Flat Systems

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Introduction

Control systems are ubiquitous in modern technology. The use of feedback control can be found in systems ranging from simple thermostats that regulate the temperature of a room, to digital engine controllers that govern the operation of engines in cars, ships, and planes, to flight control systems for high performance aircraft. The rapid advances in sensing, computation, and actuation technologies is continuing to drive this trend and the role of control theory in advanced (and even not so advanced) systems is increasing.

A typical use of control theory in many modern systems is to invert the system dynamics to compute the inputs required to perform a specific task. This inversion may involve finding appropriate inputs to steer a control system from one state to another or may involve finding inputs to follow a desired trajectory for some or all of the state variables of the system. In general, the solution to a given control problem will not be unique, if it exists at all, and so one must trade off the performance of the system for the stability and actuation effort. Often this tradeoff is described as a cost function balancing the desired performance objectives with stability and effort, resulting in an optimal control problem.

This inverse dynamics problem assumes that the dynamics for the system are known and fixed. In practice, uncertainty and noise are always present in systems and must be accounted for in order to achieve acceptable performance of this system. Feedback control formulations allow the system to respond to errors and changing operating conditions in real-time and can substantially affect the operability of the system by stabilizing the system and extending its capabilities. Again, one may formulate the feedback regulation problems as an optimization problem to allow tradeoffs between stability, performance, and actuator effort.

The basic paradigm used in most, if not all, control techniques is to exploit the mathematical structure of the system to obtain solutions to the inverse dynamics and feedback regulation problems. The most common structure to exploit is linear structure, where one approximates the given system by its linearization and then uses properties of linear control systems combined with appropriate cost function to give closed form (or at least numerically computable) solutions. By using different linearizations around different operating points, it is even possible to obtain good results when the system is nonlinear by "scheduling" the gains depending on the operating point.

As the systems that we seek to control become more complex, the use of linear structure alone is often not sufficient to solve the control problems that are arising in applications. This is especially true of the inverse dynamics problems, where the desired task may span multiple operating regions and hence the use of a single linear system is inappropriate.

In order to solve these harder problems, control theorists look for different types of structure to exploit in addition to simple linear structure. In this paper we concentrate on a specific class of systems, called "(differentially) flat systems", for which the structure of the trajectories of the (nonlinear) dynamics can be completely characterized. Flat systems are a generalization of linear systems (in the sense that all linear, controllable systems are flat), but the techniques used for controlling flat systems are much different than many of the existing techniques for linear systems. As we shall see, flatness is particularly well tuned for allowing one to solve the inverse dynamics problems and one builds off of that fundamental solution in using the structure of flatness to solve more general control problems.

Flatness was first defined by Fliess et al. [13, 16] using the formalism of differential algebra, see also [33] for a somewhat different approach. In differential algebra, a system is viewed as a differential field generated by a set of variables (states and inputs). The system is said to be flat if one can find a set of variables, called the flat outputs, such that the system is (non-differentially) algebraic over the differential field generated by the set of flat outputs. Roughly speaking, a system is flat if we can find a set of outputs (equal in number to the number of inputs) such that all states and inputs can be determined from these outputs without integration. More precisely, if the system has states $x \in \mathbb{R}^n$, and inputs $u \in \mathbb{R}^m$ then the system is flat if we can find outputs $y \in \mathbb{R}^m$ of the form

$$y = h(x, u, \dot{u}, \dots, u^{(r)})$$

such that

$$x = \varphi(y, \dot{y}, \dots, y^{(q)})$$
$$u = \alpha(y, \dot{y}, \dots, y^{(q)}).$$

More recently, flatness has been defined in a more geometric context, where tools for nonlinear control are more commonly available. One approach is to use exterior differential systems and regard a nonlinear control system as a Pfaffian system on an appropriate space [51]. In this context, flatness can be described in terms of the notion of absolute equivalence defined by E. Cartan [6, 7, 70].

In this paper we adopt a somewhat different geometric point of view, relying on a Lie-Bäcklund framework as the underlying mathematical structure. This point of view was originally described by Fliess et al. in 1993 [14] and is related to the work of Pomet et al. [57, 55] on "infinitesimal Brunovsky forms" (in the context of feedback linearization). It offers a compact framework in which to describe basic results and is also closely related to the basic techniques that are used to compute the functions that are required to characterize the solutions of flat systems (the so-called flat outputs).

Applications of flatness to problems of engineering interest have grown steadily in recent years. It is important to point out that many classes of systems commonly used in nonlinear control theory are flat, see for instance the examples in section 4. As already noted, all controllable linear systems can be shown to be flat. Indeed, any system that can be transformed into a linear system by changes of coordinates, static feedback transformations (change of coordinates plus nonlinear change of inputs), or dynamic feedback transformations is also flat. Nonlinear control systems in "pure feedback form", which have gained popularity due to the applicability of backstepping [29] to such systems, are also flat. Thus, many of the systems for which strong nonlinear control techniques are available are in fact flat systems, leading one to question how the structure of flatness plays a role in control of such systems.

One common misconception is that flatness amounts to dynamic feedback linearization. It is true that any flat system can be feedback linearized using dynamic feedback (up to some regularity conditions that are generically satisfied). However, flatness is a property of a system and does not imply that one intends to then transform the system, via a dynamic feedback and appropriate changes of coordinates, to a single linear system. Indeed, the power of flatness is precisely that it does not convert nonlinear systems into linear ones. When a system is flat it is an indication that the nonlinear structure of the system is well characterized and one can exploit that structure in designing control algorithms for motion planning, trajectory generation, and stabilization. Dynamic feedback linearization is one such technique, although it is often a poor choice if the dynamics of the system are substantially different in different operating regimes.

Another advantage of studying flatness over dynamic feedback linearization is that flatness is a *geometric* property of a system, independent of coordinate choice. Typically when one speaks of linear systems in a state space context, this does not make sense geometrically since the system is linear only in certain choices of coordinate representations. In particular, it is difficult to discuss the notion of a linear state space system on a manifold since the very definition of linearity requires an underlying linear space. In this way, flatness can be considered the proper geometric notion of linearity, even though the system may be quite nonlinear in almost any natural representation.

Finally, the notion of flatness can be extended to distributed parameters systems with boundary control, see section 3.2.2, and is useful even for controlling linear systems, whereas feedback linearization is yet to be defined in that context.

This paper provides a self-contained description of flat systems. Section 1 introduces the fundamental concepts of equivalence and flatness in a simple geometric framework. This is essentially an open-loop point of view. In section 2 we adopt a closed-loop point of view and relate equivalence and flatness to feedback design. Section 3 is devoted to open problems and new perspectives including developments on symmetries and distributed parameters systems. Finally, section 4 contains a representative catalog of flat systems arising in various fields of engineering.

1 Equivalence and flatness

1.1 Control systems as infinite dimensional vector fields

A system of differential equations

$$\dot{x} = f(x), \quad x \in X \subset \mathbb{R}^n \tag{1}$$

is by definition a pair (X, f), where X is an open set of $\mathbb{R}^n$ and f is a smooth vector field on X. A solution, or *trajectory*, of (1) is a mapping $t \mapsto x(t)$ such that

$$\dot{x}(t) = f(x(t)) \qquad \forall t \geq 0.$$

Notice that if $x \mapsto h(x)$ is a smooth function on X and $t \mapsto x(t)$ is a trajectory of (1), then

$$\frac{d}{dt}h(x(t)) = \frac{\partial h}{\partial x}(x(t)) \cdot \dot{x}(t) = \frac{\partial h}{\partial x}(x(t)) \cdot f(x(t)) \qquad \forall t \geq 0.$$

For that reason the *total derivative*, i.e., the mapping

$$x \mapsto \frac{\partial h}{\partial x}(x) \cdot f(x)$$

is somewhat abusively called the "time-derivative" of h and denoted by $\dot{h}$.

We would like to have a similar description, i.e., a "space" and a vector field on this space, for a control system

$$\dot{x} = f(x, u), \tag{2}$$

where f is smooth on an open subset $X \times U \subset \mathbb{R}^n \times \mathbb{R}^m$. Here f is no longer a vector field on X, but rather an *infinite collection* of vector fields on X parameterized by u: for all $u \in U$, the mapping

$$x \mapsto f_u(x) = f(x, u)$$

is a vector field on X. Such a description is not well-adapted when considering dynamic feedback.

It is nevertheless possible to associate to (2) a vector field with the "same" solutions using the following remarks: given a smooth solution of (2), i.e., a mapping $t \mapsto (x(t), u(t))$ with values in $X \times U$ such that

$$\dot{x}(t) = f(x(t), u(t)) \qquad \forall t \geq 0,$$

we can consider the *infinite* mapping

$$t \mapsto \xi(t) = (x(t), u(t), \dot{u}(t), \ldots)$$

taking values in $X \times U \times \mathbb{R}^\infty_m$, where $\mathbb{R}^\infty_m = \mathbb{R}^m \times \mathbb{R}^m \times \ldots$ denotes the product of an infinite (countable) number of copies of $\mathbb{R}^m$. A typical point of $\mathbb{R}^\infty_m$ is thus of the form $(u^1, u^2, \ldots)$ with $u^i \in \mathbb{R}^m$. This mapping satisfies

$$\dot{\xi}(t) = \big(f(x(t), u(t)), \dot{u}(t), \ddot{u}(t), \ldots\big) \qquad \forall t \geq 0,$$

hence it can be thought of as a trajectory of the *infinite* vector field

$$(x, u, u^1, \ldots) \mapsto F(x, u, u^1, \ldots) = (f(x, u), u^1, u^2, \ldots)$$

on $X \times U \times \mathbb{R}^\infty_m$. Conversely, any mapping

$$t \mapsto \xi(t) = (x(t), u(t), u^1(t), \ldots)$$

that is a trajectory of this infinite vector field necessarily takes the form $(x(t), u(t), \dot{u}(t), \ldots)$ with $\dot{x}(t) = f(x(t), u(t))$, hence corresponds to a solution of (2). Thus F is truly a vector field and no longer a parameterized family of vector fields.

Using this construction, the control system (2) can be seen as the data of the "space" $X \times U \times \mathbb{R}_m^\infty$ together with the "smooth" vector field F on this space. Notice that, as in the uncontrolled case, we can define the "time-derivative" of a smooth function $(x, u, u^1, \ldots) \mapsto h(x, u, u^1, \ldots, u^k)$ depending on a *finite* number of variables by

$$\begin{aligned} \dot{h}(x, u, u^1, \ldots, u^{k+1}) &:= Dh \cdot F \\ &= \frac{\partial h}{\partial x} \cdot f(x, u) + \frac{\partial h}{\partial u} \cdot u^1 + \frac{\partial h}{\partial u^1} \cdot u^2 + \cdots . \end{aligned}$$

The above sum is *finite* because h depends on finitely many variables.

Remark. To be rigorous we must say something of the underlying topology and differentiable structure of $\mathbb{R}_m^\infty$ to be able to speak of smooth objects [76]. This topology is the *Fréchet topology*, which makes things look as if we were working on the product of k copies of $\mathbb{R}^m$ for a "large enough" k. For our purpose it is enough to know that a basis of the open sets of this topology consists of infinite products $U_0 \times U_1 \times \ldots$ of open sets of $\mathbb{R}^m$, and that a function is *smooth* if it depends on a *finite* but arbitrary number of variables and is smooth in the usual sense. In the same way a mapping $\Phi : \mathbb{R}_m^\infty \to \mathbb{R}_n^\infty$ is smooth if all of its components are smooth functions.

$\mathbb{R}_m^\infty$ equipped with the Fréchet topology has very weak properties: useful theorems such as the implicit function theorem, the Frobenius theorem, and the straightening out theorem no longer hold true. This is only because $\mathbb{R}_m^\infty$ is a very big space: indeed the Fréchet topology on the product of k copies of $\mathbb{R}^m$ for any finite k coincides with the usual Euclidian topology.

We can also define manifolds modeled on $\mathbb{R}_m^\infty$ using the standard machinery. The reader not interested in these technicalities can safely ignore the details and won't loose much by replacing "manifold modeled on $\mathbb{R}_m^\infty$" by "open set of $\mathbb{R}_m^\infty$".

We are now in position to give a formal definition of a system:

Definition 1. A *system* is a pair $(\mathfrak{M}, F)$ where $\mathfrak{M}$ is a smooth manifold, possibly of infinite dimension, and F is a smooth vector field on $\mathfrak{M}$.

Locally, a control system looks like an open subset of $\mathbb{R}^\alpha$ (α not necessarily finite) with coordinates $(\xi_1, \ldots, \xi_\alpha)$ together with the vector field

$$\xi \mapsto F(\xi) = (F_1(\xi), \ldots, F_\alpha(\xi))$$

where all the components F_i depend only on a finite number of coordinates. A *trajectory* of the system is a mapping $t \mapsto \xi(t)$ such that $\dot{\xi}(t) = F(\xi(t))$.

We saw in the beginning of this section how a "traditional" control system fits into our definition. There is nevertheless an important difference: we lose the notion of *state dimension*. Indeed

$$\dot{x} = f(x,u), \quad (x,u) \in X \times U \subset \mathbb{R}^n \times \mathbb{R}^m \tag{3}$$

and

$$\dot{x} = f(x,u), \qquad \dot{u} = v \tag{4}$$

now have the same description $(X \times U \times \mathbb{R}_m^\infty, F)$, with

$$F(x,u,u^1,\ldots) = (f(x,u), u^1, u^2, \ldots),$$

in our formalism: $t \mapsto (x(t), u(t))$ is a trajectory of (3) if and only if $t \mapsto (x(t), u(t), \dot{u}(t))$ is a trajectory of (4). This situation is not surprising since the state dimension is of course not preserved by dynamic feedback. On the other hand we will see there is still a notion of *input dimension*.

Example 1 *(The trivial system).* The *trivial system* $(\mathbb{R}_m^\infty, F_m)$, with coordinates $(y, y^1, y^2, \ldots)$ and vector field

$$F_m(y, y^1, y^2, \ldots) = (y^1, y^2, y^3, \ldots)$$

describes any "traditional" system made of m chains of integrators of arbitrary lengths, and in particular the direct transfer $y = u$.

In practice we often identify the "system" $F(x, \overline{u}) := (f(x,u), u^1, u^2, \ldots)$ with the "dynamics" $\dot{x} = f(x,u)$ which defines it. Our main motivation for introducing a new formalism is that it will turn out to be a natural framework for the notions of equivalence and flatness we want to define.

Remark. It is easy to see that the manifold $\mathfrak{M}$ is finite-dimensional only when there is no input, i.e., to describe a *determined* system of differential equations one needs as many equations as variables. In the presence of inputs, the system becomes *underdetermined*, there are more variables than equations, which accounts for the infinite dimension.

Remark. Our definition of a system is adapted from the notion of *diffiety* introduced in [76] to deal with systems of (partial) differential equations. By definition a diffiety is a pair $(\mathfrak{M}, CT\mathfrak{M})$ where $\mathfrak{M}$ is smooth manifold, possibly of infinite dimension, and $CT\mathfrak{M}$ is an involutive finite-dimensional distribution on $\mathfrak{M}$, i.e., the Lie bracket of any two vector fields of $CT\mathfrak{M}$ is itself in $CT\mathfrak{M}$. The dimension of $CT\mathfrak{M}$ is equal to the number of independent variables.

As we are only working with systems with lumped parameters, hence governed by ordinary differential equations, we consider diffieties with one dimensional distributions. For our purpose we have also chosen to single out a particular vector field rather than work with the distribution it spans.

1.2 Equivalence of systems

In this section we define an equivalence relation formalizing the idea that two systems are "equivalent" if there is an invertible transformation exchanging their trajectories. As we will see later, the relevance of this rather natural equivalence notion lies in the fact that it admits an interpretation in terms of dynamic feedback.

Consider two systems $(\mathfrak{M}, F)$ and $(\mathfrak{N}, G)$ and a smooth mapping $\Psi : \mathfrak{M} \to \mathfrak{N}$ (remember that by definition every component of a smooth mapping depends only on finitely many coordinates). If $t \mapsto \xi(t)$ is a trajectory of $(\mathfrak{M}, F)$, i.e.,

$$\forall \xi, \quad \dot{\xi}(t) = F(\xi(t)),$$

the composed mapping $t \mapsto \zeta(t) = \Psi(\xi(t))$ satisfies the chain rule

$$\dot{\zeta}(t) = \frac{\partial \Psi}{\partial \xi}(\xi(t)) \cdot \dot{\xi}(t) = \frac{\partial \Psi}{\partial \xi}(\xi(t)) \cdot F(\xi(t)).$$

The above expressions involve only finite sums even if the matrices and vectors have infinite sizes: indeed a row of $\frac{\partial \Psi}{\partial \xi}$ contains only a finite number of non zero terms because a component of Ψ depends only on finitely many coordinates. Now, if the vector fields F and G are Ψ-*related*, i.e.,

$$\forall \xi, \quad G(\Psi(\xi)) = \frac{\partial \Psi}{\partial \xi}(\xi) \cdot F(\xi)$$

then

$$\dot{\zeta}(t) = G(\Psi(\xi(t)) = G(\zeta(t)),$$

which means that $t \mapsto \zeta(t) = \Psi(\xi(t))$ is a trajectory of $(\mathfrak{N}, G)$. If moreover Ψ has a smooth inverse Φ then obviously F, G are also Φ-related, and there is a one-to-one correspondence between the trajectories of the two systems. We call such an invertible Ψ relating F and G an *endogenous transformation*.

Definition 2. Two systems $(\mathfrak{M}, F)$ and $(\mathfrak{N}, G)$ are *equivalent at* $(p, q) \in \mathfrak{M} \times \mathfrak{N}$ if there exists an endogenous transformation from a neighborhood of p to a neighborhood of q. $(\mathfrak{M}, F)$ and $(\mathfrak{N}, G)$ are *equivalent* if they are equivalent at every pair of points (p, q) of a dense open subset of $\mathfrak{M} \times \mathfrak{N}$.

Notice that when $\mathfrak{M}$ and $\mathfrak{N}$ have the same *finite* dimension, the systems are necessarily equivalent by the straightening out theorem. This is no longer true in infinite dimensions.

Consider the two systems $(X \times U \times \mathbb{R}_m^\infty, F)$ and $(Y \times V \times \mathbb{R}_s^\infty, G)$ describing the dynamics

$$\dot{x} = f(x, u), \quad (x, u) \in X \times U \subset \mathbb{R}^n \times \mathbb{R}^m \tag{5}$$

$$\dot{y} = g(y, v), \quad (y, v) \in Y \times V \subset \mathbb{R}^r \times \mathbb{R}^s. \tag{6}$$

The vector fields F, G are defined by

$$F(x,u,u^1,\ldots) = (f(x,u),u^1,u^2,\ldots)$$
$$G(y,v,v^1,\ldots) = (g(y,v),v^1,v^2,\ldots).$$

If the systems are equivalent, the endogenous transformation Ψ takes the form

$$\Psi(x,u,u^1,\ldots) = (\psi(x,\overline{u}),\beta(x,\overline{u}),\dot\beta(x,\overline{u}),\ldots).$$

Here we have used the short-hand notation $\overline{u} = (u,u^1,\ldots,u^k)$, where k is some finite but otherwise arbitrary integer. Hence Ψ is completely specified by the mappings ψ and β, i.e, by the expression of y, v in terms of $x, \overline{u}$. Similarly, the inverse Φ of Ψ takes the form

$$\Phi(y,v,v^1,\ldots) = (\varphi(y,\overline{v}),\alpha(y,\overline{v}),\dot\alpha(y,\overline{v}),\ldots).$$

As Ψ and Φ are inverse mappings we have

$$\begin{aligned}\psi\big(\varphi(y,\overline{v}),\overline{\alpha}(y,\overline{v})\big) &= y\\ \beta\big(\varphi(y,\overline{v}),\overline{\alpha}(y,\overline{v})\big) &= v\end{aligned} \qquad \text{and} \qquad \begin{aligned}\varphi\big(\psi(x,\overline{u}),\overline{\beta}(x,\overline{u})\big) &= x\\ \alpha\big(\psi(x,\overline{u}),\overline{\beta}(x,\overline{u})\big) &= u.\end{aligned}$$

Moreover F and G Ψ-related implies

$$f\big(\varphi(y,\overline{v}),\alpha(y,\overline{v})\big) = D\varphi(y,\overline{v})\cdot\overline{g}(y,\overline{v})$$

where $\overline{g}$ stands for $(g,v^1,\ldots,v^k)$, i.e., a truncation of G for some large enough k. Conversely,

$$g\big(\psi(x,\overline{u}),\beta(y,\overline{u})\big) = D\psi(x,\overline{u})\cdot\overline{f}(y,\overline{u}).$$

In other words, whenever $t \mapsto (x(t),u(t))$ is a trajectory of (5)

$$t \mapsto (y(t),v(t)) = \big(\varphi(x(t),\overline{u}(t)),\ \alpha(x(t),\overline{u}(t))\big)$$

is a trajectory of (6), and vice versa.

Example 2 *(The PVTOL, see example 21).* The system generated by

$$\begin{aligned}\ddot x &= -u_1\sin\theta + \varepsilon u_2\cos\theta\\ \ddot z &= u_1\cos\theta + \varepsilon u_2\sin\theta - 1\\ \ddot\theta &= u_2.\end{aligned}$$

is globally equivalent to the systems generated by

$$\ddot y_1 = -\xi\sin\theta, \qquad \ddot y_2 = \xi\cos\theta - 1,$$

where ξ and θ are the control inputs. Indeed, setting

$$\begin{matrix} X := (x, z, \dot{x}, \dot{z}, \theta, \dot{\theta}) \\ U := (u_1, u_2) \end{matrix} \quad \text{and} \quad \begin{matrix} Y := (y_1, y_2, \dot{y}_1, \dot{y}_2) \\ V := (\xi, \theta) \end{matrix}$$

and using the notations in the discussion after definition 2, we define the mappings $Y = \psi(X, \overline{U})$ and $V = \beta(X, \overline{U})$ by

$$\psi(X, \overline{U}) := \begin{pmatrix} x - \varepsilon \sin\theta \\ z + \varepsilon \cos\theta \\ \dot{x} - \varepsilon\dot{\theta}\cos\theta \\ \dot{z} - \varepsilon\dot{\theta}\sin\theta \end{pmatrix} \quad \text{and} \quad \beta(X, \overline{U}) := \begin{pmatrix} u_1 - \varepsilon\dot{\theta}^2 \\ \theta \end{pmatrix}$$

to generate the mapping Ψ. The inverse mapping Φ is generated by the mappings $X = \varphi(Y, \overline{V})$ and $U = \alpha(Y, \overline{V})$ defined by

$$\varphi(Y, \overline{V}) := \begin{pmatrix} y_1 + \varepsilon \sin\theta \\ y_2 - \varepsilon \cos\theta \\ \dot{y}_1 + \varepsilon\dot{\theta}\cos\theta \\ \dot{y}_2 - \varepsilon\dot{\theta}\sin\theta \\ \theta \\ \dot{\theta} \end{pmatrix} \quad \text{and} \quad \alpha(Y, \overline{V}) := \begin{pmatrix} \xi + \varepsilon\dot{\theta}^2 \\ \ddot{\theta} \end{pmatrix}$$

An important property of endogenous transformations is that they preserve the input dimension:

Theorem 1. *If two systems $(X \times U \times \mathbb{R}^\infty_m, F)$ and $(Y \times V \times \mathbb{R}^\infty_s, G)$ are equivalent, then they have the same number of inputs, i.e., $m = s$.*

Proof. Consider the truncation Φ_μ of Φ on $X \times U \times (\mathbb{R}^m)^\mu$,

$$\begin{aligned} \Phi_\mu : X \times U \times (\mathbb{R}^{m+k})^\mu &\rightarrow Y \times V \times (\mathbb{R}^s)^\mu \\ (x, u, u^1, \ldots, u^{k+\mu}) &\mapsto (\varphi, \alpha, \dot{\alpha}, \ldots, \alpha^{(\mu)}), \end{aligned}$$

i.e., the first $\mu + 2$ blocks of components of Ψ; k is just a fixed "large enough" integer. Because Ψ is invertible, Ψ_μ is a submersion for all μ. Hence the dimension of the domain is greater than or equal to the dimension of the range,

$$n + m(k + \mu + 1) \geq s(\mu + 1) \qquad \forall \mu > 0,$$

which implies $m \geq s$. Using the same idea with Ψ leads to $s \geq m$. □

Remark. Our definition of equivalence is adapted from the notion of equivalence between diffieties. Given two diffieties $(\mathfrak{M}, CT\mathfrak{M})$ and $(\mathfrak{N}, CT\mathfrak{N})$, we say that a smooth mapping Ψ from (an open subset of) $\mathfrak{M}$ to $\mathfrak{N}$ is *Lie-Bäcklund* if its tangent mapping $T\Psi$ satisfies $T\Phi(CT\mathfrak{M}) \subset CT\mathfrak{N}$. If moreover Ψ has a smooth inverse Φ such that $T\Psi(CT\mathfrak{N}) \subset CT\mathfrak{M}$, we say it is a *Lie-Bäcklund isomorphism*. When such an isomorphism exists, the diffieties are said to be *equivalent*. An endogenous transformation is just a special Lie-Bäcklund isomorphism, which preserves the time parameterization of the integral curves. It is possible to define the more general concept of *orbital* equivalence [14, 12] by considering general Lie-Bäcklund isomorphisms, which preserve only the geometric locus of the integral curves (see an example in section 26).

1.3 Differential Flatness

We single out a very important class of systems, namely systems equivalent to a trivial system $(\mathbb{R}^\infty_s, F_s)$ (see example 1):

Definition 3. The system $(\mathfrak{M}, F)$ is *flat at* $p \in \mathfrak{M}$ (resp. *flat*) if it equivalent at p (resp. equivalent) to a trivial system.

We specialize the discussion after definition 2 to a flat system $(X \times U \times \mathbb{R}^\infty_m, F)$ describing the dynamics

$$\dot{x} = f(x,u), \quad (x,u) \in X \times U \subset \mathbb{R}^n \times \mathbb{R}^m.$$

By definition the system is equivalent to the trivial system $(\mathbb{R}^\infty_s, F_s)$ where the endogenous transformation Ψ takes the form

$$\Psi(x,u,u^1,\dots) = (h(x,\overline{u}), \dot{h}(x,\overline{u}), \ddot{h}(x,\overline{u}), \dots). \tag{7}$$

In other words Ψ is the infinite prolongation of the mapping h. The inverse Φ of Ψ takes the form

$$\Psi(\overline{y}) = (\psi(\overline{y}), \beta(\overline{y}), \dot{\beta}(\overline{y}), \dots).$$

As Φ and Ψ are inverse mappings we have in particular

$$\varphi\big(\overline{h}(x,\overline{u})\big) = x \quad \text{and} \quad \alpha\big(\overline{h}(x,\overline{u})\big) = u.$$

Moreover F and G Φ-related implies that whenever $t \mapsto y(t)$ is a trajectory of $y = v$ –i.e., nothing but an *arbitrary* mapping–

$$t \mapsto \big(x(t), u(t)\big) = \big(\psi(\overline{y}(t)),\ \beta(\overline{y}(t))\big)$$

is a trajectory of $\dot{x} = f(x,u)$, and vice versa.

We single out the importance of the mapping h of the previous example:

Definition 4. Let $(\mathfrak{M}, F)$ be a flat system and Ψ the endogenous transformation putting it into a trivial system. The first block of components of Ψ, i.e., the mapping h in (7), is called a *flat (*or *linearizing) output*.

With this definition, an obvious consequence of theorem 1 is:

Corollary 1. *Consider a flat system. The dimension of a flat output is equal to the input dimension, i.e., $s = m$.*

Example 3 *(The PVTOL).* The system studied in example 2 is flat, with

$$y = h(X, \overline{U}) := (x - \varepsilon \sin\theta, z + \varepsilon \cos\theta)$$

as a flat output. Indeed, the mappings $X = \varphi(\overline{y})$ and $U = \alpha(\overline{y})$ which generate the inverse mapping Φ can be obtained from the implicit equations

$$\begin{aligned} (y_1 - x)^2 + (y_2 - z)^2 &= \varepsilon^2 \\ (y_1 - x)(\ddot{y}_2 + 1) - (y_2 - z)\ddot{y}_1 &= 0 \\ (\ddot{y}_2 + 1)\sin\theta + \ddot{y}_1 \cos\theta &= 0. \end{aligned}$$

We first solve for x, z, θ,

$$\begin{aligned} x &= y_1 + \varepsilon \frac{\ddot{y}_1}{\sqrt{\ddot{y}_1^2 + (\ddot{y}_2 + 1)^2}} \\ z &= y_2 + \varepsilon \frac{(\ddot{y}_2 + 1)}{\sqrt{\ddot{y}_1^2 + (\ddot{y}_2 + 1)^2}} \\ \theta &= \arg(\ddot{y}_1, \ddot{y}_2 + 1), \end{aligned}$$

and then differentiate to get $\dot{x}, \dot{z}, \dot{\theta}, u$ in function of the derivatives of y. Notice the only singularity is $\ddot{y}_1^2 + (\ddot{y}_2 + 1)^2 = 0$.

1.4 Application to motion planning

We now illustrate how flatness can be used for solving control problems. Consider a nonlinear control system of the form

$$\dot{x} = f(x, u) \qquad x \in \mathbb{R}^n,\ u \in \mathbb{R}^m$$

with flat output

$$y = h(x, u, \dot{u}, \ldots, u^{(r)}).$$

By virtue of the system being flat, we can write all trajectories $(x(t), u(t))$ satisfying the differential equation in terms of the flat output and its derivatives:

$$\begin{aligned} x &= \varphi(y, \dot{y}, \ldots, y^{(q)}) \\ u &= \alpha(y, \dot{y}, \ldots, y^{(q)}). \end{aligned}$$

We begin by considering the problem of steering from an initial state to a final state. We parameterize the components of the flat output y_i, $i = 1, \ldots, m$ by

$$y_i(t) := \sum_j A_{ij} \lambda_j(t), \tag{8}$$

where the $\lambda_j(t)$, $j = 1, \ldots, N$ are basis functions. This reduces the problem from finding a function in an infinite dimensional space to finding a finite set of parameters.

Suppose we have available to us an initial state x_0 at time τ_0 and a final state x_f at time τ_f. Steering from an initial point in state space to a desired point in state space is trivial for flat systems. We have to calculate the values of the flat output and its derivatives from the desired points in state space and then solve for the coefficients A_{ij} in the following system of equations:

$$\begin{array}{rclcrcl} y_i(\tau_0) & = & \sum_j A_{ij}\lambda_j(\tau_0) & \quad & y_i(\tau_f) & = & \sum_j A_{ij}\lambda_j(\tau_f) \\ & \vdots & & & & \vdots & \\ y_i^{(q)}(\tau_0) & = & \sum_j A_{ij}\lambda_j^{(q)}(\tau_0) & & y_i^{(q)}(\tau_f) & = & \sum_j A_{ij}\lambda_j^{(q)}(\tau_f). \end{array} \tag{9}$$

To streamline notation we write the following expressions for the case of a *one*-dimensional flat output only. The multi-dimensional case follows by repeatedly applying the one-dimensional case, since the algorithm is decoupled in the component of the flat output. Let $\Lambda(t)$ be the $q+1$ by N matrix $\Lambda_{ij}(t) = \lambda_j^{(i)}(t)$ and let

$$\begin{aligned} \bar{y}_0 &= (y_1(\tau_0), \ldots, y_1^{(q)}(\tau_0)) \\ \bar{y}_f &= (y_1(\tau_f), \ldots, y_1^{(q)}(\tau_f)) \\ \bar{y} &= (\bar{y}_0, \bar{y}_f). \end{aligned} \tag{10}$$

Then the constraint in equation (9) can be written as

$$\bar{y} = \begin{pmatrix} \Lambda(\tau_0) \\ \Lambda(\tau_f) \end{pmatrix} A =: \Lambda A. \tag{11}$$

That is, we require the coefficients A to be in an affine sub-space defined by equation (11). The only condition on the basis functions is that Λ is full rank, in order for equation (11) to have a solution.

The implications of flatness is that the trajectory generation problem can be reduced to simple algebra, in theory, and computationally attractive algorithms in practice. In the case of the towed cable system of example 25, a reasonable state space representation of the system consists of approximately 128 states. Traditional approaches to trajectory generation, such as optimal control, cannot be easily applied in this case. However, it follows from the fact that the system is flat that the feasible trajectories of the system are completely characterized by the motion of the point at the bottom of the cable. By converting the input constraints on the system to constraints on the curvature and higher derivatives of the motion of the bottom of the cable, it is possible to compute efficient techniques for trajectory generation.

1.5 Motion planning with singularities

In the previous section we assumed the endogenous transformation

$$\Psi(x,u,u_1,\dots) := \big(h(x,\overline{u}),\dot{h}(x,\overline{u}),\ddot{h}(x,\overline{u}),\dots\big)$$

generated by the flat output $y = h(x,\overline{u})$ everywhere nonsingular, so that we could invert it and express x and u in function of y and its derivatives,

$$(y,\dot{y},\dots,y^{(q)}) \mapsto (x,u) = \phi(y,\dot{y},\dots,y^{(q)}).$$

But it may well be that a singularity is in fact an interesting point of operation. As ϕ is not defined at such a point, the previous computations do not apply. A way to overcome the problem is to "blow up" the singularity by considering trajectories $t \mapsto y(t)$ such that

$$t \mapsto \phi\big(y(t),\dot{y}(t),\dots,y^{(q)}(t)\big)$$

can be prolonged into a smooth mapping at points where ϕ is not defined. To do so requires a detailed study of the singularity. A general statement is beyond the scope of this paper and we simply illustrate the idea with an example.

Example 4. Consider the flat dynamics

$$\dot{x}_1 = u_1, \quad \dot{x}_2 = u_2 u_1, \quad \dot{x}_3 = x_2 u_1,$$

with flat output $y := (x_1, x_3)$. When $u_1 = 0$, i.e., $\dot{y}_1 = 0$ the endogenous transformation generated by the flat output is singular and the inverse mapping

$$(y,\dot{y},\ddot{y}) \xmapsto{\phi} (x_1,x_2,x_3,u_1,u_2) = \left(y_1, \frac{\dot{y}_2}{\dot{y}_1}, y_2, \dot{y}_1, \frac{\ddot{y}_2\dot{y}_1 - \ddot{y}_1\dot{y}_2}{\dot{y}_1^3}\right),$$

is undefined. But if we consider trajectories $t \mapsto y(t) := \big(\sigma(t), p(\sigma(t))\big)$, with σ and p smooth functions, we find that

$$\frac{\dot{y}_2(t)}{\dot{y}_1(t)} = \frac{\frac{dp}{d\sigma}\big(\sigma(t)\big)\cdot\dot{\sigma}(t)}{\dot{\sigma}(t)} \qquad \text{and} \qquad \frac{\ddot{y}_2\dot{y}_1 - \ddot{y}_1\dot{y}_2}{\dot{y}_1^3} = \frac{\frac{d^2p}{d\sigma^2}\big(\sigma(t)\big)\cdot\dot{\sigma}^3(t)}{\dot{\sigma}^3(t)},$$

hence we can prolong $t \mapsto \phi(y(t), \dot{y}(t), \ddot{y}(t))$ everywhere by

$$t \mapsto \left(\sigma(t),\ \frac{dp}{d\sigma}\big(\sigma(t)\big),\ p(\sigma(t)),\ \dot{\sigma}(t),\ \frac{d^2 p}{d\sigma^2}\big(\sigma(t)\big)\right).$$

The motion planning can now be done as in the previous section: indeed, the functions σ and p and their derivatives are constrained at the initial (resp. final) time by the initial (resp. final) point but otherwise arbitrary.

For a more substantial application see [66, 67, 16], where the same idea was applied to nonholonomic mechanical systems by taking advantage of the "natural" geometry of the problem.

2 Feedback design with equivalence

2.1 From equivalence to feedback

The equivalence relation we have defined is very natural since it is essentially a $1-1$ correspondence between trajectories of systems. We had mainly an open-loop point of view. We now turn to a closed-loop point of view by interpreting equivalence in terms of feedback. For that, consider the two dynamics

$$\begin{aligned} \dot{x} &= f(x,u), \quad (x,u) \in X \times U \subset \mathbb{R}^n \times \mathbb{R}^m \\ \dot{y} &= g(y,v), \quad (y,v) \in Y \times V \subset \mathbb{R}^r \times \mathbb{R}^s. \end{aligned}$$

They are described in our formalism by the systems $(X \times U \times \mathbb{R}^\infty_m, F)$ and $(Y \times V \times \mathbb{R}^\infty_s, G)$, with F and G defined by

$$\begin{aligned} F(x,u,u^1,\ldots) &:= (f(x,u), u^1, u^2, \ldots) \\ G(y,v,v^1,\ldots) &:= (g(y,v), v^1, v^2, \ldots). \end{aligned}$$

Assume now the two systems are equivalent, i.e., they have the same trajectories. Does it imply that it is possible to go from $\dot{x} = f(x,u)$ to $\dot{y} = g(y,v)$ by a (possibly) dynamic feedback

$$\begin{aligned} \dot{z} &= a(x,z,v), \quad z \in Z \subset \mathbb{R}^q \\ u &= \kappa(x,z,v), \end{aligned}$$

and *vice versa*? The question might look stupid at first glance since such a feedback can only increase the state dimension. Yet, we can give it some sense if we agree to work "up to pure integrators" (remember this does not change the system in our formalism, see the remark after definition 1).

Theorem 2. *Assume $\dot{x} = f(x,u)$ and $\dot{y} = g(y,v)$ are equivalent. Then $\dot{x} = f(x,u)$ can be transformed by (dynamic) feedback and coordinate change into*

$$\dot{y} = g(y,v), \quad \dot{v} = v^1, \quad \dot{v}^1 = v^2, \quad \ldots \quad , \quad \dot{v}^\mu = w$$

for some large enough integer μ. Conversely, $\dot{y} = g(y,v)$ can be transformed by (dynamic) feedback and coordinate change into

$$\dot{x} = f(x,u), \quad \dot{u} = u^1, \quad \dot{u}^1 = u^2, \quad \ldots \quad , \quad \dot{u}^\nu = w$$

for some large enough integer ν.

Proof [33]. Denote by F and G the infinite vector fields representing the two dynamics. Equivalence means there is an invertible mapping

$$\Phi(y,\overline{v}) = (\varphi(y,\overline{v}), \alpha(y,\overline{v}), \dot{\alpha}(y,\overline{v}), \ldots)$$

such that

$$F(\Phi(y,\overline{v})) = D\Phi(y,\overline{v}).G(y,\overline{v}). \tag{12}$$

Let $\tilde{y} := (y, v, v^1, \ldots, v^\mu)$ and $w := v^{\mu+1}$. For μ large enough, φ (resp. α) depends only on $\tilde{y}$ (resp. on $\tilde{y}$ and w). With these notations, Φ reads

$$\Phi(\tilde{y},\overline{w}) = (\varphi(\tilde{y}), \alpha(\tilde{y},w), \dot{\alpha}(y,\overline{w}), \ldots),$$

and equation (12) implies in particular

$$f(\varphi(\tilde{y}), \alpha(\tilde{y},w)) = D\varphi(\tilde{y}).\tilde{g}(\tilde{y},w), \tag{13}$$

where $\tilde{g} := (g, v^1, \ldots, v^k)$. Because Φ is invertible, φ is full rank hence can be completed by some map π to a coordinate change

$$\tilde{y} \mapsto \phi(\tilde{y}) = (\varphi(\tilde{y}), \pi(\tilde{y})).$$

Consider now the dynamic feedback

$$\begin{aligned} u &= \alpha(\phi^{-1}(x,z), w)) \\ \dot{z} &= D\pi(\phi^{-1}(x,z)).\tilde{g}(\phi^{-1}(x,z), w)), \end{aligned}$$

which transforms $\dot{x} = f(x,u)$ into

$$\begin{pmatrix} \dot{x} \\ \dot{z} \end{pmatrix} = \tilde{f}(x,z,w) := \begin{pmatrix} f(x, \alpha(\phi^{-1}(x,z), w)) \\ D\pi(\phi^{-1}(x,z)).\tilde{g}(\phi^{-1}(x,z), w)) \end{pmatrix}.$$

Using (13), we have

$$\tilde{f}\big(\phi(\tilde{y}), w\big) = \begin{pmatrix} f\big(\varphi(\tilde{y}), \alpha(\tilde{y},w)\big) \\ D\pi(\tilde{y}).\tilde{g}(\tilde{y},w) \end{pmatrix} = \begin{pmatrix} D\varphi(\tilde{y}) \\ D\pi(\tilde{y}) \end{pmatrix} \cdot \tilde{g}(\tilde{y},w) = D\phi(\tilde{y}).\tilde{g}(\tilde{y},w).$$

Therefore $\tilde{f}$ and $\tilde{g}$ are ϕ-related, which ends the proof. Exchanging the roles of f and g proves the converse statement. □

As a flat system is equivalent to a trivial one, we get as an immediate consequence of the theorem:

Corollary 2. *A flat dynamics can be linearized by (dynamic) feedback and coordinate change.*

Remark. As can be seen in the proof of the theorem there are many feedbacks realizing the equivalence, as many as suitable mappings π. Notice all these feedback explode at points where φ is singular (i.e., where its rank collapses).

Further details about the construction of a linearizing feedback from an output and the links with extension algorithms can be found in [35].

Example 5 *(The PVTOL).* We know from example 3 that the dynamics

$$\begin{aligned}\ddot{x} &= -u_1 \sin\theta + \varepsilon u_2 \cos\theta \\ \ddot{z} &= u_1 \cos\theta + \varepsilon u_2 \sin\theta - 1 \\ \ddot{\theta} &= u_2\end{aligned}$$

admits the flat output

$$y = (x - \varepsilon \sin\theta, z + \varepsilon \cos\theta).$$

It is transformed into the linear dynamics

$$y_1^{(4)} = v_1, \qquad y_2^{(4)} = v_2$$

by the feedback

$$\begin{aligned}\ddot{\xi} &= -v_1 \sin\theta + v_2 \cos\theta + \xi\dot{\theta}^2 \\ u_1 &= \xi + \varepsilon\dot{\theta}^2 \\ u_2 &= \frac{-1}{\xi}(v_1 \cos\theta + v_2 \sin\theta + 2\dot{\xi}\dot{\theta})\end{aligned}$$

and the coordinate change

$$(x, z, \theta, \dot{x}, \dot{z}, \dot{\theta}, \xi, \dot{\xi}) \mapsto (y, \dot{y}, \ddot{y}, y^{(3)}).$$

The only singularity of this transformation is $\xi = 0$, i.e., $\ddot{y}_1^2 + (\ddot{y}_2 + 1)^2 = 0$. Notice the PVTOL is not linearizable by static feedback (see section 3.1.2).

2.2 Endogenous feedback

Theorem 2 asserts the existence of a feedback such that

$$\begin{aligned}\dot{x} &= f(x, \kappa(x, z, w)) \\ \dot{z} &= a(x, z, w).\end{aligned} \tag{14}$$

reads, up to a coordinate change,

$$\dot{y} = g(y, v), \quad \dot{v} = v^1, \quad \ldots \quad , \quad \dot{v}^\mu = w. \tag{15}$$

But (15) is trivially equivalent to $\dot{y} = g(y, v)$ (see the remark after definition 1), which is itself equivalent to $\dot{x} = f(x, u)$. Hence, (14) is equivalent to $\dot{x} = f(x, u)$. This leads to

Definition 5. Consider the dynamics $\dot{x} = f(x, u)$. We say the feedback

$$\begin{aligned} u &= \kappa(x, z, w) \\ \dot{z} &= a(x, z, w) \end{aligned}$$

is *endogenous* if the open-loop dynamics $\dot{x} = f(x, u)$ is equivalent to the closed-loop dynamics

$$\begin{aligned} \dot{x} &= f(x, \kappa(x, z, w)) \\ \dot{z} &= a(x, z, w). \end{aligned}$$

The word "endogenous" reflects the fact that the feedback variables z and w are in loose sense "generated" by the original variables $x, \overline{u}$ (see [33, 36] for further details and a characterization of such feedbacks)

Remark. It is also possible to consider at no extra cost "generalized" feedbacks depending not only on w but also on derivatives of w.

We thus have a more precise characterization of equivalence and flatness:

Theorem 3. *Two dynamics $\dot{x} = f(x, u)$ and $\dot{y} = g(y, v)$ are equivalent if and only if $\dot{x} = f(x, u)$ can be transformed by endogenous feedback and coordinate change into*

$$\dot{y} = g(y, v), \quad \dot{v} = v^1, \quad \ldots \quad , \quad \dot{v}^\mu = w. \tag{16}$$

for some large enough integer ν, and vice versa.

Corollary 3. *A dynamics is flat if and only if it is linearizable by endogenous feedback and coordinate change.*

Another trivial but important consequence of theorem 2 is that an endogenous feedback can be "unraveled" by another endogenous feedback:

Corollary 4. *Consider a dynamics*

$$\begin{aligned} \dot{x} &= f(x, \kappa(x, z, w)) \\ \dot{z} &= a(x, z, w) \end{aligned}$$

where

$$u = \kappa(x, z, w)$$
$$\dot{z} = a(x, z, w)$$

is an endogenous feedback. Then it can be transformed by endogenous feedback and coordinate change into

$$\dot{x} = f(x, u), \quad \dot{u} = u^1, \quad \dots \quad , \quad \dot{u}^\mu = w. \tag{17}$$

for some large enough integer μ.

This clearly shows which properties are preserved by equivalence: properties that are preserved by adding pure integrators and coordinate changes, in particular controllability.

An endogenous feedback is thus truly "reversible", up to pure integrators. It is worth pointing out that a feedback which is *invertible* in the sense of the standard –but maybe unfortunate– terminology [52] is not necessarily endogenous. For instance the invertible feedback $\dot{z} = v,\ u = v$ acting on the scalar dynamics $\dot{x} = u$ is not endogenous. Indeed, the closed-loop dynamics $\dot{x} = v,\ \dot{z} = v$ is no longer controllable, and there is no way to change that by another feedback!

2.3 Tracking: feedback linearization

One of the central problems of control theory is *trajectory tracking*: given a dynamics $\dot{x} = f(x, u)$, we want to design a controller able to track any reference trajectory $t \mapsto \big(x_r(t), u_r(t)\big)$. If this dynamics admits a flat output $y = h(x, \overline{u})$, we can use corollary 2 to transform it by (endogenous) feedback and coordinate change into the linear dynamics $y^{(\mu+1)} = w$. Assigning then

$$v := y_r^{(\mu+1)}(t) - K\Delta\tilde{y}$$

with a suitable gain matrix K, we get the stable closed-loop error dynamics

$$\Delta y^{(\mu+1)} = -K\Delta\tilde{y},$$

where $y_r(t) := \big(x_r(t), \overline{u}_r(t)\big)$ and $\tilde{y} := (y, \dot{y}, \dots, y^\mu)$ and $\Delta\xi$ stands for $\xi - \xi_{r(t)}$. This control law meets the design objective. Indeed, there is by the definition of flatness an invertible mapping

$$\Phi(\overline{y}) = (\varphi(\overline{y}), \alpha(\overline{y}), \dot{\alpha}(\overline{y}), \dots)$$

relating the infinite dimension vector fields $F(x,\overline{u}) := (f(x,u), u, u^1, \dots)$ and $G(\overline{y}) := (y, y^1, \dots)$. From the proof of theorem 2, this means in particular

$$\begin{aligned} x &= \varphi(\tilde{y}_r(t) + \Delta\tilde{y}) \\ &= \varphi(\tilde{y}_r(t)) + R_\varphi(y_r(t), \Delta\tilde{y}).\Delta\tilde{y} \\ &= x_r(t) + R_\varphi(y_r(t), \Delta\tilde{y}).\Delta\tilde{y} \end{aligned}$$

and

$$\begin{aligned} u &= \alpha(\tilde{y}_r(t) + \Delta\tilde{y}, -K\Delta\tilde{y}) \\ &= \alpha(\tilde{y}_r(t)) + R_\alpha(y_r^{(\mu+1)}(t), \Delta\tilde{y}).\begin{pmatrix} \Delta\tilde{y} \\ -K\Delta\tilde{y} \end{pmatrix} \\ &= u_r(t) + R_\alpha(\tilde{y}_r(t), y_r^{(\mu+1)}(t), \Delta\tilde{y}, \Delta w).\begin{pmatrix} \Delta\tilde{y} \\ -K\Delta\tilde{y} \end{pmatrix}, \end{aligned}$$

where we have used the fundamental theorem of calculus to define

$$\begin{aligned} R_\varphi(Y, \Delta Y) &:= \int_0^1 D\varphi(Y + t\Delta Y)dt \\ R_\alpha(Y, w, \Delta Y, \Delta w) &:= \int_0^1 D\alpha(Y + t\Delta Y, w + t\Delta w)dt. \end{aligned}$$

Since $\Delta y \to 0$ as $t \to \infty$, this means $x \to x_r(t)$ and $u \to u_r(t)$. Of course the tracking gets poorer and poorer as the ball of center $\tilde{y}_r(t)$ and radius Δy approaches a singularity of φ. At the same time the control effort gets larger and larger, since the feedback explodes at such a point (see the remark after theorem 2). Notice the tracking quality and control effort depend only on the mapping Φ, hence on the flat output, and not on the feedback itself.

We end this section with some comments on the use of feedback linearization. A linearizing feedback should always be fed by a *trajectory generator*, even if the original problem is not stated in terms of tracking. For instance, if it is desired to *stabilize* an equilibrium point, applying directly feedback linearization without first planning a reference trajectory yields very large control effort when starting from a distant initial point. The role of the trajectory generator is to define an *open-loop* “reasonable” trajectory –i.e., satisfying some state and/or control constraints– that the linearizing feedback will then track.

2.4 Tracking: singularities and time scaling

Tracking by feedback linearization is possible only far from singularities of the endogenous transformation generated by the flat output. If the reference trajectory passes through or near a singularity, then feedback linearization cannot be directly applied, as is the case for motion planning, see section 1.5. Nevertheless, it can be used after a *time scaling*, at least in the presence of "simple" singularities. The interest is that it allows exponential tracking, though in a new "singular" time.

Example 6. Take a reference trajectory $t \mapsto y_r(t) = (\sigma(t), p(\sigma(t))$ for example 4. Consider the dynamic time-varying compensator $u_1 = \xi \dot{\sigma}(t)$ and $\dot{\xi} = v_1 \dot{\sigma}(t)$. The closed loop system reads

$$x_1' = \xi, \quad x_2' = u_2 \xi, \quad x_3' = x_2 \xi \quad \xi' = v_1.$$

where $'$ stands for $d/d\sigma$, the extended state is (x_1, x_2, x_3, ξ), the new control is (v_1, v_2). An equivalent second order formulation is

$$x_1'' = v_1, \quad x_3'' = u_2 \xi^2 + x_2 v_1.$$

When ξ is far from zero, the static feedback $u_2 = (v_2 - x_2 v_1)/\xi^2$ linearizes the dynamics,

$$x_1'' = v_1, \quad x_3'' = v_2$$

in σ scale. When the system remains close to the reference, $\xi \approx 1$, even if for some t, $\dot{\sigma}(t) = 0$. Take

$$\begin{aligned} v_1 &= 0 - \operatorname{sign}(\sigma) a_1 (\xi - 1) - a_2 (x_1 - \sigma) \\ v_2 &= \tfrac{d^2 p}{d\sigma^2} - \operatorname{sign}(\sigma) a_1 \left(x_2 \xi - \tfrac{dp}{d\sigma} \right)) - a_2 (x_3 - p) \end{aligned} \tag{18}$$

with $a_1 > 0$ and $a_2 > 0$, then the error dynamics becomes exponentially stable in σ-scale (the term $\operatorname{sign}(\sigma)$ is for dealing with $\dot{\sigma} < 0$).

Similar computations for trailer systems can be found in [15, 12].

2.5 Tracking: flatness and backstepping

2.5.1 Some drawbacks of feedback linearization

We illustrate on two simple (and caricatural) examples that feedback linearization may not lead to the best tracking controller in terms of control effort.

Example 7. Assume we want to track any trajectory $t \mapsto \big(x_r(t), u_r(t)\big)$ of

$$\dot{x} = -x - x^3 + u, \quad x \in \mathbb{R}.$$

The linearizing feedback

$$\begin{aligned} u &= x + x^3 - k\Delta x + \dot{x}_r(t) \\ &= u_r(t) + 3x_r(t)\Delta x^2 + \big(1 + 3x_r^2(t) - k\big)\Delta x + \Delta x^3 \end{aligned}$$

meets this objective by imposing the closed-loop dynamics $\Delta\dot{x} = -k\Delta x$.

But a closer inspection shows the open-loop error dynamics

$$\begin{aligned}\Delta\dot{x} &= -\big(1 + 3x_r^2(t)\big)\Delta x - \Delta x^3 + 3x_r(t)\Delta x^2 + \Delta u\\ &= -\Delta x\big(1 + 3x_r^2(t) - 3x_r(t)\Delta x + \Delta x^2\big) + \Delta u\end{aligned}$$

is naturally stable when the open-loop control $u := u_r(t)$ is applied (indeed $1 + 3x_r^2(t) - 3x_r(t)\Delta x + \Delta x^2$ is always strictly positive). In other words, the linearizing feedback does not take advantage of the natural damping effects.

Example 8. Consider the dynamics

$$\dot{x}_1 = u_1, \qquad \dot{x}_2 = u_2(1 - u_1),$$

for which it is required to track an arbitrary trajectory $t \mapsto \big(x_r(t), u_r(t)\big)$ (notice $u_r(t)$ may not be so easy to define because of the singularity $u_1 = 1$). The linearizing feedback

$$\begin{aligned}u_1 &= -k\Delta x_1 + \dot{x}_{1r}(t)\\ u_2 &= \frac{-k\Delta x_2 + \dot{x}_{2r}(t)}{1 + k\Delta x_1 - \dot{x}_{1r}(t)}\end{aligned}$$

meets this objective by imposing the closed-loop dynamics $\Delta\dot{x} = -k\Delta x$. Unfortunately u_2 grows unbounded as u_1 approaches one. This means we must in practice restrict to reference trajectories such that $|1 - u_{1r}(t)|$ is always "large" –in particular it is impossible to cross the singularity– and to a "small" gain k.

A smarter control law can do away with these limitations. Indeed, considering the error dynamics

$$\begin{aligned}\Delta\dot{x}_1 &= \Delta u_1\\ \Delta\dot{x}_2 &= (1 - u_{1r}(t) - \Delta u_1)\Delta u_2 - u_{2r}(t)\Delta u_1,\end{aligned}$$

and differentiating the positive function $V(\Delta x) := \frac{1}{2}(\Delta x_1^2 + \Delta x_2^2)$ we get

$$\dot{V} = \Delta u_1(\Delta x_1 - u_{2r}(t)\Delta x_2) + (1 - u_{1r}(t) - \Delta u_1)\Delta u_1 \Delta u_2.$$

The control law

$$\begin{aligned}\Delta u_1 &= -k(\Delta x_1 - u_{2r}(t)\Delta x_2)\\ \Delta u_2 &= -(1 - u_{1r}(t) - \Delta u_1)\Delta x_2\end{aligned}$$

does the job since

$$\dot{V} = -\big(\Delta x_1 - u_{2r}(t)\Delta x_2\big)^2 - \big((1 - u_{1r}(t) - \Delta u_1)\Delta x_2\big)^2 \leq 0.$$

Moreover, when $u_{1r}(t) \neq 0$, $\dot{V}$ is zero if and only if $\|\Delta x\|$ is zero. It is thus possible to cross the singularity –which has been made an unstable equilibrium of the closed-loop error dynamics– and to choose the gain k as large as desired. Notice the singularity is overcome by a "truly" multi-input design.

It should not be inferred from the previous examples that feedback linearization necessarily leads to inefficient tracking controllers. Indeed, when the trajectory generator is well-designed, the system is always close to the reference trajectory. Singularities are avoided by restricting to reference trajectories which stay away from them. This makes sense in practice when singularities do not correspond to interesting regions of operations. In this case, designing a tracking controller "smarter" than a linearizing feedback often turns out to be rather complicated, if possible at all.

2.5.2 Backstepping

The previous examples are rather trivial because the control input has the same dimension as the state. More complicated systems can be handled by *backstepping*. Backstepping is a versatile design tool which can be helpful in a variety of situations: stabilization, adaptive or output feedback, etc ([29] for a complete survey). It relies on the simple yet powerful following idea: consider the system

$$\begin{aligned}\dot{x} &= f(x,\xi), \quad f(x_0,\xi_0) = 0\\ \dot{\xi} &= u,\end{aligned}$$

where $(x,\xi) \in \mathbb{R}^n \times \mathbb{R}$ is the state and $u \in \mathbb{R}$ the control input, and assume we can asymptotically stabilize the equilibrium x_0 of the subsystem $\dot{x} = f(x,\xi)$, i.e., we know a control law $\xi = \alpha(x),\ \alpha(x_0) = \xi_0$ and a positive function $V(x)$ such that

$$\dot{V} = DV(x).f(x,\alpha(x)) \leq 0.$$

A key observation is that the "virtual" control input ξ can then "backstepped" to stabilize the equilibrium (x_0,ξ_0) of the complete system. Indeed, introducing the positive function

$$W(x,\xi) := V(x) + \frac{1}{2}(\xi - \alpha(x))^2$$

and the error variable $z := \xi - \alpha(x)$, we have

$$\begin{aligned}\dot{W} &= DV(x).f(x,\alpha(x)+z) + z\big(u - \dot{\alpha}(x,\xi)\big)\\ &= DV(x).\big(f(x,\alpha(x)) + R(x,z).z\big) + z\big(u - D\alpha(x).f(x,\xi)\big)\\ &= \dot{V} + z\big(u - D\alpha(x).f(x,\xi) + DV(x).R(x,z)\big),\end{aligned}$$

where we have used the fundamental theorem of calculus to define

$$R(x,h) := \int_0^1 \frac{\partial f}{\partial \xi}(x, x+th)dt$$

(notice $R(x,h)$ is trivially computed when f is linear in ξ). As $\dot{V}$ is negative by assumption, we can make $\dot{W}$ negative, hence stabilize the system, by choosing for instance

$$u := -z + D\alpha(x).f(x,\xi) - DV(x).R(x,z).$$

2.5.3 Blending equivalence with backstepping

Consider a dynamics $\dot{y} = g(y,v)$ for which we would like to solve the tracking problem. Assume it is equivalent to another dynamics $\dot{x} = f(x,u)$ for which we can solve this problem, i.e., we know a tracking control law together with a Lyapunov function. How can we use this property to control $\dot{y} = g(y,v)$? Another formulation of the question is: assume we know a controller for $\dot{x} = f(x,u)$. How can we derive a controller for

$$\begin{aligned}\dot{x} &= f(x, \kappa(x,z,v))\\ \dot{z} &= a(x,z,v),\end{aligned}$$

where $u = \kappa(x,z,v), \dot{z} = a(x,z,v)$ is an endogenous feedback? Notice backstepping answers the question for the elementary case where the feedback in question is a pure integrator.

By theorem 2, we can transform $\dot{x} = f(x,u)$ by (dynamic) feedback and coordinate change into

$$\dot{y} = g(y,v), \quad \dot{v} = v^1, \quad \ldots \quad , \quad \dot{v}^\mu = w. \tag{19}$$

for some large enough integer μ. We can then trivially backstep the control from v to w and change coordinates. Using the same reasoning as in section 2.3, it is easy to prove this leads to a control law solving the tracking problem for $\dot{x} = f(x,u)$. In fact, this is essentially the method we followed in section 2.3 on the special case of a flat $\dot{x} = f(x,u)$. We illustrated in section 2.5.1 potential drawbacks of this approach.

However, it is often possible to design better –though in general more complicated– tracking controllers by suitably using backstepping. This point of view is extensively developed in [29], though essentially in the single-input case, where general equivalence boils down to equivalence by coordinate change. In the multi-input case new phenomena occur as illustrated by the following examples.

Example 9 *(The PVTOL).* We know from example 2 that

$$\begin{aligned}\ddot{x} &= -u_1 \sin\theta + \varepsilon u_2 \cos\theta \\ \ddot{z} &= u_1 \cos\theta + \varepsilon u_2 \sin\theta - 1 \\ \ddot{\theta} &= u_2\end{aligned} \tag{20}$$

is globally equivalent to

$$\ddot{y}_1 = -\xi \sin\theta, \qquad \ddot{y}_2 = \xi \cos\theta - 1,$$

where $\xi = u_1 + \varepsilon\dot{\theta}^2$. This latter form is rather appealing for designing a tracking controller and leads to the error dynamics

$$\begin{aligned}\Delta\ddot{y}_1 &= -\xi \sin\theta + \xi_r(t) \sin\theta_r(t) \\ \Delta\ddot{y}_2 &= \xi \cos\theta - \xi_r(t) \cos\theta_r(t)\end{aligned}$$

Clearly, if θ were a control input, we could track trajectories by assigning

$$\begin{aligned}-\xi \sin\theta &= \alpha_1(\Delta y_1, \Delta\dot{y}_1) + \ddot{y}_{1r}(t) \\ \xi \cos\theta &= \alpha_2(\Delta y_2, \Delta\dot{y}_2) + \ddot{y}_{2r}(t)\end{aligned}$$

for suitable functions α_1, α_2 and find a Lyapunov function $V(\Delta y, \Delta\dot{y})$ for the system. In other words, we would assign

$$\begin{aligned}\xi &= \Xi\big(\Delta y, \Delta\dot{y}, \ddot{y}_r(t)\big) := \sqrt{(\alpha_1 + \ddot{y}_{1r})^2 + (\alpha_2 + \ddot{y}_{2r})^2} \\ \theta &= \Theta\big(\Delta y, \Delta\dot{y}, \ddot{y}_r(t)\big) := \arg(\alpha_1 + \ddot{y}_{1r}, \alpha_2 + \ddot{y}_{2r}).\end{aligned} \tag{21}$$

The angle θ is a priori not defined when $\xi = 0$, i.e., at the singularity of the flat output y. We will not discuss the possibility of overcoming this singularity and simply assume we stay away from it. Aside from that, there remains a big problem: how should the "virtual" control law (21) be understood? Indeed, it seems to be a differential equation: because y depends on θ, hence Ξ and Θ are in fact functions of the variables

$$x, \dot{x}, z, \dot{z}, \theta, \dot{\theta}, y_r(t), \dot{y}_r(t), \ddot{y}_r(t).$$

Notice ξ is related to the actual control u_1 by a relation that also depends on $\dot{\theta}$.

Let us forget this apparent difficulty for the time being and backstep (21) the usual way. Introducing the error variable $\kappa_1 := \theta - \Theta\big(\Delta y, \Delta\dot{y}, \ddot{y}_r(t)\big)$ and using the fundamental theorem of calculus, the error dynamics becomes

$$\begin{aligned}\Delta\ddot{y}_1 &= \alpha_1(\Delta y_1, \Delta\dot{y}_1) - \kappa_1\ R_{\sin}\big(\Theta(\Delta y, \Delta\dot{y}, \ddot{y}_r(t)), \kappa_1\big)\ \Xi\big(\Delta y, \Delta\dot{y}, \ddot{y}_r(t)\big) \\ \Delta\ddot{y}_2 &= \alpha_2(\Delta y_1, \Delta\dot{y}_1) + \kappa_1\ R_{\cos}\big(\Theta(\Delta y, \Delta\dot{y}, \ddot{y}_r(t)), \kappa_1\big)\ \Xi\big(\Delta y, \Delta\dot{y}, \ddot{y}_r(t)\big) \\ \dot{\kappa}_1 &= \dot{\theta} - \dot{\Theta}\big(\kappa_1, \Delta y, \Delta\dot{y}, \ddot{y}_r(t), y_r^{(3)}(t)\big)\end{aligned}$$

Notice the functions

$$R_{\sin}(x,h) = \sin x \frac{\cos h - 1}{h} + \cos x \frac{\sin h}{h}$$
$$R_{\cos}(x,h) = \cos x \frac{\cos h - 1}{h} - \sin x \frac{\sin h}{h}$$

are bounded and analytic. Differentiate now the positive function

$$V_1(\Delta y, \Delta\dot{y}, \kappa_1) := V(\Delta y, \Delta\dot{y}) + \frac{1}{2}\kappa_1^2$$

to get

$$\begin{aligned}\dot{V}_1 = {} & \frac{\partial V}{\partial \Delta y_1}\Delta\dot{y}_1 + \frac{\partial V}{\partial \Delta\dot{y}_1}(\alpha_1 - \kappa_1 R_{\sin}\Xi) + \\ & \frac{\partial V}{\partial \Delta y_2}\Delta\dot{y}_2 + \frac{\partial V}{\partial \Delta\dot{y}_2}(\alpha_2 + \kappa_1 R_{\cos}\Xi) + \kappa_1\ (\dot{\theta} - \dot{\Theta}) \\ = {} & \dot{V} + \kappa_1\ \Big(\dot{\theta} - \dot{\Theta} + \kappa_1\ \big(R_{\cos}\frac{\partial V}{\partial \Delta y_1} - R_{\sin}\frac{\partial V}{\partial \Delta y_2}\big)\ \Xi\Big),\end{aligned}$$

where we have omitted arguments of all the functions for the sake of clarity. If $\dot{\theta}$ were a control input, we could for instance assign

$$\begin{aligned}\dot{\theta} &:= -\kappa_1 + \dot{\Theta} - \kappa_1\ \big(R_{\cos}\frac{\partial V}{\partial \Delta y_1} - R_{\sin}\frac{\partial V}{\partial \Delta y_2}\big)\ \Xi \\ &:= \Theta_1\big(\kappa_1, \Delta y, \Delta\dot{y}, \ddot{y}_r(t), y_r^{(3)}(t)\big),\end{aligned}$$

to get $\dot{V}_1 = \dot{V} - \kappa_1^2 \leq 0$. We thus backstep this "virtual" control law: we introduce the error variable

$$\kappa_2 := \dot{\theta} - \Theta_1\big(\kappa_1, \Delta y, \Delta\dot{y}, \ddot{y}_r(t), y_r^{(3)}(t)\big)$$

together with the positive function

$$V_2(\Delta y, \Delta\dot{y}, \kappa_1, \kappa_2) := V_1(\Delta y, \Delta\dot{y}, \kappa_1) + \frac{1}{2}\kappa_2^2.$$

Differentiating

$$\begin{aligned}V_2 &= \dot{V} + \kappa_1(-\kappa_1 + \kappa_2) + \kappa_2(v_2 - \dot{\Theta}_1) \\ &= \dot{V}_1 + \kappa_2(u_2 - \dot{\Theta}_1 + \kappa_2),\end{aligned}$$

and we can easily make $\dot{V}_1$ negative by assigning

$$u_2 := \Theta_2\big(\kappa_1, \kappa_2, \Delta y, \Delta\dot{y}, \ddot{y}_r(t), y_r^{(3)}(t), y_r^{(4)}(t)\big) \tag{22}$$

for some suitable function Θ_2.

A key observation is that Θ_2 and V_2 are in fact functions of the variables

$$x, \dot{x}, z, \dot{z}, \theta, \dot{\theta}, y_r(t), \ldots, y_r^{(4)}(t),$$

which means (22) makes sense. We have thus built a static control law

$$u_1 = \Xi\big(x, \dot{x}, z, \dot{z}, \theta, \dot{\theta}, y_r(t), \dot{y}_r(t), \ddot{y}_r(t)\big) + \varepsilon\dot{\theta}^2$$
$$u_2 = \Theta_2\big(x, \dot{x}, z, \dot{z}, \theta, \dot{\theta}, y_r(t), \ldots, y_r^{(4)}(t)\big)$$

that does the tracking for (20). Notice it depends on $y_r(t)$ up to the fourth derivative.

Example 10. The dynamics

$$\dot{x}_1 = u_1, \quad \dot{x}_2 = x_3(1 - u_1), \quad \dot{x}_3 = u_2,$$

admits (x_1, x_2) as a flat output. The corresponding endogenous transformation is singular, hence any linearizing feedback blows up, when $u_1 = 1$. However, it is easy to backstep the controller of example 8 to build a globally tracking static controller

Remark. Notice that none the of two previous examples can be linearized by *static* feedback (see section 3.1.2). *Dynamic* feedback is necessary for that. Nevertheless we were able to derive *static* tracking control laws for them. An explanation of why this is possible is that a flat system can in theory be linearized by a *quasistatic* feedback [10] –provided the flat output does not depend on derivatives of the input–.

2.5.4 Backstepping and time-scaling

Backstepping can be combined with linearization and time-scaling, as illustrated in the following example.

Example 11. Consider example 4 and its tracking control defined in example 6. Assume, for example, that $\dot{\sigma} \geq 0$. With the dynamic controller

$$\dot{\xi} = v_1\dot{\sigma}, \quad u_1 = \xi\dot{\sigma}, \quad u_2 = (v_2 - x_2 v_1)/\xi^2$$

where v_1 and v_2 are given by equation (18), we have, for the error $e = y - y_r$, a Lyapunov function $V(e, de/d\sigma)$ satisfying

$$dV/d\sigma \leq -aV \tag{23}$$

with some constant $a > 0$. Remember that $de/d\sigma$ corresponds to $(\xi - 1, x_2\xi - dp/d\sigma)$. Assume now that the real control is not (u_1, u_2) but $(\dot{u}_1 := w_1, u_2)$. With the extended Lyapunov function

$$W = V(e, de/d\sigma) + \frac{1}{2}(u_1 - \xi\dot{\sigma})^2$$

we have

$$\dot{W} = \dot{V} + (w_1 - \dot{\xi}\dot{\sigma} - \xi\ddot{\sigma})((u_1 - \xi\dot{\sigma}).$$

Some manipulations show that

$$\dot{V} = (u_1 - \dot{\sigma}\xi)\left(\frac{\partial V}{\partial e_1} + \frac{\partial V}{\partial e_2}x_2 + \frac{\partial V}{\partial e_2'}u_2\xi\right) + \dot{\sigma}\frac{dV}{d\sigma}$$

(remember $\dot{\xi} = v_1\dot{\sigma}$ and (v_1, v_2) are given by (18)). The feedback $(b > 0)$

$$w_1 = -\left(\frac{\partial V}{\partial e_1} + \frac{\partial V}{\partial e_2}x_2 + \frac{\partial V}{\partial e_2'}u_2\xi\right) + \dot{\xi}\dot{\sigma} + \xi\ddot{\sigma} - b(u_1 - \xi\dot{\sigma})$$

achieves asymptotic tracking since $\dot{W} \leq -a\dot{\sigma}V - b(u_1 - \xi\dot{\sigma})^2$.

2.5.5 Conclusion

It is possible to generalize the previous examples to prove that a control law can be backstepped "through" any endogenous feedback. In particular a flat dynamics can be seen as a (generalized) endogenous feedback acting on the flat output; hence we can backstep a control law for the flat output through the whole dynamics. In other words the flat output serves as a first "virtual" control in the backstepping process. It is another illustration of the fact that a flat output "summarizes" the dynamical behavior.

Notice also that in a tracking problem the knowledge of a flat output is extremely useful not only for the tracking itself (i.e., the closed-loop problem) but also for the trajectory generation (i.e., the open-loop problem)

3 Open problems and new perspectives

3.1 Checking flatness: an overview

3.1.1 The general problem

Devising a general computable test for checking whether $\dot{x} = f(x, u), x \in \mathbb{R}^n, u \in \mathbb{R}^m$ is flat remains up to now an open problem. This means there are no systematic methods for constructing flat outputs. This does not make flatness a useless concept: for instance Lyapunov functions and uniform first integrals of dynamical systems are extremely helpful notions both from a theoretical and practical point of view though they cannot be systematically computed.

The main difficulty in checking flatness is that a candidate flat output $y = h(x, u, \ldots, u^{(r)})$ may a priori depend on derivatives of u of arbitrary order r. Whether this order r admits an upper bound (in terms of n and m) is at the moment completely unknown. Hence we do not know whether a finite bound exists at all. In the sequel, we say a system is r-flat if it admits a flat output depending on derivatives of u of order at most r.

To illustrate this upper bound might be at least linear in the state dimension, consider the system

$$x_1^{(\alpha_1)} = u_1, \quad x_2^{(\alpha_2)} = u_2, \quad \dot{x}_3 = u_1 u_2$$

with $\alpha_1 > 0$ and $\alpha_2 > 0$. It admits the flat output

$$y_1 = x_3 + \sum_{i=1}^{\alpha_1} (-1)^i x_1^{(\alpha_1 - i)} u_2^{(i-1)}, \quad y_2 = x_2,$$

hence is r-flat with $r := \min(\alpha_1, \alpha_2) - 1$. We suspect (without proof) there is no flat output depending on derivatives of u of order less than $r - 1$.

If such a bound $\kappa(n, m)$ were known, the problem would amount to checking p-flatness for a *given* $p \leq \kappa(n, m)$ and could be solved in theory. Indeed, it consists [33] in finding m functions $h_1, \ldots, h_m$ depending on $(x, u, \ldots, u^{(p)})$ such that

$$\dim \operatorname{span} \Big\{ dx_1, \ldots, dx_n, du_1, \ldots, du_m, dh_1^{(\mu)}, \ldots, dh_m^{(\mu)} \Big\}_{0 \leq \mu \leq \nu} = m(\nu + 1),$$

where $\nu := n + pm$. This means checking the integrability of the partial differential system with a transversality condition

$$\begin{aligned} dx_i \wedge dh \wedge \ldots \wedge dh^{(\nu)} &= 0, & i &= 1, \ldots, n \\ du_j \wedge dh \wedge \ldots \wedge dh^{(\nu)} &= 0, & j &= 1, \ldots, m \\ dh \wedge \ldots \wedge dh^{(\nu)} &\neq 0, & & \end{aligned}$$

where $dh^{(\mu)}$ stands for $dh_1^{(\mu)} \wedge \ldots \wedge dh_m^{(\mu)}$. It is in theory possible to conclude by using a computable criterion [3, 58], though this seems to lead to practically intractable calculations. Nevertheless it can be hoped that, due to the special structure of the above equations, major simplifications might appear.

3.1.2 Known results

Systems linearizable by static feedback. A system which is linearizable by static feedback and coordinate change is clearly flat. Hence the geometric necessary and sufficient conditions in [26, 25] provide sufficient conditions for flatness. Notice a flat system is in general not linearizable by static feedback (see for instance example 3), with the major exception of the single-input case.

Single-input systems. When there is only one control input flatness reduces to static feedback linearizability [8] and is thus completely characterized by the test in [26, 25].

Affine systems of codimension 1. A system of the form

$$\dot{x} = f_0(x) + \sum_{j=1}^{n-1} u_j g_j(x), \qquad x \in \mathbb{R}^n,$$

i.e., with one input less than states and linear w.r.t. the inputs is 0-flat as soon as it is controllable [8] (more precisely strongly accessible for almost every x).

The picture is much more complicated when the system is not linear w.r.t. the control, see [34] for a geometric sufficient condition.

Affine systems with 2 inputs and 4 states. Necessary and sufficient conditions for 1-flatness of the system can be found in [56]. They give a good idea of the complexity of checking r-flatness even for r small.

Driftless systems. For driftless systems of the form $\dot{x} = \sum_{i=1}^{m} f_i(x) u_i$ additional results are available.

Theorem 4 (Driftless systems with two inputs [38]). *The system*

$$\dot{x} = f_1(x)u_1 + f_2(x)u_2$$

is flat if and only if the generic rank of E_k is equal to $k+2$ for $k = 0, \ldots, n-2n$ where $E_0 := span\{f_1, f_2\}$, $E_{k+1} := span\{E_k, [E_k, E_k]\}$, $k \geq 0$.

A flat two-input driftless system is always 0-flat. As a consequence of a result in [46], a flat two-input driftless system satisfying some additional regularity conditions can be put by *static* feedback and coordinate change into the *chained system* [47]

$$\dot{x}_1 = u_1, \quad \dot{x}_2 = u_2, \quad \dot{x}_3 = x_2 u_1, \quad \ldots, \quad \dot{x}_n = x_{n-1} u_1.$$

Theorem 5 (Driftless systems, n states, and $n-2$ inputs [39, 40]).

$$\dot{x} = \sum_{i=1}^{n-2} u_i f_i(x), \quad x \in \mathbb{R}^n$$

is flat as soon as it is controllable (i.e., strongly accessible for almost every x). More precisely it is 0-flat when n is odd, and 1-flat when n is even.

All the results mentioned above rely on the use of exterior differential systems. Additional results on driftless systems, with applications to nonholonomic systems, can be found in [74, 73, 70].

Mechanical systems. For mechanical systems with one control input less than configuration variables, [62] provides a geometric characterization, in terms of the metric derived form the kinetic energy and the control codistribution, of flat outputs depending only on the configuration variables.

A necessary condition. Because it is not known whether flatness can be checked with a finite test, see section 3.1.1, it is very difficult to prove that a system is *not* flat. The following result provides a simple necessary condition.

Theorem 6 (The ruled-manifold criterion [65, 16]). *Assume $\dot{x} = f(x,u)$ is flat. The projection on the p-space of the submanifold $p = f(x,u)$, where x is considered as a parameter, is a ruled submanifold for all x.*

The criterion just means that eliminating u from $\dot{x} = f(x,u)$ yields a set of equations $F(x,\dot{x}) = 0$ with the following property: for all (x,p) such that $F(x,p) = 0$, there exists $a \in \mathbb{R}^n$, $a \neq 0$ such that

$$\forall \lambda \in \mathbb{R}, \quad F(x, p + \lambda a) = 0.$$

$F(x,p) = 0$ is thus a ruled manifold containing straight lines of direction a.

The proof directly derives from the method used by Hilbert [23] to prove the second order Monge equation $\frac{d^2 z}{dx^2} = \left(\frac{dy}{dx}\right)^2$ is not solvable without integrals.

A restricted version of this result was proposed in [71] for systems linearizable by a special class of dynamic feedbacks.

As crude as it may look, this criterion is up to now the only way –except for two-input driftless systems– to prove a multi-input system is not flat.

Example 12. The system

$$\dot{x}_1 = u_1, \quad \dot{x}_2 = u_2, \quad \dot{x}_3 = (u_1)^2 + (u_2)^3$$

is not flat, since the submanifold $p_3 = p_1^2 + p_2^3$ is not ruled: there is no $a \in \mathbb{R}^3$, $a \neq 0$, such that

$$\forall \lambda \in \mathbb{R}, p_3 + \lambda a_3 = (p_1 + \lambda a_1)^2 + (p_2 + \lambda a_2)^3.$$

Indeed, the cubic term in λ implies $a_2 = 0$, the quadratic term $a_1 = 0$ hence $a_3 = 0$.

Example 13. The system $\dot{x}_3 = \dot{x}_1^2 + \dot{x}_2^2$ does not define a ruled submanifold of $\mathbb{R}^3$: it is not flat in $\mathbb{R}$. But it defines a ruled submanifold in $\mathbb{C}^3$: in fact it is flat in $\mathbb{C}$, with the flat output

$$y = \big(x_3 - (\dot{x}_1 - \dot{x}_2\sqrt{-1})(x_1 + x_2\sqrt{-1}),\ x_1 + x_2\sqrt{-1}\big).$$

Example 14 *(The ball and beam [21]).* We now prove by the ruled manifold criterion that

$$\ddot{r} = -Bg\sin\theta + Br\dot{\theta}^2$$
$$(mr^2 + J + J_b)\ddot{\theta} = \tau - 2mr\dot{r}\dot{\theta} - mgr\cos\theta,$$

where $(r, \dot{r}, \theta, \dot{\theta})$ is the state and τ the input, is not flat (as it is a single-input system, we could also prove it is not static feedback linearizable, see section 3.1.2). Eliminating the input τ yields

$$\dot{r} = v_r, \quad \dot{v}_r = -Bg\sin\theta + Br\dot{\theta}^2, \quad \dot{\theta} = v_\theta$$

which defines a ruled manifold in the $(\dot{r}, \dot{v}_r, \dot{\theta}, \dot{v}_\theta)$-space for any r, v_r, θ, v_θ, and we cannot conclude directly. Yet, the system is obviously equivalent to

$$\dot{r} = v_r, \quad \dot{v}_r = -Bg\sin\theta + Br\dot{\theta}^2,$$

which clearly does not define a ruled submanifold for any (r, v_r, θ). Hence the system is not flat.

3.2 Infinite dimension "flat" systems

The idea underlying equivalence and flatness –a one-to-one correspondence between trajectories of systems– is not restricted to control systems described by *ordinary* differential equations. It can be adapted to delay differential systems and to partial differential equations with boundary control. Of course, there are many more technicalities and the picture is far from clear. Nevertheless, this new point of view seems promising for the design of control laws. In this section, we sketch some recent developments in this direction.

3.2.1 Delay systems

Consider for instance the simple differential delay system

$$\dot{x}_1(t) = x_2(t), \quad \dot{x}_2(t) = x_1(t) - x_2(t) + u(t-1).$$

Setting $y(t) := x_1(t)$, we can clearly explicitly parameterize its trajectories by

$$x_1(t) = y(t), \quad x_2(t) = \dot{y}(t), \quad u(t) = \ddot{y}(t+1) + \dot{y}(t+1) - y(t+1).$$

In other words, $y(t) := x_1(t)$ plays the role of a"flat" output. This idea is investigated in detail in [42], where the class of δ-*free* systems is defined (δ is the delay operator). More precisely, [42] considers linear differential delay systems

$$M(d/dt, \delta)w = 0$$

where M is a $(n-m) \times n$ matrix with entries polynomials in d/dt and δ and $w = (w_1, \ldots, w_n)$ are the system variables. Such a system is said to be δ-free if it can be related to the "free" system $y = (y_1, \ldots, y_m)$ consisting of arbitrary functions of time by

$$w = P(d/dt, \delta, \delta^{-1})y$$
$$y = Q(d/dt, \delta, \delta^{-1})w,$$

where P (resp. Q) is a $n \times m$ (resp. $m \times n$) matrix the entries of which are polynomial in d/dt, δ and δ^{-1}.

Many linear delay systems are δ-free. For example, $\dot{x}(t) = Ax(t)+Bu(t-1)$, (A, B) controllable, is δ-free, with the Brunovski output of $\dot{x} = Ax + Bv$ as a "δ-free" output.

The following systems, commonly used in process control,

$$z_i(s) = \sum_{j=1}^{m} \left\{ \frac{K_i^j \exp(-s\delta_i^j)}{1 + \tau_i^j s} \right\} u_j(s), \quad i = 1, \ldots p$$

(s Laplace variable, gains K_i^j, delays δ_i^j and time constants τ_i^j between u_j and z_i) are δ-free [54]. Other interesting examples of δ-free systems arise from partial differential equations:

Example 15 *(Torsion beam system).* The torsion motion of a beam (figure 1) can be modeled in the linear elastic domain by

$$\partial_t^2 \theta(x,t) = \partial_x^2 \theta(x,t), \quad x \in [0,1]$$
$$\partial_x \theta(0,t) = u(t)$$
$$\partial_x \theta(1,t) = \partial_t^2 \theta(1,t),$$

where $\theta(x,t)$ is the torsion of the beam and $u(t)$ the control input. From d'Alembert's formula, $\theta(x,t) = \phi(x+t) + \psi(x-t)$, we easily deduce

$$2\theta(t,x) = \dot{y}(t+x-1) - \dot{y}(t-x+1) + y(t+x-1) + y(t-x+1)$$
$$2u(t) = \ddot{y}(t+1) + \ddot{y}(t-1) - \dot{y}(t+1) + \dot{y}(t-1),$$

where we have set $y(t) := \theta(1,t)$. This proves the system is δ-free with $\theta(1,t)$ as a "δ-flat" output. See [43, 17] for details and an application to motion planning.

3.2.2 Distributed parameters systems

For partial differential equations with boundary control and mixed systems of partial and ordinary differential equations, it seems possible to describe the

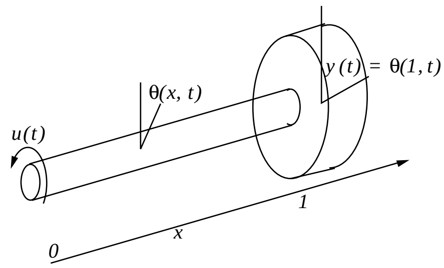

Figure 1: torsion of a flexible beam

one-to-one correspondence via series expansion, though a sound theoretical framework is yet to be found. We illustrate this original approach to control design on the following two "flat" systems.

Example 16 *(Heat equation).* Consider the linear heat equation

$$\partial_t\theta(x,t) = \partial_x^2\theta(x,t), \quad x \in [0,1] \tag{24}$$

$$\partial_x\theta(0,t) = 0 \tag{25}$$

$$\theta(1,t) = u(t), \tag{26}$$

where $\theta(x,t)$ is the temperature and $u(t)$ is the control input. We claim that

$$y(t) := \theta(0,t)$$

is a "flat" output. Indeed, the equation in the Laplace variable s reads

$$s\hat{\theta}(x,s) = \hat{\theta}''(x,s) \quad \text{with} \quad \hat{\theta}'(0,s) = 0, \quad \hat{\theta}(1,s) = \hat{u}(s)$$

($'$ stands for ∂_x and $\hat{}$ for the Laplace transform), and the solution is clearly $\hat{\theta}(x,s) = \cosh(x\sqrt{s})\hat{u}(s)/\cosh(\sqrt{s})$. As $\hat{\theta}(0,s) = \hat{u}(s)/\cosh(\sqrt{s})$, this implies

$$\hat{u}(s) = \cosh(\sqrt{s})\ \hat{y}(s) \quad \text{and} \quad \hat{\theta}(x,s) = \cosh(x\sqrt{s})\ \hat{y}(s).$$

Since $\cosh\sqrt{s} = \sum_{i=0}^{+\infty} s^i/(2i)!$, we eventually get

$$\theta(x,t) = \sum_{i=1}^{+\infty} x^{2i}\ \frac{y^{(i)}(t)}{(2i)!} \tag{27}$$

$$u(t) = \sum_{i=1}^{+\infty} \frac{y^{(i)}(t)}{(2i)!}. \tag{28}$$

In other words, whenever $t \mapsto y(t)$ is an arbitrary function (i.e., a trajectory of the trivial system $y = v$), $t \mapsto \big(\theta(x,t), u(t)\big)$ defined by (27)-(28) is a (formal) trajectory of (24)–(26), and vice versa. This is exactly the idea underlying our definition of flatness in section 1.3. Notice these calculations have been known for a long time, see [75, pp. 588 and 594].

To make the statement precise, we now turn to convergence issues. On the one hand, $t \mapsto y(t)$ must be a smooth function such that

$$\exists\, K, M > 0, \quad \forall i \geq 0, \forall t \in [t_0, t_1], \quad |y^{(i)}(t)| \leq M(Ki)^{2i}$$

to ensure the convergence of the series (27)-(28).

On the other hand $t \mapsto y(t)$ cannot in general be analytic. Indeed, if the system is to be steered from an initial temperature profile $\theta(x, t_0) = \alpha_0(x)$ at time t_0 to a final profile $\theta(x, t_1) = \alpha_1(x)$ at time t_1, equation (24) implies

$$\forall t \in [0,1], \forall i \geq 0, \quad y^{(i)}(t) = \partial_t^i \theta(0,t) = \partial_x^{2i}\theta(0,t),$$

and in particular

$$\forall i \geq 0, \qquad y^{(i)}(t_0) = \partial_x^{2i}\alpha_0(0) \quad \text{and} \quad y^{(i)}(t_1) = \partial_x^{2i}\alpha_1(1).$$

If for instance $\alpha_0(x) = c$ for all $x \in [0,1]$ (i.e., uniform temperature profile), then $y(t_0) = c$ and $y^{(i)}(t_0) = 0$ for all $i \geq 1$, which implies $y(t) = c$ for all t when the function is analytic. It is thus impossible to reach any final profile but $\alpha_1(x) = c$ for all $x \in [0,1]$.

Smooth functions $t \in [t_0, t_1] \mapsto y(t)$ that satisfy

$$\exists\, K, M > 0, \quad \forall i \geq 0, \qquad |y^{(i)}(t)| \leq M(Ki)^{\sigma i}$$

are known as Gevrey-Roumieu functions of order σ [61] (they are also closely related to class S functions [20]). The Taylor expansion of such functions is convergent for $\sigma \leq 1$ and divergent for $\sigma > 1$ (the larger σ is, the "more divergent" the Taylor expansion is). Analytic functions are thus Gevrey-Roumieu of order ≤ 1.

In other words we need a Gevrey-Roumieu function on $[t_0, t_1]$ of order > 1 but ≤ 2, with initial and final Taylor expansions imposed by the initial and final temperature profiles. With such a function, we can then compute open-loop control steering the system from one profile to the other by the formula (27).

For instance, we steered the system from uniform temperature 0 at $t = 0$ to uniform temperature 1 at $t = 1$ by using the function

$$\mathbb{R} \ni t \mapsto y(t) := \begin{cases} 0 & \text{if } t < 0 \\ 1 & \text{if } t > 1 \\ \dfrac{\int_0^t \exp\bigl(-1/(\tau(1-\tau))^\gamma\bigr) d\tau}{\int_0^1 \exp\bigl(-1/(\tau(1-\tau))^\gamma\bigr) d\tau} & \text{if } t \in [0,1], \end{cases}$$

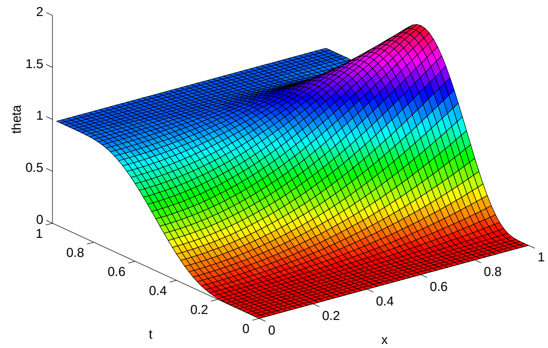

Figure 2: evolution of the temperature profile for $t \in [0, 1]$.

with $\gamma = 1$ (this function is Gevrey-Roumieu functions of order $1 + 1/\gamma$). The evolution of the temperature profile $\theta(x, t)$ is displayed on figure 2 (the Matlab simulation is available upon request at rouchon@cas.ensmp.fr).

Similar but more involved calculations with convergent series corresponding to Mikunsiński operators are used in [18] to control a flexible rod modeled by an Euler-Bernoulli equation. For nonlinear systems, convergence issues are more involved and are currently under investigation. Yet, it is possible to work –at least formally– along the same line.

Example 17 *(Flexion beam system).* Consider with [30] the mixed system

$$\rho \partial_t^2 u(x,t) = \rho \omega^2(t) u(x,t) - EI \partial_x^4 u(x,t), \quad x \in [0,1]$$
$$\dot{\omega}(t) = \frac{\Gamma_3(t) - 2\omega(t) <u, \partial_t u>(t)}{I_d + <u,u>(t)}$$

with boundary conditions

$$u(0,t) = \partial_x u(0,t) = 0, \qquad \partial_x^2 u(1,t) = \Gamma_1(t), \qquad \partial_x^3 u(1,t) = \Gamma_2(t),$$

where ρ, EI, I_d are constant parameters, $u(x,t)$ is the deformation of the beam, $\omega(t)$ is the angular velocity of the body and $<f,g>(t) := \int_0^1 \rho f(x,t) g(x,t) dx$. The three control inputs are $\Gamma_1(t)$, $\Gamma_2(t)$, $\Gamma_3(t)$. We claim that

$$y(t) := \left(\partial_x^2 u(0,t),\ \partial_x^3 u(0,t), \omega(t)\right)$$

is a "flat" output. Indeed, $\omega(t)$, $\Gamma_1(t)$, $\Gamma_2(t)$ and $\Gamma_3(t)$ can clearly be expressed in terms of $y(t)$ and $u(x,t)$, which transforms the system into the equivalent

Cauchy-Kovalevskaya form

$$EI\partial_x^4 u(x,t) = \rho y_3^2(t)u(x,t) - \rho\partial_t^2 u(x,t) \qquad \text{and} \qquad \begin{cases} u(0,t) = 0 \\ \partial_x u(0,t) = 0 \\ \partial_x^2 u(0,t) = y_1(t) \\ \partial_x^3 u(0,t) = y_2(t). \end{cases}$$

Set then formally $u(x,t) = \sum_{i=0}^{+\infty} a_i(t)\frac{x^i}{i!}$, plug this series into the above system and identify term by term. This yields

$$a_0 = 0, \qquad a_1 = 0, \qquad a_2 = y_1, \qquad a_3 = y_2,$$

and the iterative relation $\forall i \geq 0,\ EIa_{i+4} = \rho y_3^2 a_i - \rho\ddot{a}_i$. Hence for all $i \geq 1$,

$$\begin{aligned} a_{4i} &= 0 & a_{4i+2} &= \frac{\rho}{EI}(y_3^2 a_{4i-2} - \ddot{a}_{4i-2}) \\ a_{4i+1} &= 0 & a_{4i+3} &= \frac{\rho}{EI}(y_3^2 a_{4i-1} - \ddot{a}_{4i-1}). \end{aligned}$$

There is thus a 1–1 correspondence between (formal) solutions of the system and arbitrary mappings $t \mapsto y(t)$: the system is formally flat.

3.3 State constraints and optimal control

3.3.1 Optimal control

Consider the standard optimal control problem

$$\min_u J(u) = \int_0^T L(x(t), u(t))dt$$

together with $\dot{x} = f(x,u)$, $x(0) = a$ and $x(T) = b$, for known a, b and T.

Assume that $\dot{x} = f(x,u)$ is flat with $y = h(x, u, \ldots, u^{(r)})$ as flat output,

$$x = \varphi(y, \ldots, y^{(q)}), \quad u = \alpha(y, \ldots, y^{(q)}).$$

A numerical resolution of $\min_u J(u)$ a priori requires a discretization of the state space, i.e., a finite dimensional approximation. A better way is to discretize the flat output space. As in section 1.4, set $y_i(t) = \sum_1^N A_{ij}\lambda_j(t)$. The initial and final conditions on x provide then initial and final constraints on y and its derivatives up to order q. These constraints define an affine sub-space V of the vector space spanned by the the A_{ij}'s. We are thus left with the nonlinear programming problem

$$\min_{A \in V} J(A) = \int_0^T L(\varphi(y, \ldots, y^{(q)}), \alpha(y, \ldots, y^{(q)}))dt,$$

where the y_i's must be replaced by $\sum_1^N A_{ij}\lambda_j(t)$.

This methodology is used in [50] for trajectory generation and optimal control. It should also be very useful for predictive control. The main expected benefit is a dramatic improvement in computing time and numerical stability. Indeed the exact quadrature of the dynamics –corresponding here to exact discretization via well chosen input signals through the mapping α– avoids the usual numerical sensitivity troubles during integration of $\dot{x} = f(x,u)$ and the problem of satisfying $x(T) = b$.

3.3.2 State constraints

In the previous section, we did not consider state constraints. We now turn to the problem of planning a trajectory steering the state from a to b while satisfying the constraint $k(x, u, \dots, u^{(p)}) \leq 0$. In the flat output "coordinates" this yields the following problem: find $T > 0$ and a smooth function $[0,T] \ni t \mapsto y(t)$ such that $(y, \dots, y^{(q)})$ has prescribed value at $t = 0$ and T and such that $\forall t \in [0,T]$, $K(y, \dots, y^{(\nu)})(t) \leq 0$ for some ν. When $q = \nu = 0$ this problem, known as the *piano mover problem*, is already very difficult.

Assume for simplicity sake that the initial and final states are equilibrium points. Assume also there is a quasistatic motion strictly satisfying the constraints: there exists a *path* (not a trajectory) $[0,1] \ni \sigma \mapsto Y(\sigma)$ such that $Y(0)$ and $Y(1)$ correspond to the initial and final point and for any $\sigma \in [0,1]$, $K(Y(\sigma), 0, \dots, 0) < 0$. Then, there exists $T > 0$ and $[0,T] \ni t \mapsto y(t)$ solution of the original problem. It suffices to take $Y(\eta(t/T))$ where T is large enough, and where η is a smooth increasing function $[0,1] \ni s \mapsto \eta(s) \in [0,1]$, with $\eta(0) = 0$, $\eta(1) = 1$ and $\frac{d^i\eta}{ds^i}(0,1) = 0$ for $i = 1, \dots, \max(q, \nu)$.

In [64] this method is applied to a two-input chemical reactor. In [60] the minimum-time problem under state constraints is investigated for several mechanical systems. [68] considers, in the context of non holonomic systems, the path planning problem with obstacles. Due to the nonholonomic constraints, the above quasistatic method fails: one cannot set the y-derivative to zero since they do not correspond to time derivatives but to arc-length derivatives. However, several numerical experiments clearly show that sorting the constraints with respect to the order of y-derivatives plays a crucial role in the computing performance.

3.4 Symmetries

3.4.1 Symmetry preserving flat output

Consider the dynamics $\dot{x} = f(x,u)$, $(x,u) \in X \times U \subset \mathbb{R}^n \times \mathbb{R}^m$. According to section 1 it generates a system $(F, \mathfrak{M})$, where $\mathfrak{M} := X \times U \times \mathbb{R}_m^\infty$ and

$F(x, u, u^1, \dots) := (f(x, u), u^1, u^2, \dots)$. At the heart of our notion of equivalence are endogenous transformations, which map solutions of a system to solutions of another system. We single out here the important class of transformations mapping solutions of a system onto solutions of the *same* system:

Definition 6. An endogenous transformation $\Phi_g : \mathfrak{M} \longmapsto \mathfrak{M}$ is a *symmetry* of the system $(F, \mathfrak{M})$ if

$$\forall \xi := (x, u, u^1, \dots) \in \mathfrak{M}, \quad F(\Phi_g(\xi)) = D\Phi_g(\xi) \cdot F(\xi).$$

More generally, we can consider a *symmetry group*, i.e., a collection $\left(\Phi_g\right)_{g \in G}$ of symmetries such that $\forall g_1, g_2 \in G, \Phi_{g_1} \circ \Phi_{g_2} = \Phi_{g_1 * g_2}$, where $(G, *)$ is a group.

Assume now the system is flat. The choice of a flat output is by no means unique, since any endogenous transformation on a flat output gives rise to another flat output.

Example 18 *(The kinematic car).* The system generated by

$$\dot{x} = u_1 \cos\theta, \quad \dot{y} = u_1 \sin\theta, \quad \dot{\theta} = u_2,$$

admits the 3-parameter symmetry group of planar (orientation-preserving) isometries: for all translation $(a, b)'$ and rotation α , the endogenous mapping generated by

$$\begin{aligned} X &= x\cos\alpha - y\sin\alpha + a \\ Y &= x\sin\alpha + y\cos\alpha + b \\ \Theta &= \theta + \alpha \\ U^1 &= u^1 \\ U^2 &= u^2 \end{aligned}$$

is a symmetry, since the state equations remain unchanged,

$$\dot{X} = U_1 \cos\Theta, \quad \dot{Y} = U_1 \sin\Theta, \quad \dot{\Theta} = U_2.$$

This system is flat $z := (x, y)$ as a flat output. Of course, there are infinitely many other flat outputs, for instance $\tilde{z} := (x, y + \dot{x})$. Yet, z is obviously a more "natural" choice than $\tilde{z}$, because it "respects" the symmetries of the system. Indeed, each symmetry of the system induces a transformation on the flat output z

$$\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \longmapsto \begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix} = \begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} z_1 \cos\alpha - z_2 \sin\alpha + a \\ z_1 \sin\alpha + z_2 \cos\alpha + b \end{pmatrix}$$

which does not involve *derivatives* of z, i.e., a *point* transformation. This point transformation generates an endogenous transformation $(z, \dot{z}, \dots) \mapsto$

$(Z, \dot{Z}, \dots)$. Following [19], we say such an endogenous transformation which is the total prolongation of a point transformation is *holonomic*.

On the contrary, the induced transformation on $\tilde{z}$

$$\begin{pmatrix} \tilde{z}_1 \\ \tilde{z}_2 \end{pmatrix} \longmapsto \begin{pmatrix} \tilde{Z}_1 \\ \tilde{Z}_2 \end{pmatrix} = \begin{pmatrix} X \\ Y + \dot{X} \end{pmatrix} = \begin{pmatrix} \tilde{z}_1 \cos\alpha + (\dot{\tilde{z}}_1 - \tilde{z}_2)\sin\alpha + a \\ \tilde{z}_1 \sin\alpha + \tilde{z}_2 \cos\alpha + (\ddot{\tilde{z}}_1 - \dot{\tilde{z}}_2)\sin\alpha + b \end{pmatrix}$$

is *not* a point transformation (it involves derivatives of $\tilde{z}$) and does not give to a holonomic transformation.

Consider the system $(F, \mathfrak{M})$ admitting a symmetry Φ_g (or a symmetry group $\left(\Phi_g\right)_{g\in G}$). Assume moreover the system is flat with h as a flat output and denotes by $\Psi := (h, \dot{h}, \ddot{h}, \dots)$ the endogenous transformation generated by h. We then have:

Definition 7 (Symmetry-preserving flat output). The flat output h *preserves* the symmetry Φ_g if the composite transformation $\Psi \circ \Phi_g \circ \Psi^{-1}$ is holonomic.

This leads naturally to a fundamental question: assume a flat system admits the symmetry group $\left(\Phi_g\right)_{g\in G}$. Is there a flat output which preserves $\left(\Phi_g\right)_{g\in G}$?

This question can in turn be seen as a special case of the following problem: view a dynamics $\dot{x} - f(x,u) = 0$ as an *underdetermined differential system* and assume it admits a symmetry group; can it then be reduced to a "smaller" differential system? Whereas this problem has been studied for a long time and received a positive answer in the *determined* case, the underdetermined case seems to have been barely untouched [53].

3.4.2 Flat outputs as potentials and gauge degree of freedom

Symmetries and the quest for potentials are at the heart of physics. To end the paper, we would like to show that flatness fits into this broader scheme.

Maxwell's equations in an empty medium imply that the magnetic field H is divergent free, $\nabla \cdot H = 0$. In Euclidian coordinates (x_1, x_2, x_3), it gives the underdetermined partial differential equation

$$\frac{\partial H_1}{\partial x_1} + \frac{\partial H_2}{\partial x_2} + \frac{\partial H_3}{\partial x_3} = 0$$

A key observation is that the solutions to this equation derive from a vector potential $H = \nabla \times A$: the constraint $\nabla \cdot H = 0$ is automatically satisfied whatever the potential A. This potential parameterizes all the solutions of the underdetermined system $\nabla \cdot H = 0$, see [59] for a general theory. A is a priori not uniquely defined, but up to an arbitrary gradient field, the gauge degree of freedom. The symmetries of the problem indicate how to use this degree of freedom to fix a "natural" potential.

The picture is similar for flat systems. A flat output is a "potential" for the underdetermined differential equation $\dot{x} - f(x,u) = 0$. Endogenous transformations on the flat output correspond to gauge degrees of freedom. The "natural" flat output is determined by symmetries of the system. Hence controllers designed from this flat output can also preserve the physics.

A slightly less esoteric way to convince the reader that flatness is an interesting notion is to take a look at the following small catalog of flat systems.

4 A catalog of flat systems

We give here a (partial) list of flat systems encountered in applications.

4.1 Holonomic mechanical systems

Example 19 *(Fully actuated holonomic systems).* The dynamics of a holonomic system with as many independent inputs as configuration variables is

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} = M(q)u + D(q,\dot{q}),$$

with $M(q)$ invertible. It admits q as a flat output –even when $\frac{\partial^2 L}{\partial \dot{q}^2}$ is singular–: indeed, u can be expressed in function of $q, \dot{q}$ by the *computed torque* formula

$$u = M(q)^{-1}\left(\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} - D(q,\dot{q})\right).$$

If q is constrained by $c(q) = 0$ the system remains flat, and the flat output corresponds to the configuration point in $c(q) = 0$.

Example 20 *(Planar rigid body with forces).* Consider a planar rigid body moving in a vertical plane under the influence of gravity and controlled by two forces having lines of action that are fixed with respect to the body and intersect at a single point (see figure 3). Let (x,y) represent the horizontal and vertical coordinates of center of mass G of the body with respect to a stationary frame, and let θ be the counterclockwise orientation of a body fixed line through the center of mass. Take m as the mass of the body and J as the moment of inertia. Let $g \approx 9.8$ m/sec^2 represent the acceleration due to gravity.

Without loss of generality, we will assume that the lines of action for F_1 and F_2 intersect the y axis of the rigid body and that F_1 and F_2 are perpendicular.

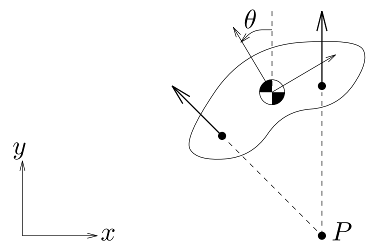

Figure 3: A rigid body controlled by two body fixed forces.

The equations of motion for the system can be written as

$$\begin{aligned} m\ddot{x} &= F_1 \cos\theta - F_2 \sin\theta \\ m\ddot{y} &= F_1 \sin\theta + F_2 \cos\theta - mg \\ J\ddot{\theta} &= rF_1. \end{aligned}$$

The flat output of this system corresponds to *Huyghens center of oscillation* [16]

$$(x - \frac{J}{mr}\sin\theta, \quad y + \frac{J}{mr}\cos\theta).$$

This example has some practical importance. The PVTOL system, the gantry crane and the robot $2k\pi$ (see below) are of this form, as is the simplified *planar ducted fan* [49]. Variations of this example can be formed by changing the number and type of the inputs [45].

Example 21 *(PVTOL aircraft).* A simplified Vertical Take Off and Landing aircraft moving in a vertical Plane [22] can be described by

$$\begin{aligned} \ddot{x} &= -u_1 \sin\theta + \varepsilon u_2 \cos\theta \\ \ddot{z} &= u_1 \cos\theta + \varepsilon u_2 \sin\theta - 1 \\ \ddot{\theta} &= u_2. \end{aligned}$$

A flat output is $y = (x - \varepsilon \sin\theta, z + \varepsilon \cos\theta)$, see [37] more more details and a discussion in relation with unstable zero dynamics.

Example 22 *(The robot* $2k\pi$ *of Ecole des Mines).* It is a robot arm carrying a pendulum, see figure 4. The control objective is to flip the pendulum from its natural downward rest position to the upward position and maintains it there. The first three degrees of freedom (the angles $\theta_1, \theta_2, \theta_3$) are actuated by electric motors, while the two degrees of freedom of the pendulum are not actuated.

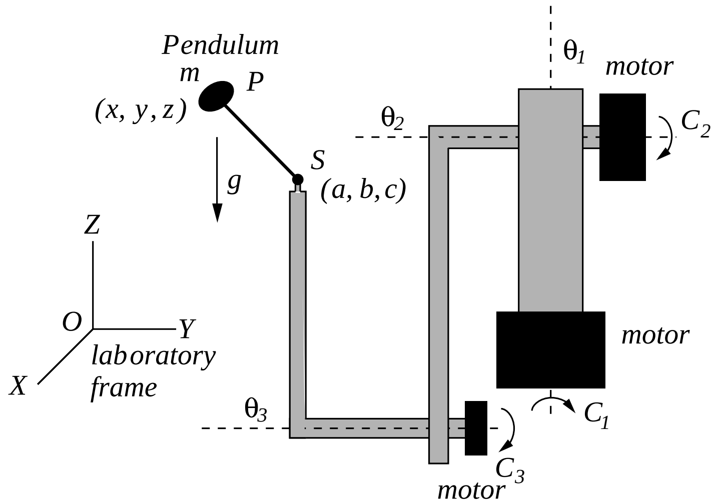

Figure 4: The robot $2k\pi$ carrying its pendulum.

The position $P = (x, y, z)$ of the pendulum oscillation center is a flat output. Indeed, it is related to the position $S = (a, b, c)$ of the suspension point by

$$\begin{aligned}(x-a)(\ddot{z}+g) &= \ddot{x}(z-c)\\ (y-b)(\ddot{z}+g) &= \ddot{y}(z-c)\\ (x-a)^2+(y-b)^2+(z-c)^2 &= l^2,\end{aligned}$$

where l is the distance between S and P. On the other hand the geometry of the robot defines a relation $(a, b, c) = \mathcal{T}(\theta_1, \theta_2, \theta_3)$ between the position of S and the robot configuration. This relation is locally invertible for almost all configurations but is not globally invertible.

Example 23 *(Gantry crane [16]).* A direct application of Newton's laws provides the implicit equations of motion

$$\begin{aligned}m\ddot{x} &= -T\sin\theta & x &= R\sin\theta + D\\ m\ddot{z} &= -T\cos\theta + mg & z &= R\cos\theta,\end{aligned}$$

where x, z, θ are the configuration variables and T is the tension in the cable. The control inputs are the trolley position D and the cable length R. This system is flat, with the position (x, z) of the load as a flat output.

Example 24 *(Conventional aircraft).* A conventional aircraft is flat, provided some small aerodynamic effects are neglected, with the coordinates of the center of mass and side-slip angle as a flat output. See [33] for a detailed study.

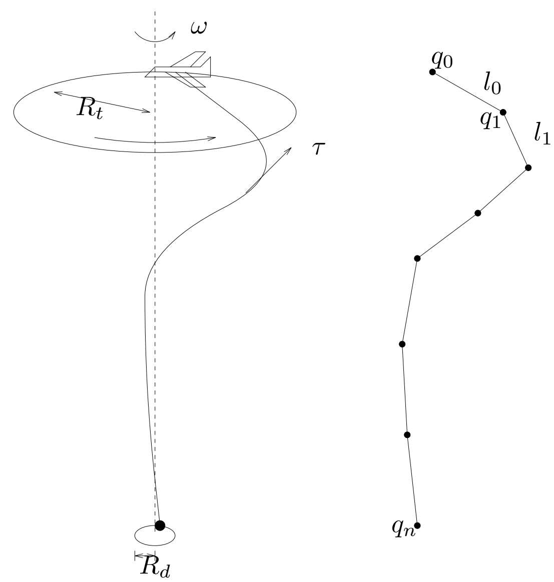

Figure 5: Towed cable system and finite link approximate model.

Example 25 *(Towed cable system).* Consider the dynamics of a system consisting of an aircraft flying in a circular pattern while towing a cable with a tow body (drogue) attached at the bottom. Under suitable conditions, the cable reaches a relative equilibrium in which the cable maintains its shape as it rotates. By choosing the parameters of the system appropriately, it is possible to make the radius at the bottom of the cable much smaller than the radius at the top of the cable. This is illustrated in Figure 5. The motion of the towed cable system can be approximately represented using a finite element model in which segments of the cable are replaced by rigid links connected by spherical joints. The forces acting on the segment (tension, aerodynamic drag and gravity) are lumped and applied at the end of each rigid link. In addition to the forces on the cable, we must also consider the forces on the drogue and the towplane. The drogue is modeled as a sphere and essentially acts as a mass attached to the last link of the cable, so that the forces acting on it are included in the cable dynamics. The external forces on the drogue again consist of gravity and aerodynamic drag. The towplane is attached to the top of the cable and is subject to drag, gravity, and the force of the attached cable. For simplicity, we simply model the towplane as a pure force applied at the top of the cable. Our goal is to generate trajectories for this system that allow operation away from relative equilibria as well as transition between one equilibrium point and another. Due to the high dimension of the model for the system (128 states is typical), traditional approaches to solving this problem, such as optimal control theory, cannot be easily applied. However, it can be shown that this system is differentially flat using the position of the bottom of the cable as the differentially flat output. Thus all feasible trajectories for the system are characterized by the trajectory of the bottom of the cable. See [44] for a more complete description and additional references.

We end this section with a system which is not known to be flat for generic parameter value but still enjoys the weaker property of being *orbitally* flat [14].

Example 26 *(Satellite with two controls).* Consider with [4] a satellite with two control inputs u_1, u_2 described by

$$\begin{aligned}
\dot\omega_1 &= u_1\\
\dot\omega_2 &= u_2\\
\dot\omega_3 &= a\omega_1\omega_2\\
\dot\varphi &= \omega_1\cos\theta + \omega_3\sin\theta\\
\dot\theta &= (\omega_1\sin\theta - \omega_3\cos\theta)\tan\varphi + \omega_2\\
\dot\psi &= \frac{(\omega_3\cos\theta - \omega_1\sin\theta)}{\cos\varphi},
\end{aligned} \tag{29}$$

where $a = (J_1 - J_2)/J_3$ (J_i are the principal moments of inertia); physical sense imposes $|a| \leq 1$. Eliminating u_1, u_2 and ω_1, ω_2 by

$$\omega_1 = \frac{\dot\varphi - \omega_3\sin\theta}{\cos\theta} \quad \text{and} \quad \omega_2 = \dot\theta + \dot\psi\sin\varphi$$

yields the equivalent system

$$\dot\omega_3 = a(\dot\theta + \dot\psi\sin\varphi)\frac{\dot\varphi - \omega_3\sin\theta}{\cos\theta} \tag{30}$$

$$\dot\psi = \frac{\omega_3 - \dot\varphi\sin\theta}{\cos\varphi\cos\theta}. \tag{31}$$

But this system is in turn equivalent to

$$\cos\theta\big(\ddot\psi\cos\varphi - (1+a)\dot\psi\dot\varphi\sin\varphi\big) + \sin\theta\big(\ddot\varphi + a\dot\psi^2\sin\varphi\cos\varphi\big) + \dot\theta(1-a)(\dot\varphi\cos\theta - \dot\psi\sin\theta\cos\varphi) = 0$$

by substituting $\omega_3 = \dot\psi\cos\varphi\cos\theta + \dot\varphi\sin\theta$ in (30).

When $a = 1$, θ can clearly be expressed in function of φ, ψ and their derivatives. We have proved that (29) is flat with (φ, ψ) as a flat output. A similar calculation can be performed when $a = -1$.

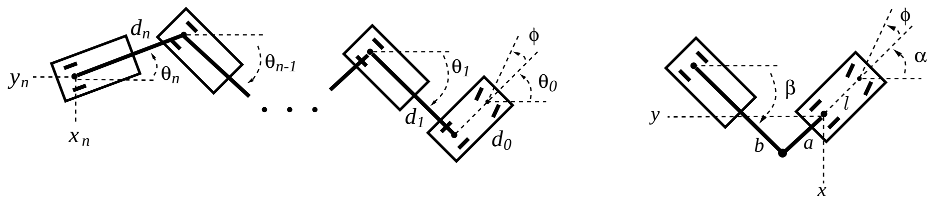

Figure 6: n-trailer system (left) and 1-trailer system with kingpin hitch (right).

When $|a| < 1$, whether (29) is flat is unknown. Yet, it is *orbitally* flat [63]. To see that, rescale time by $\dot{\sigma} = \omega_3$; by the chain rule $\dot{x} = \dot{\sigma} x'$ whatever the variable x, where $'$ denotes the derivation with respect to σ. Setting then

$$\bar{\omega}_1 := \omega_1/\omega_3, \quad \bar{\omega}_2 := \omega_2/\omega_3, \quad \bar{\omega}_3 := -1/a\omega_3,$$

and eliminating the controls transforms (29) into

$$\begin{aligned}
\omega_3' &= \bar{\omega}_1 \bar{\omega}_2 \\
\varphi' &= \bar{\omega}_1 \cos\theta + \sin\theta \\
\theta' &= (\bar{\omega}_1 \sin\theta - \cos\theta)\tan\varphi + \bar{\omega}_2 \\
\psi' &= \frac{(\cos\theta - \bar{\omega}_1 \sin\theta)}{\cos\varphi}.
\end{aligned}$$

The equations are now independent of a. This implies the satellite with $a \neq 1$ is orbitally equivalent to the satellite with $a = 1$. Since it is flat when $a = 1$ it is orbitally flat when $a \neq 1$, with (φ, ψ) as an orbitally flat output.

4.2 Nonholonomic mechanical systems

Example 27 *(Kinematics generated by* two *nonholonomic constraints).* Such systems are flat by theorem 5 since they correspond to driftless systems with n states and $n-2$ inputs. For instance the rolling disc (p. 4), the rolling sphere (p. 96) and the bicycle (p. 330) considered in the classical treatise on nonholonomic mechanics [48] are flat.

Example 28 *(Mobile robots).* Many mobile robots modeled by rolling without sliding constraints, such as those considered in [5, 47, 74] are flat. In particular, the n-trailer system (figure 6) has for flat output the mid-point P_n of the last trailer axle [67, 16]. The 1-trailer system with kingpin hitch is also flat, with a rather complicated flat output involving elliptic integrals [66, 12], but by theorem 4 the system is *not* flat when there is more than one trailer.

Example 29 *(The rolling penny).* The dynamics of this Lagrangian system submitted to a nonholonomic constraint is described by

$$\begin{aligned}\ddot{x} &= \lambda \sin\varphi + u_1 \cos\varphi \\ \ddot{y} &= -\lambda \cos\varphi + u_1 \sin\varphi \\ \ddot{\varphi} &= u_2 \\ \dot{x}\sin\varphi &= \dot{y}\cos\varphi\end{aligned}$$

where x, y, φ are the configuration variables, λ is the Lagrange multiplier of the constraint and u_1, u_2 are the control inputs. A flat output is (x, y): indeed, parameterizing time by the arclength s of the curve $t \mapsto (x(t), y(t))$ we find

$$\cos\varphi = \frac{\mathrm{d}x}{\mathrm{d}s}, \quad \sin\varphi = \frac{\mathrm{d}y}{\mathrm{d}s}, \quad u_1 = \dot{s}, \quad u_2 = \kappa(s)\ \ddot{s} + \frac{\mathrm{d}\kappa}{\mathrm{d}s}\ \dot{s}^2,$$

where κ is the curvature. These formulas remain valid even if $u_1 = u_2 = 0$.

This example can be generalized to any mechanical system subject to m flat nonholonomic constraints, provided there are $n - m$ control forces independent of the constraint forces (n the number of configuration variables), i.e., a "fully-actuated" nonholonomic system as in [5].

All these flat nonholonomic systems have a controllability singularity at rest. Yet, it is possible to "blow up" the singularity by reparameterizing time with the arclength of the curve described by the flat output, hence to plan and track trajectories starting from and stopping at rest as explained in sections 1.5 and 2.4, see [16, 67, 12] for more details.

4.3 Electromechanical systems

Example 30 *(DC-to-DC converter).* A Pulse Width Modulation DC-to-DC converter can be modeled by

$$\dot{x}_1 = (u-1)\frac{x_2}{L} + \frac{E}{L}, \qquad \dot{x}_2 = (1-u)\frac{x_1}{LC} - \frac{x_2}{RC},$$

where the duty ratio $u \in [0,1]$ is the control input. The electrical stored energy $y := \dfrac{x_1^2}{2C} + \dfrac{x_2^2}{2L}$ is a flat output [69, 27].

Example 31 *(Magnetic bearings).* A simple flatness-based solution to motion planning and tracking is proposed in [32]. The control law ensures that only one electromagnet in each actuator works at a time and permits to reduce the number of electromagnets by a better placement of actuators.

Example 32 *(Induction motor).* The standard two-phase model of the induction motor reads in complex notation (see [31] for a complete derivation)

$$R_s i_s + \dot{\psi}_s = u_s \qquad \psi_s = L_s i_s + M e^{jn\theta} i_r$$
$$R_r i_r + \dot{\psi}_r = 0 \qquad \psi_r = M e^{-jn\theta} i_s + L_r i_r,$$

where ψ_s and i_s (resp. ψ_r and i_r) are the complex stator (resp. rotor) flux and current, θ is the rotor position and $j = \sqrt{-1}$. The control input is the voltage u_s applied to the stator. Setting $\psi_r = \rho e^{j\alpha}$, the rotor motion is described by

$$J\frac{d^2\theta}{dt^2} = \frac{n}{R_r}\rho^2\dot{\alpha} - \tau_L(\theta,\dot{\theta}),$$

where τ_L is the load torque.

This system is flat with the two angles (θ, α) as a flat output [41] (see [9] also for a related result).

4.4 Chemical systems

Example 33 *(CSTRs).* Many simple models of Continuous Stirred Tank Reactors (CSTRs) admit flats outputs with a direct physical interpretation in terms of temperatures or product concentrations [24, 1], as do closely related biochemical processes [2, 11]. In [64] flatness is used to steer a reactor model from a steady state to another one while respecting some physical constraints.

A basic model of a CSTR with *two* chemical species and any number of exothermic or endothermic reactions is

$$\dot{x}_1 = f_1(x_1, x_2) + g_1(x_1, x_2)u$$
$$\dot{x}_2 = f_2(x_1, x_2) + g_2(x_1, x_2)u,$$

where x_1 is a concentration, x_2 a temperature and u the control input (feedflow or heat exchange). It is obviously linearizable by static feedback, hence flat.

When more chemical species are involved, a single-input CSTR is in general not flat, see [28]. Yet, the addition of another manipulated variable often renders it flat, see [1] for an example on a free-radical polymerization CSTR. For instance basic model of a CSTR with *three* chemical species, any number of exothermic or and two control inputs is

$$\dot{x}_1 = f_1(x) + g_1^1(x)u_1 + g_1^2(x)u_2$$
$$\dot{x}_2 = f_2(x) + g_2^1(x)u_1 + g_2^2(x)u_2$$
$$\dot{x}_3 = f_3(x) + g_3^1(x)u_1 + g_3^2(x)u_2,$$

where x_1, x_2 are concentrations and x_3 is a temperature temperature and u_1, u_2 are the control inputs (feed-flow, heat exchange, feed-composition,...). Such a system is always flat, see section 3.1.2.

Example 34 *(Polymerization reactor).* Consider with [72] the reactor

$$\begin{aligned}
\dot{C}_m &= \frac{C_{m_{m_s}}}{\tau} - \left(1 + \overline{\varepsilon}\frac{\mu_1}{\mu_1 + M_m C_m}\right)\frac{C_m}{\tau} + R_m(C_m, C_i, C_s, T) \\
\dot{C}_i &= -k_i(T)C_i + u_2\frac{C_{i_{i_s}}}{V} - \left(1 + \overline{\varepsilon}\frac{\mu_1}{\mu_1 + M_m C_m}\right)\frac{C_i}{\tau} \\
\dot{C}_s &= u_2\frac{C_{s_{i_s}}}{V} + \frac{C_{s_{m_s}}}{\tau} - \left(1 + \overline{\varepsilon}\frac{\mu_1}{\mu_1 + M_m C_m}\right)\frac{C_s}{\tau} \\
\dot{\mu}_1 &= -M_m R_m(C_m, C_i, C_s, T) - \left(1 + \overline{\varepsilon}\frac{\mu_1}{\mu_1 + M_m C_m}\right)\frac{\mu_1}{\tau} \\
\dot{T} &= \phi(C_m, C_i, C_s, \mu_1, T) + \alpha_1 T_j \\
\dot{T}_j &= f_6(T, T_j) + \alpha_4 u_1,
\end{aligned}$$

where u_1, u_2 are the control inputs and $C_{m_{m_s}}$, M_m, $\overline{\varepsilon}$, $\tau, C_{i_{i_s}}$, $C_{s_{m_s}}$, $C_{s_{i_s}}$, V, α_1, α_4 are constant parameters. The functions R_m, k_i, ϕ and f_6 are not well-known and derive from experimental data and semi-empirical considerations, involving kinetic laws, heat transfer coefficients and reaction enthalpies.

The polymerization reactor is flat whatever the functions R_m, k_i, ϕ, f_6 and admits $(C_{s_{i_s}}C_i - C_{i_{i_s}}C_s,\ M_m C_m + \mu_1)$ as a flat output [65].

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