphisms are also denoted by σ_a. Finally we define the homomorphism $\bar{}\colon \mathbb{F}_2 \to \mathbb{F}_2$ given by $\bar{a} = a$ and $\bar{b} = b^{-1}$. Note that it induces homomorphisms $\bar{}\colon \mathrm{BS}(m,n) \to \mathrm{BS}(m,n)$ for $m,n \in \mathbb{Z}^*$ and $\bar{}\colon \overline{\mathrm{BS}}(m,\xi) \to \overline{\mathrm{BS}}(m,\xi)$ for $m \in \mathbb{Z}^*, \xi \in \mathbb{Z}_m$.

2. Parametrization of limits

This section is devoted to the study of the map

$$\overline{\mathrm{BS}}_m \colon \mathbb{Z}_m \to \mathscr{G}_2;\ \xi \mapsto \overline{\mathrm{BS}}(m,\xi).$$

First, we characterize its lack of injectivity, thus giving a classification of groups $\overline{\mathrm{BS}}(m,\xi)$ up to isomorphism of marked groups (Theorem 2.1). Second, we prove that it is continuous (Corollary 2.11).

2.1. Lack of injectivity. Let us summarize the key points of this set-theoretic part. Our aim is the classification (up to isomorphism of marked groups) of the groups $\overline{\mathrm{BS}}(m,\xi)$ marked by the pair (a,b).

Theorem 2.1. *Let $m \in \mathbb{Z}^*$ and $\xi, \eta \in \mathbb{Z}_m$. The equality of marked groups*

$$\overline{\mathrm{BS}}(m, \xi) = \overline{\mathrm{BS}}(m, \eta)$$

holds if and only if there is some $d \in \mathbb{N}^*$ such that $\gcd(\xi, m) = \gcd(\eta, m) = d$ and the images of ξ/d and η/d in $\mathbb{Z}_{m/d}$ are equal.

As a consequence, we obtain the injectivity of $\overline{\mathrm{BS}}_m$, when restricted to the set invertible m-adic integers.

Corollary 2.2. *Let $m \in \mathbb{Z}^*$ and let $\xi, \eta \in \mathbb{Z}_m$ with $\xi \neq \eta$. If no prime factor of m divides both ξ and η, then one has $\overline{\mathrm{BS}}(m, \xi) \neq \overline{\mathrm{BS}}(m, \eta)$. Hence the map $\overline{\mathrm{BS}}_m$ becomes injective when restricted to invertible m-adic integers.*

Proof of Corollary 2.2. By Theorem 2.1, we may suppose that $\gcd(m, \xi) = \gcd(m, \eta) = d$. Then $d = 1$ by assumption and $\pi(\xi) = \xi \neq \eta = \pi(\eta)$ where $\pi : \mathbb{Z}_m \twoheadrightarrow \mathbb{Z}_{m/d}$ is the canonical ring morphism. By Theorem 2.1, one gets $\overline{\mathrm{BS}}(m, \xi) \neq \overline{\mathrm{BS}}(m, \eta)$. □

To prove Theorem 2.1, we first need to characterize the converging sequences of Baumslag–Solitar groups marked by the standard pair (a, b). This is done in the following statement.

Theorem 2.3. *Let $m \in \mathbb{Z}^*$ and let $(\xi_n)_n$ be a sequence of rational integers such that $|\xi_n|$ tends to infinity as n goes to infinity. The sequence $(\mathrm{BS}(m, \xi_n))_n$ converges in $\mathcal{G}_2$ if and only if the two following conditions hold:*

(i) *There is $d \in \mathbb{N}^*$ such that $\gcd(m, \xi_n) = d$ for all n large enough.*

(ii) *$(\xi_n/d)_n$ converges in $\mathbb{Z}_{m/d}$.*

One implication of Theorem 2.3 was proved by the second-named author [Sta06] for $d = 1$. The purpose of the following proposition is to generalize it to any d:

Proposition 2.4. *Let $m \in \mathbb{Z}^*$ and let $(\xi_n)_n$ be a sequence of rational integers such that $|\xi_n|$ tends to infinity as n goes to infinity. If $(\xi_n)_n$ defines a converging sequence in $\mathbb{Z}_m$ then the sequence of marked groups $(\mathrm{BS}(md, \xi_n d))_n$ converges in $\mathcal{G}_2$ for any $d \in \mathbb{Z}^*$.*

We obtain this proposition by performing minor technical modifications on [Sta06], Theorem 6, and the related lemmas. Nevertheless, the process being not completely obvious, we give a detailed proof of it. As the proof of Proposition 2.4 is more complex than the first two statements, we postpone it to the end of this section.

We first show that Theorem 2.3 implies Theorem 2.1.

Proof of Theorem 2.1. Choose a sequence of rational integers $(\xi_n)_n$ such that

$$\xi_{2n} \xrightarrow[\mathbb{Z}_m]{} \xi,\ \xi_{2n+1} \xrightarrow[\mathbb{Z}_m]{} \eta \text{ and } |\xi_n| \to \infty \text{ as } n \text{ goes to infinity.}$$

One has $\overline{\mathrm{BS}}(m,\xi) = \overline{\mathrm{BS}}(m,\eta)$ if and only if the sequence $(\mathrm{BS}(m,\xi_n))_n$ converges. By Theorem 2.3 this is equivalent to have $\gcd(m,\xi_n) = d$ for n large enough (for some d in $\mathbb{N}^*$) and the sequence $(\pi(\xi_n/d))_n$ converging in $\mathbb{Z}_{m/d}$ (where $\pi : \mathbb{Z}_m \twoheadrightarrow \mathbb{Z}_{m/d}$ is the canonical ring morphism). Finally, the first condition is equivalent to $\gcd(m,\xi) = d = \gcd(m,\eta)$ by Lemma 1.4, and the second condition boils down to $\pi(\xi/d) = \pi(\eta/d)$. □

We now prove that Proposition 2.4 and the following lemma imply Theorem 2.3. This lemma is actually independent and is only used for the converse implication.

Lemma 2.5. *Let $m_i, d_i, k_i \in \mathbb{Z}^*$ for $i = 1, 2$ such that*

$$m_1 d_1 = m_2 d_2,\ |k_2 d_2| \neq 1,\ \gcd(m_2, k_2) = 1 \text{ and } d_1 \text{ does not divide } d_2.$$

Then the distance between $\mathrm{BS}(m_1 d_1, k_1 d_1)$ *and* $\mathrm{BS}(m_2 d_2, k_2 d_2)$ *in* $\mathscr{G}_2$ *is not less than* $e^{-\delta}$ *with* $\delta = 10 + 2 d_1 m_1^2$.

Proof of Lemma 2.5. Consider $r = a^2 b^{d_1 m_1^2} a^{-2} b$ and let $w = r\bar{r}$. On one hand, we have $r = b^{d_1 k_1^2 + 1}$ in $\mathrm{BS}(m_1 d_1, k_1 d_1)$, which implies that $w = 1$ in $\mathrm{BS}(m_1 d_1, k_1 d_1)$. On the other hand, $r = a b^{m_1 d_2 k_2} a^{-1} b$ in $\mathrm{BS}(m_2 d_2, k_2 d_2)$. As m_2 and k_2 are coprime integers, we deduce that $m_2 d_2$ divides $m_1 d_2 k_2$ if and only if d_1 divides d_2. Under the assumptions of the lemma, the writing $a b^{m_1 d_2 k_2} a^{-1} b a b^{-m_1 d_2 k_2} a^{-1} b^{-1}$ is then a reduced form for w in $\mathrm{BS}(m_2 d_2, k_2 d_2)$. By Britton's Lemma, we have $w \neq 1$ in $\mathrm{BS}(m_2 d_2, k_2 d_2)$. As $|w| = 10 + 2 d_1 m_1^2$, we get the conclusion. □

Proof of Theorem 2.3. Proposition 2.4 implies immediately that conditions (i) and (ii) are sufficient.

Let us show that conditions (i) and (ii) are necessary. For this we assume that the sequence $(\mathrm{BS}(m,\xi_n))_n$ converges in $\mathscr{G}_2$. If (i) does not hold, we can find two subsequences $(\xi'_n)_n$ and $(\xi''_n)_n$ of $(\xi_n)_n$ such that $\gcd(m,\xi'_n) = d_1$, $\gcd(m,\xi''_n) = d_2$, $|\xi''_n| > 1$ for all n and d_1 does not divide d_2. Then Lemma 2.5 clearly shows that $(\mathrm{BS}(m,\xi_n))_n$ is not a converging sequence in $\mathscr{G}_2$, a contradiction. Hence (i) holds; let d be a rational integer satisfying it. The marked subgroup $\Gamma_{\xi_n,d}$ of $\mathrm{BS}(m,\xi_n)$ generated by (a, b^d) is equal to $\mathrm{BS}(m/d, \xi_n/d)$ endowed with its standard marking (a,b). The sequence of the $\mathrm{BS}(m/d, \xi_n/d)$ is then also converging in $\mathscr{G}_2$. By Theorem 3 of [Sta06], the sequence of rational integers $(\xi_n/d)_n$ defines a converging sequence in $\mathbb{Z}_{m/d}$. Hence (ii) holds. □

We now turn to the proof of Proposition 2.4, which occupies the end of this subsection. Because of Lemma 1.5, it boils down to show that for any w in $\mathbb{F}_2$ we have the implication: if w reduces to 1 in $G_n = \mathrm{BS}(md, \xi_n d)$ for infinitely many n then w reduces to 1 in G_n for all n large enough. Our strategy is to prove that under the hypothesis of Proposition 2.4, any word w undergoes the same sequence of cancelations in G_n for all n large enough.

Let $(G, (a, b))$ be a marked group on two generators a and b. We call *a-length* of an element $g \in G$ the minimal number of letters a, a^{-1} occurring in a word on $\{a, a^{-1}, b, b^{-1}\}$ which represents g.

The precise statement is

Lemma 2.6. *Let $w = b^{e_0}a^{\varepsilon_1}b^{e_1}\dots a^{\varepsilon_h}b^{e_h}$ with $\varepsilon_j = \pm 1$ for $j = 1, \dots, h$. Let $0 \le t \le h/2$ and let C be a class modulo m^t. Assume that there are infinitely many $n \in C$ such that the a-length of (the image of) w in $\mathrm{BS}(md, nd)$ is at most $h-2t$. Then there exist $\delta_1, \dots, \delta_{h-2t} \in \{\pm 1\}$ and polynomial functions $\alpha_0, \dots, \alpha_{h-2t}$, depending only on w and C, such that*

$$w =_{\mathrm{BS}(md,nd)} \bar{w}(n) := b^{\alpha_0(n)}a^{\delta_1}b^{\alpha_1(n)}\dots a^{\delta_{h-2t}}b^{\alpha_{h-2t}(n)}$$

for all $n \in C$ with $|n|$ large enough.

Proposition 2.4 then follows.

Proof of Proposition 2.4. Assume that $w = b^{e_0}a^{\varepsilon_1}b^{e_1}\dots a^{\varepsilon_h}b^{e_h}$ is trivial in $\mathrm{BS}(md, \xi_n d)$ for infinitely many n. The sum $\varepsilon_1 + \dots + \varepsilon_h$ has to be zero. Thus h is even and we can set $t = h/2$. By hypothesis, there exists a class modulo m^t, say C, such that $\xi_n \in C$ for all n large enough. We apply Lemma 2.6: there exists a polynomial function $\alpha = \alpha_0$ depending only on w and C such that w reduces to $\bar{w}(n) = b^{\alpha(\xi_n)}$ in $\mathrm{BS}(md, \xi_n d)$ for all n large enough. As $\alpha(\xi_n)$ is zero for infinitely many n, the polynomial function α is the zero function. Hence $w = 1$ in $\mathrm{BS}(md, \xi_n d)$ for all n large enough, which proves that $(\mathrm{BS}(md, \xi_n d))_n$ converges in $\mathscr{G}_2$ by Lemma 1.5. □

It only remains to prove Lemma 2.6. Reductions of w in $\mathrm{BS}(md, \xi_n d)$ are ruled by the congruence classes of its b exponents modulo md and $\xi_n d$. By Britton's Lemma, we have indeed: $w = b^{e_0}a^{\varepsilon_1}b^{e_1}\dots a^{\varepsilon_h}b^{e_h}$ is not reduced in $\mathrm{BS}(md, \xi_n d)$ if and only if there is some i in $\{1, \dots, h\}$ such that

(1) either $\varepsilon_i = -\varepsilon_{i+1} = 1$ and $e_i \equiv 0 \pmod{md}$, or

(2) $\varepsilon_i = -\varepsilon_{i+1} = -1$ and $e_i \equiv 0 \pmod{\xi_n d}$.

In both cases, the word w reduces in $\mathrm{BS}(md, \xi_n d)$ to a word w' such that its length with respect to $\{a^{-1}, a\}$ decreases by 2. We define below integers $r_0(\xi_n), \dots, r_t(\xi_n)$, $s_0(\xi_n), \dots, s_t(\xi_n)$ to keep track of the b exponents in the reductions $w', w'', \dots, w^{(t)}$

of w in $\mathrm{BS}(md, \xi_n d)$ along a sequence of t cancelations of type (1) or (2). These integers will be crucially used in the proof of Lemma 2.6 to give a description of such b exponents as functions of n. The integers $r_i(\xi_n)$ (respectively $s_i(\xi_n)$ modulo m) are shown to depend only on the congruence class of ξ_n modulo m^t.

Lemma 2.7 ([Sta06], Lemma 4). *Fix $m \in \mathbb{Z}^*$. We define recursively two sequences $s_0, s_1, \ldots$ and $r_0, r_1, \ldots$ of functions from $\mathbb{Z}^*$ to $\mathbb{Z}$ by*

(i) *$r_0(n) = 0$ and $s_0(n) = 1$ for all n;*

(ii) *$s_{i-1}(n)n = s_i(n)m + r_i(n)$ and $0 \leqslant r_i(n) < m$ for $i \geqslant 1$* (Euclidean division).

Then, for any n, n' in $\mathbb{Z}^*$ and $t \geq 1$ such that $n' \equiv n \pmod{m^t}$, we have $r_i(n) = r_i(n')$ and $s_i(n) \equiv s_i(n') \pmod{m^{t-i}}$ for all $0 \leqslant i \leqslant t$.

Remark 2.8. Let us observe that the functions $s_0, \ldots, s_t$ are polynomial functions when restricted to a given class C modulo m^t. Let $c \in \{0, \ldots, m^t - 1\}$ and let C be the set of rational integers n such that $n \equiv c \pmod{m^t}$. Define recursively the polynomial $P_{i,C}(X)$ by $P_{0,C}(X) = 1$ and $P_{i,C}(X) = XP_{i-1,C}(X) - \frac{r_i(n)}{m}$ with $n \in C$ and $i \geq 1$. Clearly, $mP_{i,C}(X) = mX^i - r_1(n)X^{i-1} - \cdots - r_i(n)$. The previous lemma implies that $P_{i,C}$ is a polynomial whose coefficients do not depend on $n \in C$. As $s_i(n) = P_{i,C}(\frac{n}{m})$ for all $0 \leq i \leq t$ and all $n \in C$, s_i is a polynomial function of degree i on C.

The following lemma describes how the b exponents in w transform through a cancelation of type (1) or (2) in $\mathrm{BS}(md, nd)$. These exponents are put into a particular form by means of the s_i and this form is shown to be (fortunately) preserved under reductions. The result coincides with [Sta06], Lemma 5, when $d = 1$. The proof is very similar, it is provided for completeness.

Lemma 2.9. *Fix $m, d \in \mathbb{Z}^*$ and $t \geq 1$. Let C be a class modulo m^t. Let $k_0, \ldots, k_t \in \mathbb{Z}$ and let $\alpha\colon C \to \mathbb{Z}$ be the function defined by*

$$\alpha(n) = k_0 + k_1 nd + k_2 s_1(n)nd + \cdots + k_t s_{t-1}(n)nd$$

where $s_0, s_1, \ldots$ are given by Lemma 2.7. Let us also take $r_0, r_1, \ldots$ as in Lemma 2.7.

(i) *The class of $\alpha(n)$ modulo md does not depend on $n \in C$. If $\alpha(n) \equiv 0 \pmod{md}$ for some $n \in C$, then d divides k_0 and we get $ab^{\alpha(n)}a^{-1} = b^{\beta(n)}$ in $\mathrm{BS}(md, nd)$ for all $n \in C$, with*

$$\beta(n) = l_1 nd + l_2 s_1(n)nd + \cdots + l_{t+1} s_t(n)nd$$

and

$$l_1 = \frac{1}{m}\left(\frac{k_0}{d} + k_1 r_1(n) + \cdots + k_t r_t(n)\right), \quad l_i = k_{i-1} \text{ for } 2 \leq i \leq t+1.$$

(ii) *We have either* $\alpha(n) \equiv 0 \pmod{nd}$ *for all* $n \in C$ *such that* $|n| > k_0$ *or* $\alpha(n) \not\equiv 0 \pmod{nd}$ *for all* $n \in C$ *such that* $|n| > k_0$.

In the first case we get $a^{-1}b^{\alpha(n)}a = b^{\beta(n)}$ *in* $\mathrm{BS}(md, nd)$ *for all* $n \in C$ *such that* $|n| > k_0$, *with*

$$\beta(n) = l_0 d + l_1 nd + l_2 s_1(n) nd + \cdots + l_{t-1} s_{t-2}(n) nd$$

and

$$l_0 = k_1 m - k_2 r_1(n) - \cdots - k_t r_{t-1}(n), \quad l_i = k_{i+1} \text{ for } 1 \le i \le t-1.$$

Moreover, the l_i*'s are constant rational integers in both cases.*

Proof. (i) Lemma 2.7 ensures that the class of $s_i(n)$ modulo m does not depend on $n \in C$ for all $i = 1, \ldots, t-1$. Hence the class of $\alpha(n)$ modulo md does not depend on $n \in C$. Assume now that $\alpha(n) \equiv 0 \pmod{md}$ for some $n \in C$. Then, we have $k_0 \equiv 0 \pmod{md}$, $ab^{\alpha(n)}a^{-1} = b^{n\alpha(n)/m}$ in $\mathrm{BS}(md, nd)$ with

$$\alpha(n) = \left(\frac{k_0}{d} + k_1 r_1(n) + \cdots + k_t r_t(n) + k_1 s_1(n) m + \cdots + k_t s_t(n) m\right) d$$

since $s_{i-1}(n)n = s_i(n)m + r_i(n)$ for all $i \ge 1$. Hence we obtain

$$\begin{aligned} \frac{n\alpha(n)}{m} &= \frac{1}{m}\left(\frac{k_0}{d} + k_1 r_1(n) + \cdots + k_t r_t(n)\right) nd + k_1 s_1(n) nd + \cdots + k_t s_t(n) nd \\ &= l_1 nd + l_2 s_1(n) nd + \cdots + l_{t+1} s_t(n) nd = \beta(n). \end{aligned}$$

By Lemma 2.7, $r_i(n)$ does not depend on $n \in C$ for all $i = 1, \ldots, t$. Consequently, the rational integers $l_1, \ldots, l_{t+1}$ do not depend on $n \in C$.

(ii) Assume that $|n| > |k_0|$. We have $\alpha(n) \equiv 0 \pmod{nd}$ if and only if $k_0 = 0$. Suppose now that it is the case. We have then $ab^{\alpha(n)}a^{-1} = b^{m\alpha(n)/n}$ in $\mathrm{BS}(md, nd)$ with

$$\begin{aligned} \frac{m\alpha(n)}{n} &= k_1 md + k_2 s_1(n) md + \cdots + k_t s_{t-1}(n) md \\ &= k_1 md - k_2 r_1(n) d - \ldots - k_t r_{t-1}(n) d + k_2 s_0(n) nd + \cdots + k_t s_{t-2}(n) nd \\ &= l_0 d + l_1 nd + l_2 s_1(n) nd + \cdots + l_{t-1} s_{t-2}(n) nd = \beta(n) \end{aligned}$$

since $s_i(n)m = s_{i-1}(n)n - r_i(n)$ for all $i \ge 1$. By Lemma 2.7 again, the rational integers $l_0, \ldots, l_{t-1}$ do not depend on $n \in C$. □

We are now able to prove Lemma 2.6.

Proof of Lemma 2.6. Set $G_n = \mathrm{BS}(md, nd)$. We show by induction on $t \ge 0$ that, provided the assumption in the lemma is satisfied, the functions $\alpha_i : C \to \mathbb{Z}$ exist

and satisfy

$$\begin{cases} \alpha_i(n) = k_{0,i} & \text{if } t = 0, \\ \alpha_i(n) = k_{0,i} + k_{1,i}nd + k_{2,i}s_2(n)nd + \cdots + k_{t,i}s_{t-1}(n)nd & \text{if } t > 0, \end{cases} \quad (*)$$

where the s_j are the functions defined in Lemma 2.7 and $k_{0,i}, \ldots, k_{t,i}$ are integers that depend only on w and C. Functions of the form $(*)$ depend only on w and C. These functions are polynomial functions of $n \in C$ by Remark 2.8.

Case $t = 0$: It suffices to take $\alpha_i(n) = e_i$ to satisfy $(*)$ and $\bar{w}(n) = w$ for all n.

Induction step $(0 < t \leq h/2)$: We denote by C' the class modulo m^{t-1} defined by C. By assumption, there exists an infinite subset $I \subseteq C \subset C'$ such that the a-length of w in $\mathrm{BS}(md, nd)$ is at most $h - 2t$ for all $n \in I$. By induction hypothesis, there exist $\delta'_1, \ldots, \delta'_{h-2t+2} \in \{\pm 1\}$ and polynomial functions $\alpha'_0, \ldots, \alpha'_{h-2t+2}$ satisfying

$$\begin{cases} \alpha'_i(n) = k_{0,i} & \text{if } t = 1, \\ \alpha'_i(n) = k_{0,i} + k_{1,i}nd + k_{2,i}s_2(n)nd + \cdots + k_{t-1,i}s_{t-2}(n)nd & \text{if } t > 1, \end{cases}$$

and such that $w = w'(n) := b^{\alpha'_0(n)}a^{\delta'_1}b^{\alpha'_1(n)} \ldots a^{\delta'_{h-2t+2}}b^{\alpha'_{h-2t+2}(n)}$ in $\mathrm{BS}(md, nd)$ for all $n \in C'$ with $|n|$ large enough.

Hence, for all $n \in I$ with $|n|$ large enough, the word $w'(n)$ is not reduced in $\mathrm{BS}(md, nd)$ since its a-length in this group is at most $h - 2t$. For such n's, Britton's Lemma gives an index $i = i(n) \in \{1, \ldots, h - 2t + 1\}$ such that

(a) either $a^{\delta'_i}b^{\alpha'_i(n)}a^{\delta'_{i+1}} = ab^{\alpha'_i(n)}a^{-1}$ and $\alpha'_i(n) \equiv 0 \pmod{md}$, or

(b) $a^{\delta'_i}b^{\alpha'_i(n)}a^{\delta'_{i+1}} = a^{-1}b^{\alpha'_i(n)}a$ and $\alpha'_i(n) \equiv 0 \pmod{nd}$.

Therefore, there exists an infinite subset $I' \subseteq I \subseteq C$ and a *fixed* index i, such that one of the cases (a),(b) occurs for all $n \in I'$. Applying Lemma 2.9, we see that a reduction will occur at index i for all $n \in C$ with $|n|$ large enough. Hence, we get $a^{\delta'_i}b^{\alpha'_i(n)}a^{\delta'_{i+1}} = b^{\beta(n)}$ in $\mathrm{BS}(md, nd)$, where

$$\beta(n) = l_0 d + l_1 nd + l_2 s_1(n)nd + \cdots + l_{t+1}s_t(n)nd,$$

for all $n \in C$ with $|n|$ large enough (two last terms vanishing if we were in case (b)). Finally, we are done by setting

$$\alpha_j(n) := \begin{cases} \alpha'_j(n) & \text{if } j < i - 1, \\ \alpha'_{i-1}(n) + \beta(n) + \alpha'_{i+1}(n) & \text{if } j = i - 1, \\ \alpha'_{j-2}(n) & \text{if } j \geq i \end{cases}$$

for $j = 0, \ldots, h - 2t$. □

2.2. Continuity. We now turn to the topologist's point of view. Namely, we are going to prove the following analogue of Theorem 2.3.

Theorem 2.10. *Let m be in $\mathbb{Z}^*$ and let $(\xi_n)_n$ be a sequence in $\mathbb{Z}_m$. The sequence $(\overline{\mathrm{BS}}(m,\xi_n))_n$ converges in $\mathscr{G}_2$ if and only if the following conditions both hold:*

(i) *There is $d \in \mathbb{N}^*$ such that $\gcd(m,\xi_n) = d$ for all n large enough.*

(ii) *The images of ξ_n/d define a converging sequence in $\mathbb{Z}_{m/d}$.*

In particular, if $(\xi_n)_n$ converges to ξ in $\mathbb{Z}_m$, then the sequence $(\overline{\mathrm{BS}}(m,\xi_n))_n$ converges to $\overline{\mathrm{BS}}(m,\xi)$ in $\mathscr{G}_2$.

The following corollary is immediate.

Corollary 2.11. *For all $m \in \mathbb{Z}^*$, the map $\overline{\mathrm{BS}}_m \colon \mathbb{Z}_m \to \mathscr{G}_2$ is continuous.*

For $|m| \geqslant 2$, note that if we endow $\mathbb{Z}$ with the m-adic ultrametric, the analogue map

$$\mathbb{Z}^* \to \mathscr{G}_2;\ n \mapsto \mathrm{BS}(m,n)$$

is nowhere continuous. Indeed, $n + m^k$ tends to n as k tends to infinity, while $\mathrm{BS}(m, n+m^k) \to \overline{\mathrm{BS}}(m,n)$. We recall that one has $\overline{\mathrm{BS}}(m,n) \neq \mathrm{BS}(m,n)$, since the word $ab^m a^{-1} b^{-n}$ defines the trivial element in $\mathrm{BS}(m,n)$ but not in $\overline{\mathrm{BS}}(m,n)$.

Proof of Theorem 2.10. Let $(\xi_n)_n$ be a sequence of m-adic integers. By an easy argument of diagonal extraction, we can find a rational integer sequence $(\eta_n)_n$ such that

(1) $d(\mathrm{BS}(m,\eta_n), \overline{\mathrm{BS}}(m,\xi_n)) < \frac{1}{n}$,

(2) $d_m(\eta_n,\xi_n) < \frac{1}{n}$,

(3) $|\eta_n| \geq n$.

The sequence $(\overline{\mathrm{BS}}(m,\xi_n))_n$ converges if and only if $(\mathrm{BS}(m,\eta_n))_n$ converges because of inequality (1). By Theorem 2.3, the sequence $(\mathrm{BS}(m,\eta_n))_n$ converges if and only if there is some d in $\mathbb{Z}^*$ such that $\gcd(\eta_n, m) = d$ for all n large enough and $(\eta_n/d)_n$ converges in $\mathbb{Z}_{m/d}$. By (2) and Lemma 1.4, we have $\gcd(\xi_n, m) = d$ for all n large enough if and only if $\gcd(\eta_n, m) = d$ for all n large enough. Since $d_{m/d}(\eta_n/d, \pi(\xi_n/d)) \leq d_m(\eta_n,\xi_n)$, we deduce from (2) that the sequence $(\eta_n/d)_n$ converges in $\mathbb{Z}_{m/d}$ if and only if $(\pi(\xi_n/d))_n$ converges in $\mathbb{Z}_{m/d}$.

Now let us turn to the particular case $\xi_n \to \xi$ in $\mathbb{Z}_m$. We get condition (i) by Lemma 1.4, and condition (ii) is obvious. The sequence $(\overline{\mathrm{BS}}(\xi_n, m))_n$ is thus converging. Moreover, its limit is $\overline{\mathrm{BS}}(m,\xi)$, since it does not change if one intercalates a ξ-constant subsequence in $(\xi_n)_n$. □

We now "specialize to the case of invertible elements" and show that, in this case, the $\overline{\mathrm{BS}}$ groups form the set of accumulation points of the BS groups. Precise statements are as follows:

Definition 2.12. For $m \in \mathbb{Z}^*$, we define

$$X_m = \{\mathrm{BS}(m,n) \mid n \text{ is relatively prime to } m\},$$
$$Z_m^\times = \{\overline{\mathrm{BS}}(m,\xi) \mid \xi \in \mathbb{Z}_m^\times\}.$$

By convention, we say that $Z_{\pm 1}^\times$ is empty.

Corollary 2.13. *For all $m \in \mathbb{Z}^*$, the boundary of X_m in $\mathcal{G}_2$ is $Z_m^\times$. It is homeomorphic to the set of invertible m-adic integers.*

Proof. Theorem 3 of [Sta06] implies that the elements of $\overline{X_m}$ are the $\mathrm{BS}(m,n)$ with n relatively prime to m and the $\overline{\mathrm{BS}}(m,\xi)$ with $\xi \in \mathbb{Z}_m^\times$. One sees easily that the groups $\mathrm{BS}(m,n)$ are isolated points in $\overline{X_m}$ (consider the word $ab^m a^{-1} b^{-n}$). The equality $\partial X_m = Z_m^\times$ follows immediately. The second statement is a direct consequence of Corollary 2.11. □

3. Actions and structure of the limits

It is well known that Baumslag–Solitar groups act on their Bass–Serre trees and (affinely) on the real line or on $\mathbb{Q}$. In this section, we study such actions (which are "trivial" in the affine case) and the structure of the limits $\overline{\mathrm{BS}}(m,\xi)$.

3.1. A common quotient. We recall that $\Gamma(m,n) = \mathbb{Z} \ltimes_{\frac{n}{m}} \mathbb{Z}\left[\frac{\gcd(m,n)}{\mathrm{lcm}(m,n)}\right]$ acts affinely on $\mathbb{R}$ and that it is a marked quotient of both $\mathrm{BS}(m,n)$ and $\mathbb{Z} \wr \mathbb{Z}$ (see Section 1.3). Notice that these groups are distinct from the groups $\Gamma_{\xi_n,d}$ introduced in Section 2.

The following lemma enables us to define converging sequences of homomorphisms induced by the same endomorphism of $\mathbb{F}_m$.

Lemma 3.1. *Let $(G_n)_n$ and $(H_n)_n$ be sequences of marked groups and let G and H be their limits in $\mathcal{G}_m$. Let ϕ be an endomorphism of $\mathbb{F}_m$ which induces an homomorphism from G_n to H_n for all n. Then ϕ induces an homomorphism from G to H. Moreover, if $\phi\colon G_n \to H_n$ is injective for infinitely many n, then $\phi\colon G \to H$ is injective.*

Proof. Let $w \in \mathbb{F}_m$ be such that $w =_G 1$. For all n large enough we have $w =_{G_n} 1$ (Lemma 1.5), and hence $\phi(w) =_{H_n} 1$. Consequently, $\phi(w) =_H 1$, which shows that ϕ induces an homomorphism from G to H. Assume ϕ is injective for infinitely many n and consider $w \in \mathbb{F}_m$ such that its image in G belongs to $\ker(\phi\colon G \to H)$. There are infinitely many n such that the following hold:

(i) $\phi\colon G_n \to H_n$ is injective,

(ii) $\phi(w) =_{H_n} 1$.

It follows that $w =_{G_n} 1$ holds for infinitely many n. As a consequence, $w =_G 1$. □

Proposition 3.2. *For any $m \in \mathbb{Z}^*$ and $\xi \in \mathbb{Z}_m$, the morphism of marked groups $q\colon \mathbb{F}_2 \twoheadrightarrow \mathbb{Z} \wr \mathbb{Z}$ factors through a morphism $q_{m,\xi}\colon \overline{\mathrm{BS}}(m,\xi) \twoheadrightarrow \mathbb{Z} \wr \mathbb{Z}$.*

Proof. Take $(\xi_n)_n$ a sequence of rational integers such that $|\xi_n|$ tends to infinity and ξ_n tends to ξ in $\mathbb{Z}_m$ as n goes to infinity. By Definition 1.7, the sequence $(\mathrm{BS}(m,\xi_n))_n$ converges to $\overline{BS}(m,\xi)$. We now claim that $\Gamma(m,\xi_n)$ tends to $\mathbb{Z} \wr \mathbb{Z}$ as n goes to infinity, a statement analogous to Theorem 2 in [Sta06]. As each $\Gamma(m,\xi_n)$ is a marked quotient of $\mathbb{Z} \wr \mathbb{Z}$, we only have to prove that any non-trivial element $(\sigma, P(t)) \in \mathbb{Z} \wr \mathbb{Z}$ defines eventually non-trivial elements in the $\Gamma(m,\xi_n)$'s. This is obvious if $\sigma \neq 0$. Assume $\sigma = 0$. The Laurent polynomial P has only finitely many roots so that $P(\xi_n/m) \neq 0$ for n sufficiently large. This tells us that (σ, P) is eventually non-trivial in the $\Gamma(m,\xi_n)$'s, which proves the claim.

Hence one has the following "diagram":

$$\begin{array}{ccc} \mathrm{BS}(m,\xi_n) & \xrightarrow{n\to\infty} & \overline{\mathrm{BS}}(m,\xi) \\ \downarrow & & \\ \Gamma(m,\xi_n) & \xrightarrow{n\to\infty} & \mathbb{Z} \wr \mathbb{Z} \end{array}$$

As vertical arrows are induced by the identity of $\mathbb{F}_2$, the result follows from the previous lemma. □

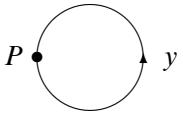
3.2. Affine actions. Let R be a ring and $\mathrm{Aff}(R) \cong R^\times \ltimes R$ be the affine group on R. For any $\alpha \in R^\times$, it is straightforward to see that formulae $a \cdot x = \alpha x$ and $b \cdot x = x + 1$ define a homomorphism $\mathbb{Z} \wr \mathbb{Z} \to \mathrm{Aff}(R)$. Thus Proposition 3.2 immediately gives the following

Proposition 3.3. *Let R be a ring and let $\alpha \in R^\times$. For any $m \in \mathbb{Z}^*$ and any $\xi \in \mathbb{Z}_m$, the group $\overline{\mathrm{BS}}(m,\xi)$ acts affinely on R via formulae*

$$a \cdot x = \alpha x \quad \text{and} \quad b \cdot x = x + 1.$$

3.3. Actions on trees. We are to produce a tree on which the group $\overline{\mathrm{BS}}(m,\xi)$ acts transitively. This tree will be constructed from the Bass–Serre trees of the groups $\mathrm{BS}(m,\xi_n)$. It will be shown that the tree we construct does not depend on the auxiliary sequence $(\xi_n)_n$.

We recall that $\mathrm{BS}(m,n)$ is the fundamental group of the graph of groups (G,Y) shown in Figure 1, see [Ser77], Section 5.1. Notice that a is the element of $\pi_1(G,Y,P)$



$$G_P = \langle b\rangle \cong \mathbb{Z};\ G_y = \langle c\rangle \cong \mathbb{Z}$$
$$c^y = b^m;\ c^{\bar{y}} = b^n$$

Figure 1. Baumslag–Solitar groups as graphs of groups.

associated to the edge y. To be precise, we set the *Bass–Serre tree* of $\mathrm{BS}(m,n)$ to be the universal covering associated to (G,Y), the maximal subtree P and the orientation given by the edge $\bar{y}$ [Ser77], Section 5.3. We choose the edge $\bar{y}$ instead of y to minimize the dependence on n of the set of tree edges. Denoting by T the Bass–Serre tree of $\mathrm{BS}(m,n)$, one has

$$V(T) = \mathrm{BS}(m,n)/\langle b\rangle, \qquad E_+(T) = \mathrm{BS}(m,n)/\langle b^m\rangle,$$
$$o(w\langle b^m\rangle) = w\langle b\rangle, \qquad t(w\langle b^m\rangle) = wa^{-1}\langle b\rangle,$$

where $o(e)$ and $t(e)$ denote the origin and terminal vertex of the edge e. The chosen orientation on T is preserved by the $\mathrm{BS}(m,n)$-action.

Given $m \in \mathbb{Z}^*$, $\xi \in \mathbb{Z}_m$ and $(\xi_n)_n$ a sequence of rational integers such that $|\xi_n| \to \infty$ and $\xi_n \to \xi$ in $\mathbb{Z}_m$, we denote by H_n (respectively H_n^m) the subgroup of $\mathrm{BS}(m,\xi_n)$ generated by b (respectively b^m) and by T_n the Bass–Serre tree of $\mathrm{BS}(m,\xi_n)$. We set

$$Y = \Big(\prod_{n\in\mathbb{N}} V(T_n)\Big)/\!\sim = \Big(\prod_{n\in\mathbb{N}} \mathrm{BS}(m,\xi_n)/H_n\Big)/\!\sim,$$
$$Y^m = \Big(\prod_{n\in\mathbb{N}} E_+(T_n)\Big)/\!\sim = \Big(\prod_{n\in\mathbb{N}} \mathrm{BS}(m,\xi_n)/H_n^m\Big)/\!\sim,$$

where $\sim$ is defined in both cases by $(x_n)_n \sim (y_n)_n \Leftrightarrow$ there exists n_0 such that $x_n = y_n$ for all $n \geq n_0$. We now define an oriented graph $X = X_{m,\xi}$ by

$$V(X) = \{x \in Y \mid \text{there exists } w \in \mathbb{F}_2 \text{ such that } (x_n)_n \sim (wH_n)_n\},$$
$$E_+(X) = \{y \in Y^m \mid \text{there exists } w \in \mathbb{F}_2 \text{ such that } (y_n)_n \sim (wH_n^m)_n\},$$
$$o((wH_n^m)_n) = (wH_n)_n = (o(wH_n^m))_n,$$
$$t((wH_n^m)_n) = (wa^{-1}H_n)_n = (t(wH_n^m))_n.$$

The map o is well defined since $(vH_n^m)_n \sim (wH_n^m)_n$ implies $(vH_n)_n \sim (wH_n)_n$. On the other hand, the map t is well defined since $(vH_n^m)_n \sim (wH_n^m)_n$ implies $v^{-1}w \in H_n^m$ for n large enough, whence $(va^{-1})^{-1}(wa^{-1}) = av^{-1}wa^{-1} \in H_n$ for those values of n. It follows that $(va^{-1}H_n)_n \sim (wa^{-1}H_n)_n$. The graph X is thus well defined and the free group $\mathbb{F}_2$ acts obviously on it by left multiplications.

Remark 3.4. There is an alternative way to define X, which we now describe briefly. Consider the $\mathbb{Z}$-trees $V(T_n)$ and define the ultraproduct $V = \left(\prod V(T_n)\right)/\omega$, which is a ${}^{\omega}\mathbb{Z}$-tree (where ω is some non-principal ultrafilter over $\mathbb{N}$). The ultraproduct group $\left(\prod \mathrm{BS}(m,\xi_n)\right)/\omega$ contains $\overline{\mathrm{BS}}(m,\xi)$ and acts on V. Now the set of vertices of X is equal to subtree of V spanned by the orbit $\overline{\mathrm{BS}}(m,\xi)\cdot v_0$, with $v_0 = [H_n]_\omega$. However, even if this approach would immediately tell us that X is a tree endowed with a $\overline{\mathrm{BS}}(m,\xi)$-action (one has only to check that the subtree is a $\mathbb{Z}$-tree), we prefer the more down-to-earth one described above for the sake of a self-contained and explicit construction. See [CG05], [Chi01], [Pau04] for more information on this second approach.

It is almost obvious that X is simple (i.e., it has no loop and no bigon). Indeed a loop (or bigon) in X would immediately provide a loop (or bigon) in some tree T_n. The statement we want to prove is the following.

Theorem 3.5. *Let $m \in \mathbb{Z}^*$, $\xi \in \mathbb{Z}_m$ and $(\xi_n)_n$ a sequence of rational integers such that $|\xi_n| \to \infty$ and $\xi_n \to \xi$ in $\mathbb{Z}_m$. The graph $X = X_{m,\xi}$ (seen here as unoriented) satisfies the following properties:*

(a) *It is a tree.*

(b) *It does not depend (up to equivariant isomorphism) on the choice of the auxiliary sequence $(\xi_n)_n$.*

(c) *The obvious action of the free group $\mathbb{F}_2$ on $X_{m,\xi}$ factors through the canonical projection $\mathbb{F}_2 \to \overline{\mathrm{BS}}(m,\xi)$.*

Before the proof, we give a simple consequence of [Sta06], Lemma 6, or of Lemma 2.6 (with $t = h/2$).

Lemma 3.6. *Let $(vH_n)_n$ and $(wH_n)_n$ be two vertices of the graph X. If $vH_n = wH_n$ for infinitely many values of n, then $(vH_n)_n = (wH_n)_n$ in X.*

Proof of Theorem 3.5. (a) Let us show first that the graph X is connected. We show by induction on $|w|$ that any vertex $(wH_n)_n$, with $w \in \mathbb{F}_2$, is connected to $(H_n)_n$. The case $|w| = 0$ is trivial. If $|w| = \ell > 0$, there exists $x \in \{a^{\pm 1}, b^{\pm 1}\}$ such that wx has length $\ell - 1$. By induction hypothesis, it is sufficient to show that $(wH_n)_n$ is connected to $(wxH_n)_n$. If $x = a$, then the edge $(wxH_n^m)_n$ connects $(wxH_n)_n$ to $(wH_n)_n$. If $x = a^{-1}$, then the edge $(wH_n^m)_n$ connects $(wH_n)_n$ to $(wxH_n)_n$ and if $x = b^{\pm 1}$, then one has even $(wH_n)_n = (wxH_n)_n$.

Second, we show that X has no closed path without backtracking. We take a closed path in X, with vertices

$$(v_0H_n)_n, (v_1H_n)_n, \ldots, (v_\ell H_n)_n = (v_0H_n)_n,$$

and we want to prove the existence of some backtracking.

For any n, the sequence $v_0H_n, v_1H_n, \dots, v_\ell H_n$ defines a path in T_n. For n large enough, one has $v_\ell H_n = v_0 H_n$, so that the path is closed. As the T_n's are trees, these closed paths all have at least one backtracking. Thus, there exists $i \in \{0, \dots, \ell - 1\}$ such that $v_{i-1}H_n = v_{i+1}H_n$ for infinitely many values of n (v-indexes are taken modulo ℓ). Now, by Lemma 3.6, we obtain that $(v_{i-1}H_n)_n = (v_{i+1}H_n)_n$ in X. As X is a simple graph, this means that the original closed path in X has some backtracking, as desired.

(b) We show now that X does not depend (up to equivariant isomorphism) on the choice of the sequence $(\xi_n)_n$. Take another sequence $(\xi'_n)_n$ satisfying both $|\xi'_n| \to \infty$ and $\xi'_n \to \xi$ in $\mathbb{Z}_m$ and consider the associated tree X'. We construct the sequence $(\xi''_n)_n$ given by

$$\xi''_n = \begin{cases} \xi_{\frac{n}{2}} & \text{if } n \text{ is even,} \\ \xi'_{\frac{n-1}{2}} & \text{if } n \text{ is odd,} \end{cases}$$

which satisfies again both $|\xi''_n| \to \infty$ and $\xi''_n \to \xi$ in $\mathbb{Z}_m$ and the associated tree X''. There are obvious equivariant surjective graph morphisms $X'' \to X$ and $X'' \to X'$. We have to show the injectivity of these morphisms, which we can check on vertices only, for we are dealing with trees. But Lemma 3.6 precisely implies the injectivity on vertices.

(c) Take $w \in \mathbb{F}_2$ such that $w = 1$ in $\overline{\mathrm{BS}}(m, \xi)$. We have to prove that w acts trivially on X. As X is a simple graph, we only have to prove that w acts trivially on $V(X)$. Let $(vH_n)_n$ be a vertex of X. For n large enough, we have $w = 1$ in $\mathrm{BS}(m, \xi_n)$, so that $wvH_n = vH_n$. Hence we have $w \cdot (vH_n)_n \sim (vH_n)_n$, as desired. □

Remark 3.7. The action of $\overline{\mathrm{BS}}(m, \xi)$ on X is transitive and the stabilizer of the vertex $v_0 = (H_n)_n$ is the subgroup of elements which are powers of b in all but finitely many $\mathrm{BS}(m, \xi_n)$. It does not coincide with the subgroup of $\overline{\mathrm{BS}}(m, \xi)$ generated by b, since the element $ab^m a^{-1}$ is not in the latter subgroup but stabilizes the vertex.

Remark 3.8. The $\overline{\mathrm{BS}}(m, \xi)$-action on X being transitive on vertices and on edges, $\overline{\mathrm{BS}}(m, \xi)$ is a HNN-extension with basis $\mathrm{Stab}(v_0)$. We shall not focus on this structure in the present article but leave it for another paper.

We end this section with statements about the structure of the tree $X_{m,\xi}$. The first one is the analogue of Lemma 3.6 for edges.

Lemma 3.9. *Let $(vH_n^m)_n$ and $(wH_n^m)_n$ be two edges of the graph X. If one has $vH_n^m = wH_n^m$ for infinitely many values of n, then $(vH_n^m)_n = (wH_n^m)_n$ in X.*

Proof. By assumption, one has $vH_n = wH_n$ and $va^{-1}H_n = wa^{-1}H_n$ for infinitely many values of n. By Lemma 3.6, we get $(vH_n)_n = (wH_n)_n$ and $(va^{-1}H_n)_n =$

$(wa^{-1}H_n)_n$. As the edges $(vH_n^m)_n$ and $(wH_n^m)_n$ have the same origin and terminal vertex, they are equal since X is a simple graph. □

Proposition 3.10. *Let $m \in \mathbb{Z}^*$ and $\xi \in \mathbb{Z}_m$. Each vertex of the tree $X_{m,\xi}$ has exactly $|m|$ outgoing edges. More precisely, (given a sequence $(\xi_n)_n$ of non-zero rational integers such that $|\xi_n| \to \infty$ and $\xi_n \to \xi$) the edges outgoing from the vertex $(wH_n)_n$ are exactly $(wH_n^m)_n, (wbH_n^m)_n, \ldots, (wb^{|m|-1}H_n^m)_n$.*

Proof. It suffices to treat the case $w = 1$. The edges $(H_n^m)_n, (bH_n^m)_n, \ldots, (b^{|m|-1}H_n^m)_n$ are clearly outgoing from $(H_n)_n$ and distinct. Let now $(vH_n^m)_n$ be an edge outgoing from $(H_n)_n$. In particular, we have $(vH_n)_n = (H_n)_n$, so that $v = b^{\lambda_n}$ in $\mathrm{BS}(m, \xi_n)$ for n large enough. There exists necessarily $\lambda \in \{0, \ldots, |m| - 1\}$ such that we have $\lambda_n \equiv \lambda \pmod{m}$ for infinitely many values of n, so that $vH_n^m = b^\lambda H_n^m$ for infinitely many values of n. By Lemma 3.9, we get $(vH_n^m)_n = (b^\lambda H_n^m)_n$, and we are done. □

3.4. A structure theorem. We denote by N the kernel of the map q appearing in Proposition 3.2. We now are able to state the main results of this section.

Theorem 3.11. *Consider the exact sequence (where $N_{m,\xi}$ is the image of N in $\overline{\mathrm{BS}}(m, \xi)$)*

$$1 \to N_{m,\xi} \to \overline{\mathrm{BS}}(m, \xi) \xrightarrow{q_{m,\xi}} \mathbb{Z} \wr \mathbb{Z} \to 1.$$

For any $m \in \mathbb{Z}^*$ and $\xi \in \mathbb{Z}_m$, the group $N_{m,\xi} = \ker q_{m,\xi}$ is free.

Remark 3.12. Since $\mathbb{Z} \wr \mathbb{Z}$ is metabelian, the second derived subgroup of $\overline{\mathrm{BS}}(m, \xi)$ is a free group. Thus $\overline{\mathrm{BS}}(m, \xi)$ enjoys the same property as the generalized Baumslag–Solitar groups; see [Kro90], Corollary 2.

Proof of Theorem 3.11. Take $m \in \mathbb{Z}^*$, $\xi \in \mathbb{Z}_m$ and $(\xi_n)_n$ a sequence of rational integers such that $|\xi_n| \to \infty$ and $\xi_n \to \xi$ in $\mathbb{Z}_m$. Set X to be the tree constructed in Subsection 3.3. By [Ser77], Section 3.3, Theorem 4, it is sufficient to prove that $N_{m,\xi}$ acts freely on X, i.e., that any $w' \in N$ which stabilizes a vertex satisfies $w' = 1$ in $\overline{\mathrm{BS}}(m, \xi)$.

Let us take $w' \in N$ and $(vH_n)_n$ a vertex of X which is stabilized by w'. Thus $w = v^{-1}w'v$ stabilizes the vertex $(H_n)_n$, i.e., w is a power of b, say $w = b^{\lambda_n}$, in all but finitely many $\mathrm{BS}(m, \xi_n)$. Then the image of w in $\Gamma(m, \xi_n) = \mathbb{Z} \ltimes_{\frac{n}{m}} \mathbb{Z}\left[\frac{\gcd(m,\xi_n)}{\mathrm{lcm}(m,\xi_n)}\right]$ is equal to $(0, \lambda_n)$ for all but finitely many values of n. But, on the other hand, one has $w \in N$, that is, $w = 1$ in $\mathbb{Z} \wr \mathbb{Z}$, which implies that $\lambda_n = 0$ for those values of n (since the $\Gamma(m, \xi_n)$ are marked quotients of $\mathbb{Z} \wr \mathbb{Z}$). Thus, one has $w = b^0 = 1$ in all but finitely many $\mathrm{BS}(m, \xi_n)$, which gives $w = 1 = w'$ in $\overline{\mathrm{BS}}(m, \xi)$. □

Corollary 3.13. *For any $m \in \mathbb{Z}^*$ and $\xi \in \mathbb{Z}_m$, the group $\overline{\mathrm{BS}}(m, \xi)$*

(1) *has the Haagerup property,*

(2) *is residually solvable.*

Remark 3.14. The Haagerup property for Baumslag–Solitar groups was already known [GJ03]. It may also be deduced from the fact that the second derived subgroup is free [Kro90], Corollary 2.

Proof of Corollary 3.13. Looking at the exact sequence

$$1 \to N_{m,\xi} \to \overline{\mathrm{BS}}(m, \xi) \xrightarrow{q} \mathbb{Z} \wr \mathbb{Z} \to 1,$$

we see that the quotient group is amenable (it is even metabelian) and the kernel group has the Haagerup property by Theorem 3.11. By [CCJ+01], Example 6.1.6, we conclude that $\overline{\mathrm{BS}}(m, \xi)$ has the Haagerup property. As a free group is residually solvable, $\overline{\mathrm{BS}}(m, \xi)$ is then the extension of a residually solvable group by a solvable one and hence is residually solvable by [Gru57], Lemma 1.5. □

4. Presentations of the limits

The purpose of this section is, first, to prove that the groups $\overline{\mathrm{BS}}(m, \xi)$ are almost never finitely presented (Theorem 4.1), and second, to give presentations, whose construction is based on actions on trees described in Section 3.3 (Theorem 4.4).

4.1. Most of the limits are not finitely presented. Our first goal in this section is to prove

Theorem 4.1. *For any $m \in \mathbb{Z}^*$ and $\xi \in \mathbb{Z}_m \setminus m\mathbb{Z}_m$, the group $\overline{\mathrm{BS}}(m, \xi)$ is not finitely presented.*

Notice that the theorem excludes also the existence of a finite presentation of such a group with another generating set; see for instance [dlH00], Proposition V.2. By Theorem 2.1, there is only one remaining case, the case $\xi = 0$, where it is still unknown whether or not $\overline{\mathrm{BS}}(m, 0)$ is finitely presented. We nevertheless make the following remark.

Remark 4.2. For $|m| = 1$, the limits $\overline{\mathrm{BS}}(\pm 1, \xi)$ are not finitely presented.

Proof. The result [Sta06], Theorem 2, implies that $\overline{\mathrm{BS}}(\pm 1, \xi) = \mathbb{Z} \wr \mathbb{Z}$ for the unique element $\xi \in \mathbb{Z}_{\pm 1}$ and Baumslag's result [Bau61] on the presentations of wreath products ensures that $\mathbb{Z} \wr \mathbb{Z}$ is not finitely presented. □

Lemma 4.3. *Let $m \in \mathbb{Z}^*$ and $\xi \in \mathbb{Z}_m \setminus m\mathbb{Z}_m$. Let ℓ be the maximal exponent in the decomposition of m in prime factors and set $d = \gcd(m, \xi)$, $m_1 = m/d$.*

(a) *There exists a sequence $(\xi_n)_n$ in $\mathbb{Z}^*$ such that for all $n \geqslant 1$ one has $|\xi_n| > |\xi_{n-1}|$ and*

$$\xi_n \equiv \xi \pmod{m^n \mathbb{Z}_m},$$
$$\xi_n \not\equiv \xi \pmod{m_1^{\ell n+1} d\, \mathbb{Z}_m}.$$

(b) *This sequence satisfies $|\xi_n| \to \infty$, $\xi_n \to \xi$ and*

$$\xi_n \equiv \xi_r \pmod{m^n} \text{ for all } r \geqslant n,$$
$$\xi_n \not\equiv \xi_{\ell n+1} \pmod{m_1^{\ell n+1} d} \text{ for all } n.$$

Proof. (a) Let p be a prime factor of m_1 (there exists one, for $\xi \notin m\mathbb{Z}_m$). The sequence $(\xi_n)_n$ is constructed inductively. We choose for ξ_0 any non-zero rational integer such that $\xi_0 - \xi \notin m\mathbb{Z}_m$. At the n-th step, we begin by noticing that the exponent of p in the decomposition of m^n (respectively $m_1^{\ell n+1} d$) is at most ℓn (respectively at least $\ell n + 1$). Hence, $m_1^{\ell n+1} d$ is not a divisor of m^n so that there exists $\alpha \in \mathbb{Z}$ with $\xi \equiv \alpha \pmod{m^n \mathbb{Z}_m}$ but $\xi \not\equiv \alpha \pmod{m_1^{\ell n+1} d\, \mathbb{Z}_m}$.

Notice now that we may replace α by any element of the class $\alpha + m^n m_1^{\ell n+1} d\, \mathbb{Z}$, so that it suffices to choose ξ_n among the elements β in the latter class which satisfy $|\beta| > |\xi_{n-1}|$.

(b) The properties $\xi_n \equiv \xi \pmod{m^n \mathbb{Z}_m}$ and $|\xi_n| > |\xi_{n-1}|$ clearly imply that $\xi_n \to \xi$, $|\xi_n| \to \infty$ and $\xi_n \equiv \xi_r \pmod{m^n}$ for $r \geqslant n$ (for the latter one, Proposition 1.1 (d) is used).

Finally, combining $\xi_{\ell n+1} \equiv \xi \pmod{m^{\ell n+1} \mathbb{Z}_m}$ and $\xi_n \not\equiv \xi \pmod{m_1^{\ell n+1} d\, \mathbb{Z}_m}$ gives $\xi_n \not\equiv \xi_{\ell n+1} \pmod{m_1^{\ell n+1} d}$. □

Proof of Theorem 4.1. The hypothesis $\xi \in \mathbb{Z}_m \setminus m\mathbb{Z}_m$ implies that $|m| \geqslant 2$. Take ℓ, d, m_1 and a sequence $(\xi_n)_n$ as in Lemma 4.3. Then $\mathrm{BS}(m, \xi_n) \to \overline{\mathrm{BS}}(m, \xi)$. It is thus sufficient by Lemma 1.6 to prove that the $\mathrm{BS}(m, \xi_n)$ are not marked quotients of $\overline{\mathrm{BS}}(m, \xi)$ (for n large enough).

Notice now that (for n large enough) one has $\gcd(m, \xi_n) = d$ since $\xi_n \equiv \xi \pmod{m\mathbb{Z}_m}$ holds. Then, for $n \geqslant 1$, we define the words

$$w_n = a^{n+1} b^m a^{-1} b^{-\xi_n} a^{-n} b a^{n+1} b^{-m} a^{-1} b^{\xi_n} a^{-n} b^{-1}.$$

By Lemma 3 of [Sta06], we have $w_n = 1$ in $\mathrm{BS}(m, k)$ if and only if $k \equiv \xi_n \pmod{m_1^n d}$. By part (b) of Lemma 4.3, it then follows that $w_n = 1$ in $\mathrm{BS}(m, \xi_r)$ for all $r \geqslant n$, hence $w_n = 1$ in $\overline{\mathrm{BS}}(m, \xi)$ (for n large enough). On the other hand we get $w_{\ell n+1} \neq 1$ in $\mathrm{BS}(m, \xi_n)$ in the same way, so that $\mathrm{BS}(m, \xi_n)$ is not a marked quotient of $\overline{\mathrm{BS}}(m, \xi)$ (for n large enough). □

4.2. Presentations of the limits. As we noticed before (see Remark 3.8), the limits $\overline{\mathrm{BS}}(m,\xi)$ are HNN-extensions. This would give us presentations of the limits, provided that we would have presentations for the vertex and edges stabilizers. Nevertheless, rather than to follow this approach and look for presentations for the stabilizers, we establish directly a presentation for the whole group.

For $m \in \mathbb{Z}^*$ and $\xi \in \mathbb{Z}_m$, we define the set $\mathcal{R} = \mathcal{R}_{m,\xi}$ by

$$\mathcal{R}_{m,\xi} = \{w\bar{w} \mid w = ab^{\alpha_1} \dots ab^{\alpha_k} a^{-1} b^{\alpha_{k+1}} \dots a^{-1} b^{\alpha_{2k}} \\ \text{with } k \in \mathbb{N}^*,\ \alpha_i \in \mathbb{Z}\ (i = 1, \dots, 2k) \text{ and } w \cdot v_0 = v_0\},$$

where v_0 is the favoured vertex $(H_n)_n$ of the tree $X_{m,\xi}$ defined in Section 3.3.

Recall that the stabilizer of the vertex v_0 consists of elements which are powers of b in all but finitely many $\mathrm{BS}(m,\xi_n)$, where $(\xi_n)_n$ is any sequence of rational integers such that $|\xi_n| \to \infty$ and $\xi_n \to \xi$ in $\mathbb{Z}_m$ (see Remark 3.7). It follows that we have $w \cdot v_0 = v_0 \iff \bar{w} \cdot v_0 = v_0$. The aim is now to prove the following result.

Theorem 4.4. *For all $m \in \mathbb{Z}^*$ and $\xi \in \mathbb{Z}_m$, the marked group $\overline{\mathrm{BS}}(m,\xi)$ admits the presentation $\langle a, b \mid \mathcal{R}_{m,\xi}\rangle$.*

Set $\Gamma = \langle a, b \mid \mathcal{R}_{m,\xi}\rangle$ until the end of this section. The elements of $\mathcal{R}_{m,\xi}$ are trivial in $\overline{\mathrm{BS}}(m,\xi)$: indeed, if w fixes the favoured vertex H_n in the Bass–Serre tree of $\mathrm{BS}(m,\xi_n)$, then $w\bar{w}$ is trivial in $\mathrm{BS}(m,\xi_n)$. This gives a marked (hence surjective) homomorphism $\Gamma \to \overline{\mathrm{BS}}(m,\xi)$. Theorem 4.4 is thus reduced to the following proposition, which gives the injectivity.

Proposition 4.5. *Let w be a word on the alphabet $\{a, a^{-1}, b, b^{-1}\}$. If one has $w = 1$ in $\overline{\mathrm{BS}}(m,\xi)$, then the equality $w = 1$ also holds in Γ.*

Before proving Proposition 4.5, we need to introduce some notions which admit geometric interpretations.

From words to paths. Let us call *path* (in a graph) any finite sequence of vertices such that each of them is adjacent to the preceding one. Let w be any word on the alphabet $\{a, a^{-1}, b, b^{-1}\}$. It defines canonically a path in the Cayley graph of Γ (or $\overline{\mathrm{BS}}(m,\xi)$) which starts at the trivial vertex. Let us denote those paths by $p_\Gamma(w)$ and $p_{\overline{\mathrm{BS}}}(w)$. The map $\Gamma \to \overline{\mathrm{BS}}(m,\xi)$ defines a graph morphism which sends the path $p_\Gamma(w)$ onto the path $p_{\overline{\mathrm{BS}}}(w)$.

The word w defines the same way a finite sequence of vertices in $X_{m,\xi}$ starting at v_0 and such that each of them is equal or adjacent to the preceding one. Indeed, let f be the map $V(\mathrm{Cay}(\overline{\mathrm{BS}}(m,\xi),(a,b))) \to V(X_{m,\xi})$ defined by $f(g) = g \cdot v_0$ for any $g \in \overline{\mathrm{BS}}(m,\xi)$. If g, g' are adjacent vertices in $\mathrm{Cay}(\overline{\mathrm{BS}}(m,\xi),(a,b))$, then $f(g)$ and $f(g')$ are either adjacent (case $g' = ga^{\pm 1}$), or equal (case $g' = gb^{\pm 1}$).

The sequence associated to w is the image by f of the path $p_{\overline{\mathrm{BS}}}(w)$. Now, deleting consecutive repetitions in this sequence, we obtain a path that we denote by $p_X(w)$.

It follows that if the word w satisfies $w = 1$ in $\overline{\mathrm{BS}}(m,\xi)$ (or, stronger, $w = 1$ in Γ), then the path $p_X(w)$ is closed (i.e., its last vertex is v_0).

Height and valleys. Recall that one has a homomorphism σ_a from $\overline{\mathrm{BS}}(m,\xi)$ onto $\mathbb{Z}$ given by $\sigma_a(a) = 1$ and $\sigma_a(b) = 0$. Given a vertex v in $X_{m,\xi}$, we call *height* of v the number $h(v) = \sigma_a(g)$ where g is any element of $\overline{\mathrm{BS}}(m,\xi)$ such that $g \cdot v_0 = v$. It is easy to check that any element g' of $\overline{\mathrm{BS}}(m,\xi)$ which defines an elliptic automorphism of $X_{m,\xi}$ satisfies $\sigma_a(g') = 0$, so that the height function is well defined. It is clear from construction that the height difference between two adjacent vertices is 1.

Given $L \geqslant 1$ and $k \geqslant 1$, we call (L,k)-*valley* any path p in $X_{m,\xi}$ such that one has:

- $p = (v_0, v_1, \ldots, v_L = \nu_0, \nu_1, \ldots, \nu_{2k})$, where v_0 is the favoured vertex;
- $h(v_0) = 0 = h(\nu_k)$ and $h(v_0) = -k = h(\nu_{2k})$;
- $h(v) < 0$ for any other vertex v of p;
- $\nu_0 = \nu_{2k}$.

Given an (L,k)-valley $p = (v_0, v_1, \ldots, v_L = \nu_0, \nu_1, \ldots, \nu_{2k})$, the subpaths $(\nu_0, \ldots, \nu_k)$ and $(\nu_k, \ldots, \nu_{2k} = \nu_0)$ have to be geodesic, for the height difference between $\nu_0 = \nu_{2k}$ and ν_k is k. Thus, one has $\nu_1 = \nu_{2k-1}, \ldots, \nu_{k-1} = \nu_{k+1}$.

Lemma 4.6. *Let w be a word on the alphabet $\{a, a^{-1}, b, b^{-1}\}$ such that the path $p_X(w)$ is an (L,k)-valley, say $p_X(w) = (v_0, v_1, \ldots, v_L = \nu_0, \nu_1, \ldots, \nu_{2k})$. There exists a word w' such that the equality $w' = w$ holds in Γ and the path $p_X(w')$ is $(v_0, v_1, \ldots, v_L)$.*

Proof. We argue by induction on L.

Case $L = 1$: Then one has $k = 1$, $p_X(w) = (v_0, v_1 = \nu_0, \nu_1, \nu_2)$. Up to replacing w by a word which defines the same element in $\mathbb{F}_2$ (hence in Γ) and the same path in X, we may assume to have $w = b^{\alpha_0}a^{-1}b^{\alpha_1}ab^{\beta_1}a^{-1}b^{\beta_2}$. Set $r = ab^{-\beta_1}a^{-1}b^{-\alpha_1}ab^{\beta_1}a^{-1}b^{\alpha_1}$. Since ν_0 and ν_2 are equal, the subword $ab^{\beta_1}a^{-1}$ (of w) defines a closed subpath in X, so that we obtain $ab^{-\beta_1}a^{-1}b^{-\alpha_1} \cdot v_0 = v_0$. Consequently, we get $r \in \mathcal{R}$, whence $r = 1$ in Γ. Inserting r next to the last position, we obtain

$$w \underset{\Gamma}{=} b^{\alpha_0+\beta_1}a^{-1}b^{\alpha_1+\beta_2} =: w'.$$

This equality also implies that the paths $p_X(w)$ and $p_X(w')$ have the same endpoint. Hence one has $p_X(w') = (v_0, \nu_2) = (v_0, v_1)$ and we are done.

Induction step: We assume $L > 1$ to hold. Up to replacing w by a word which defines the same element in $\mathbb{F}_2$ (hence in Γ) and the same path in X, we may write

$$w = b^{\alpha_0}a^{\varepsilon_1}b^{\alpha_1}\ldots a^{\varepsilon_L}b^{\alpha_L} \cdot ab^{\beta_1}\ldots ab^{\beta_k}a^{-1}b^{\beta_{k+1}}\ldots a^{-1}b^{\beta_{2k}}$$

with $\varepsilon_i = \pm 1$ and $\alpha_i \in \mathbb{Z}$ for all i. We distinguish two cases:

(1) the vertex v_{L-1} is higher than v_L (i.e. $\varepsilon_L = -1$);

(2) the vertex v_{L-1} is lower than v_L (i.e., $\varepsilon_L = 1$).

Case (1): Set

$$z = ab^{-\beta_{2k-1}} \dots ab^{-\beta_k} a^{-1} b^{-\beta_{k-1}} \dots a^{-1} b^{-\beta_1} a^{-1} b^{-\alpha_L} \quad \text{and} \quad r = z\bar{z}.$$

Since v_0 and v_{2k} are equal, the subword $ab^{\beta_1} \dots ab^{\beta_k} a^{-1} b^{\beta_{k+1}} \dots a^{-1} b^{\beta_{2k-1}} a^{-1}$ (of w) defines a closed subpath in X, so that (when considered as a word on its own right) it stabilizes the vertex v_0. Inverting it, we get

$$ab^{-\beta_{2k-1}} \dots ab^{-\beta_k} a^{-1} b^{-\beta_{k-1}} \dots a^{-1} b^{-\beta_1} a^{-1} \cdot v_0 = v_0.$$

It implies $r \in \mathcal{R}$, whence $r = 1$ in Γ. Inserting r in next to last position, we obtain

$$\begin{aligned} w =_\Gamma w^* := {} & b^{\alpha_0} a^{\varepsilon_1} b^{\alpha_1} \dots a^{\varepsilon_{L-1}} b^{\alpha_{L-1}+\beta_{2k-1}} \\ & \cdot ab^{\beta_{2k-2}} \dots ab^{\beta_k} a^{-1} b^{\beta_{k-1}} \dots a^{-1} b^{\beta_1} \cdot a^{-1} b^{\alpha_L+\beta_{2k}}. \end{aligned}$$

We write $w^* = w'' a^{-1} b^{\alpha_L+\beta_{2k}}$. Since the words w and w^* begin the same way and since one has $w = w^*$ in Γ, the path $p_X(w^*)$ has the form

$$(v_0, v_1, \dots, v_{L-1} = \omega_0, \omega_1, \dots, \omega_{2k-2}, v_L).$$

Since v_{L-1} is higher than v_L, we have $h(v_{L-1}) = -(k-1)$. Contemplating w^*, one sees that we have $h(\omega_{k-1}) = 0$, $h(\omega_{2k-2}) = -(k-1)$, so that the subpaths $(\omega_0, \dots, \omega_{k-1})$ and $(\omega_{k-1}, \dots, \omega_{2k-2}, v_L)$ are geodesic. On the other hand, the geodesic between ω_{k-1} and v_L passes through $v_{L-1} = \omega_0$, so $\omega_{2k-2} = v_{L-1}$. It follows that $p_X(w'')$ has the form $(v_0, v_1, \dots, v_{L-1} = \omega_0, \omega_1, \dots, \omega_{2k-2} = v_{L-1})$, thus it is an $(L-1, k-1)$-valley. We apply the induction hypothesis to w'' and get a word w''' such that $w'' = w'''$ holds in Γ and $p_X(w''') = (v_0, \dots, v_{L-1})$. It suffices to set $w' = w''' a^{-1} b^{\alpha_L+\beta_{2k}}$ to conclude.

Case (2): In this case, we have $h(v_L) = -k$ and $h(v_{L-1}) = -(k+1)$, so that the edge linking these vertices goes from v_L to v_{L-1}. By Proposition 3.10, there exists $\lambda \in \{0, \dots, |m|-1\}$ such that $wb^\lambda a^{-1} \cdot v_0 = v_{L-1}$. Set $w^* = wb^\lambda a^{-1}$ and $\tilde{w} = w^* a b^{-\lambda}$, and so $w = \tilde{w}$ holds in Γ. The path $p_X(w^*)$ is an $(L-1, k+1)$-valley, so that we may apply the induction hypothesis to w^*. This gives a word w'' such that $w'' = w^*$ in Γ and $p_X(w'') = (v_0, \dots, v_{L-1})$. We conclude by setting $w' = w'' a b^{-\lambda}$. □

Proof of Proposition 4.5. Let w be a word which defines the trivial element in $\overline{\mathrm{BS}}(m, \xi)$. The path $p_X(w)$ is closed; we denote its length by ℓ, which is equal to the number of letters $a^{\pm 1}$ occurring in w. We argue by induction on ℓ.

Case $\ell = 0$: In this case, w has no $a^{\pm 1}$ letter; thus, one has $w = b^{\alpha_0}$ in $\mathbb{F}_2$, so that $b^{\alpha_0} = 1$ in $\overline{\mathrm{BS}}(m, \xi)$. It follows $\alpha_0 = 0$, hence $w = 1$ in Γ.

Induction step ($\ell > 0$): Let us write $p_X(w) = (v_0, v_1, \ldots, v_{\ell-1}, v_\ell = v_0)$. If w is not freely cyclically reduced, let w_r denote the word obtained by freely cyclically reducing w. Then either some letter a or a^{-1} has been deleted, in which case $p_X(w_r)$ is strictly shorter than $p_X(w)$ and we may apply the induction hypothesis to get $w = w_r = 1$ in Γ, or $p_X(w_r)$ has length ℓ and it is equivalent to deal with w_r instead of w.

Hence, we may assume w to be freely cyclically reduced and write

$$w = b^{\alpha_0} a^{\varepsilon_1} b^{\alpha_1} \ldots a^{\varepsilon_\ell} b^{\alpha_\ell}$$

with $\varepsilon_i = \pm 1$ and $\alpha_i \in \mathbb{Z}$ for all i. We may moreover assume $h(v_i) \leqslant 0$ for all i: indeed, if the vertex v_k has maximal height (among the v_i), then it suffices to replace w by the free reduction of the conjugate $x^{-1} w x$, where x is the prefix $b^{\alpha_0} \ldots b^{\alpha_{k-1}}$ of w. Note that this operation preserves the length of $p_X(w)$ and the fact that w is freely cyclically reduced.

Denote by $k_0 = 0 < k_1 < \cdots < k_s = \ell$ the indices k' such that $h(v_{k'}) = 0$. We now distinguish two possibilities:

(1) the path $p_X(w)$ backtracks at some v_{k_i} (i.e., there exists i with $1 \leqslant i \leqslant s-1$ such that $v_{k_i+1} = v_{k_i-1}$),

(2) the path $p_X(w)$ does not backtrack at any v_{k_i} (i.e., for all i with $1 \leqslant i \leqslant s-1$ one has $v_{k_i+1} \neq v_{k_i-1}$).

Case (1): Let i be an index (with $1 \leqslant i \leqslant s-1$) such that $v_{k_i+1} = v_{k_i-1}$. The subword $w' = a^{\varepsilon_{k_i-1+1}} b^{\alpha_{k_i-1+1}} \ldots a^{\varepsilon_{k_i}} b^{\alpha_{k_i}} a^{\varepsilon_{k_i+1}}$ defines by construction a $(k_i - k_{i-1} - 1, 1)$-valley in the tree X. Lemma 4.6 furnishes then a word w'' such that $w'' = w'$ holds in Γ and the path $p_X(w'')$ is strictly shorter than $p_X(w')$. We construct a word w^* by replacing w' by w'' in w. The path $p_X(w^*)$ is strictly shorter than $p_X(w)$ and one has $w^* = w$ in Γ. Applying the induction hypothesis to w^*, we get $w^* = 1$ in Γ, hence $w = 1$ in Γ.

Case (2): Let us recall that for all $n \in \mathbb{Z}^*$, the following diagram of marked morphisms is commutative (see Section 1.3 and Proposition 3.2).

$$\begin{array}{ccccc} \mathbb{F}_2 & \twoheadrightarrow & \Gamma & \twoheadrightarrow & \overline{\mathrm{BS}}(m,\xi) \longrightarrow \mathbb{Z} \wr \mathbb{Z} \\ \downarrow & & & & \downarrow \\ \mathrm{BS}(m,n) & & \longrightarrow & & \Gamma(m,n) = \mathbb{Z} \ltimes_{\frac{n}{m}} \mathbb{Z}\left[\frac{\gcd(m,n)}{\mathrm{lcm}(m,n)}\right] \end{array}$$

The image of w in $\mathbb{Z} \wr \mathbb{Z}$ is $\left(\sum_{i=1}^{\ell} \varepsilon_i, \sum_{i=0}^{\ell} \alpha_i t^{h(v_i)}\right)$. Since $w = 1$ in $\overline{\mathrm{BS}}(m, \xi)$, this implies that $\sum_{i=1}^{\ell} \varepsilon_i = 0$ and $\sum_{i=0}^{\ell} \alpha_i t^{h(v_i)} = 0$.

Suppose first (by contradiction) that $s = 1$. All vertices of $p_X(w)$ but v_0 and v_ℓ have strictly negative height. The image of w in $\mathbb{Z} \wr \mathbb{Z}$ has then the form $(0, \alpha_0 + \alpha_\ell + \sum_{j<0} \beta_j t^j)$, so that $\alpha_0 = -\alpha_\ell$. Since $h(v_1) = -1 = h(v_{\ell-1})$, we also have $\varepsilon_1 = -1$ and $\varepsilon_\ell = 1$. We thus see that w is not freely cyclically reduced, a contradiction.

Hence, we have $s \geqslant 2$. There exists some i in $\{0, \dots, s-1\}$ such that $v_{k_i} = v_{k_{i+1}}$: either one has $v_{k_i} = v_0$ for all i, or we take i such that v_{k_i} is farthest from v_0. We set

$$w^* := a^{\varepsilon_{k_i+1}} b^{\alpha_{k_i+1}} \dots a^{\varepsilon_{k_{i+1}-1}} b^{\alpha_{k_{i+1}-1}} a^{\varepsilon_{k_i+1}} .$$

This is a subword of w, not containing all $a^{\pm 1}$ letters since $s \geqslant 2$. Moreover, the path $p_X(w^*)$ has the form $(v_0^* = v_0, v_1^*, \dots, v_{\ell^*-1}^*, v_{\ell^*}^* = v_0)$. Let us now fix a sequence $(\xi_n)_n$ of non-zero rational integers such that $|\xi_n| \to \infty$ and $\xi_n \to \xi$ in $\mathbb{Z}_m$. By Theorem 3.5, we may assume the tree X to be constructed from the sequence $(\xi_n)_n$. The word w^* stabilizing v_0, we get that $w^* = b^{\lambda_n}$ in $\mathrm{BS}(m, \xi_n)$ for some $\lambda_n \in \mathbb{Z}$, which implies that $w^* = b^{\lambda_n}$ in $\Gamma(m, \xi_n)$ for n large enough. Now the above diagram implies that

$$\lambda_n = \sum_{j=1}^{\ell^*-1} \alpha_{k_i+j} \left(\frac{\xi_n}{m}\right)^{h(v_j^*)}$$

for those values of n. Then, taking absolute values, this gives

$$|\lambda_n| \leqslant \sum_{j=1}^{\ell^*-1} \left|\alpha_{k_i+j} \left(\frac{\xi_n}{m}\right)^{h(v_j^*)}\right| \leqslant \frac{|m|}{|\xi_n|} \sum_{j=1}^{\ell^*-1} |\alpha_{k_i+j}|.$$

It follows that $|\lambda_n| < 1$ holds for n large enough, since $|\xi_n|$ tends to ∞. For those values of n, we get $w^* = b^0 = 1$ in $\mathrm{BS}(m, \xi_n)$. Consequently, we get $w^* = 1$ in $\overline{\mathrm{BS}}(m, \xi)$.

Since w^* does not contain all $a^{\pm 1}$ letters of w, the path $p_X(w^*)$ is strictly shorter than $p_X(w)$, so that we apply the induction hypothesis to w^* and get $w^* = 1$ in Γ. Erasing w^* in w, we obtain that

$$w =_\Gamma w' = b^{\alpha_0} a^{\varepsilon_1} b^{\alpha_1} \dots a^{\varepsilon_{k_i}} b^{\alpha_{k_i}+\alpha_{k_i+1}} a^{\varepsilon_{k_{i+1}+1}} b^{\alpha_{k_{i+1}+1}} \dots a^{\varepsilon_\ell} b^{\alpha_\ell}.$$

Applying the induction hypothesis to w', we get $w = 1$ in Γ. □

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