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Smoothness Characterization and Stability of Nonlinear and Non-Separable Multi-scale Representations

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Abstract The aim of the paper is the construction and the analysis of nonlinear and non-separable multi-scale representations for multivariate functions. The described multi-scale representation is associated with an isotropic dilation matrix. We show that the smoothness of a function can be characterized by the rate of decay of its multi-scale coefficients. We also study the stability of these representations, a key issue in the designing of adaptive algorithms.

Keywords Nonlinear Multi-scale Representations · Stability · Smoothness

Mathematics Subject Classification (2000) MSC 41A46 · MSC 41A60 · 41A63

1 Introduction

A multi-scale representation of an abstract object v (e.g. a function representing the grey level of an image) is defined as $\mathcal{M}v := (v^0, d^0, d^1, d^2, \dots)$, where v^0 is the coarsest approximation of v in some sense and d^j , with $j \geq 0$, are additional detail coefficients representing the fluctuations between two successive levels.

Several strategies exist to build such representations: wavelet basis, lifting schemes and also the discrete framework of Harten [8]. Using a wavelet basis, we compute $(v^0, d^0, d^1, d^2, \dots)$ through linear filtering and thus the multi-scale representation corresponds to a change of basis. Although wavelet bases are optimal for one-dimensional

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functions, this is no longer the case for multivariate objects such as images where the presence of singularities requires special treatments. Nevertheless, the approximation property of wavelet bases and their use in image processing are now well understood (see [5] and [12] for details).

Overcoming this "course of dimensionality" for wavelet basis was in the past decade the subject of active research. We mention here several strategies developed from the wavelets theory: the curvelets transforms [3], the directionlets transforms [6] and the bandelets transform [11]. Another approach proposed in [13] and studied in [2] uses the discrete framework of Harten, which allows a better treatment of singularities and consequently better approximation results.

The applications of all these methods to image processing are numerous: let us mention some of these works in [2], [1] and [4]. In [2], the extension of univariate methods using tensor product representations is studied. Although this extension is natural and simple the results are not optimal.

We propose in the present paper a nonlinear multi-scale representations based on the general framework of A. Harten (see [8] and [9]). Our representation is non-separable and is associated to an isotropic dilation matrix M . Since the details are computed adaptively, the multi-scale transform is completely nonlinear and is no more equivalent to a change of basis. Moreover, the point of view of Harten is essentially discrete and is not based on the study of scaling equations which implies that the results of wavelet theory cannot be used. To study these representations, we develop some new analysis tools and we prove that these representations give the same approximation order as for wavelet basis. This strategy is fruitful in applications since it allows to cope up with the deficiencies of wavelet bases without losing the approximation order.

The outline of the paper is as follows : after recalling some notations and definitions, we detail the construction of nonlinear and non-separable multi-scale representations. In section 3 and section 4, extending upon [14], we characterize the smoothness of a function v belonging to some Besov spaces by means of the decay of its detail coefficients. In section 5 we study the stability of the underlying subdivision scheme and we then switch on to the stability of the multi-scale representation in section 6. For similar, one-dimensional results see [18] and [14]. Finally, we give a illustration of such a multi-scale representation in two dimensions (see section 7).

2 Notations and Generalities

We start by introducing some notations that will be used throughout the paper. Let us consider a multi-index $\mu = (\mu_1, \mu_2, \dots, \mu_d) \in \mathbb{N}^d$ and a vector $x = (x_1, x_2, \dots, x_d) \in \mathbb{R}^d$. We define $|\mu| = \sum_{i=1}^d \mu_i$ and $x^\mu = \prod_{i=1}^d x_i^{\mu_i}$. For two multi-indices $m, \mu \in \mathbb{N}^d$ we define

$$\binom{\mu}{m} = \binom{\mu_1}{m_1} \cdots \binom{\mu_d}{m_d}.$$

For a fixed integer $N \in \mathbb{N}$, we define

$$q_N = \#\{\mu, |\mu| = N\} \tag{2.1}$$

where $\#Q$ stands for the cardinal of the set Q . Let $\ell(\mathbb{Z}^d)$ be the space of all sequences indexed by \mathbb{Z}^d . The subspace of bounded sequences is denoted by $\ell^\infty(\mathbb{Z}^d)$ and $\|u\|_{\ell^\infty(\mathbb{Z}^d)}$

is the supremum of $\{|u_k| : k \in \mathbb{Z}^d\}$. We denote $\ell^0(\mathbb{Z}^d)$ the subspace of all sequences with finite support (i.e. the number of non-zero components of a sequence is finite). As usual, let $\ell^p(\mathbb{Z}^d)$ be the Banach space of sequences u on \mathbb{Z}^d such that $\|u\|_{\ell^p(\mathbb{Z}^d)} < \infty$, where

$$\|u\|_{\ell^p(\mathbb{Z}^d)} := \left(\sum_{k \in \mathbb{Z}^d} |u_k|^p \right)^{\frac{1}{p}} \quad \text{for } 1 \leq p < \infty.$$

By $L^p(\mathbb{R}^d)$ we denote the space of all measurable functions v such that $\|v\|_{L^p(\mathbb{R}^d)} < \infty$, where

$$\|v\|_{L^p(\mathbb{R}^d)} := \left(\int_{\mathbb{R}^d} |v(x)|^p dx \right)^{\frac{1}{p}} \quad \text{for } 1 \leq p < \infty,$$

$$\|v\|_{L^\infty(\mathbb{R}^d)} := \operatorname{ess\,sup}_{x \in \mathbb{R}^d} |v(x)|.$$

Throughout the paper, the symbol $\|\cdot\|_\infty$ is the sup norm in \mathbb{Z}^d when applied either to a vector or a matrix. Let us recall that for a function v the finite difference of order $N \in \mathbb{N}$, in the direction $h \in \mathbb{R}^d$ is defined by:

$$\nabla_h^N v(x) := \sum_{k=0}^N (-1)^k \binom{N}{k} v(x - kh).$$

and the mixed finite difference of order $n = (n_1, \dots, n_d) \in \mathbb{N}^d$ in the direction $h = (h_1, \dots, h_d) \in \mathbb{R}^d$ by:

$$\nabla_h^n v(x) := \nabla_{h_1 e_1}^{n_1} \cdots \nabla_{h_d e_d}^{n_d} v(x) = \sum_{k_1, \dots, k_d=0}^{\max(n_1, \dots, n_d)} (-1)^{|n|} \binom{n}{k} v(x - k \cdot h),$$

where $k \cdot h := \sum_{i=1}^d k_i h_i$ is the usual inner product while (e_1, \dots, e_d) is the canonical basis on \mathbb{Z}^d . For any invertible matrix B we put

$$\nabla_B^n v(x) := \nabla_{B e_1}^{n_1} \cdots \nabla_{B e_d}^{n_d} v(x).$$

Similarly, we define $D^\mu v(x) = D_1^{\mu_1} \cdots D_d^{\mu_d} v(x)$, where D_j is the differential operator with respect to the j th coordinate of the canonical basis. For a sequence $(u_p)_{p \in \mathbb{Z}^d}$, we will use the mixed finite differences of order N defined by the formulas

$$\nabla^n u := \nabla_{e_1}^{n_1} \nabla_{e_2}^{n_2} \cdots \nabla_{e_d}^{n_d} u,$$

$$\Delta^N u := \{\nabla^n u, |n| = N, n \in \mathbb{N}^d\}.$$

In the following, we will consider dilation matrices to define inter-scale operators. A dilation matrix is an invertible integer-valued matrix M satisfying $\lim_{n \rightarrow \infty} M^{-n} = 0$, and $m := |\det(M)|$. Besides, we will suppose that the dilation matrix is isotropic:

Definition 1 A matrix M is called isotropic if it is similar to the diagonal matrix $\operatorname{diag}(\sigma_1, \dots, \sigma_d)$, i.e. there exists an invertible matrix A such that

$$M = A^{-1} \operatorname{diag}(\sigma_1, \dots, \sigma_d) A, \quad (2.2)$$

with $\sigma_1, \dots, \sigma_d$ being the eigenvalues of matrix M , $|\sigma_1| = \dots = |\sigma_d| = m^{\frac{1}{d}}$.

Moreover, for any given norm in \mathbb{R}^d there exist constants $C_1, C_2 > 0$ such that for any integer n and for any $v \in \mathbb{R}^d$

$$C_1 m^{\frac{n}{d}} \|v\| \leq \|M^n v\| \leq C_2 m^{\frac{n}{d}} \|v\|.$$

We end this section with the following remark on notations: for two positive quantities A and B depending on a set of parameters, the relation $A \lesssim B$ implies the existence of a positive constant C , independent of the parameters, such that $A \leq CB$. Also $A \sim B$ means $A \lesssim B$ and $B \lesssim A$.

2.1 Besov Spaces

In the following sections, we will study the smoothness of the limit function v in Besov spaces. Let us recall the definition of Besov spaces. Let $p, q \geq 1$, s be a positive real number and N be any integer such that $N > s$. The Besov space $B_{p,q}^s(\mathbb{R}^d)$ consists of those functions $v \in L^p(\mathbb{R}^d)$ satisfying

$$(2^{js} \omega_N(v, 2^{-j})_{L^p})_{j \geq 0} \in \ell^q(\mathbb{Z}^d),$$

where $\omega_N(v, t)_{L^p}$ is the modulus of smoothness of v of order $N \in \mathbb{N} \setminus \{0\}$ in $L^p(\mathbb{R}^d)$:

$$\omega_N(v, t)_{L^p} = \sup_{\substack{h \in \mathbb{R}^d \\ \|h\|_2 \leq t}} \|\nabla_h^N v\|_{L^p(\mathbb{R}^d)}, \quad t \geq 0,$$

where $\|\cdot\|_2$ is the Euclidean norm. The norm in $B_{p,q}^s(\mathbb{R}^d)$ is then given by

$$\|v\|_{B_{p,q}^s(\mathbb{R}^d)} := \|v\|_{L^p(\mathbb{R}^d)} + \|(2^{js} \omega_N(v, 2^{-j})_{L^p})_{j \geq 0}\|_{\ell^q(\mathbb{Z}^d)}.$$

Let us now introduce a new modulus of smoothness $\tilde{\omega}_N$ that uses mixed finite differences of order N :

$$\tilde{\omega}_N(v, t)_{L^p} = \sup_{\substack{n \in \mathbb{N}^d \\ |n|=N}} \sup_{\substack{h \in \mathbb{R}^d \\ \|h\|_2 \leq t}} \|\nabla_h^n v\|_{L^p(\mathbb{R}^d)}, \quad t > 0.$$

It is easy to see that for any v in $L^p(\mathbb{R}^d)$, $\|\nabla_h^N v\|_{L^p(\mathbb{R}^d)} \lesssim \sum_{|n|=N} \|\nabla_h^n v\|_{L^p(\mathbb{R}^d)}$, thus

$\omega_N(v, t)_{L^p} \lesssim \tilde{\omega}_N(v, t)_{L^p}$. The inverse inequality $\tilde{\omega}_N(v, t)_{L^p} \lesssim \omega_N(v, t)_{L^p}$ immediately follows from Lemma 4 of [18]. It implies that:

$$\|v\|_{B_{p,q}^s(\mathbb{R}^d)} \sim \|v\|_{L^p} + \|(2^{js} \tilde{\omega}_N(v, 2^{-j})_{L^p})_{j \geq 0}\|_{\ell^q(\mathbb{Z}^d)}.$$

Going further, there exists a family of equivalent norms on $B_{p,q}^s(\mathbb{R}^d)$.

Lemma 1 For all $\sigma > 1$, $\|v\|_{B_{p,q}^s(\mathbb{R}^d)} \sim \|v\|_{L^p(\mathbb{R}^d)} + \|(\sigma^{js} \tilde{\omega}_N(v, \sigma^{-j})_{L^p})_{j \geq 0}\|_{\ell^q(\mathbb{Z}^d)}$.

Proof : Since $\sigma > 1$, for any $j > 0$ there exists $j' > 0$ such that $2^{j'} \leq \sigma^j \leq 2^{j'+1}$. According to this, we have the inequalities

$$2^{j's} \tilde{\omega}_N(v, 2^{-j'-1})_{L^p} \leq \sigma^{js} \tilde{\omega}_N(v, \sigma^{-j})_{L^p} \leq 2^{(j'+1)s} \tilde{\omega}_N(v, 2^{-j'})_{L^p},$$

from which the norm equivalence follows. \square

3 Multi-scale Representations

Let us recall the concept of multiresolution analysis (MRA) for some d -dimensional Hilbert space V . To this end, let M be a $d \times d$ dilation matrix (not necessarily isotropic).

Definition 1 A multiresolution analysis of V is a sequence $(V_j)_{j \in \mathbb{Z}^d}$ of closed subspaces of V satisfying the following properties:

1. The subspaces are embedded: $V_j \subset V_{j+1}$;
2. $f \in V_j$ if and only if $f(M \cdot) \in V_{j+1}$;
3. $\bigcup_{j \in \mathbb{Z}} V_j = V$;
4. $\bigcap_{j \in \mathbb{Z}} V_j = \{0\}$;
5. There exists a compactly supported function $\varphi \in V_0$ such that the family $\{\varphi(\cdot - k)\}_{k \in \mathbb{Z}^d}$ forms a Riesz basis of V_0 .

The function φ is called the *scaling function*. Since $V_0 \subset V_1$, φ satisfies the following equation:

$$\varphi = \sum_{k \in \mathbb{Z}^d} g_k \varphi(M \cdot - k), \text{ with } \sum_k g_k = m. \quad (3.1)$$

To get the approximation of a given function v at the level j , we assume the existence of a compactly supported function $\tilde{\varphi}$ dual to φ (i.e. for all $k, n \in \mathbb{Z}^d$ $\langle \tilde{\varphi}(\cdot - n), \varphi(\cdot - k) \rangle = \delta_{n,k}$, where $\delta_{n,k}$ denotes the Kronecker symbol and $\langle \cdot, \cdot \rangle$ the inner product on V), which satisfies a so-called scaling equation

$$\tilde{\varphi} = \sum_{n \in \mathbb{Z}^d: \|n\|_\infty \leq P} \tilde{h}_n \tilde{\varphi}(M \cdot - n), \text{ with } \sum_k \tilde{h}_k = m. \quad (3.2)$$

The approximation v_j of v we consider is then obtained by projection of v on V_j as follows:

$$v_j = \sum_{n \in \mathbb{Z}^d} v_n^j \varphi(M^j \cdot - n). \quad (3.3)$$

where

$$v_n^j = \int v(x) m^j \tilde{\varphi}(M^j x - n) dx, \quad n \in \mathbb{Z}^d. \quad (3.4)$$

Multi-scale representations based on specific type of choice for $\tilde{\varphi}$ are commonly used in image processing or numerical analysis. We now mention two of them: the first is the point value case obtained when $\tilde{\varphi}$ is the Dirac distribution and the second case is the cell average case obtained when $\tilde{\varphi}$ is the indicator function of some domain on \mathbb{R}^d . In the theoretical study that follows, we assume that the data are obtained through a projection of a functional v as in (3.4).

A strategy which allows to build multi-scale representations based on such projection can be done in terms of a very general discrete framework based on the concept of inter-scale operators introduced by A. Harten in [8], which we now recall. Assume that we have a sequence of index sets Γ_j , $j \geq 0$ and two inter-scale discrete operators associated to this sequence: the *projection* operator P_{j-1}^j and the *prediction* operator P_j^{j-1} . The projection operator P_{j-1}^j acts from fine to coarse level, that is, $v^{j-1} = P_{j-1}^j v^j$. This operator is always assumed to be *linear*. The *prediction* operator

P_j^{j-1} acts from coarse to fine level. It computes the 'approximation' \hat{v}^j of v^j from the vector $(v_k^{j-1})_{k \in \mathbb{Z}^d}$ which is associated to $v_{j-1} \in V_{j-1}$:

$$\hat{v}^j = P_j^{j-1} v^{j-1}.$$

This operator may be *nonlinear*. Besides, we assume that these operators satisfy the *consistency* property:

$$P_{j-1}^j P_j^{j-1} = I, \quad (3.5)$$

i.e., the projection of \hat{v}^j coincides with v^{j-1} . Having defined the prediction error $e^j := v^j - \hat{v}^j$, we obtain a redundant representation of vector v^j :

$$v^j = \hat{v}^j + e^j. \quad (3.6)$$

By the consistency property, one has

$$P_{j-1}^j e^j = P_{j-1}^j v^j - P_{j-1}^j \hat{v}^j = v^{j-1} - v^{j-1} = 0.$$

Hence, $e^j \in \text{Ker}(P_{j-1}^j)$. Using a basis of this kernel, we write down the error e^j in a non-redundant way and get the detail vector $d^{j-1} = E e^{j-1}$. The data v^j is thus completely equivalent to the data (v^{j-1}, d^{j-1}) . Iterating this process from the initial data v^J , we obtain its *nonlinear multi-scale representation*

$$\mathcal{M}v^J = (v^0, d^0, \dots, d^{J-1}). \quad (3.7)$$

From here on, we assume the equivalence

$$\|e^j\|_{\ell^p(\mathbb{Z}^d)} \sim \|d^{j-1}\|_{\ell^p(\mathbb{Z}^d)}. \quad (3.8)$$

Note that the projection operator is characterized by the function $\tilde{\varphi}$. Namely, if we consider the discretization defined by (3.4) then, in view of (3.2), we may write the projection operator as follows:

$$v_k^{j-1} = m^{-1} \sum_{\|n\|_\infty \leq P} \tilde{h}_n v_{Mk+n}^j = m^{-1} \sum_{\|n-Mk\|_\infty \leq P} \tilde{h}_{n-Mk} v_n^j := (P_{j-1}^j v^j)_k. \quad (3.9)$$

To describe the prediction operator, for every $w \in \ell^\infty(\mathbb{Z}^d)$ we consider a linear operator $S(w)$ defined on $\ell^\infty(\mathbb{Z}^d)$ by

$$(S(w)u)_k := \sum_{l \in \mathbb{Z}^d} a_{k-Ml}(w) u_l, \quad k \in \mathbb{Z}^d. \quad (3.10)$$

Note that the coefficients $a_k(w)$ depend on w . We assume that S is local:

$$\exists K > 0 \quad \text{such that} \quad a_{k-Ml}(w) = 0 \quad \text{if} \quad \|k - Ml\|_\infty > K \quad (3.11)$$

and that $a_k(w)$ is bounded independently of w :

$$\exists C > 0 \quad \text{such that} \quad \forall w \in \ell^\infty(\mathbb{Z}^d) \quad \forall k, l \in \mathbb{Z}^d \quad |a_{k-Ml}(w)| < C. \quad (3.12)$$

Remark 1 From (3.12) it immediately follows that for any $p \geq 1$ the norms $\|S(w)\|_{\ell^p(\mathbb{Z}^d)}$ are bounded independently of w .

The prediction operator is then defined by

$$\hat{v}^j = P_j^{j-1} v^{j-1} = S(v^{j-1}) v^{j-1}. \quad (3.13)$$

If for all $k, l \in \mathbb{Z}^d$ and all $w \in \ell^\infty(\mathbb{Z}^d)$ we put $a_{k-Ml}(w) = g_{k-Ml}$, g_{k-Ml} being defined in the scaling equation (3.1) for φ , we get the so-called *linear* prediction operator. In the general case, the prediction operator could be considered as a perturbation of the linear prediction operator because of the consistency property, that is why we will call them *quasi-linear* prediction operators. The operator-valued function which associates to any w an operator $S(w)$ is called a quasi-linear prediction rule.

For what follows, we need to introduce the notion of polynomial reproduction for quasi-linear prediction rules. A polynomial q of total degree N is defined as a linear combination $q(x) = \sum_{|n| \leq N} c_n x^n$. Let us denote by Π the linear space of all polynomials, by Π_N the linear space of all polynomials of total degree N . With this in mind, we have the following definition for polynomial reproduction:

Definition 2 We will say that the quasi-linear prediction rule $S(w)$ reproduces polynomials of total degree N if for any $w \in \ell^\infty(\mathbb{Z}^d)$ and any $u \in \ell^\infty(\mathbb{Z}^d)$ such that $u_k = p(k) \forall k \in \mathbb{Z}^d$ and $p \in \Pi_N$, we have:

$$(S(w)u)_k = p(M^{-1}k) + q(k),$$

where $\deg(q) < \deg(p)$. If $q = 0$, we say that the quasi-linear prediction rule S exactly reproduces polynomials of total degree N .

Note that the property is required for *any data* w , and not only for $w = u$.

4 Smoothness of Nonlinear Multi-scale Representations

In this section, we prove the equivalence between the norm of a function v belonging to $B_{p,q}^s(\mathbb{R}^d)$ and a quantity computed using its nonlinear detail coefficients d^j . Lower estimates of the Besov norm are associated to so-called direct theorems while upper estimates are associated to so-called inverse theorems. Note that a similar technique applied in a wavelet setting was used in [5].

4.1 Direct Theorem

Let v be a function in some Besov space $B_{p,q}^s(\mathbb{R}^d)$ with $p, q \geq 1$ and $s > 0$, $(v^0, (d^j)_{j \geq 0})$ be its nonlinear multi-scale representation. We now show under what conditions we are able to get a lower estimate of $\|v\|_{B_{p,q}^s(\mathbb{R}^d)}$ using $(v^0, (d^j)_{j \geq 0})$. To prove such a result, we need to have first an estimate of the norm of the prediction error:

Lemma 2 *Assume that the quasi-linear prediction rule exactly reproduces polynomials up to degree $N - 1$ then the following estimation holds*

$$\|e^j\|_{\ell^p(\mathbb{Z}^d)} \lesssim \sum_{|n|=N} \|\nabla^n v^j\|_{\ell^p(\mathbb{Z}^d)}. \quad (4.1)$$

Proof : Let us compute

$$e_k^j(w) := v_k^j - \sum_{\|k-Ml\|_\infty \leq K} a_{k-Ml}(w) v_l^{j-1}.$$

Using (3.9), we can write it down as

$$\begin{aligned} e_k^j(w) &= v_k^j - m^{-1} \sum_{\substack{l \in \mathbb{Z}^d \\ \|k-Ml\|_\infty \leq K}} a_{k-Ml}(w) \sum_{\substack{n \in \mathbb{Z}^d \\ \|n-Ml\|_\infty \leq P}} \tilde{h}_{n-Ml} v_n^j \\ &= v_k^j - m^{-1} \sum_{\substack{n \in \mathbb{Z}^d \\ \|k-n\|_\infty \leq K+P}} v_n^j \sum_{\substack{l \in \mathbb{Z}^d \\ \|k-Ml\|_\infty \leq K}} a_{k-Ml}(w) \tilde{h}_{n-Ml} = \sum_{n \in F(k)} b_{k,n}(w) v_n^j, \end{aligned}$$

where $b_{k,n}(w) = \sum_{\substack{l \in \mathbb{Z}^d \\ \|k-Ml\|_\infty \leq K}} a_{k-Ml}(w) \tilde{h}_{n-Ml}$, and $F(k) = \{n \in \mathbb{Z}^d : \|n-k\|_\infty \leq$

$P+K\}$ is a finite set for any given k . For any $k \in \mathbb{Z}^d$ let us define a vector $b_k(w) := (b_{k,n}(w))_{n \in F(k)}$. By hypothesis, $e^j(w) = 0$ if there exists $p \in \Pi_{N'}$, $0 \leq N' < N$ such that $v_k = p(k)$. Consequently, for any $q \in \mathbb{Z}^d$, $|q| < N$, $b_k(w)$ is orthogonal to any polynomial sequence associated to the polynomial $l^q = l_1^{q_1} \cdots l_d^{q_d}$, thus it can be written in terms of a basis of the space orthogonal to the space spanned by these vectors. According to [10], Theorem 4.3, we can take $\{\nabla^\mu \delta_{\cdot-l}, |\mu| = N, l \in \mathbb{Z}^d\}$ as a basis of this space. By denoting $c_l^\mu(w)$ the coordinates of $b_k(w)$ in this basis:

$$b_{k,n}(w) = \sum_{|\mu|=N} \sum_{l \in \mathbb{Z}^d} c_l^\mu(w) \nabla^\mu \delta_{n-l}$$

and taking $w = v^{j-1}$ we get

$$e_j^k := e_j^k(v^{j-1}) = \sum_{n \in F(k)} \sum_{|\mu|=N} \sum_{l \in \mathbb{Z}^d} c_l^\mu(v^{j-1}) \nabla^\mu \delta_{n-l} v_n^j = \sum_{n \in F(k)} \sum_{|\mu|=N} c_n^\mu(v^{j-1}) \nabla^\mu v_n^j. \quad (4.2)$$

Finally, we use (3.12) to conclude that the coefficients $b_{k,n}(v^{j-1})$ and $c_l^\mu(v^{j-1})$ are bounded independently of k , n and w , and (4.1) follows from (4.2). \square

The preceding lemma enables us to compute the lower estimate:

Theorem 1 *If for $p, q \geq 1$ and some positive s , v belongs to $B_{p,q}^s(\mathbb{R}^d)$, if the quasi-linear prediction rule exactly reproduces polynomials up to degree $N-1$ with $N > s$, and if the equivalence (3.8) is satisfied, then*

$$\|v^0\|_{\ell^p(\mathbb{Z}^d)} + \|(m^{(s/d-1/p)j} \|(d_k^j)_{k \in \mathbb{Z}^d}\|_{\ell^p(\mathbb{Z}^d)})_{j \geq 0}\|_{\ell^q(\mathbb{Z}^d)} \lesssim \|v\|_{B_{p,q}^s(\mathbb{R}^d)}. \quad (4.3)$$

Proof: Using the Hölder inequality and the fact that $\tilde{\varphi}$ is compactly supported, we first obtain

$$\|v^0\|_{\ell^p(\mathbb{Z}^d)} = \|(\langle v, \tilde{\varphi}(\cdot - k) \rangle)_{k \in \mathbb{Z}^d}\|_{\ell^p(\mathbb{Z}^d)} \lesssim \|(\|v\|_{L^p(\text{Supp}(\tilde{\varphi}(\cdot - k)))})_{k \in \mathbb{Z}^d}\|_{\ell^p(\mathbb{Z}^d)} \lesssim \|v\|_{L^p(\mathbb{R}^d)}.$$

Let us then consider a quasi-linear prediction rule which exactly reproduces polynomials up to degree $N-1$. Since $\|e^j\|_{\ell^p(\mathbb{Z}^d)} \sim \|d^{j-1}\|_{\ell^p(\mathbb{Z}^d)}$, by Lemma 2 we get

$$\|(m^{(s/d-1/p)j} \|(d_k^j)_{k \in \mathbb{Z}^d}\|_{\ell^p(\mathbb{Z}^d)})_{j \geq 0}\|_{\ell^q(\mathbb{Z}^d)} \lesssim \|(m^{(s/d-1/p)j} \sum_{|n|=N} \|(\nabla^n v_k^j)_{k \in \mathbb{Z}^d}\|_{\ell^p(\mathbb{Z}^d)})_{j \geq 0}\|_{\ell^q(\mathbb{Z}^d)}.$$

We have successively

$$\begin{aligned}
\sum_{|n|=N} \|\nabla^n v^j\|_{\ell^p(\mathbb{Z}^d)} &= \sum_{|n|=N} \|\nabla^n (\langle v, m^j \tilde{\varphi}(M^j \cdot -k) \rangle)_{k \in \mathbb{Z}^d}\|_{\ell^p(\mathbb{Z}^d)} \\
&= \sum_{|n|=N} \|(\langle \nabla_{M^{-j}}^n v, m^j \tilde{\varphi}(M^j \cdot -k) \rangle)_{k \in \mathbb{Z}^d}\|_{\ell^p(\mathbb{Z}^d)} \\
&\lesssim m^{j/p} \sum_{|n|=N} (\|\nabla_{M^{-j}}^n v\|_{L^p(\text{Supp}(\tilde{\varphi}(M^j \cdot -k)))}_{k \in \mathbb{Z}^d})_{\ell^p(\mathbb{Z}^d)} \\
&\lesssim m^{j/p} \sum_{|n|=N} \|\nabla_{M^{-j}}^n v\|_{L^p(\mathbb{R}^d)} \\
&\lesssim m^{j/p} \tilde{\omega}_N(v, C_2 m^{-j/d})_{L^p},
\end{aligned}$$

Since M is isotropic, for any integer $C > 0$ and any $t > 0$, $\tilde{\omega}_N(v, Ct)_{L^p} \leq C \tilde{\omega}_N(v, t)_{L^p}$. Thus,

$$\sum_{|n|=N} \|\nabla^n v^j\|_{\ell^p(\mathbb{Z}^d)} \lesssim m^{j/p} \tilde{\omega}_N(v, m^{-j/d})_{L^p},$$

which implies (4.3). \square

4.2 Inverse Theorem

Now, we are given the sequence $(v^0, (d^j)_{j \leq 0})$ and we study the smoothness of the reconstruction process:

$$v^j = S(v^{j-1})v^{j-1} + E d^{j-1}$$

by considering limit of the sequence of functions

$$v_j(x) = \sum_{k \in \mathbb{Z}^d} v_k^j \varphi(M^j x - k) \quad (4.4)$$

where φ is defined in (3.1), and we show that under certain conditions on the sequence $(v^0, (d^j)_{j \leq 0})$ and φ , v_j converges to some function v belonging to a Besov space.

For that purpose, we need to establish that if the quasi-linear prediction rule S reproduces polynomials up to total degree $N - 1$ then all the mixed finite differences of order less or equal to N can be described using local and bounded difference operators:

Proposition 1 *Let S be a quasi-linear prediction rule reproducing polynomials up to total degree $N - 1$. Then for any l , $0 < l \leq N$ there exists a local bounded difference operator S_l such that:*

$$\Delta^l S(w)u := S_l(w)\Delta^l u,$$

for all $u, w \in \ell^\infty(\mathbb{Z}^d)$.

The proof is available in [16], Proposition 1. Note that contrary to the tensor product case studied in [14], the operator $S_l(w)$ is multi-dimensional and is defined from $(\ell^\infty(\mathbb{Z}^d))^{q_l}$ onto $(\ell^\infty(\mathbb{Z}^d))^{q_l}$ where we recall that $q_l = \#\{l, |\mu| = l\}$ and cannot be reduced to a set of difference operators in some given directions.

The inverse theorem we show at the end of this section is based on some property of the joint spectral radius of the difference operators introduced above. It is defined as follows:

Definition 3 Let us consider a set of local and bounded difference operators $(S_l)_{l \geq 0}$, defined in Proposition 1 with $S_0 := S$. The joint spectral radius in $(\ell^p(\mathbb{Z}^d))^{q_l}$ of S_l is given by

$$\begin{aligned} \rho_p(S_l) &:= \inf_{j \geq 0} \sup_{(w^0, \dots, w^{j-1}) \in (\ell^p(\mathbb{Z}^d))^j} \|S_l(w^{j-1}) \cdots S_l(w^0)\|_{(\ell^p(\mathbb{Z}^d))^{q_l} \rightarrow (\ell^p(\mathbb{Z}^d))^{q_l}}^{1/j} \\ &= \inf\{\rho, \|S_l(w^{j-1}) \cdots S_l(w^0) \Delta^l v\|_{(\ell^p(\mathbb{Z}^d))^{q_l}} \lesssim \rho^j \|\Delta^l v\|_{(\ell^p(\mathbb{Z}^d))^{q_l}}, \forall v \in \ell^p(\mathbb{Z}^d)\}. \end{aligned}$$

Remark 2 When $v^j = S(v^{j-1})v^{j-1}$, for all $j > 0$ we may write:

$$\Delta^l S(v^j)v^j = S_l(S(v^{j-1})v^{j-1})\Delta^l v^j = S_l(S(v^{j-1})v^{j-1})S_l(v^{j-1})\Delta^l v^{j-1} = \dots := (S_l)^j v^0.$$

This naturally leads to another definition of joint spectral radius putting $w^j = S^j v^0$ in the above definition. In [17], the following definition was introduced to study the convergence and stability of one dimensional power-P scheme: Let us consider a set of local and bounded difference operators $(S_l)_{l \geq 0}$, defined in Proposition 1 where $S_0 := S$. The joint spectral radius in $(\ell^p(\mathbb{Z}^d))^{q_l}$ of S_l is given by

$$\begin{aligned} \tilde{\rho}_p(S_l) &:= \inf_{j \geq 0} \| (S_l)^j \|_{(\ell^p(\mathbb{Z}^d))^{q_l} \rightarrow (\ell^p(\mathbb{Z}^d))^{q_l}}^{1/j} \\ &= \inf\{\rho, \|\Delta^l S^j v\|_{(\ell^p(\mathbb{Z}^d))^{q_l}} \lesssim \rho^j \|\Delta^l v\|_{(\ell^p(\mathbb{Z}^d))^{q_l}}, \forall v \in \ell^p(\mathbb{Z}^d)\}. \end{aligned}$$

Since our prediction operator is quasi-linear, the definition (3) is more appropriate.

Another ingredient that we will use in the proof of the inverse theorem is the following proposition on joint spectral radius that holds for any matrix M , $\det M \neq 0$:

Proposition 2 *Assume that $S_0 = S$ and that S reproduces polynomials up to total degree $N - 1$. Then*

$$\rho_p(S_{n+1}) \geq \frac{1}{\|M\|_\infty} \rho_p(S_n) \text{ for all } n = 0, \dots, N - 1.$$

In particular, if M is an isotropic matrix and S reproduces polynomials up to total degree $N - 1$, then

$$\rho_p(S_{n+1}) \geq m^{-\frac{1}{d}} \rho_p(S_n) \text{ for all } n = 0, \dots, N - 1.$$

The proof is available in [16]. Considering that φ exactly reproduces polynomials when the associated subdivision scheme does, we are ready to state the inverse theorem:

Theorem 2 *Let S be a quasi-linear prediction rule reproducing exactly polynomials up to total degree $N - 1$ and let φ exactly reproduces polynomials up to degree $N - 1$. Assume that $\rho_p(S_N) < m^{1/p-s/d}$ for some $s \geq N - 1$.*

– *If $\|v^0\|_{\ell^p(\mathbb{Z}^d)} + \sum_{j \geq 0} m^{-j/p} \|d^j\|_{\ell^p(\mathbb{Z}^d)} < \infty$, then the limit function v belongs to $L^p(\mathbb{R}^d)$ and*

$$\|v\|_{L^p(\mathbb{R}^d)} \leq \|v^0\|_{\ell^p(\mathbb{Z}^d)} + \sum_{j \geq 0} m^{-j/p} \|d^j\|_{\ell^p(\mathbb{Z}^d)} \quad (4.5)$$

– If (v^0, d^0, d^1, \dots) are such that

$$\|v^0\|_{\ell^p(\mathbb{Z}^d)} + \|(m^{(s/d-1/p)j} \|(d_k^j)_{k \in \mathbb{Z}^d}\|_{\ell^p(\mathbb{Z}^d)})_{j \geq 0}\|_{\ell^q(\mathbb{Z}^d)} < \infty,$$

the limit function v belongs to $B_{p,q}^s(\mathbb{R}^d)$ and

$$\|v\|_{B_{p,q}^s(\mathbb{R}^d)} \lesssim \|v^0\|_{\ell^p(\mathbb{Z}^d)} + \|(m^{(s/d-1/p)j} \|(d_k^j)_{k \in \mathbb{Z}^d}\|_{\ell^p(\mathbb{Z}^d)})_{j \geq 0}\|_{\ell^q(\mathbb{Z}^d)}. \quad (4.6)$$

Before we prove the theorem, we need to establish some extensions to the non-separable case of results obtained in [14]:

Lemma 3 *Let S be a data dependent prediction rule exactly reproducing polynomials of total degree $N - 1$. Then,*

$$\|v_{j+1} - v_j\|_{L^p(\mathbb{R}^d)} \lesssim m^{-j/p} \left(\|\Delta^N v^j\|_{(\ell^p(\mathbb{Z}^d))^{q_N}} + \|d^j\|_{\ell^p(\mathbb{Z}^d)} \right). \quad (4.7)$$

Moreover, if $\rho > \rho_p(S_N)$,

$$\|\Delta^N v^j\|_{(\ell^p(\mathbb{Z}^d))^{q_N}} \lesssim \rho^j \left(\|v^0\|_{\ell^p(\mathbb{Z}^d)} + \sum_{l=0}^{j-1} \rho^{-l} \|d^l\|_{\ell^p(\mathbb{Z}^d)} \right). \quad (4.8)$$

Proof: Using the definition of functions $v_j(x)$ and the scaling equation (3.1), we get

$$\begin{aligned} v_{j+1}(x) - v_j(x) &= \sum_k v_k^{j+1} \varphi(M^{j+1}x - k) - \sum_k v_k^j \varphi(M^j x - k) \\ &= \sum_k ((S(v^j)v^j)_k + d_k^j) \varphi(M^{j+1}x - k) - \sum_k v_k^j \sum_l g_{l-Mk} \varphi(M^{j+1}x - l) \\ &= \sum_k ((S(v^j)v^j)_k - \sum_l g_{k-Ml} v_l^j) \varphi(M^{j+1}x - k) + \sum_k d_k^j \varphi(M^{j+1}x - k). \end{aligned}$$

If S exactly reproduces polynomials up to total degree $N - 1$ then having in mind that $\sum_l g_{k-Ml} v_l^j$ is a linear prediction that exactly reproduces polynomials up to total degree $N - 1$ and using the same arguments as in Lemma 2, we get

$$\left\| \sum_k ((Sv^j)_k - \sum_l g_{k-Ml} v_l^j) \varphi(M^{j+1}x - k) \right\|_{L^p(\mathbb{R}^d)} \lesssim m^{-j/p} \|\Delta^N v^j\|_{\ell^p(\mathbb{Z}^d)}.$$

The proof of (4.7) is thus complete. To prove (4.8), we note that:

$$\begin{aligned} \|\Delta^N v^j\|_{(\ell^p(\mathbb{Z}^d))^{q_N}} &\lesssim \|S_N(v^{j-1})\Delta^N v^{j-1}\|_{(\ell^p(\mathbb{Z}^d))^{q_N}} + \|\Delta^N d^{j-1}\|_{(\ell^p(\mathbb{Z}^d))^{q_N}} \\ &\lesssim \rho \|\Delta^N v^{j-1}\|_{(\ell^p(\mathbb{Z}^d))^{q_N}} + \|\Delta^N d^{j-1}\|_{(\ell^p(\mathbb{Z}^d))^{q_N}} \\ &\lesssim \rho^j \left(\|v^0\|_{\ell^p(\mathbb{Z}^d)} + \sum_{l=0}^{j-1} \rho^{-l} \|d^l\|_{\ell^p(\mathbb{Z}^d)} \right). \end{aligned}$$

□

Proof of Theorem 2: From estimates (4.7) and (4.8) one has, in particular

$$\|v_{j+1} - v_j\|_{L^p(\mathbb{R}^d)} \lesssim m^{-j/p} \rho^j \left(\|v^0\|_{\ell^p(\mathbb{Z}^d)} + \sum_{l=0}^j \rho^{-l} \|d^l\|_{\ell^p(\mathbb{Z}^d)} \right) \quad (4.9)$$

for $\rho > \rho_p(S_1)$. From Proposition 2 it follows that for any $k \leq N$, $\rho_p(S_k) < m^{1/p-s/d+(N-k)/d}$, and in particular, since $s \geq N-1$, $\rho_p(S_1) < m^{1/p}$. We then choose ρ such that $\rho_p(S_1) < \rho < m^{1/p}$ to obtain:

$$\begin{aligned} \|v\|_{L^p(\mathbb{R}^d)} &\leq \|v_0\|_{L^p(\mathbb{R}^d)} + \sum_{j \geq 0} \|v_{j+1} - v_j\|_{L^p(\mathbb{R}^d)} \\ &\lesssim_{(4.9)} \|v^0\|_{\ell^p(\mathbb{Z}^d)} + \sum_{j \geq 0} m^{-j/p} \rho^j \left(\|v^0\|_{\ell^p(\mathbb{Z}^d)} + \sum_{l=0}^{j-1} \rho^{-l} \|d^l\|_{\ell^p(\mathbb{Z}^d)} \right) \\ &\lesssim \|v^0\|_{\ell^p(\mathbb{Z}^d)} \left(\sum_{j \geq 0} (m^{-1/p} \rho)^j + 1 \right) + \sum_{l \geq 0} \|d^l\|_{\ell^p(\mathbb{Z}^d)} \rho^{-l} \sum_{j > l} (m^{-1/p} \rho)^j \\ &\lesssim \|v^0\|_{\ell^p(\mathbb{Z}^d)} + \sum_{l \geq 0} \|d^l\|_{\ell^p(\mathbb{Z}^d)} m^{-l/p}. \end{aligned}$$

This proves (4.5).

Now let us prove (4.6). Since by Hölder inequality for any $q, q' > 0$, $\frac{1}{q} + \frac{1}{q'} = 1$, it holds that

$$\begin{aligned} \sum_{l \geq 0} \|d^l\|_{\ell^p(\mathbb{Z}^d)} m^{-l/p} &\leq \|(m^{(s/d-1/p)j} \|(d_k^j)_{k \in \mathbb{Z}^d}\|_{\ell^p(\mathbb{Z}^d)})_{j \geq 0}\|_{\ell^q(\mathbb{Z}^d)} \|(m^{-js/d})_{j \geq 0}\|_{\ell^{q'}(\mathbb{R}^d)} \\ &\lesssim \|(m^{(s/d-1/p)j} \|(d_k^j)_{k \in \mathbb{Z}^d}\|_{\ell^p(\mathbb{Z}^d)})_{j \geq 0}\|_{\ell^q(\mathbb{Z}^d)}, \end{aligned}$$

and finally,

$$\|v\|_{L^p(\mathbb{R}^d)} \lesssim \|v^0\|_{\ell^p(\mathbb{Z}^d)} + \|(m^{(s/d-1/p)j} \|(d_k^j)_{k \in \mathbb{Z}^d}\|_{\ell^p(\mathbb{Z}^d)})_{j \geq 0}\|_{\ell^q(\mathbb{Z}^d)}.$$

It remains to evaluate the semi-norm $|v|_{B_{p,q}^s(\mathbb{R}^d)} := \|(m^{js/d} \tilde{\omega}_N(v, m^{-j/d})_{L^p})_{j \geq 0}\|_{\ell^q(\mathbb{Z}^d)}$. For each $j \geq 0$, we have

$$\tilde{\omega}_N(v, m^{-j/d})_{L^p} \leq \tilde{\omega}_N(v - v_j, m^{-j/d})_{L^p} + \tilde{\omega}_N(v_j, m^{-j/d})_{L^p}. \quad (4.10)$$

For the first term on the right hand side of (4.10), one has using Lemma 3

$$\begin{aligned} \tilde{\omega}_N(v - v_j, m^{-j/d})_{L^p} &\lesssim \sum_{l \geq j} \|v_{l+1} - v_l\|_{L^p(\mathbb{R}^d)} \\ &\lesssim \sum_{l \geq j} m^{-l/p} (\|\Delta^N v^l\|_{\ell^p(\mathbb{Z}^d)} + \|d^l\|_{\ell^p(\mathbb{Z}^d)^{q_N}}) \\ &\lesssim \sum_{l \geq j} m^{-l/p} \left(\rho^l \|v^0\|_{\ell^p(\mathbb{Z}^d)} + \sum_{k=0}^l \rho^{l-k} \|d^k\|_{\ell^p(\mathbb{Z}^d)} \right). \end{aligned}$$

For the first term, choosing $\rho_p(S_N) < \rho < m^{1/p}$, we have

$$\sum_{l \geq j} m^{-l/p} \rho^l \|v^0\|_{\ell^p(\mathbb{Z}^d)} \sim m^{-j/p} \rho^j \|v^0\|_{\ell^p(\mathbb{Z}^d)}.$$

For the second term, we get

$$\begin{aligned}
& \sum_{l \geq j} m^{-l/p} \sum_{k=0}^l \rho^{l-k} \|d^k\|_{\ell^p(\mathbb{Z}^d)} \\
&= m^{-j/p} \sum_{k=0}^j \rho^{j-k} \|d^k\|_{\ell^p(\mathbb{Z}^d)} + \sum_{l > j} m^{-l/p} \sum_{k=0}^l \rho^{l-k} \|d^k\|_{\ell^p(\mathbb{Z}^d)} \\
&\lesssim m^{-j/p} \sum_{k=0}^j \rho^{j-k} \|d^k\|_{\ell^p(\mathbb{Z}^d)} + \sum_{k \geq 0} \sum_{l > \max(k, j)} m^{-l/p} \rho^{l-k} \|d^k\|_{\ell^p(\mathbb{Z}^d)} \\
&\lesssim m^{-j/p} \sum_{k=0}^j \rho^{j-k} \|d^k\|_{\ell^p(\mathbb{Z}^d)} + \sum_{k \geq 0} \|d^k\|_{\ell^p(\mathbb{Z}^d)} \rho^{-k} \sum_{l > \max(k, j)} \rho^l m^{-l/p} \\
&\lesssim m^{-j/p} \sum_{k=0}^j \rho^{j-k} \|d^k\|_{\ell^p(\mathbb{Z}^d)} + \sum_{k > j} m^{-k/p} \|d^k\|_{\ell^p(\mathbb{Z}^d)}.
\end{aligned}$$

Similarly, for the second term on the right hand side of (4.10), one has

$$\tilde{\omega}_N(v_j, m^{-j/d})_{L^p} \lesssim \|v_j\|_{L^p(\mathbb{R}^d)} \lesssim \|v\|_{L^p(\mathbb{R}^d)}$$

The estimate of the semi-norm $|v|_{B_{p,q}^s}$ is then reduced to the estimates of $\|(m^{j^s/d} a_j)_{j \geq 0}\|_{\ell^q(\mathbb{Z}^d)}$, $\|(m^{j^s/d} b_j)_{j \geq 0}\|_{\ell^q(\mathbb{Z}^d)}$ and $\|(m^{j^s/d} c_j)_{j \geq 0}\|_{\ell^q(\mathbb{Z}^d)}$, with

$$\begin{aligned}
a_j &:= m^{-j/p} \rho^j \|v^0\|_{\ell^p(\mathbb{Z}^d)}, \\
b_j &:= m^{-j/p} \rho^j \sum_{l=0}^j \rho^{-l} \|d^l\|_{\ell^p(\mathbb{Z}^d)}, \\
c_j &:= \sum_{l > j} m^{-l/p} \|d^l\|_{\ell^p(\mathbb{Z}^d)}.
\end{aligned}$$

The main point here is that $\rho_p(S_N)(m^{s/d-1/p}) < 1$, thus, we can choose $\rho > \rho_p(S_N)$ such that

$$\alpha = m^{s/d-1/p} \rho < 1.$$

Therefore

$$\|(\sigma^{j^s} a_j)_{j \geq 0}\|_{\ell^q(\mathbb{Z}^d)} = \|v^0\|_{\ell^p(\mathbb{Z}^d)} \|(\alpha^j)_{j \geq 0}\|_{\ell^q(\mathbb{Z}^d)} \lesssim \|v^0\|_{\ell^p(\mathbb{Z}^d)}. \quad (4.11)$$

In order to estimate $\|(m^{j^s/d} b_j)_{j \geq 0}\|_{\ell^q(\mathbb{Z}^d)}$, we rewrite it in the following form:

$$\begin{aligned}
\|(m^{j^s/d} b_j)_{j \geq 0}\|_{\ell^q(\mathbb{Z}^d)} &= \|(m^{j(s/d-1/p)} \rho^j \sum_{l=0}^j \rho^{-l} \|d^l\|_{\ell^p(\mathbb{Z}^d)})_{j \geq 0}\|_{\ell^q(\mathbb{Z}^d)} \\
&= \|(\alpha^j \sum_{l=0}^j \alpha^{-l} m^{(s/d-1/p)l} \|d^l\|_{\ell^p(\mathbb{Z}^d)})_{j \geq 0}\|_{\ell^q(\mathbb{Z}^d)}.
\end{aligned}$$

We now make use of the following discrete Hardy inequality: if $0 < \alpha < 1$, then

$$\|(\alpha^j \sum_{l=0}^j \alpha^{-l} x_l)_{j \geq 0}\|_{\ell^q(\mathbb{Z}^d)} \lesssim \|(x_j)_{j \geq 0}\|_{\ell^q(\mathbb{Z}^d)}.$$

Applying it to $x_l = m^{(s/d-1/p)l} \|d^l\|_{\ell^p(\mathbb{Z}^d)}$ yields

$$\|(m^{j s/d} b_j)_{j \geq 0}\|_{\ell^q(\mathbb{Z}^d)} \lesssim \|(m^{(s/d-1/p)j} \|(d_k^j)_{k \in \mathbb{Z}^d}\|_{\ell^p(\mathbb{Z}^d)})_{j \geq 0}\|_{\ell^q(\mathbb{Z}^d)}. \quad (4.12)$$

To estimate $\|(m^{j s/d} c_j)_{j \geq 0}\|_{\ell^q(\mathbb{Z}^d)}$, we rewrite it as follows

$$\|(m^{j s/d} c_j)_{j \geq 0}\|_{\ell^q(\mathbb{Z}^d)} = \|m^{j s/d} \sum_{l>j} m^{-ls/d} \left(m^{l(s/d-1/p)} \|(d_k^l)_{k \in \mathbb{Z}^d}\|_{\ell^p(\mathbb{Z}^d)} \right)\|_{\ell^q(\mathbb{Z}^d)}$$

and make use of another discrete Hardy inequality: if $\beta > 1$, then

$$\|(\beta^j \sum_{l>j} \beta^{-l} y_l)_{j \geq 0}\|_{\ell^q(\mathbb{Z}^d)} \lesssim \|(y_j)_{j \geq 0}\|_{\ell^q(\mathbb{Z}^d)}.$$

Taking $y_l = m^{l(s/d-1/p)} \|d^l\|_{\ell^p(\mathbb{Z}^d)}$, we obtain, since $s > N - 1$,

$$\|(m^{j s/d} c_j)_{j \geq 0}\|_{\ell^q(\mathbb{Z}^d)} \lesssim \|(m^{j(s/d-1/p)} \|(d_k^j)_{k \in \mathbb{Z}^d}\|_{\ell^p(\mathbb{Z}^d)})_{j \geq 0}\|_{\ell^q(\mathbb{Z}^d)}. \quad (4.13)$$

Then (4.6) follows by combining (4.11), (4.12) and (4.13). \square

5 Stability of Nonlinear Subdivision Schemes

We study the stability of the iteration $v^j = S(v^{j-1})v^{j-1}$ in Sobolev spaces $W_N^p(\mathbb{R}^d)$. We recall that the elements of $W_N^p(\mathbb{R}^d)$ are those functions of $L^p(\mathbb{R}^d)$ having their differential up to order N in $L^p(\mathbb{R}^d)$, the norm on $W_N^p(\mathbb{R}^d)$ being defined as follows:

$$\|v\|_{W_N^p(\mathbb{R}^d)} = \|v\|_{L^p} + \sum_{|\mu| \leq N} \|D^\mu v\|_{L^p(\mathbb{R}^d)}.$$

We use the following definition for the stability of subdivision scheme in Sobolev spaces:

Definition 4 Let S be a quasi-linear prediction rule, the subdivision scheme $v^j = S(v^{j-1})v^{j-1}$ is said to be stable in $W_N^p(\mathbb{R}^d)$ if for all v^0 and \tilde{v}^0 in $\ell^\infty(\mathbb{Z}^d)$, we have:

$$\|D^\mu v_j - D^\mu \tilde{v}_j\|_{L^p(\mathbb{R}^d)} \lesssim \|v^0 - \tilde{v}^0\|_{\ell^p(\mathbb{Z}^d)} \quad \forall |\mu| \leq N$$

where $v_j = \sum_{k \in \mathbb{Z}^d} v_k^j \varphi(M^j x - k)$, φ belonging to $W_N^p(\mathbb{R}^d)$

We now state the following theorem on the stability of the subdivision scheme in $W_N^p(\mathbb{R}^d)$.

Theorem 3 Let S be a quasi-linear prediction rule that exactly reproduces polynomials up to total degree N . Assume that S_N satisfies $\rho_p(S_{N+1}) < m^{1/p-N/d}$ and that

$$\|\Delta^{N+1}(v^j - \tilde{v}^j)\|_{(\ell^p(\mathbb{Z}^d))^{q_{N+1}}} \lesssim \rho^j \|v^0 - \tilde{v}^0\|_{\ell^p(\mathbb{Z}^d)} \quad (5.1)$$

for some $\rho_p(S_{N+1}) < \rho < m^{1/p-N/d}$, and also that φ is in $W_N^p(\mathbb{R}^d)$ and exactly reproduces polynomial up to total degree N then the subdivision scheme is stable in $W_N^p(\mathbb{R}^d)$.

Before proving the theorem we need to show the following lemma:

Lemma 4 Assume that S is a quasi-linear prediction rule exactly reproducing polynomials up to total degree N and that φ exactly reproduces polynomials up to total degree N then

$$\|v^j - \tilde{v}^j\|_{\ell^p(\mathbb{Z}^d)} \leq m^{1/p-N/d} \|v^{j-1} - \tilde{v}^{j-1}\|_{\ell^p(\mathbb{Z}^d)} + C \|\Delta^{N+1}(v^j - \tilde{v}^j)\|_{(\ell^p(\mathbb{Z}^d))^{q_{N+1}}} \quad (5.2)$$

PROOF: Let us consider φ that exactly reproduces polynomials up to total degree N ; it defines a subdivision scheme \tilde{S} that exactly reproduces polynomials up to total degree N . Then let us define for any $x = (x_1, \dots, x_d) \in \mathbb{R}^d$

$$r_i(x) = \sum_{l=1}^d \lambda_{i,l} x_l, \quad i = 1, \dots, d,$$

where the matrix $\Lambda = (\lambda_{i,l})_{i,l=1}^d$ is the same as in (2.2). For a multi-index $\mu = (\mu_1, \dots, \mu_d) \in \mathbb{N}^d$ let

$$r_\mu(x) := r_1^{\mu_1}(x) \cdots r_d^{\mu_d}(x)$$

and let us consider the differential operator

$$r_\mu(D) := r_1^{\mu_1}(D) \cdots r_d^{\mu_d}(D), \quad \text{where } r_i(D) = \sum_{l=1}^d \lambda_{i,l} D_{e_l}, \quad i = 1, \dots, d.$$

Let us for any Since Λ is invertible, the set $\{r_\mu : |\mu| = N\}$ forms a basis of the space of all polynomials of exact degree N , which proves that

$$\sum_{|\mu|=N} \|D_{e_1}^{\mu_1} \cdots D_{e_d}^{\mu_d}(v - v_j)\|_{L^p(\mathbb{R}^d)} \sum_{|\mu|=N} \sum_{l \geq j} \|r_\mu(D)(v_{l+1} - v_l)\|_{L^p(\mathbb{R}^d)}.$$

Now, we use the fact that M is isotropic, then $r_\mu(D)(f(M^l x)) = \sigma^{l\mu/d}(r_\mu(D)f)(M^l x)$ ([10]), where $\sigma^\mu = \prod_{i=1}^d \sigma_i^{\mu_i}$. With this in mind, we may write:

$$\|r_\mu(D)(v_j - v_{j-1})\|_{L^p(\mathbb{R}^d)} \sim m^{j(-1/p+N/d)} \|v^j - \tilde{S}v^{j-1}\|_{\ell^p(\mathbb{Z}^d)}$$

Using the consistency property and the fact that \tilde{S} exactly reproduces polynomials up to total degree N and also since the projection operator is linear, we may deduce

$$\begin{aligned} \|r_\mu(D)(v_j - \tilde{v}_j)\|_{L^p(\mathbb{R}^d)} &\leq \|r_\mu(D)(v_{j-1} - \tilde{v}_{j-1})\|_{L^p(\mathbb{R}^d)} + \\ &C m^{j(-1/p+N/d)} \|\Delta^{N+1}(v^j - \tilde{v}^j)\|_{(\ell^p(\mathbb{Z}^d))^{q_{N+1}}} \end{aligned}$$

and we finally get:

$$\|v^j - \tilde{v}^j\|_{\ell^p(\mathbb{Z}^d)} \leq m^{1/p-N/d} \|v^{j-1} - \tilde{v}^{j-1}\|_{\ell^p(\mathbb{Z}^d)} + C \|\Delta^{N+1}(v^j - \tilde{v}^j)\|_{(\ell^p(\mathbb{Z}^d))^{q_{N+1}}}$$

□

PROOF OF THEOREM (3): First, remark that:

$$\sum_{|\mu|=N} \|D^\mu(v_j - \tilde{v}_j)\|_{\ell^p(\mathbb{Z}^d)} \sim \sum_{|\mu|=N} \|r_\mu(D)(v_j - \tilde{v}_j)\|_{\ell^p(\mathbb{Z}^d)}$$

Then since the matrix M is isotropic we may write

$$\|r_\mu(D)(v_j - \tilde{v}_j)\|_{\ell^p(\mathbb{Z}^d)} \sim m^{j(-1/p+N/d)} \|v^j - \tilde{v}^j\|_{\ell^p(\mathbb{Z}^d)}.$$

Let ρ be such that $\rho_p(S_{N+1}) < \rho < m^{1/p-N/d}$. Let us consider the sequences $\alpha^j := m^{j(-1/p+N/d)} \|v^j - \tilde{v}^j\|_{\ell^p(\mathbb{Z}^d)}$ and $\beta^j = m^{j(-1/p+N/d)} \|\Delta^{N+1}(v^j - \tilde{v}^j)\|_{(\ell^p(\mathbb{Z}^d))^{q_{N+1}}}$. By (5.1) and Lemma 4, these sequences satisfy the following inequalities:

$$\begin{aligned} \alpha^j &\leq \alpha^{j-1} + D\beta^j \\ \beta^j &\leq C(\rho m^{(-1/p+N/d)})^j \alpha^0, \end{aligned}$$

the constant C being independent of j . From the above inequality we deduce the following estimation:

$$m^{j(-1/p+N/d)} \|v^j - \tilde{v}^j\|_{\ell^p(\mathbb{Z}^d)} \lesssim \|v^0 - \tilde{v}^0\|_{\ell^p(\mathbb{Z}^d)}$$

□

6 Stability of the Multi-Scale Representation

Let us consider two data sets (v^0, d^0, d^1, \dots) and $(\tilde{v}^0, \tilde{d}^0, \tilde{d}^1, \dots)$ with corresponding reconstruction processes

$$v^j = S(v^{j-1})v^{j-1} + e^j = S(v^{j-1})v^{j-1} + Ed^{j-1} \quad (6.1)$$

and

$$\tilde{v}^j = S(\tilde{v}^{j-1})\tilde{v}^{j-1} + \tilde{e}^j = S(\tilde{v}^{j-1})\tilde{v}^{j-1} + E\tilde{d}^{j-1}. \quad (6.2)$$

In that context, \tilde{v} is the limit of $\tilde{v}_j(x) = \sum_{k \in \mathbb{Z}^d} \tilde{v}_k^j \varphi(M^j x - k)$ (and similarly for v).

6.1 Stability in L^p Spaces

First, we study the stability of the multi-scale representation in $L^p(\mathbb{R}^d)$, which is stated in the following theorem:

Theorem 4 *Let S be a quasi-linear prediction rule that reproduces the constants and suppose that $\rho_p(S_1) < m^{\frac{1}{p}}$. Assume that*

$$\|\nabla v^j - \nabla \tilde{v}^j\|_{(\ell^p(\mathbb{Z}^d))^d} \lesssim \rho^j \left(\|v^0 - \tilde{v}^0\|_{\ell^p(\mathbb{Z}^d)} + \sum_{l=0}^{j-1} \rho^{-l} \|d^l - \tilde{d}^l\|_{\ell^p(\mathbb{Z}^d)} \right). \quad (6.3)$$

holds for some $\rho_p(S_1) < \rho < m^{1/p}$, then we have:

$$\|v - \tilde{v}\|_{L^p(\mathbb{R}^d)} \lesssim \|v^0 - \tilde{v}^0\|_{\ell^p(\mathbb{Z}^d)} + \sum_{l \geq 0} \rho^{-l} \|d^l - \tilde{d}^l\|_{\ell^p(\mathbb{Z}^d)}. \quad (6.4)$$

Proof: Let us take ρ such that $\rho_p(S_1) < \rho < m^{\frac{1}{p}}$. By letting $j \rightarrow \infty$, it is sufficient to prove that

$$\|v_j - \tilde{v}_j\|_{L^p(\mathbb{R}^d)} \lesssim \|v_0 - \tilde{v}_0\|_{L^p(\mathbb{R}^d)} + \sum_{l=1}^j \rho^{-l} \|d^l - \tilde{d}^l\|_{\ell^p(\mathbb{Z}^d)}, \quad (6.5)$$

with a constant independent of j . For simplicity, let

$$\begin{aligned} \alpha^j &:= m^{-j/p} \|v^j - \tilde{v}^j\|_{\ell^p(\mathbb{Z}^d)}, \\ \beta^j &:= m^{-j/p} \|\nabla v^j - \nabla \tilde{v}^j\|_{(\ell^p(\mathbb{Z}^d))^d}, \\ \gamma^j &:= \rho^{-j} \|d^j - \tilde{d}^j\|_{\ell^p(\mathbb{Z}^d)}. \end{aligned}$$

Using estimate (6.3), we get

$$\beta^j \lesssim \bar{\rho}^j \left(\alpha^0 + \sum_{l=0}^{j-1} \gamma^l \right),$$

where $\bar{\rho} = \rho m^{-1/p} < 1$. On the other hand, using the same technique as in the proof of Lemma 3 one can show that

$$\|v_j - \tilde{v}_j\|_{L^p(\mathbb{R}^d)} \leq \|v_{j-1} - \tilde{v}_{j-1}\|_{L^p(\mathbb{R}^d)} + D\beta^j.$$

i.e., $\alpha^j \leq \alpha^{j-1} + D\beta^j$. From this, we immediately deduce that:

$$\alpha^j \lesssim \alpha^0 + \sum_{l=1}^j \sum_{k=0}^{l-1} \bar{\rho}^l \gamma^k \lesssim \alpha^0 + \sum_{k=0}^{j-1} \gamma^k.$$

This ends the proof. \square

6.2 Stability in Besov spaces

In view of the inverse inequality (4.6), it seems natural to seek an inequality of type

$$\|v - \tilde{v}\|_{B_{p,q}^s(\mathbb{R}^d)} \lesssim \|v^0 - \tilde{v}^0\|_{\ell^p(\mathbb{Z}^d)} + \|(m^{(s/d-1/p)j} \|d^j - \tilde{d}^j\|_{\ell^p(\mathbb{Z}^d)})_{j \geq 0}\|_{\ell^q(\mathbb{Z}^d)}. \quad (6.6)$$

We now prove a stability theorem in Besov space $B_{p,q}^s(\mathbb{R}^d)$ in the following theorem:

Theorem 5 *Let us assume that S is a quasi-linear prediction rule which reproduces polynomials up to degree $N - 1$, that S is such that $\rho_p(S_N) < m^{1/p-s/d}$ for some $s > N - 1$ and assume that*

$$\|\Delta^N(v^j - \tilde{v}^j)\|_{(\ell^p(\mathbb{Z}^d))^{q_N}} \lesssim \rho^j \left(\|v^0 - \tilde{v}^0\|_{\ell^p(\mathbb{Z}^d)} + \sum_{l=0}^{j-1} \rho^{-l} \|d^l - \tilde{d}^l\|_{\ell^p(\mathbb{Z}^d)} \right) \quad (6.7)$$

for some $\rho_p(S_N) < \rho < m^{1/p-s/d}$. Then the function $v - \tilde{v}$ belongs to $B_{p,q}^s(\mathbb{R}^d)$. Moreover, we obtain for that ρ

$$\|v - \tilde{v}\|_{B_{p,q}^s(\mathbb{R}^d)} \lesssim \|v^0 - \tilde{v}^0\|_{\ell^p(\mathbb{Z}^d)} + \|(\rho^{-j} \|(d_k^j - \tilde{d}_k^j)_{k \in \mathbb{Z}^d}\|_{\ell^p(\mathbb{Z}^d)})_{j \geq 0}\|_{\ell^q(\mathbb{Z}^d)}. \quad (6.8)$$

Remark 3 With the hypothesis (6.7), which is a natural extension of (4.8) for the stability issue, we have no specific condition on s, p, q which was not the case with the tensor product method proposed in [14]. Furthermore, we shall note that a similar study was proposed in the one dimensional case to study the stability of so-called r-shift invariant subdivision operators [17] with a slightly lighter hypothesis than (6.7). However, in [17] the stability is obtained only when $\rho(S_N) < 1$, while with our approach the condition on the joint spectral radius is the same both for the stability and the smoothness.

Proof of Theorem (5): Using the same technique as in Lemma 3, we may write that:

$$\begin{aligned} \|v^j - \tilde{v}^j\|_{L^p(\mathbb{R}^d)} &\lesssim \|v^{j-1} - \tilde{v}^{j-1}\|_{L^p(\mathbb{R}^d)} + \\ &\quad C m^{-j/p} \left(\|\Delta^N(v^j - \tilde{v}^j)\|_{(\ell^p(\mathbb{Z}^d))^{q_N}} + \|d^j - \tilde{d}^j\|_{\ell^p(\mathbb{Z}^d)} \right) \end{aligned}$$

Then using the hypothesis (6.7), we get:

$$\begin{aligned} \|v^j - \tilde{v}^j\|_{L^p(\mathbb{R}^d)} &\lesssim \|v^{j-1} - \tilde{v}^{j-1}\|_{L^p(\mathbb{R}^d)} + \\ &\quad C(\rho^j m^{-j/p}) \left(\|v^0 - \tilde{v}^0\|_{\ell^p(\mathbb{Z}^d)} + \sum_{l=0}^{j-1} \rho^{-l} \|d^l - \tilde{d}^l\|_{\ell^p(\mathbb{Z}^d)} \right) \end{aligned}$$

Using the fact that $\rho m^{-1/p} < 1$ we immediately get:

$$\|v^j - \tilde{v}^j\|_{L^p} \lesssim \|v^0 - \tilde{v}^0\|_{\ell^p(\mathbb{Z}^d)} + \sum_{l=0}^{j-1} \rho^{-l} \|d^l - \tilde{d}^l\|_{\ell^p(\mathbb{Z}^d)},$$

from which we obtain when $j \rightarrow +\infty$:

$$\|v - \tilde{v}\|_{L^p} \lesssim \|v^0 - \tilde{v}^0\|_{\ell^p(\mathbb{Z}^d)} + \sum_{l \geq 0} \rho^{-l} \|d^l - \tilde{d}^l\|_{\ell^p(\mathbb{Z}^d)},$$

Since by the constraints on s we can put $\rho = m^{1/p + (-s-\varepsilon)/d}$ for some positive ε . This enables us write

$$\begin{aligned} \|v - \tilde{v}\|_{L^p(\mathbb{R}^d)} &\leq \|v^0 - \tilde{v}^0\|_{\ell^p(\mathbb{Z}^d)} + \|(m^{(s/d-1/p)j} \|d^j - \tilde{d}^j\|_{\ell^p(\mathbb{Z}^d)})_{j \geq 0}\|_{\ell^q(\mathbb{Z}^d)} \|(m^{-\varepsilon/dj})_{j \geq 0}\|_{\ell^{q'}(\mathbb{R}^d)} \\ &\lesssim \|v^0 - \tilde{v}^0\|_{\ell^p(\mathbb{Z}^d)} + \|(m^{(s/d-1/p)j} \|d^j - \tilde{d}^j\|_{\ell^p(\mathbb{Z}^d)})_{j \geq 0}\|_{\ell^q(\mathbb{Z}^d)}, \end{aligned}$$

It remains to estimate the semi-norm

$$|w|_{B_{p,q}^s(\mathbb{R}^d)} := \|(m^{js/d} \omega_N(w, m^{-j/d})_{L^p})_{j \geq 0}\|_{\ell^q(\mathbb{Z}^d)},$$

for $w := v - \tilde{v}$. For every $j \geq 0$, denoting $w_j = v_j - \tilde{v}_j$, we have

$$\omega_N(w, m^{-j/d})_{L^p} \leq \omega_N(w - w_j, m^{-j/d})_{L^p} + \omega_N(w_j, m^{-j/d})_{L^p}. \quad (6.9)$$

For the first term, using successively Lemma (3) and hypothesis (6.7), one has

$$\begin{aligned} \omega_N(w - w_j, m^{-j/d})_{L^p} &\lesssim \sum_{l \geq j} \|w_{l+1} - w_l\|_{L^p(\mathbb{R}^d)} \\ &\lesssim \sum_{l \geq j} m^{-l/p} \left(\|\Delta^N(v^l - \tilde{v}^l)\|_{(\ell^p(\mathbb{Z}^d))^{q_N}} + \|d^l - \tilde{d}^l\|_{\ell^p(\mathbb{Z}^d)} \right) \\ &\lesssim \sum_{l \geq j} m^{-l/p} \left(\rho^l \|v^0 - \tilde{v}^0\|_{\ell^p(\mathbb{Z}^d)} + \sum_{k=0}^l \rho^{l-k} \|d^k - \tilde{d}^k\|_{\ell^p(\mathbb{Z}^d)} \right). \end{aligned}$$

Since $\rho < m^{1/p}$, then for the first term we have

$$\sum_{l \geq j} m^{-l/p} \rho^l \|v^0 - \tilde{v}^0\|_{\ell^p(\mathbb{Z}^d)} \sim m^{-j/p} \rho^j \|v^0 - \tilde{v}^0\|_{\ell^p(\mathbb{Z}^d)}.$$

For the second term, we get

$$\begin{aligned} \sum_{l \geq j} m^{-l/p} \sum_{k=0}^l \rho^{l-k} \|d^k - \tilde{d}^k\|_{\ell^p(\mathbb{Z}^d)} &= \\ &= m^{-j/p} \sum_{k=0}^j \rho^{j-k} \|d^k - \tilde{d}^k\|_{\ell^p(\mathbb{Z}^d)} + \sum_{l > j} m^{-l/p} \sum_{k=0}^l \rho^{l-k} \|d^k - \tilde{d}^k\|_{\ell^p(\mathbb{Z}^d)} \\ &\lesssim m^{-j/p} \sum_{k=0}^j \rho^{j-k} \|d^k - \tilde{d}^k\|_{\ell^p(\mathbb{Z}^d)} + \sum_{k \geq 0} \sum_{l \geq \max(k, j+1)} m^{-l/p} \rho^{l-k} \|d^k - \tilde{d}^k\|_{\ell^p(\mathbb{Z}^d)} \\ &\lesssim m^{-j/p} \sum_{k=0}^j \rho^{j-k} \|d^k - \tilde{d}^k\|_{\ell^p(\mathbb{Z}^d)} + \sum_{k > j} m^{-k/p} \|d^k - \tilde{d}^k\|_{\ell^p(\mathbb{Z}^d)}. \end{aligned}$$

The second term in (6.9) is evaluated as follows:

$$\begin{aligned} \omega_N(v_j - \tilde{v}_j, m^{-j/d})_{L^p} &\lesssim \|v_j - \tilde{v}_j\|_{L^p(\mathbb{R}^d)} \\ &\lesssim m^{-j/p} \rho^j \|v^0 - \tilde{v}^0\|_{\ell^p(\mathbb{Z}^d)} + m^{-j/p} \sum_{l=0}^j \rho^{j-l} \|d^l - \tilde{d}^l\|_{\ell^p(\mathbb{Z}^d)}. \end{aligned}$$

The second term on the right hand side of (6.9), can be evaluated the same way. We have thus reduced the estimate of $|w|_{B_{p,q}^s(\mathbb{R}^d)}$, to the estimates of the discrete norms $\|(m^{j_s/d} a_j)_{j \geq 0}\|_{\ell^q(\mathbb{Z}^d)}$, and $\|(m^{j_s/d} b_j)_{j \geq 0}\|_{\ell^q(\mathbb{Z}^d)}$, where the sequences are defined as follows:

$$\begin{aligned} a_j &:= \rho^j m^{-j/p} \|v^0 - \tilde{v}^0\|_{\ell^p(\mathbb{Z}^d)}, \\ b_j &:= m^{-j/p} \sum_{l=0}^j \rho^{j-l} \|d^l - \tilde{d}^l\|_{\ell^p(\mathbb{Z}^d)}, \\ c_j &:= \sum_{l>j} m^{-l/p} \|d^l - \tilde{d}^l\|_{\ell^p(\mathbb{Z}^d)}. \end{aligned}$$

Note that this quantities are identical to that obtained in the convergence theorem replacing v^l by $v^l - \tilde{v}^l$ and d^l by $d^l - \tilde{d}^l$, so that the end of the proof is identical. \square

7 Bi-dimensional Nonlinear Affine Prediction Using the Hexagonal Dilation Matrix

We now focus on the construction of nonlinear multi-scale representations using as dilation matrix the hexagonal matrix

$$M = \begin{pmatrix} 2 & 1 \\ 0 & -2 \end{pmatrix}.$$

In that context, we make up a partition of \mathbb{Z}^2 using the coset vectors of the matrix M . For the matrix M , there are m coset vectors which are the following for the hexagonal matrix: $\varepsilon_0 = (0, 0)^T$, $\varepsilon_1 = (1, 0)^T$, $\varepsilon_2 = (1, -1)^T$, $\varepsilon_3 = (2, -1)^T$. These coset vectors satisfy $\mathbb{Z}^2 = \bigcup_{i=0}^3 \{Mk + \varepsilon_i, k \in \mathbb{Z}^2\}$. We now build the prediction \hat{v}^j for the different coset points $Mk + \varepsilon_i$ using an affine interpolant defined on the coarse grid $\Gamma^{j-1} = \{M^{-j+1}k, k \in \mathbb{Z}^2\}$ corresponding to the location of the values $(v_k^{j-1})_{k \in \mathbb{Z}^2}$ (see (3.3)). To build the affine interpolant, we use one of the following four different stencils on the grid Γ^{j-1} :

$$\begin{aligned} V_k^{j,1} &= M^{-j+1} \{k, k + e_1, k + e_2\}, \\ V_k^{j,2} &= M^{-j+1} \{k, k + e_2, k + e_1 + e_2\}, \\ W_k^{j,1} &= M^{-j+1} \{k + e_1, k + e_2, k + e_1 + e_2\}, \\ W_k^{j,2} &= M^{-j+1} \{k, k + e_1, k + e_1 + e_2\}. \end{aligned}$$

Each approximated value v_k^j being associated to the location $M^{-j}k$, we determine the stencil this point belongs to, and we then define the prediction as the value of the affine

interpolant at $M^{-j}k$ using the selected stencil. In that context, the prediction rules at Mk and $Mk + \varepsilon_1$ are independent of the choice of the stencil, and we always have:

$$v_{Mk}^j = v_k^{j-1} \text{ and } v_{Mk+\varepsilon_1}^j = \frac{1}{2}v_k^{j-1} + \frac{1}{2}v_{k+\varepsilon_1}^{j-1}. \quad (7.1)$$

For the prediction at $Mk + \varepsilon_2$, we use stencils $V_k^{j,1}$ or $V_k^{j,2}$ leading respectively to:

$$\begin{aligned} v_{Mk+\varepsilon_2}^{j,1} &= \frac{1}{4}v_{k+\varepsilon_1}^{j-1} + \frac{1}{2}v_{k+\varepsilon_2}^{j-1} + \frac{1}{4}v_k^{j-1} \\ v_{Mk+\varepsilon_2}^{j,2} &= \frac{1}{2}v_k^{j-1} + \frac{1}{4}v_{k+\varepsilon_2}^{j-1} + \frac{1}{4}v_{k+\varepsilon_1+\varepsilon_2}^{j-1}. \end{aligned} \quad (7.2)$$

For the points $Mk + \varepsilon_3$, we use $W_k^{j,1}$ or $W_k^{j,2}$ leading respectively to:

$$\begin{aligned} v_{Mk+\varepsilon_3}^{j,1} &= \frac{1}{4}v_{k+\varepsilon_2}^{j-1} + \frac{1}{4}v_{k+\varepsilon_1+\varepsilon_2}^{j-1} + \frac{1}{2}v_{k+\varepsilon_1}^{j-1} \\ v_{Mk+\varepsilon_3}^{j,2} &= \frac{1}{4}v_k^{j-1} + \frac{1}{4}v_{k+\varepsilon_1}^{j-1} + \frac{1}{2}v_{k+\varepsilon_1+\varepsilon_2}^{j-1} \end{aligned} \quad (7.3)$$

when the stencils $W_k^{j,1}$ and $W_k^{j,2}$ are used respectively.

This leads to four different linear prediction rules depending on the choice for the prediction operator for coset vector ε_2 and ε_3 . The described quasi-linear prediction operator satisfies $\rho_\infty(S_1) < 1$ since we have:

Proposition 3 *The prediction defined by (7.1), (7.2), (7.3) satisfies:*

$$\|\nabla \hat{v}_{M+\varepsilon_i}^j\|_{(l^\infty(\mathbb{Z}^2))^2} \leq \frac{3}{4} \|\nabla v^{j-1}\|_{(l^\infty(\mathbb{Z}^2))^2}.$$

The proof of the above proposition is carried out considering all the possibilities for the choice of stencil. This example shows how the tensor product approach developed in [14] can be generalized to a non-diagonal dilation matrix M .

Now, to generalize the ENO (essentially non-oscillatory [14]) method to our context, the choice of stencil has to aim at a sparser representation of edges. For instance, this could be interesting for image compression purpose. The choice of stencil can be determined by minimizing a certain cost function that is built using first order differences in such a way that the cost is high for stencil intersecting an edge and low otherwise. In that framework, we can consider the following cost function to predict at $Mk + \varepsilon_2$:

$$\begin{aligned} C_H^{j,\varepsilon_2}(k) &= \min(|v_{k+\varepsilon_1}^{j-1} - v_{k+\varepsilon_2}^{j-1}| + |v_{k+\varepsilon_1}^{j-1} - v_k^{j-1}|, |v_k^{j-1} - v_{k+\varepsilon_1+\varepsilon_2}^{j-1}| + |v_{k+\varepsilon_1+\varepsilon_2}^{j-1} - v_{k+\varepsilon_1}^{j-1}|) \\ &= \min(C_H^{j,\varepsilon_2,1}(k), C_H^{j,\varepsilon_2,2}(k)). \end{aligned}$$

When the minimum corresponds to the first (resp. second) argument, the stencil V_k^1 (resp. V_k^2) is used. Similarly, when one considers the prediction at $Mk + \varepsilon_3$, we can consider:

$$\begin{aligned} C_H^{j,\varepsilon_3}(k) &= \min(|v_{k+\varepsilon_1}^{j-1} - v_{k+\varepsilon_2}^{j-1}| + |v_{k+\varepsilon_1+\varepsilon_2}^{j-1} - v_{k+\varepsilon_2}^{j-1}|, |v_k^{j-1} - v_{k+\varepsilon_1}^{j-1}| + |v_{k+\varepsilon_1+\varepsilon_2}^{j-1} - v_k^{j-1}|) \\ &= \min(C_H^{j,\varepsilon_3,1}(k), C_H^{j,\varepsilon_3,2}(k)). \end{aligned}$$

When the minimum corresponds to the first (resp. second) argument, the stencil W_k^1 (resp. W_k^2) is used. This technique has been recently used for image compression [15].

Finally, note that making a convex combination of potential operators given by the ENO method, we obtain an extension of the so-called WENO method, that is for $i = 2, 3$:

$$\hat{v}_{Mk+\varepsilon_i}^j := \alpha_1 \hat{v}_{Mk+\varepsilon_i}^{j,1} + \alpha_2 \hat{v}_{Mk+\varepsilon_i}^{j,2}$$

A possible form for the weights is given by $\alpha_r := \frac{a_r}{a_1+a_2}$ where $a_r := \frac{1}{\varepsilon + C^{j,\varepsilon_i,r}(k)}$, and ε is used to avoid $a_r = +\infty$ when $C^{j,\varepsilon_i,r}(k) = 0$. Finally, we shall conclude by saying that our approach could be extended to other type of matrix of non-diagonal matrix M (an example is available in [15]) with a noticeable improvement on compression performance over linear method.

8 Conclusion

In this paper, we have presented a new nonlinear multi-scale representation based on the use of non-diagonal dilation matrices. We have shown that the non-linear scheme proposed by Harten naturally extends in that context and that convergence and stability in Besov spaces can be obtained when the matrix M is isotropic. These results are deeply related to the existence of difference operators associated to the prediction operator we use. The main difference with the tensor product approach consists in the fact these difference operators use mixed finite differences and thus cannot be reduced to one dimensional difference operators. This specificity obliged us to consider a definition of Besov spaces using mixed finite differences. After we have shown these theoretical results, we have ended the paper by giving some applications of the new multi-scale representation we propose when the underlying subdivision scheme is interpolatory. Future work should involve the designing of non-interpolatory multi-scale representations and also possible extension of our method to more general prediction operators than quasi-linear ones.

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