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Yoann Gelineau, Heesung Shin, Jiang Zeng

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BIJECTIONS FOR ENTRINGER FAMILIES

YOANN GELINEAU, HEESUNG SHIN, AND JIANG ZENG

ABSTRACT. André proved that the number of alternating permutations on $\{1, 2, \dots, n\}$ is equal to the Euler number E_n . A refinement of André's result was given by Entringer, who proved that counting alternating permutations according to the first element gives rise to Seidel's triangle $(E_{n,k})$ for computing the Euler numbers. In a series of papers, using generating function method and induction, Poupard gave several further combinatorial interpretations for $E_{n,k}$ both in alternating permutations and increasing trees. Kuznetsov, Pak, and Postnikov have given more combinatorial interpretations of $E_{n,k}$ in the model of trees. The aim of this paper is to provide bijections between the different models for $E_{n,k}$ as well as some new interpretations. In particular, we give the first explicit one-to-one correspondence between Entringer's alternating permutation model and Poupard's increasing tree model.

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1. INTRODUCTION

The Euler numbers E_n are defined by the generating function

$$\begin{aligned} \sum_{n \geq 0} E_n \frac{x^n}{n!} &= \tan(x) + \sec(x) \\ &= 1 + x + \frac{x^2}{2!} + 2 \frac{x^3}{3!} + 5 \frac{x^4}{4!} + 16 \frac{x^5}{5!} + 61 \frac{x^6}{6!} + 272 \frac{x^7}{7!} + 1385 \frac{x^8}{8!} + \dots \end{aligned}$$

Let \mathcal{A}_n be the set of *alternating permutations* of $[n] := \{1, 2, \dots, n\}$, that is, the permutations $\pi = \pi_1 \pi_2 \dots \pi_n$ on $[n]$ satisfying $\pi_1 > \pi_2 < \pi_3 > \pi_4 < \dots$, in an alternating way. For example, the alternating permutations of $[4]$ are:

$$2143, \quad 3241, \quad 3142, \quad 4231, \quad 4132.$$

André [And79] proved that E_n is the cardinality of the set \mathcal{A}_n . A recent survey on alternating permutations and Euler numbers is given by Stanley [Sta09].

The *Entringer numbers*[Ent66] were introduced to enumerate the alternating permutations according to the first term. More precisely, let $\mathcal{A}_{n,k}$ be the set of permutations $\pi \in \mathcal{A}_n$ such that $\pi_1 = k$ and $E_{n,k}$ the cardinality of $\mathcal{A}_{n,k}$. The first values of $E_{n,k}$ are given in Table 1.

$n \setminus k$	1	2	3	4	5	6	7
1	1						
2	0	1					
3	0	1	1				
4	0	1	2	2			
5	0	2	4	5	5		
6	0	5	10	14	16	16	
7	0	16	32	46	56	61	61

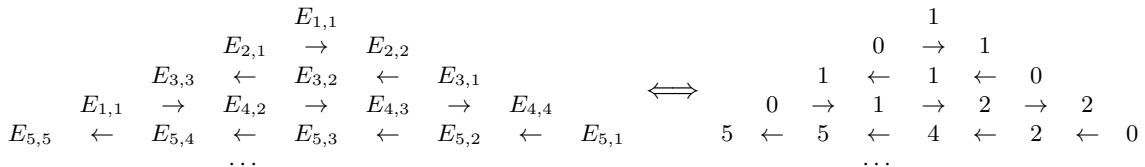
TABLE 1. The first values of Entringer numbers $E_{n,k}$

Theorem 1.1 (Entringer). *The numbers $(E_{n,k})$ ($n \geq k \geq 1$) satisfy*

$$E_{1,1} = 1, \quad E_{n,1} = 0 \ (n \geq 2), \quad E_{n,k} = E_{n,k-1} + E_{n-1,n-k+1}. \tag{1}$$

Iterating the above recurrence, we get $E_{n+1,n+1} = E_{n,n} + E_{n,n-1} + \dots + E_{n,1}$, which is equal to E_n by André’s result. Hence the Euler numbers $E_n = E_{n+1,n+1}$ are the diagonal entries in Table 1.

If we display the Entringer numbers $E_{n,k}$ in a suitable way as follows,



then the recurrence relation (1) leads to Seidel’s scheme [Sei77] for computing Euler numbers, which was rediscovered later by Kempner [Kem33].

A sequence of sets $(X_{n,k})_{1 \leq k \leq n}$ is called an *Entringer family* if for $1 \leq k \leq n$, the cardinality of $X_{n,k}$ is equal to $E_{n,k}$.

Let $X = \{x_1, \dots, x_n\}_<$ be an ordered set such that $x_1 < \dots < x_n$. An *increasing tree* on X is a spanning tree of the complete graph on X , rooted at x_1 and oriented from the smallest vertex x_1 , such that the vertices increase along the edges. Let \mathcal{T}_n be the set of *0-1-2 increasing trees* T on $[n]$, i.e. the increasing trees such that at most two edges go out from every vertex (see Figure 1).

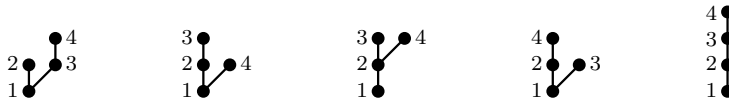


FIGURE 1. The 0-1-2 trees on $[4]$

Foata and Schützenberger proved in [FS73, §5] that the Euler number E_n is the cardinality of \mathcal{T}_n . A one-to-one correspondance between \mathcal{A}_n and \mathcal{T}_n was soon constructed by Donaghey [Don75] (see also [Cal05]). However the tree counterpart of Entringer’s result was found only in 1982 by Poupard [Pou82]. If T is a 0-1-2 increasing tree and if (i, j) is an edge in T , $i < j$, we call i the *father* of j , and j a *child* of i . If i has no child, we say that i is an *endpoint* of T . A *path* in T is a sequence of vertices (a_i) such that a_i is a child of a_{i-1} in T , and the *main path* of T is the path $(a_i)_{1 \leq i \leq \ell}$ such that $a_1 = 1$, a_i ($i = 2 \dots \ell$) is the smallest child of a_{i-1} and a_ℓ is an endpoint, denoted by $p(T)$. Let’s denote by $\mathcal{T}_{n,k}$ the set of trees $T \in \mathcal{T}_n$ such that $p(T) = k$.

Theorem 1.2 (Poupard). *The sequence $(\mathcal{T}_{n,k})_{1 \leq k \leq n}$ is an Entringer family.*

Note that contrary to the case of alternating permutations, it is not easy to interpret identity (1) in the model of 0-1-2 increasing trees. Indeed, Donaghey’s bijection doesn’t induce a bijection between $\mathcal{A}_{n,k}$ and $\mathcal{T}_{n,k}$ and Poupard’s proof in [Pou82] was analytic in nature. Finding a direct explanation in the model of trees was then raised as an open problem in [KPP94]. The first aim of this paper is to build a bijection between $\mathcal{A}_{n,k}$ and $\mathcal{T}_{n,k}$ and answer the above open problem.

Theorem 1.3. *For all $n \geq 1$ and $k \in [n]$, there is an explicit bijection $\Psi : \mathcal{A}_{n,k} \rightarrow \mathcal{T}_{n,k}$ satisfying*

$$\forall \pi \in \mathcal{A}_{n,k}, \quad \pi_1 = p(\Psi(\pi)).$$

Poupard [Pou82, Pou97] gave also other interpretations for Entringer numbers $E_{n,k}$ (see Section 4) in increasing trees and alternating permutations with induction proofs. Our second aim is to provide simple bijections between the other interpretations of Poupard in alternating permutations and the original interpretation in $\mathcal{A}_{n,k}$. Note that some other interpretations of Entringer numbers $E_{n,k}$ in the model of increasing trees were given in [KPP94]. Recently, two new interpretations of Euler numbers were given by Martin and Wagner [MW09] in the model of G-words and R-words. We shall give the corresponding interpretations of the Entringer number $E_{n,k}$ in the later models.

The rest of this paper is organized as follows. In Section 2, we introduce an intermediate model $\mathcal{D}_{n,k}$ and present a bijection ψ between $\mathcal{A}_{n,k}$ and $\mathcal{D}_{n,k}$. In Section 3, we describe a bijection φ between $\mathcal{D}_{n,k}$ and $\mathcal{T}_{n,k}$ so that $\Psi = \varphi \circ \psi$ provides the bijection for Theorem 1.3. As an application, in Subsection 3.2, we give a direct interpretation of (1) in the model of increasing trees. In Section 4, we recall the other interpretations of $E_{n,k}$ found by Poupard

and establish simple bijections between these models. In Section 5, we give some new interpretations for $E_{n,k}$, first refining the results of Martin and Wagner [MW09] in their model of G-words and R-words, and secondly introducing the new model of U-words.

2. THE LEFT-TO-RIGHT CODING ψ OF ALTERNATING PERMUTATIONS

For $n \geq 2$, let \mathbb{A}_n be the set consisting of ordered pairs (j, i) ($1 \leq i < j \leq n$), starred ordered pairs $(j, i)^*$ ($1 \leq i < j \leq n$), and the element $(n)^* = (n, n)^*$:

$$\mathbb{A}_n = \{(j, i), (j, i)^*, (n)^* : 1 \leq i < j \leq n\}.$$

Definition 2.1. A sequence (Δ_k) of elements in \mathbb{A}_n is an *encoding sequence of $[n]$* if the following conditions are verified:

- (i) if $\Delta_k = (j, i)^*$ and $\Delta_{k+1} \in \{(m, n), (m, n)^*\}$, then $i < m$,
- (ii) if $\Delta_k = (j, i)$ and $\Delta_{k+1} \in \{(m, n), (m, n)^*\}$, then $i = m$,
- (iii) if $\Delta_k = (j, i)^*$, then the integers i and j don't appear in every Δ_ℓ , for $\ell > k$,
- (iv) if $\Delta_k = (j, i)$, then the integers ℓ such that $i < \ell < j$ appear in an element Δ_ℓ , for $\ell > k$, with $\Delta_\ell \in \{(j, i)^* : 1 \leq i < j \leq n\}$.

We denote by \mathcal{D}_n the set of encoding sequences of $[n]$, and by $\mathcal{D}_{n,k}$ the subset of \mathcal{D}_n consisting of encoding sequences starting with an element in $\{(k, i), (k, i)^*, 1 \leq i < k - 1\}$.

For example, the elements in \mathcal{D}_4 are: $((2, 1)^*, (4, 3)^*), ((3, 2)^*, (4, 1)^*)$ and

$$((3, 2), (2, 1)^*, (4, 3)^*), ((4, 3), (3, 2)^*, (4, 1)^*), ((4, 3), (3, 2), (2, 1)^*, (4, 3)^*).$$

Let π be an alternating permutation on an ordered set $I = \{a_1, a_2, \dots, a_n\}_<$ such that $\pi_1 > \pi_2 < \pi_3 > \pi_4 < \dots$ (that can be interpreted by an element of \mathcal{A}_n). Suppose $\pi_1 = a_k$ and $\pi_2 = a_j$ with $a_k > a_j$. If we apply successively the elementary transpositions $(a_k, a_{k-1}), (a_{k-1}, a_{k-2}), \dots, (a_{j+2}, a_{j+1})$ to π :

$$\begin{aligned} \pi^{(1)} &= (a_k, a_{k-1}) \circ \pi, \\ \pi^{(2)} &= (a_{k-1}, a_{k-2}) \circ \pi^{(1)}, \\ &\vdots \\ \pi^{(k-j-1)} &= (a_{j+2}, a_{j+1}) \circ \pi^{(k-j-2)}, \end{aligned}$$

then we obtain a permutation starting with $a_{j+1}a_j$ (from left-to-right). Clearly all the permutations $\pi^{(1)}, \dots, \pi^{(k-j-1)}$ are alternating permutations. Deleting the first two elements in $\pi^{(k-j-1)}$, noted as $(a_{j+1}, a_j)^*$, we get an alternating permutation, say $\pi^{(k-j)}$, on $I \setminus \{a_{j+1}, a_j\}$. So we can apply the same procedure to the resulted permutation, and iterate this algorithm until we obtain the empty permutation. Clearly the last deletion is $(n)^*$ if n is odd. In others words, we can write the algorithm as follows:

- (1) Start with $\Delta(\pi) = \emptyset$ and I
- (2) While $\text{Card}(A) \geq 2$, do:
 - (a) While there is no i such that $(\pi_1, \pi_2) \neq (a_{i+1}, a_i)$, do:
$$\begin{aligned} \Delta(\pi) &\leftarrow (\Delta(\pi), (a_{i+1}, a_i)), \\ \pi &\leftarrow (a_i, a_{i+1}) \circ \pi. \end{aligned}$$
 - (b) If there is i such that $(\pi_1, \pi_2) = (a_{i+1}, a_i)$, do:
$$\Delta(\pi) \leftarrow (\Delta(\pi), (a_{i+1}, a_i)^*),$$

- $$\pi \leftarrow \pi_3 \pi_4 \dots \pi_n \text{ (eventually } \pi = \emptyset),$$
- $$I \leftarrow I \setminus \{a_i, a_{i+1}\}.$$
- (3) If $\text{Card}(I) = 1$ with $I = \{a_n\}$, do:
- $$D_\pi \leftarrow (\Delta(\pi), (a_n)^*),$$
- $$\pi \leftarrow \emptyset,$$
- $$I \leftarrow \emptyset.$$

We call the sequence of the successive operations in the above algorithm the LR-code of π , which is a sequence $\psi(\pi) = \Delta(\pi)$ made of

- the transpositions (j, i) , $1 \leq i < j \leq n$,
- the deletions $(j, i)^*$, $1 \leq i < j \leq n$ (or the deletion $(n)^*$).

Example 2.2. Let's take $\pi = 748591623 \in \mathcal{A}_{9,7}$.

Step	π	$\Delta(\pi)_i$
0	748591623	\emptyset
1	648591723	$(7, 6)$
2	548691723	$(6, 5)$
3	8691723	$(5, 4)^*$
4	7691823	$(8, 7)$
5	91823	$(7, 6)^*$
6	81923	$(9, 8)$
7	31928	$(8, 3)$
8	21938	$(3, 2)$
9	938	$(2, 1)^*$
10	839	$(9, 8)$
11	9	$(8, 3)^*$
12	\emptyset	$(9)^*$

Thus, the LR-code of π is

$$\Delta(\pi) = ((7, 6), (6, 5), (5, 4)^*, (8, 7), (7, 6)^*, (9, 8), (8, 3), (3, 2), (2, 1)^*, (9, 8), (8, 3)^*, (9)^*).$$

Theorem 2.3. For all $n \geq 1$ and $k \in [n]$, the LR-coding mapping ψ is a bijection between $\mathcal{A}_{n,k}$ and $\mathcal{D}_{n,k}$.

Proof. It suffices to show that the sequence $\psi(\pi)$ verifies the points (i)-(iv) of Definition 2.1.

- When $\Delta_k = (j, i)^*$, we have $\pi_1 = j$ and $\pi_2 = i$. Since π is alternating, we know that $m = \pi_3 > i$. The first operation that follows is either a transposition (m, n) , or a deletion $(m, n)^*$, thus (i) is verified.
- When $\Delta_k = (j, i)$, we are in the while loop of the algorithm, and π_1 becomes i . Either the loop is not over, and then i is changed with a transposition (i, k) , or the loop is over, and then i is deleted with a deletion $(i, k)^*$. Thus (ii) is verified.
- When $\Delta_k = (j, i)^*$, in the algorithm, i and j are erased from I . Then, they don't appear in any operation later, thus (iii) is verified.
- When $\Delta_k = (j, i)$, we are in the while loop of the algorithm, and i and j must be consecutive elements in I . Then, the integers ℓ such that $i < \ell < j$ must have been erased from I before, thus (iv) is verified.

It results that $\psi(\pi) \in \mathcal{D}_n$. Moreover, $\psi(\pi)$ starts with (π_1, j) or $(\pi_1, j)^*$ with $j < \pi_1$, so $\psi(\pi) \in \mathcal{D}_{n,k}$. The process is clearly invertible so ψ is a bijection between $\mathcal{A}_{n,k}$ and $\mathcal{D}_{n,k}$. ■

The previous theorem yields immediately a new interpretation for Entringer numbers.

Corollary 2.4. *The sequence $(\mathcal{D}_{n,k})_{1 \leq k \leq n}$ is an Entringer family.*

Remark 2.5. Denote the largest integer less than x by $\lfloor x \rfloor$ and the number of ordered pairs $(i, j) \in \{1, \dots, n\}$ such that $i + 1 < j$ and $\pi_i > \pi_{i+1} < \pi_j < \pi_i$ by $31-2(\pi)$. Then, one can show that the length of the sequence $\psi(\pi)$ is equal to

$$31-2(\pi) + \left\lfloor \frac{n+1}{2} \right\rfloor.$$

Indeed, $31-2(\pi)$ corresponds to the number of occurrences of terms (j, i) , $j > i$, in $\psi(\pi)$, and there are $\left\lfloor \frac{n+1}{2} \right\rfloor$ occurrences of terms $(j, i)^*$, $j > i$, in $\psi(\pi)$. Note that various formulae for counting 31-2-patterns in alternating permutations are given in [Che08, JV10, SZ10].

Remark 2.6. Let $n \geq 2$ and $k \geq 2$. An element $\Delta \in \mathcal{D}_{n,k}$ begins either with $(k, k-1)$ or $(k, k-1)^*$.

If $\Delta_1 = (k, k-1)$, then, the remaining sequence $(\Delta_2, \Delta_3, \dots)$ is still an encoding sequence of $[n]$, starting with $\Delta_2 \in \{(k-1, i), (k-1, i), 1 \leq i \leq k-2\}$. Thus, there are $E_{n,k-1}$ encoding sequences starting by $(k, k-1)$.

If $\Delta_1 = (k, k-1)^*$, then, the remaining sequence $(\Delta_2, \Delta_3, \dots)$ doesn't contain the elements k and $k+1$ and starts with an element in $\{(i, j), (i, j)^*, 1 \leq j \leq i-1\}$ with $i \geq k+1$. In other words, this is an encoding sequence of $n-2$ elements, starting with an integer i that must be greater than the $k-2$ first elements. Thus, there are $E_{n-2,k-1} + E_{n-2,k} + \dots + E_{n-2,n-2} = E_{n-1,n-k+1}$ encoding sequences starting by $(k, k-1)^*$.

Finally, the decomposition according to the nature of the first element of $\Delta \in \mathcal{D}_{n,k}$ gives(1).

3. THE LEFT-TO-RIGHT CODING OF INCREASING TREES

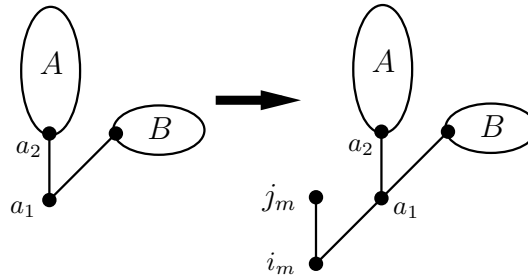
3.1. The bijection $\varphi : \mathcal{D}_{n,k} \rightarrow \mathcal{T}_{n,k}$. Starting from an encoding sequence $\Delta = (\Delta_i)_{1 \leq i \leq \ell} \in \mathcal{D}_{n,k}$, we construct a tree $T = \varphi(\Delta) \in \mathcal{T}_{n,k}$ by reading the sequence Δ in reverse order, i.e., from right to left. More precisely, for $m = \ell, \ell-1, \dots, 1$, we shall construct an increasing tree T_m corresponding to each sequence $(\Delta_m, \dots, \Delta_\ell)$ such that

$$\Delta_m = (j_m, i_m) \text{ or } (j_m, i_m)^* \implies p(T_m) = j_m, \quad (2)$$

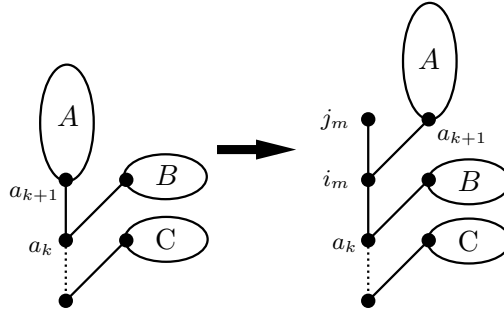
and define $T = T_1 := \varphi(\Delta)$. The algorithm goes as follows:

- If the last element of Δ is a singleton $(n)^*$, construct the tree T_ℓ with only one vertex n ; if the last element of Δ is a deletion $(n, i)^*$, construct the increasing tree T_ℓ with only one edge $i \rightarrow n$. Clearly (2) is verified. Now, suppose that we have constructed a tree T_{m-1} corresponding to the sequence $(\Delta_{m-1}, \dots, \Delta_\ell)$ with the property (2).
- $\Delta_m = (j_m, i_m)^*$ with $j_m > i_m$, we add i_m and j_m in the tree T_{m-1} to obtain T_m . Suppose that the main path of T_{m-1} is (a_1, \dots, a_{p_m}) .

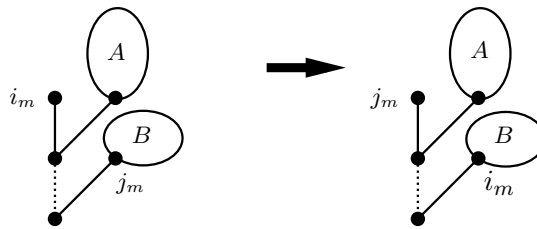
- If $i_m < a_1$, add the edges (i_m, a_1) and (i_m, j_m) to the tree T_{m-1} . Then, the tree T_m is an increasing tree rooted at i_m with (i_m, j_m) as the main path.



- If $i_m > a_1$, by induction hypothesis and property (ii) of encoding sequences, we see that $a_1 < m$. Hence, there exists $k \in \{1, \dots, p_m - 1\}$ such that $a_k < i_m < a_{k+1}$. Then, erase the edge (a_k, a_{k+1}) , create the edges (a_k, i_m) , (i_m, a_{k+1}) and (i_m, j_m) . Clearly, the tree T_m is an increasing tree with (i_m, j_m) as the last edge of the main path.

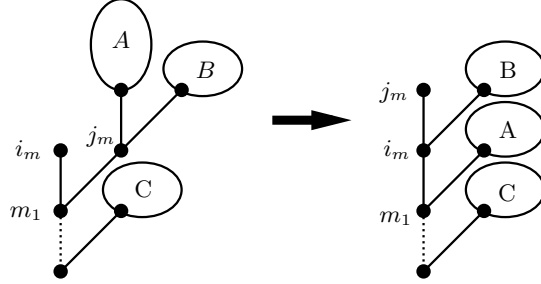


- $\Delta_m = (j_m, i_m)$ with $j_m > i_m$, by induction hypothesis and property (iii) of encoding sequences, we derive that i_m is at the end of the main path. Then, we transform the tree T_{m-1} according to the two following situations:
 - Case A : if i_m and j_m are not siblings in T , just exchange the places of i_m and j_m in T . The tree remains increasing because Then j_m is at the end of the main path in T .



- Case B : if i_m and j_m are siblings in T , transform T with the following procedure. If m_1 denotes the parent of i_m and j_m in T , erase the edge (m_1, j_m) , create an edge (i_m, j_m) , then if A and B are the two subtrees starting from j_m with $\min(A) < \min(B)$ (eventually B is empty), cut the subtree A from j_m and add

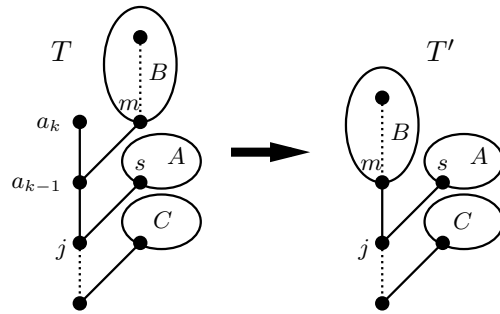
it as a direct subtree of m_1 , cut the subtree B from j_m and add it as a direct subtree of i_m . The procedure can be illustrated with the following picture:



Theorem 3.1. For all $n \geq 1$ and $k \in [n]$, the mapping $\varphi : \mathcal{D}_{n,k} \rightarrow \mathcal{T}_{n,k}$ is a bijection.

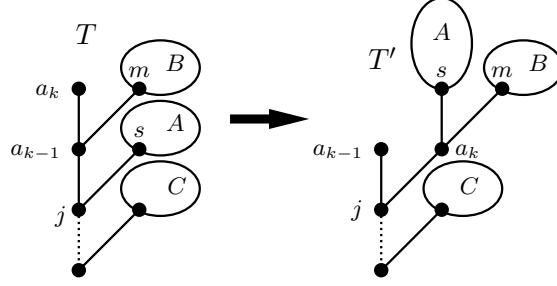
Proof. It is sufficient to construct the inverse mapping of φ to show that this is a bijection. Given T an increasing tree on the ordered set $\{a_1, \dots, a_n\}$ with $a_1 < \dots < a_n$, such that $p(T) = a_k$ (that can be interpreted by an element of $\mathcal{T}_{n,k}$), we construct an encoding sequence $\Delta = \varphi^{-1}(T)$ of $[n]$ recursively as follows:

- Case A.** If a_{k-1} is the father of a_k in T , then let m ($m > a_k$) be the other child of a_{k-1} ($m = \infty$ if a_k is the only child of a_{k-1}) and s ($s > k$) be a sibling of a_{k-1} ($s = \infty$ if a_{k-1} has no sibling), and j the father of a_{k-1} in T .
- Case A-1.** If $m < \infty$ and $m < s$, then define $\varphi^{-1}(T) = ((a_k, a_{k-1})^*, \varphi^{-1}(T'))$, where T' is the tree obtained from T by deleting the vertices a_{k-1} , a_k and their adjacent edges in T , and adding a new edge between m and j .

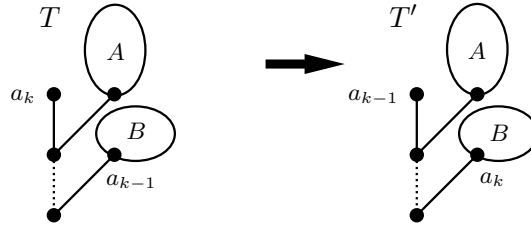


- Case A-2.** In the other cases ($m = \infty$ or $m > s$), then define $\varphi^{-1}(T) = ((a_k, a_{k-1}), \varphi^{-1}(T'))$, where T' is the tree obtained from T by erasing the edges (a_{k-1}, a_k) , (a_{k-1}, m) and (j, s) in T , and adding the edges (j, a_k) , (a_k, s) , (a_k, m) . The procedure can

be illustrated with the following picture:



Case B. If a_{k-1} is not the father of a_k in T , then define $\varphi^{-1}(T) = ((a_k, a_{k-1}), \varphi^{-1}(T'))$, where T' is the tree obtained from T by exchanging the labels a_{k-1} and a_k in T .



It remains to prove that the obtained sequence Δ verifies the points (i)-(iv) of Definition 2.1.

- If an element $(j, i)^*$ appears in Δ , that corresponds to the case A-1, when we delete the vertices i and j from the tree T . Then the next elements in Δ don't contain neither i nor j since they corresponds to $\varphi^{-1}(T')$. Moreover, if we are in the case A-1, the main path in the tree T' contains at least one element m with $m > j > i$, so the next element in Δ must be (m, k) with $m > k$. Thus (i) and (iii) are verified.
- If an element (j, i) appears in Δ , in both Case A-2 or Case B, the tree T' has i as endpoint of the main path. Then, the next element in Δ must be (i, k) with $i > k$. Moreover, i and j must be consecutive elements in the ordered set of labels in T . Then the elements ℓ such that $i < \ell < j$ don't appear in T . Thus (ii) and (iv) are verified. ■

Let $\Psi = \varphi \circ \psi$. Then $\Psi : \mathcal{A}_{n,k} \rightarrow \mathcal{T}_{n,k}$ is a bijection satisfying $\pi_1 = p(\Psi(\pi))$ for all $\pi \in \mathcal{A}_{n,k}$. Thus Theorem 1.3 is proved.

Example 3.2. Continuing the Example 2.2, we apply Ψ to π by using the known LR-code of $\pi = 748591623$. The details are given in Figure 2.

3.2. Interpretation of Entringer's formula (1) in \mathcal{T}_n . Following the interpretation of (1) in \mathcal{D}_n (cf Remark 2.6) and the bijection φ , we must consider the decomposition of the set $\mathcal{T}_{n,k}$ either the first step in the construction of φ^{-1} would consist in either removing the elements $k-1$ and k , either transform the tree to obtain another tree of \mathcal{T}_n .

For T be an element of $\mathcal{T}_{n,k}$, we say that the edge $(k-1, k)$ is *removable* if $k-1$ is the parent of k and if $k-1$ has another child m that is not greater than the sibling of $k-1$ (if such a sibling exists). For a visual representation, a tree T has its edge $(k-1, k)$ removable if it corresponds to the case A-1 in the proof of Theorem 3.1.

π	9	839	938	21938	31928	81923
π_1	9	8	9	2	3	8
$\Delta(\pi)_1$	$(9)^*$	$(8, 3)^*$	$(9, 8)$	$(2, 1)^*$	$(3, 2)$	$(8, 3)$
$\varphi \circ \psi(\pi)$						
$p(\varphi \circ \psi(\pi))$	9	8	9	2	3	8
π	91823	7691823	8691723	548691723	648591723	748591623
π_1	9	7	8	5	6	7
$\Delta(\pi)_1$	$(9, 8)$	$(7, 6)^*$	$(8, 7)$	$(5, 4)^*$	$(6, 5)$	$(7, 6)$
$\varphi \circ \psi(\pi)$						
$p(\varphi \circ \psi(\pi))$	9	7	8	5	6	7

FIGURE 2. The construction of the tree $\Psi(748591623)$

If the edge $(k-1, k)$ is not removable, the tree obtained after the first operation in the construction of φ^{-1} will be an increasing tree with n elements such that $k-1$ is the endpoint of the main chain. Then, there are exactly $E_{n, k-1}$ trees such that the edge $(k-1, k)$ is not removable.

If the edge is removable, the tree obtained with the first operation in the construction of φ^{-1} will be an increasing tree with $n-2$ elements (without the elements $k-1$ and k), and the end of the main path must be an element i greater than the $k-2$ first elements. Thus, there are $E_{n-2, k-1} + E_{n-2, k} + \dots + E_{n-2, n-2} = E_{n-1, n-k+1}$ increasing trees such that the edge $(k-1, k)$ is removable.

Finally, an interpretation of (1) appears in the model of \mathcal{T}_n . The decomposition according to the removability of the edge $(k-1, k)$ in $T \in \mathcal{T}_{n, k}$ gives (1).

4. POUPARD'S OTHER ENTRINGER FAMILIES

4.1. Another interpretation in increasing trees. Let $\mathcal{T}'_{n, k}$ be the set of trees $T \in \mathcal{T}_n$ such that the father of n in T is $k-1$. By using recurrence relations Poupard proved that $E_{n, k}$ is also the number of trees in $\mathcal{T}'_{n, k}$. A bijection φ' between $\mathcal{T}_{n, k}$ and $\mathcal{T}'_{n, k}$ was given in [KPP94, §6] for a more general class of increasing trees that they call geometric.

4.2. Another interpretation in alternating permutations. If π is a permutation of $\mathcal{A}_{n, k}$, define $\theta(\pi)$ as follows :

- if $k < n - k + 1 + \pi_2$, then $\theta(\pi) = (n - k + 1 + \pi_2, n - k + \pi_2, \dots, k + 1, k) \circ \pi$,
- if $k > n - k + 1 + \pi_2$, then $\theta(\pi) = (n - k + 1 + \pi_2, n - k + 2 + \pi_2, \dots, k - 1, k) \circ \pi$.

Since π is alternating, $\pi_2 < k = \pi_1$. If $k < n - k + 1 + \pi_2$, π_2 is unchanged by the cycle and then $\sigma(\pi)_2 = \pi_2$. Thus $\sigma(\pi)_2 < k < n - k + 1 + \pi_2 = \sigma(\pi)_1$ and $\theta(\pi)$ is still alternating.

If $k > n - k + 1 + \pi_2$, since $k \leq n$, then $n - k + 1 + \pi_2 \geq \pi_2 + 1$, so π_2 is unchanged by the cycle, $\sigma(\pi)_1 = n - k + 1 + \pi_2 > \pi_2 = \sigma(\pi)_2$ and $\theta(\pi)$ is still alternating.

Let's denote by $\mathcal{A}'_{n,k}$ the set of permutations $\pi \in \mathcal{A}_n$ such that $\pi_1 - \pi_2 = n + 1 - k$.

Theorem 4.1. *For all $n \geq 1$ and $k \in [n]$, the mapping θ is a bijection from $\mathcal{A}_{n,k}$ to $\mathcal{A}'_{n,k}$. Moreover, for every $\pi \in \mathcal{A}_{n,k}$, $\theta(\pi)_2 = \pi_2$.*

Proof. By construction, θ is clearly invertible. Moreover, if $\sigma \in \mathcal{A}_n$ with $\sigma_1 - \sigma_2 = n - k + 1$,

- if $k < n - k + 1 + \sigma_2$, then $\theta^{-1}(\sigma) = (k, k + 1, \dots, n - k + \sigma_2, n - k + 1 + \sigma_2) \circ \sigma$,
- if $k > n - k + 1 + \sigma_2$, then $\theta^{-1}(\sigma) = (k, k - 1, \dots, n - k + 2 + \sigma_2, n - k + 1 + \sigma_2) \circ \sigma$.

and then $\theta^{-1}(\sigma) \in \mathcal{A}_{n,k}$. \blacksquare

With Theorem 4.1, the following interpretation of Poupard, proved in [Pou97] by recurrence relations, can be recovered.

Corollary 4.2. *The sequence $(\mathcal{A}'_{n,k})_{1 \leq k \leq n}$ is an Entringer family.*

Note that $\mathcal{A}'_{n,k} \subset \mathcal{A}_n$. Then, define $\theta^2(\pi)$ for $\pi \in \mathcal{A}_n$. Actually, θ appears to be an involution on \mathcal{A}_n .

The result can also be generalized with the following observation. For any $\pi \in \mathcal{A}_n$, define the *complement permutation* $\bar{\pi}$ with $\bar{\pi}_i = n + 1 - \pi_i$ for $i \in [n]$. Denote by \mathcal{A}_n^* the set of permutations π such that $\bar{\pi} \in \mathcal{A}_n$.

Corollary 4.3. *For $n \geq 1$, we have*

$$\sum_{\pi \in \mathcal{A}_n^*} q^{\pi_1} p^{\pi_2 - \pi_1} = \sum_{\pi \in \mathcal{A}_n^*} p^{\pi_1} q^{\pi_2 - \pi_1}.$$

Proof. The mapping $\pi \mapsto \theta(\bar{\pi})$ is a bijection between $\{\pi \in \mathcal{A}_n^* : \pi_1 = k\}$ and $\{\pi \in \mathcal{A}_n^* : \pi_2 - \pi_1 = k\}$. Thus, the two statistics π_1 and $\pi_2 - \pi_1$ are equidistributed on \mathcal{A}_n^* . Indeed, with proof of Theorem 4.1, $\pi \mapsto \theta(\bar{\pi})$ is a bijection between $\{\pi \in \mathcal{A}_n^* : \pi_1 = k, \pi_2 - \pi_1 = \ell\}$ and $\{\pi \in \mathcal{A}_n^* : \pi_1 = \ell, \pi_2 - \pi_1 = k\}$. Thus, there is also a symmetry between both statistics. \blacksquare

4.3. Interpretations in direct alternating permutations. Let's introduce the set \mathcal{DAP}_n of *direct alternating permutations of $[n]$* , that is, the permutations π of $[n]$ such that $\pi_1^{-1} < \pi_n^{-1}$ and

$$\pi_1 > \pi_2 < \pi_3 > \dots \quad \text{or} \quad \pi_1 < \pi_2 > \pi_3 < \dots.$$

For example, the direct alternating permutations of $[4]$ are

$$1423, \quad 1324, \quad 3142, \quad 2314, \quad 2143.$$

Denote by $\mathcal{DAP}_{n,k}$ the set of $\pi \in \mathcal{DAP}_n$ such that $|\pi_1 - \pi_2| = n - k + 1$. The set $\mathcal{A}'_{n,k}$ can be split in two disjoint subsets $\mathcal{A}'_{n,k,1n}$ which is the set of permutations in $\mathcal{A}'_{n,k} \cap \mathcal{DAP}_n$ and $\mathcal{A}'_{n,k,n1} = \mathcal{A}'_{n,k} \setminus \mathcal{A}'_{n,k,1n}$. If $\pi \in \mathcal{A}'_{n,k,1n}$, define $\beta(\pi) = \pi$, and if $\pi \in \mathcal{A}'_{n,k,n1}$, define $\beta(\pi) = \bar{\pi}$. Thus $\beta(\pi) \in \mathcal{DAP}_n$ and $\beta(\pi)_1 - \beta(\pi)_2 = -(n - k + 1)$.

Theorem 4.4. *For all $n \geq 1$ and $k \in [n]$, the mapping β is a bijection between $\mathcal{A}'_{n,k}$ and $\mathcal{DAP}_{n,k}$.*

With the previous theorem, the interpretation of Poupard, proved in [Pou97] by recurrence relations, can be recovered.

Corollary 4.5. *The sequence $(\mathcal{DAP}_{n,k})_{1 \leq k \leq n}$ is an Entringer family.*

Denote by $\mathcal{DAP}'_{n,k}$ the set of $\pi \in \mathcal{DAP}_n$ such that the term immediately before 1 is k , if $k \leq n-1$, and $\mathcal{DAP}'_{n,n}$ the set of $\pi \in \mathcal{DAP}_n$ such that $\pi_1 = 1$.

We want to construct a bijection ρ between $\mathcal{A}_{n,k}$ and $\mathcal{DAP}'_{n,k}$.

If $k = n$, it suffices to define for $\pi \in \mathcal{A}_{n,n}$, $\rho(\pi) = \bar{\pi}$. Then, $\rho(\pi) \in \mathcal{DAP}'_{n,n}$.

Assume that $k \leq n-1$. The set $\mathcal{A}_{n,k}$ can be split in two disjoint subsets $\mathcal{A}_{n,k,1n}$ which is $\mathcal{A}_{n,k} \cap \mathcal{DAP}_n$ and $\mathcal{A}_{n,k,n1} := \mathcal{A}_{n,k} \setminus \mathcal{A}_{n,k,1n}$. For an ordered set $I = \{a_1, \dots, a_n\}$ with $a_1 < \dots < a_n$, denote by σ_I the permutation:

$$\sigma_I = \begin{pmatrix} a_1 & a_2 & \cdots & a_n \\ a_n & a_{n-1} & \cdots & a_1 \end{pmatrix}$$

Then, for a permutation $\pi = \pi_1 \dots \pi_n$ on the ordered set I , denote by $\bar{\pi}$ the *complement permutation on the set I* , that is $\bar{\pi} := \sigma_I \circ \pi$, and π^R the *reverse permutation*:

$$\pi^R := \pi_n \pi_{n-1} \dots \pi_1.$$

Note that when $I = [n]$, the definition of the complement permutation coincides with the one in the Remark of Subsection 4.2.

Then, for a permutation $\pi \in \mathcal{A}_{n,k}$,

- If $\pi \in \mathcal{A}_{n,k,1n}$, we can write $\pi = \sigma_1 1 \sigma_2$. Then, define $\rho(\pi) = \sigma_1^R 1 \sigma_2$. Since $1 < \pi_1 > \pi_2$, $\rho(\pi)$ is still alternating, and the term just before 1 in $\rho(\pi)$ is $\pi_1 = k$.
- If $\pi \in \mathcal{A}_{n,k,n1}$, we can write $\pi = \sigma_1 n \sigma_2$. Then, define $\rho(\pi) = \sigma_1^R 1 \bar{\sigma}_2$. Since $1 < \pi_1 > \pi_2$ and $\bar{\sigma}_2$ is alternating, $\rho(\sigma_1)$ is still alternating, and the term just before 1 in $\rho(\pi)$ is $\pi_1 = k$.

Theorem 4.6. *For all $n \geq 1$ and $k \in [n]$, the mapping ρ is a bijection between $\mathcal{A}_{n,k}$ and $\mathcal{DAP}'_{n,k}$.*

Proof. In order to prove that ρ is a bijection, it suffices to describe the inverse of ρ . Let π be an element in \mathcal{DAP}_n such that the term immediately before 1 is k . Following the construction of ρ , we have:

- If $\pi \in \mathcal{A}_{n,k}$, write $\pi = \tau_1 1 \tau_2$. Then, $\rho^{-1}(\pi) = \tau_1^R 1 \tau_2$.
- If $\pi \notin \mathcal{A}_{n,k}$, write $\pi = \tau_1 1 \tau_2$. Then, $\rho^{-1}(\pi) = \tau_1^R n \bar{\tau}_2$.

■

With the previous theorem, the following interpretation of Poupard, proved in [Pou97] by recurrence relations, can be recovered.

Corollary 4.7. *The sequence $(\mathcal{DAP}'_{n,k})_{1 \leq k \leq n}$ is an Entringer family.*

Denote by $\mathcal{DAP}''_{n,k}$ the set of $\pi \in \mathcal{DAP}_n$ such that the term immediately after n is $n+1-k$, if $k \leq n-1$, and $\mathcal{DAP}''_{n,n}$ the set of $\pi \in \mathcal{DAP}_n$ such that $\pi_n = n$.

Denote by ρ' the mapping defined for $\pi \in \mathcal{DAP}''_{n,k}$ by $\rho'(\pi) = \overline{\pi^R}$.

Theorem 4.8. *For all $n \geq 1$ and $k \in [n]$, the mapping ρ' is a bijection between $\mathcal{DAP}'_{n,k}$ and $\mathcal{DAP}''_{n,k}$.*

Proof. For $k \leq n-1$, $\pi \in \mathcal{DAP}_n$ has k just before 1 if and only if $\rho'(\pi)$ has $n+1-k$ just after n . ■

Corollary 4.9. *The sequence $(\mathcal{DAP}''_{n,k})_{1 \leq k \leq n}$ is an Entringer family.*

5. NEW ENTRINGER FAMILIES

5.1. **Interpretations in G-words and R-words.** A permutation π of $I = \{a_1, \dots, a_n\}$ with $a_1 < \dots < a_n$ is called a *G-word* if

- (i) $\pi_1 = a_n, \pi_n = a_{n-1}$,
- (ii) $\pi_2 > \pi_{n-1}$ (if $n \geq 4$).

Similarly, a permutation π of I is called an *R-word* if the previous condition (i) is verified and if (ii) is replaced by

- (ii') $\pi_2 < \pi_{n-1}$ (if $n \geq 4$).

A G-word (resp. an R-word) is said to be *primitive* if for any $(i, j) \in [n]^2$, neither the word $\pi_i \pi_{i+1} \dots \pi_j$ nor the word $\pi_j \pi_{j-1} \dots \pi_i$ is a G-word (resp. an R-word). Denote respectively by \mathcal{G}_n and \mathcal{R}_n the set of primitive G-words on $[n]$ and primitive R-words on $[n]$. For examples, the G-words in \mathcal{G}_n are:

$$634215, \quad 642315, \quad 623415, \quad 643215, \quad 624315,$$

and the R-words in \mathcal{R}_n are:

$$621435, \quad 623145, \quad 614235, \quad 631245, \quad 624135.$$

These permutations were introduced in [Mar06] with the following problem. Let I_n be the ideal of all algebraic relations on the slopes of all lines that can be formed by placing n points in a plane. Then, under two orders, I_n is generated by monomials corresponding to respectively primitive G-words and primitive R-words.

Martin and Wagner proved [MW09] that E_n is the number of primitive G-words (resp. the number of primitive R-words) on $[n + 2]$. Actually, this result can be refined to Entringer numbers, introducing a statistic on G-words.

Given a primitive G-word or an R-word π , define the *route* of π the sequence (α_i) defined with the following procedure:

- $\alpha_1 = \pi_1 = a_n, \alpha_2 = \pi_2 = a_{n-1}$,
- for $k \geq 2$, if $\alpha_k = i$, then

$$\alpha_{k+1} = \max \left[\begin{array}{l} \{j \leq i, \pi_j < \pi_i, \pi_j, \pi_{j+1}, \dots, \pi_{i-1}, \pi_i \notin \{\alpha_1, \dots, \alpha_k\}\} \\ \cup \{j \leq i, \pi_j > \pi_i, \pi_i, \pi_{i+1}, \dots, \pi_{j-1}, \pi_j \notin \{\alpha_1, \dots, \alpha_k\}\} \end{array} \right]$$

One can represent the route of a G-word or an R-word π as a graph with the vertices $\pi_1, \pi_2, \dots, \pi_n$ ordered in a line, with only one path starting from n drawn upon the line and going successively, if it's possible, to $n - 1, n - 2, \dots, 1$ without crossings (see Figure 3 for an example). Denote $\mathcal{G}_{n,k}$ (resp. $\mathcal{R}_{n,k}$) the set of primitive G-words π on $[n + 2]$ (resp. primitive R-words π on $[n + 2]$) such that $\alpha_{n+2} = n + 1 - k$.

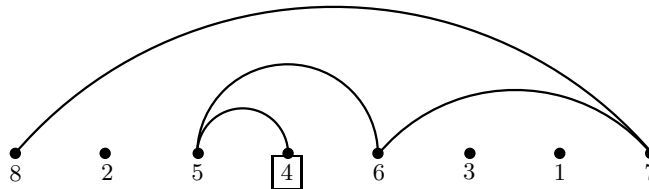


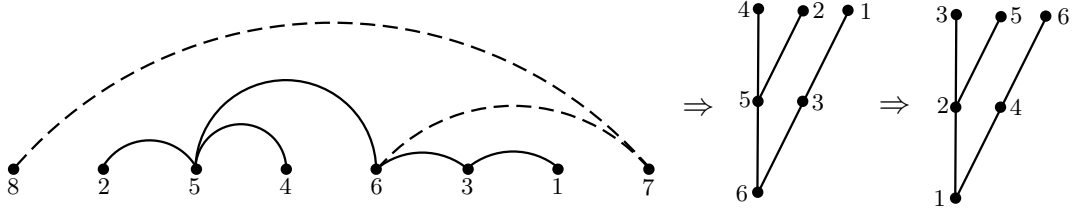
FIGURE 3. The route of the G-word $\pi = 82546317$

Theorem 5.1. *The sequences $(\mathcal{G}_{n,k})_{1 \leq k \leq n}$ and $(\mathcal{R}_{n,k})_{1 \leq k \leq n}$ are Entringer families.*

Proof. Use the bijection δ between \mathcal{G}_n and \mathcal{T}_n present in [MW09]. For π a primitive G-word on $\{a_1, \dots, a_{n+2}$ with $a_1 < \dots < a_{n+2}$, denote by π' the word $\pi_2 \dots \pi_{n+1}$. If π' is a word on $\{a_1, \dots, a_n\}$, with $a_1 < \dots < a_n$ and $a_n = \pi'_k$ for $k \in \{1, \dots, n\}$, define $T = \alpha(\pi')$ as the tree with root a_1 , from which two subgraphs go out, that are $\alpha(\pi'_1 \pi'_2 \dots \pi'_{k-1})$ and $\alpha(\pi'_{k+1} \pi'_{k+2} \dots \pi'_n)$ (eventually one of them of both are empty). The tree $\delta(\pi) = \alpha(\pi')$ is a 0-1-2 increasing tree and the application δ is a bijection from \mathcal{G}_n to \mathcal{T}_n (see [MW09] for further details).

Moreover, it is easy to see that the labels upon the main path of $T = \delta(\pi)$ are successively $(n+1-a_1), (n+1-a_2), \dots, (n+1-a_m)$, where $a_1 \dots, a_m$ ($a_1 > \dots > a_m$) are the different values that appear in the route of π . Thus, the end-point of the main path is k . Then, δ is a bijection between $\mathcal{G}_{n,k}$ and $\mathcal{T}_{n,k}$.

For example, one can construct the tree that corresponds with the G-word $\pi = 82546317$ with this construction :



The analogue result for the R-word can be proved using the same method with the bijection δ' between \mathcal{R}_n and \mathcal{T}_n present in [MW09]. \blacksquare

5.2. Interpretations in U-words.

Definition 5.2. A *U-word of length n* is a sequence $u = (u_i)_{1 \leq i \leq n}$ such that $u_1 = 1$ and $\forall i \in \{2, \dots, n\}, u_i + u_{i-1} \leq i$. We denote by \mathcal{U}_n the set of U-words of length n .

For example, the U-words of length 4 are:

$$1111, \quad 1112, \quad 1113, \quad 1121, \quad 1122.$$

Denote by $\mathcal{U}_{n,k}$ the set of U-words $(u_i) \in \mathcal{U}_n$ such that $u_n = n+1-k$.

Theorem 5.3. *The sequence $(\mathcal{U}_{n,k})_{1 \leq k \leq n}$ is an Entringer family.*

Proof. For any finite set X , let $\#X$ denotes its cardinality. For $\pi \in \mathcal{A}_{n,k}$, let $\gamma(\pi) = w^R$, where $w = w_1 \dots w_n$ is the word defined by

$$w_i = \begin{cases} \#\{j \geq \pi_i, j \notin \{\pi_1, \pi_2, \dots, \pi_{i-1}\}\}, & \text{if } i \text{ is odd,} \\ \#\{j \leq \pi_i, j \notin \{\pi_1, \pi_2, \dots, \pi_{i-1}\}\}, & \text{if } i \text{ is even.} \end{cases}$$

For example, if $\pi = 6351724 \in \mathcal{A}_{7,6}$, then the word w is computed as follows:

- $\{j \geq 6\} = \{6, 7\}$, so $w_1 = 2$,
- $\{j \leq 3, j \neq 6\} = \{1, 2, 3\}$, so $w_2 = 3$,
- $\{j \geq 5, j \notin \{3, 6\}\} = \{5, 7\}$, so $w_3 = 2$,
- $\{j \leq 1, j \notin \{3, 5, 6\}\} = \{1\}$, so $w_4 = 1$,
- $\{j \geq 7, j \notin \{1, 3, 5, 6\}\} = \{7\}$, so $w_5 = 1$,
- $\{j \leq 2, j \notin \{1, 3, 5, 6, 7\}\} = \{2\}$, so $w_6 = 1$,

- $\{j \geq 4, j \notin \{1, 2, 3, 5, 6, 7\}\} = \{4\}$, so $w_7 = 1$,

Then, $w = 2321111$ and $\gamma(\pi) = 1111232$.

We show that the mapping γ is a bijection between $\mathcal{A}_{n,k}$ and $\mathcal{U}_{n,k}$. Following the construction, $\gamma(\pi)_n = w_1 = n + 1 - \pi_1 = n + 1 - k$. Moreover, when $\gamma(\pi)_i = w_{n+1-i}$ is written, $n - i$ elements have been read in π before ; thus the number of elements counted by $\gamma(\pi)_i$ must be less than i . Moreover, the numbers counted by $\gamma(\pi)_{i-1}$ and $\gamma(\pi)_i$ are in the $n - i$ elements that have not been read in π and are two disjoint sets since π is alternating. Thus $\gamma(\pi)_i + \gamma(\pi)_{i-1}$ must be less than i . Finally, $\gamma(\pi) \in \mathcal{U}_{n,k}$.

Conversely, if $u \in \mathcal{U}_{n,k}$, the permutation $\pi = \gamma^{-1}(u) \in \mathcal{A}_{n,k}$ can be found back with:

- $\pi_1 = n + 1 - u_n$
- $\forall n \geq 1, \pi_{2i}$ is the u_{n-2i+1} -st smallest element in $[n] \setminus \{\pi_1, \dots, \pi_{2i-1}\}$.
- $\forall n \geq 1, \pi_{2i+1}$ is the u_{n-2i} -st greatest element in $[n] \setminus \{\pi_1, \dots, \pi_{2i}\}$.

■

Denote by $\mathcal{U}'_{n,k}$ the set of U-words $(u_i) \in \mathcal{U}_n$ such that $u_{n+1} + u_n = k$.

Theorem 5.4. *The sequence $(\mathcal{U}'_{n,k})_{1 \leq k \leq n}$ is an Entringer family.*

Proof. There are two possibilities to prove the previous result.

The mapping γ also induces a bijection between $\mathcal{A}'_{n,k}$ and $\mathcal{U}'_{n,k}$. For $\pi \in \mathcal{A}'_{n,k}$, there exists $j \in [n]$ such that $\pi \in \mathcal{A}_{n,j}$, so we can define $v = \gamma(\pi) \in \mathcal{U}_{n,j} \subset \mathcal{U}_n$. It suffices to show that $v \in \mathcal{U}'_{n,k}$. In the construction of $\gamma(\pi)$, v_n is the number of elements that are greater than π_1 , and v_{n-1} is the number of elements that are less than π_2 . Then $v_n = n + 1 - \pi_1$ and $v_{n-1} = \pi_2$, and $v_{n-1} + v_n = n + 1 - (\pi_1 - \pi_2) = k$ since $\pi \in \mathcal{A}'_{n,k}$.

Actually, it is easy to construct a bijection $\alpha : \mathcal{U}_{n,k} \rightarrow \mathcal{U}'_{n,k}$. For $u = (u_1, \dots, u_n) \in \mathcal{U}_{n,k}$, let $\alpha(u) = (u_1, u_2, \dots, u_{n-1}, n + 1 - u_{n-1} - u_n)$. Since $u \in \mathcal{U}_{n,k}$, $u_n - u_{n-1} \leq n + 1$, so we have $\alpha(u) \in \mathcal{U}_n$. Moreover, the last element $\alpha(u)_n = n + 1 - u_{n-1} - (n + 1 - k) = k - u_{n-1} = k - \alpha(u)_{n-1}$, so $\alpha(u) \in \mathcal{U}'_{n,k}$. The mapping α is then clearly a bijection between $\mathcal{U}_{n,k}$ and $\mathcal{U}'_{n,k}$.

■

6. CONCLUDING REMARKS

6.1. List of bijections for Entringer families. In what follows, we list all the twelve interpretations for Entringer families along with the bijections discussed in this paper:

- (1) the permutation $\pi \in \mathcal{A}_{n,k}$ such that $\pi_1 = k$,
- (2) the encoding sequence $\Delta \in \mathcal{D}_{n,k}$, obtained by $\Delta = \psi(\pi)$, where ψ is the bijection described in Section 2, then k is the first element read in Δ ,
- (3) the 0-1-2 increasing tree $T \in \mathcal{T}_{n,k}$, obtained by $T = \varphi(\Delta)$, where φ is the bijection described in Section 3, then k is the end-point of the main path of T ,
- (4) the 0-1-2 increasing tree $T' \in \mathcal{T}'_{n,k}$, obtained by $T' = \varphi'(T)$, where φ' is the bijection described in [KPP94, §6], then $k - 1$ is the father of n in T' ,
- (5) the alternating permutation $\sigma \in \mathcal{A}'_{n,k}$, obtained by $\sigma = \theta(\pi)$, where θ is the bijection described in Subsection 4.2, then $k = n + 1 - \sigma_1 + \sigma_2$,
- (6) the direct alternating permutation $\sigma' \in \mathcal{DAP}_{n,k}$, obtained by $\sigma' = \beta(\sigma)$, where β is the bijection described in Subsection 4.3, then $k = n + 1 - |\sigma_1 - \sigma_2|$,

- (7) the direct alternating permutation $\tau_1 \in \mathcal{DAP}'_{n,k}$, obtained by $\tau_1 = \rho(\pi)$, where ρ is the bijection described in Subsection 4.3, then k is the term immediately before 1 (or n if τ_1 starts with 1),
- (8) the direct alternating permutation $\tau_2 \in \mathcal{DAP}''_{n,k}$, obtained by $\tau_2 = \rho'(\tau_2)$, where ρ' is the bijection described in Subsection 4.3, then $n+1-k$ is the term immediately after n (or 1 if τ_2 ends with n),
- (9) the G-word $\pi' \in \mathcal{G}_{n,k}$, obtained by $\pi' = \delta^{-1}(T)$, where δ is the bijection described in Subsection 5.1, then $n+1-k$ is the end of the route of π' ,
- (10) the R-word $\pi'' \in \mathcal{R}_{n,k}$, obtained by $\pi'' = (\delta')^{-1}(T)$, where δ' is the bijection described in Subsection 5.1, then $n+1-k$ is the end of the route of π'' ,
- (11) the sequence $u \in \mathcal{U}_{n,k}$, obtained by $u = \gamma(\pi)$, where γ is the bijection described in Subsection 5.2, then $n+1-k$ is the last element of u ,
- (12) the sequence $v \in \mathcal{U}'_{n,k}$, obtained by $v = \gamma(\sigma) = \alpha(u)$, where α and γ are the bijections described in Subsection 5.2, then k is the sum of the two last elements of v .

We summarize the bijections of this paper in the diagram of Figure 4, where at the left we gather all the models in alternating permutations, and at the right we gather the models in the increasing trees.

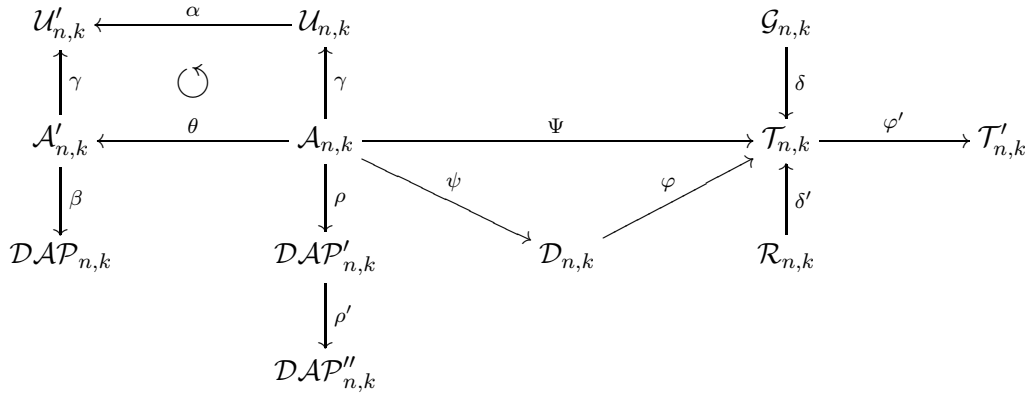


FIGURE 4. The bijections mentioned in the paper

6.2. Illustration for $n = 4$. In Figure 5, we summarize twelve interpretations for $E_{4,k}$, $k \in \{2, 3, 4\}$. In every column, the corresponding elements are described via the different bijections mentioned in the paper. Moreover, in the table, boxes point out the statistic $k = \pi_1$ if $\pi \in \mathcal{A}_{n,k}$ and the corresponding statistics in the other models.

6.3. An open problem. Consider the so-called *reduced tangent numbers* $t_n = E_{2n+1}/2^n$. Poupard [Pou89] proved that t_n is the number of 0-2 increasing trees (i.e. the trees in \mathcal{T}_n such that every vertex has 0 or 2 children). However, it seems that there is no interpretation à la André for t_n in alternating permutations. Furthermore, let $t_{n,k}$ denote the number of 0-2 increasing trees such that the end-point of the main path is k , then the sequence $(t_{n,k})$ is obviously a refinement of t_n as Entringer numbers for Euler numbers.

Let s_n (resp. $s_{n,k}$) be the number of *split-pair arrangements* of $[n]$, that are arrangements σ of the multi-set $\{0, 0, 1, 1, 2, 2, \dots, n, n\}$ such that $\sigma(1) = n$ (resp. $\sigma(1) = \sigma(k+1) = n$)











	k	2	3		4	
(1)	$\pi \in \mathcal{A}_{4,k}$	$\boxed{2}143$	$\boxed{3}241$	$\boxed{3}142$	$\boxed{4}231$	$\boxed{4}132$
(2)	$\Delta \in \mathcal{D}_{4,k}$	$(\boxed{2}, 1)^*$ $(4, 3)^*$	$(\boxed{3}, 2)^*$ $(4, 1)^*$	$(\boxed{3}, 2)$ $(2, 1)^*$ $(4, 3)^*$	$(\boxed{4}, 3)$ $(3, 2)^*$ $(4, 1)^*$	$(\boxed{4}, 3)$ $(3, 2)$ $(2, 1)^*$ $(4, 3)^*$
(3)	$T \in \mathcal{T}_{4,k}$					
(4)	$T' \in \mathcal{T}'_{4,k}$					
(5)	$\sigma \in \mathcal{A}'_{4,k}$	$\overbrace{41} \ 32$ $\boxed{3}$	$\overbrace{42} \ 31$ $\boxed{2}$	$\overbrace{31} \ 42$ $\boxed{2}$	$\overbrace{32} \ 41$ $\boxed{1}$	$\overbrace{21} \ 43$ $\boxed{1}$
(6)	$\sigma' \in \mathcal{DAP}_{4,k}$	$\overbrace{14} \ 23$ $\boxed{3}$	$\overbrace{13} \ 24$ $\boxed{2}$	$\overbrace{31} \ 42$ $\boxed{2}$	$\overbrace{23} \ 14$ $\boxed{1}$	$\overbrace{21} \ 43$ $\boxed{1}$
(7)	$\tau_1 \in \mathcal{DAP}'_{4,k}$	$\boxed{2}143$	$2\boxed{3}14$	$\boxed{3}142$	$\boxed{(4)}1324$	$\boxed{(4)}1423$
(8)	$\tau_2 \in \mathcal{DAP}''_{4,k}$	$214\boxed{3}$	$14\boxed{2}3$	$314\boxed{2}$	$1324\boxed{(1)}$	$2324\boxed{(1)}$
(9)	$\pi' \in \mathcal{G}_{4,k}$	$6\boxed{3}4215$	$64\boxed{2}315$	$6\boxed{2}3415$	$6432\boxed{1}5$	$6243\boxed{1}5$
(10)	$\pi'' \in \mathcal{R}_{4,k}$	$6214\boxed{3}5$	$6\boxed{2}3145$	$614\boxed{2}35$	$63\boxed{1}245$	$624\boxed{1}35$
(11)	$u \in \mathcal{U}_{4,k}$	$111\boxed{3}$	$112\boxed{2}$	$111\boxed{2}$	$112\boxed{1}$	$111\boxed{1}$
(12)	$v \in \mathcal{U}'_{4,k}$	$11 \overbrace{11}$ $\boxed{2}$	$11 \overbrace{21}$ $\boxed{3}$	$11 \overbrace{12}$ $\boxed{3}$	$11 \overbrace{22}$ $\boxed{4}$	$11 \overbrace{13}$ $\boxed{4}$

FIGURE 5. Twelve interpretations for $E_{4,k}$, $1 \leq k \leq 4$

and, between the two occurrences of i in σ ($0 \leq i \leq n - 1$), the number $i + 1$ appears exactly once.

Recently, Graham and Zang [GZ08] proved that for $1 \leq k \leq n$, $s_{n,k} = t_{n,k}$. In particular, $s_n = t_n$. There is no bijective proof between Poupard's model and Graham and Zang's model.

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(Yoann Gelineau) UNIVERSITÉ DE LYON; UNIVERSITÉ LYON 1; INSTITUT CAMILLE JORDAN; UMR 5208 DU CNRS; 43, BOULEVARD DU 11 NOVEMBRE 1918, F-69622 VILLEURBANNE CEDEX, FRANCE
E-mail address: `gelineau@math.univ-lyon1.fr`

(Heesung Shin) UNIVERSITÉ DE LYON; UNIVERSITÉ LYON 1; INSTITUT CAMILLE JORDAN; UMR 5208 DU CNRS; 43, BOULEVARD DU 11 NOVEMBRE 1918, F-69622 VILLEURBANNE CEDEX, FRANCE
E-mail address: `hshin@math.univ-lyon1.fr`

(Jiang Zeng) UNIVERSITÉ DE LYON; UNIVERSITÉ LYON 1; INSTITUT CAMILLE JORDAN; UMR 5208 DU CNRS; 43, BOULEVARD DU 11 NOVEMBRE 1918, F-69622 VILLEURBANNE CEDEX, FRANCE
E-mail address: `gelineau@math.univ-lyon1.fr`