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| Yves Guiraud, Philippe Malbos. Coherence in monoidal track categories. 2010. hal-00470795v1

HAL Id: hal-00470795

<https://hal.science/hal-00470795v1>

Preprint submitted on 7 Apr 2010 (v1), last revised 1 Dec 2010 (v2)

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Coherence in monoidal track categories

Yves Guiraud

INRIA

LORIA, Nancy

yves.guiraud@loria.fr

Philippe Malbos

Université Lyon 1

Institut Camille Jordan

malbos@math.univ-lyon1.fr

Abstract – We introduce homotopical methods based on rewriting on higher-dimensional categories to prove coherence results in categories with an algebraic structure. We express the coherence problem for (symmetric) monoidal categories as an asphericity problem for a track category and use rewriting methods on polygraphs to solve it. The setting is generalized to more general coherence problems, seen as 3-dimensional word problems in a track category. We prove general results that, in the case of braided monoidal categories, yield the coherence theorem for braided monoidal categories.

Keywords – coherence; monoidal category; higher-dimensional track category; rewriting; polygraph.

M.S.C. 2000 – 18C10, 18D10, 68Q42.

INTRODUCTION

A monoidal category is a category equipped with a binary tensor product, associative up to a natural isomorphism, and having a distinguished object, which is a unit for the product up to natural isomorphisms. Associativity and unity satisfy, in turn, a coherence condition: all the diagrams built from the corresponding natural isomorphisms are commutative. A cornerstone result for monoidal categories was to reduce the infinite requirement “every diagram commutes” to a finite requirement “if a specified finite set of diagrams commute then every diagram commutes”, [7, 9]. We call such a finite set of diagrams a *coherence basis*.

A symmetric monoidal category is a monoidal category whose tensor product is commutative up to an isomorphism, called symmetry. In a symmetric monoidal category the coherence problem has the same formulation as in monoidal categories, with additional coherence diagrams for the symmetry, [7].

In a symmetric monoidal category the symmetry is its own inverse. Braided monoidal categories are monoidal categories commutative up to an isomorphism which is not its own inverse. The coherence problem in braided categories has another formulation: a diagram is commutative if and only if its two sides correspond to the same braid, [6].

In this paper we formulate the coherence problem for monoidal track 2-categories in the homotopical terms for higher-dimensional categories as introduced in [4]. This formulation gives a way to reduce the coherence problem to a 3-dimensional word problem in track categories. The construction of convergent (*i.e.*, terminating and confluent) presentations of monoidal track 2-categories allows us to reduce the problem “every diagram commutes” to “if the diagrams induced by critical branchings commute then every diagram commutes”: the critical branchings form a coherence basis. Let us illustrate the methodology on a simple example.

Coherence for a category with a tensor. Let us consider a category \mathcal{C} equipped with a binary tensor product $\otimes : \mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C}$ which is associative up to a natural isomorphism, *i.e.*, there is a natural

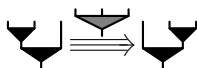
isomorphism

$$\alpha_{x,y,z} : (x \otimes y) \otimes z \rightarrow x \otimes (y \otimes z),$$

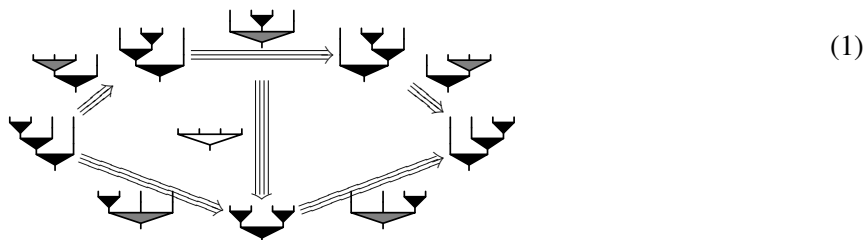
such that the following diagram commutes in \mathcal{C} :

$$\begin{array}{ccc}
 (x \otimes (y \otimes z)) \otimes t & \xrightarrow{\alpha} & x \otimes ((y \otimes z) \otimes t) \\
 \alpha \nearrow & & \searrow \alpha \\
 ((x \otimes y) \otimes z) \otimes t & \text{\textcircled{C}} & x \otimes (y \otimes (z \otimes t)) \\
 \alpha \searrow & & \nearrow \alpha \\
 (x \otimes y) \otimes (z \otimes t) & &
 \end{array}$$

Let us consider the 3-polygraph As_3 with one 0-cell, one 1-cell $|$, one 2-cell \blacktriangledown and one 3-cell



Let As_3^\top be the free track 3-category generated by As_3 . We consider the cellular extension As_4 of As_3^\top with one 4-cell:



Let \mathbf{As} be the track 3-category obtained as the quotient of As_3^\top by the cellular extension As_4 . The category of (small) categories equipped with a tensor product \otimes associative up to natural isomorphism α is isomorphic to the category $\mathcal{Alg}(\mathbf{As})$ of algebras over the 3-category \mathbf{As} : such an algebra is a 3-functor from \mathbf{As} to the monoidal 2-category Cat of small categories, functors and natural transformations, seen as a 3-category with only one 0-cell. The correspondence associates to a category $(\mathcal{C}, \otimes, \alpha)$ the algebra $A : \mathbf{As} \rightarrow Cat$ defined by:

$$A(|) = \mathcal{C}, \quad A(\blacktriangledown) = \otimes, \quad A(\text{3-cell}) = \alpha.$$

Let \approx_{As_4} be the homotopy relation on the free track 3-category As_3^\top generated by the cellular extension As_4 . The coherence problem for $(\mathcal{C}, \otimes, \alpha)$ is equivalent to the problem “does any pair (A, B) of parallel 3-cells of As_3^\top satisfy $A \approx_{As_4} B$ ”. In this way, the coherence problem is reduced to showing that As_4 forms a homotopy basis of the track 3-category As_3^\top , *i.e.*, that the 3-category \mathbf{As} is aspherical.

This new problem can be reduced to a 3-dimensional word problem in a track category. In [4], the authors prove that for a convergent (*i.e.*, terminating and confluent) n -polygraph Σ , a cellular extension of generating confluences forms a homotopy basis of the free track n -category Σ^\top . In our example, the 3-polygraph As_3 is convergent and has a unique critical branching: the two possible applications of the 3-cell \blacktriangledown on the following 2-cell:



It follows, that the cellular extension As_4 is a homotopy basis of the track 3-category As_3^\top , hence the coherence result in any algebra over \mathbf{As} .

The basic case. A 2-pro (resp. 2-prop) \mathbf{P} is a strict (resp. symmetric) monoidal category enriched in track 1-categories, whose underlying monoid of objects is the set of natural numbers \mathbb{N} . An algebra over a 2-pro (resp. 2-prop) \mathbf{P} is a strict (resp. symmetric) monoidal 2-functor from \mathbf{P} to Cat . Here Cat is considered as a 3-category with one 0-cell, categories as 1-cells, functors as 2-cells and natural transformations as 3-cell, (see 1.3.2). The image of a 3-sphere $\gamma = (f, g)$ in \mathbf{P} is a diagram $\mathcal{C}(\gamma)$ in Cat .

The coherence problem for algebras over a 2-pro(p) \mathbf{P} can be formulated as “given an algebra \mathcal{C} over \mathbf{P} , does every diagram $\mathcal{C}(\gamma)$ in Cat , for γ a 3-sphere in \mathbf{P} , commute?”, (see 1.3.3). A 2-pro(p) is *aspherical* when every 3-sphere (f, g) of \mathbf{P} satisfies $f = g$. Thus the coherence problem for algebras over a 2-pro(p) \mathbf{P} can be reformulated as “is \mathbf{P} aspherical?”.

Thus, reducing the coherence problem “every diagram commutes” to “if some diagrams commute then every diagram commutes” consists in constructing an algebraic presentation of the 2-pro(p) proving that it is aspherical. We show that a convergent presentation gives a procedure to solve the coherence problem.

In the case of 2-pros, a *convergent presentation* is a convergent 3-polygraph together with a cellular extension of generating confluences. We have:

Theorem 2.1.4. *If a 2-pro \mathbf{P} admits a convergent presentation, then \mathbf{P} is aspherical.*

In the case of 2-pros, we consider *algebraic convergent presentations*, (see 3.2.1). We have:

Theorem 3.2.4. *If a 2-prop \mathbf{P} admits an algebraic convergent presentation (Σ_3, Σ_4) such that Σ_4 is Tietze-equivalent to $\pi(\Gamma_{\Sigma_3})$, then \mathbf{P} is aspherical.*

The braided case and the generalized coherence problem. For the case of braided monoidal categories, we consider a generalized version of the coherence problem: given a 2-prop \mathbf{P} , decide, for any 3-sphere γ of \mathbf{P} , whether or not the diagram $\mathcal{C}(\gamma)$ commutes for every \mathbf{P} -algebra \mathcal{C} . To solve it, we proceed in two steps. First, we prove that coherence is preserved by equivalences of 2-pros, so that we can reduce a 2-prop to its non-aspherical part:

Theorem 4.2.4. *Let \mathbf{P} and \mathbf{Q} be 2-pros with \mathbf{Q} aspherical and $\mathbf{Q} \subseteq \mathbf{P}$. Then the canonical projection $\pi : \mathbf{P} \rightarrow \mathbf{P}/\mathbf{Q}$ is an equivalence of 2-pros.*

Then, given an algebraic 2-prop \mathbf{P} , we define the *initial \mathbf{P} -algebra* \mathcal{P} , (see 4.3), and we prove:

Theorem 4.3.2. *Let \mathbf{P} be an algebraic 2-prop and let (A, B) be a sphere of \mathbf{P} . Then $A = B$ if and only if $\mathcal{P}(A) = \mathcal{P}(B)$.*

In the case of the 2-prop of braided monoidal categories, this methodology recovers the coherence result of Joyal and Street, [6].

1. PRELIMINARIES

In this section, we recall several notions on (track) higher-dimensional categories, polygraphs and homotopy bases from [4].

1. Preliminaries

1.1. Higher-dimensional categories and polygraphs

1.1.1. Notations. We consider small, strict n -categories and strict n -functors between them. Given an n -category \mathcal{C} , we denote by \mathcal{C}_k the set of k -cells of \mathcal{C} and, abusively, the corresponding underlying k -category. If f is in \mathcal{C}_k , then $s_i(f)$ and $t_i(f)$ respectively denote the i -source and i -target of f ; we simply write $s(f)$ and $t(f)$ when $i = k - 1$. The source and target maps satisfy the *globular relations*:

$$s_i s_{i+1} = s_i t_{i+1} \quad \text{and} \quad t_i s_{i+1} = t_i t_{i+1}.$$

If f and g are i -composable k -cells, *i.e.*, when $t_i(f) = s_i(g)$, we denote by $f \star_i g$ the i -composite k -cell. The compositions satisfy the *exchange relations* given, for every $i \neq j$ and every possible cells f, g, h and k , by:

$$(f \star_i g) \star_j (h \star_i k) = (f \star_j h) \star_i (g \star_j k).$$

If f is a k -cell, we denote by 1_f its identity $(k + 1)$ -cell and, by abuse, all the higher-dimensional identity cells it generates. When 1_f is composed with cells of dimension $k + 1$ or higher, we simply denote it by f .

1.1.2. Spheres and cellular extensions. Let \mathcal{C} be an n -category. A k -sphere of \mathcal{C} is a pair $\gamma = (f, g)$ of parallel k -cells of \mathcal{C} , *i.e.*, with $s(f) = s(g)$ and $t(f) = t(g)$. We call f the *source* of γ and g its *target*. We denote by $\mathbf{S}_k \mathcal{C}$ the set of k -spheres of \mathcal{C} . An n -category \mathcal{C} is *aspherical* when every n -sphere γ of \mathcal{C} satisfies $s(\gamma) = t(\gamma)$.

A *cellular extension* of \mathcal{C} is a pair $\Gamma = (\Gamma_{n+1}, \partial)$ made of a set Γ_{n+1} and a map $\partial : \Gamma_{n+1} \rightarrow \mathbf{S} \mathcal{C}$. By considering all the formal compositions of elements of Γ_{n+1} , seen as $(n + 1)$ -cells with source and target in \mathcal{C} , and identities of n -cells of \mathcal{C} , one builds the *free $(n + 1)$ -category generated by Γ on \mathcal{C}* , denoted by $\mathcal{C}[\Gamma]$. The *quotient of \mathcal{C} by Γ* , denoted by \mathcal{C}/Γ , is the n -category one gets from \mathcal{C} by identification of source and target of every cell in Γ . Given an n -sphere (f, g) in \mathcal{C} , we write $f \equiv_{\Gamma} g$ if $\pi(f) = \pi(g)$, where $\pi : \mathcal{C} \rightarrow \mathcal{C}/\Gamma$ is the canonical projection. Two cellular extensions Γ_1 and Γ_2 of \mathcal{C} are *Tietze-equivalent* if the n -categories \mathcal{C}/Γ_1 and \mathcal{C}/Γ_2 are isomorphic.

1.1.3. Polygraphs. We define n -polygraphs and free n -categories by induction on n . A *1-polygraph* is a graph $\Sigma = (\Sigma_0, \Sigma_1)$. We denote by Σ^* the free 1-category it generates. An $(n + 1)$ -polygraph is a pair $\Sigma = (\Sigma_n, \Sigma_{n+1})$ made of an n -polygraph Σ_n and a cellular extension Σ_{n+1} of the free n -category Σ_n^* generated by the n -polygraph Σ_n . The *free $(n + 1)$ -category generated by Σ* is defined by $\Sigma^* = \Sigma_n^*[\Sigma_{n+1}]$; the *n -category presented by Σ* is defined by $\bar{\Sigma} = \Sigma_n^*/\Sigma_{n+1}$.

Let Σ be an n -polygraph. One says that an $(n - 1)$ -cell u *reduces* to some $(n - 1)$ -cell v when there exists a non-identity n -cell from u to v in Σ^* . A *reduction sequence* is a family $(u_k)_k$ of $(n - 1)$ -cells such that, for each k , u_k reduces to u_{k+1} . One says that Σ is *terminating* when there exists no infinite reduction sequence.

A *branching* (resp. *confluence*) of Σ is a pair (f, g) of n -cells of Σ^* with the same source (resp. target). A branching is (f, g) is *confluent* when there exists a confluence (f', g') such that $t(f) = s(f')$ and $t(g) = s(g')$ holds. The polygraph Σ is *confluent* when every branching is confluent. A polygraph is *convergent* when it is terminating and confluent.

1.2. Higher-dimensional track categories

1.2.1. Definition. A *track n-category* is an n-category whose n-cells are invertible, *i.e.*, an $(n - 1)$ -category enriched in groupoids. The inverse of an n-cell f is denoted by f^- .

Let \mathcal{C} be an n-category \mathcal{C} and Γ be a cellular extension of \mathcal{C} . We define the *free track $(n + 1)$ -category generated by Γ on \mathcal{C}* as the $(n + 1)$ -category

$$\mathcal{C}(\Gamma) = \mathcal{C}[\Gamma, \Gamma^-] / \text{Inv}(\Gamma),$$

where Γ^- is the following cellular extension of \mathcal{C}

$$\Gamma^- = \{ \gamma^- : t(\gamma) \rightarrow s(\gamma) \mid \gamma \in \Gamma \}$$

and $\text{Inv}(\Gamma)$ is the following cellular extension of $\mathcal{C}[\Gamma, \Gamma^-]$

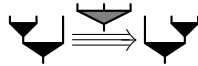
$$\text{Inv}(\Gamma) = \{ \gamma *_{\mathbb{N}} \gamma^- \rightarrow 1_{s\gamma}, \gamma^- *_{\mathbb{N}} \gamma \rightarrow 1_{t\gamma} \mid \gamma \in \Gamma \}.$$

If Σ is an n-polygraph, we denote by Σ^\top the free track n-category $\Sigma_{n-1}^*(\Sigma_n)$ generated by Σ_n on Σ_{n-1}^* .

1.2.2. Homotopy bases. A *homotopy basis* of a track n-category \mathcal{T} is a cellular extension Γ of \mathcal{T} such that the track $(n + 1)$ -category \mathcal{T}/Γ is aspherical, *i.e.*, such that, for every n-sphere γ of \mathcal{T} , there exists an $(n + 1)$ -cell from $s\gamma$ to $t\gamma$ in $\mathcal{T}(\Gamma)$.

When Σ is a convergent n-polygraph, the choice of a confluence diagram for each critical branching of Σ forms a cellular extension of Σ^\top . This construction is called a cellular extension of *generating confluences* and forms a homotopy basis of Σ^\top , see [4].

1.2.3. Example. The polygraph As_3 with one 0-cell, one 1-cell, one 2-cell \blacktriangledown and one 3-cell



is convergent and the generating confluence (1) forms a homotopy basis of the track 3-category As_3^\top .

1.3. Higher-dimensional pro(p)s

1.3.1. Definition. For $n \geq 1$, an *n-pro* is a strict monoidal category enriched in track $(n - 1)$ -categories, whose underlying monoid of objects is the monoid \mathbb{N} of natural numbers with the addition. An *n-prop* is a symmetric n-pro, *i.e.*, a strict symmetric monoidal category enriched in track $(n - 1)$ -categories, whose underlying monoid of objects is \mathbb{N} . In particular, 1-pro(p)s coincide with Mac Lane's pro(p)s, [8].

1.3.2. Algebras over 2-pro(p)s. We consider the large 2-category Cat of small categories, functors and natural transformations as a large 3-category with one 0-cell. This 3-category, still denoted Cat by abuse, has the following cells and compositions:

- one 0-cell, small categories as 1-cells, functors as 2-cells, natural transformations as 3-cells;

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- cartesian product as 0-composition, composition of functors as 1-composition, vertical composition of natural transformations as 2-composition.

If \mathbf{P} is a 2-pro (resp. 2-prop), a \mathbf{P} -algebra is a strict (resp. symmetric) monoidal 2-functor from \mathbf{P} to Cat . We denote in the same way a \mathbf{P} -algebra \mathcal{C} and the (small) category $\mathcal{C}(1)$.

Let \mathbf{P} be a 2-pro(p). If \mathcal{C} and \mathcal{D} are \mathbf{P} -algebras, a *morphism of \mathbf{P} -algebras from \mathcal{C} to \mathcal{D}* is a natural transformation from \mathcal{C} to \mathcal{D} , i.e., the data of:

- a functor $F : \mathcal{C} \rightarrow \mathcal{D}$,
- for every 2-cell $f : m \Rightarrow n$ in \mathbf{P} , a natural isomorphism

$$\begin{array}{ccc} \mathcal{C}^m & \xrightarrow{F^m} & \mathcal{D}^m \\ \mathcal{C}(f) \downarrow & \swarrow \Psi_f & \downarrow \mathcal{D}(f) \\ \mathcal{C}^n & \xrightarrow{F^n} & \mathcal{D}^n \end{array}$$

such that the following relations hold:

- $\Psi_{f \star_0 g} = \Psi_f \star_0 \Psi_g$,
- $\Psi_{f \star_1 g} = (\Psi_f \star_1 \mathcal{D}(g)) \star_2 (\mathcal{C}(f) \star_1 \Psi_g)$,
- for every 3-cell $A : f \Rightarrow g : m \Rightarrow n$ in \mathbf{P} ,

$$\Psi_f \star_2 (\mathcal{C}(A) \star_1 F^n) = (F^m \star_1 \mathcal{D}(A)) \star_2 \Psi_g.$$

The \mathbf{P} -algebras and their morphisms form a category, denoted by $\mathcal{Alg}(\mathbf{P})$.

1.3.3. Coherence problem for algebras over a 2-pro(p). Let \mathbf{P} be a 2-pro(p) and let \mathcal{C} be a \mathbf{P} -algebra. A \mathbf{P} -diagram in \mathcal{C} is the image $\mathcal{C}(\gamma)$ of a 3-sphere in \mathbf{P} . A \mathbf{P} -diagram $\mathcal{C}(\gamma)$ *commutes* if $\mathcal{C}(s\gamma) = \mathcal{C}(t\gamma)$ holds in Cat . The coherence problem for algebras over a 2-pro(p) is:

COHERENCE PROBLEM: Given a 2-pro(p) \mathbf{P} , does every \mathbf{P} -diagram commute in every \mathbf{P} -algebra?

We have:

1.3.4. Proposition. *Let \mathbf{P} be a 2-pro(p). If \mathbf{P} is aspherical then, in every \mathbf{P} -algebra \mathcal{C} , all the \mathbf{P} -diagrams commute.*

Thus, the coherence problem for algebras over a 2-pro(p) can be reformulated as follows:

COHERENCE PROBLEM: Given a 2-prop(p) \mathbf{P} , is \mathbf{P} aspherical ?

2. COHERENCE IN MONOIDAL CATEGORIES

2.1. Coherence in algebras over 2-pros

2.1.1. Definition. Let \mathbf{P} be a 2-pro. A *presentation of \mathbf{P}* is a pair (Σ, Γ) , where Σ is a 3-polygraph and Γ is a cellular extension of Σ^\top , such that $\mathbf{P} \simeq \Sigma^\top / \Gamma$. Note that, if (Σ, Γ) is a presentation of a 2-pro \mathbf{P} , then the 3-polygraph Σ has exactly one 0-cell and one 1-cell.

2.2. Identities among relations for presentations of 2-pros

2.1.2. Aspherical presentation. A presentation (Σ, Γ) is said to be *aspherical* if Γ is a homotopy base of the free track 3-category Σ^\top .

If a 2-pro \mathbf{P} admits an aspherical presentation, then it is aspherical. In this way, an aspherical presentation of a 2-pro \mathbf{P} is a solution of the coherence problem for \mathbf{P} -algebras.

2.1.3. Coherence as a 3-dimensional word problem. A presentation (Σ, Γ) of a 2-pro \mathbf{P} is *convergent* if Σ is a convergent 3-polygraph and Γ is (Tietze-equivalent to) a cellular extension of generating confluences of Σ . Any convergent presentation is aspherical, [4]. By this result, the coherence problem for a \mathbf{P} -algebra can be formulated as a 3-dimensional word problem on \mathbf{P} .

2.1.4. Theorem. *If a 2-pro \mathbf{P} admits a convergent presentation, then \mathbf{P} is aspherical.*

2.2. Identities among relations for presentations of 2-pros

This section is based on notions and results from [5]. Let \mathbf{P} be a 2-pro and Σ be a presentation of \mathbf{P} . We consider the free abelian track 3-category $(\Sigma_3)_{\text{ab}}^\top$. We denote by $\bar{\alpha}$ the image of a 3-cell α by the canonical projection $(\Sigma_3)_{\text{ab}}^\top \rightarrow \mathbf{P}$. The natural system of identities among relations of the presentation Σ is the abelian natural system on \mathbf{P} , denoted by $\Pi(\Sigma)$, defined as follows. For any 3-cell α in \mathbf{P} , the abelian group $\Pi(\Sigma)_\alpha$ is defined by generators and relations:

- It has one generator $[A]$ for every 4-cell $A : \beta \rightrightarrows \beta$ with $\bar{\beta} = \alpha$.
- Its defining relations are:
 - i) $[A \star_3 B] = [A] + [B]$, for $A, B : \beta \rightrightarrows \beta$ with $\bar{\beta} = \alpha$;
 - ii) $[A \star_3 B] = [B \star_3 A]$, for $A : \beta \rightrightarrows \gamma$ and $B : \gamma \rightrightarrows \beta$ with $\bar{\beta} = \bar{\gamma} = \alpha$.

There is an isomorphism of abelian natural systems on $(\Sigma_3)_{\text{ab}}^\top$:

$$\Phi : \widehat{\Pi(\Sigma)} \xrightarrow{\sim} \text{Aut}^{(\Sigma_3)_{\text{ab}}^\top}.$$

where $\widehat{\Pi(\Sigma)}$ denotes the natural system on $(\Sigma_3)_{\text{ab}}^\top$ defined by $\widehat{\Pi(\Sigma)}_\alpha = \Pi(\Sigma)_{\bar{\alpha}}$. The isomorphism Φ is given, for a 3-cell α of $(\Sigma_3)_{\text{ab}}^\top$, by

$$\Phi_\alpha([A]) = B^- \star_3 A \star_3 B,$$

where A is a closed 4-cell of $(\Sigma_3)_{\text{ab}}^\top$ with base β such that $\bar{\beta} = \bar{\alpha}$ and B is any 4-cell of $(\Sigma_3)_{\text{ab}}^\top$ with source β and target α .

We denote by $\tilde{\Sigma}_4$ the set of closed 3-cells of Σ_3^\top defined by:

$$\tilde{\Sigma}_4 = \{f \star_2 g^- \mid (f, g) \in \Sigma_4\}.$$

2.2.1. Proposition. *Let \mathbf{P} be a 2-pro. For any aspherical presentation Σ of \mathbf{P} , the set $[\tilde{\Sigma}_4]$ forms a generating set of the abelian natural system $\Pi(\Sigma)$.*

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Proof. This is an immediate consequence of results of [5]. Since Σ_4 is a homotopy basis of the track 3-category Σ_3^\top , it is a homotopy basis of the abelianized track 3-category $(\Sigma_3)_{ab}^\top$, thus, any closed 3-cell A in Σ_{ab}^\top can be written

$$A = (g_1 \star_{n-1} C_1[B_1^{\varepsilon_1}] \star_{n-1} g_1^-) \star_{n-1} \cdots \star_{n-1} (g_k \star_{n-1} C_k[B_k^{\varepsilon_k}] \star_{n-1} g_k^-),$$

where, for every i in $\{1, \dots, k\}$, $B_i \in \tilde{\Sigma}_4$, $\varepsilon_i \in \{-, +\}$, C_i is a whisker of Σ_3^* and $g_i \in \Sigma_3^*$. Hence, any identity among relations $[A]$ in $\Pi(\Sigma)$, can be written:

$$[A] = \sum_{i=1}^k \varepsilon_i [g_i \star_{n-1} C_i[B_i] \star_{n-1} g_i^-] = \sum_{i=1}^k \varepsilon_i C_i[B_i].$$

Thus, the elements of $[\tilde{\Sigma}_4]$ form a generating set for $\Pi(\Sigma)$. \square

Let Σ and Σ' be two finite presentations of a 2-pro \mathbf{P} and let $\Pi(\Sigma)$ and $\Pi(\Sigma')$ the associated abelian natural systems on \mathbf{P} of identities among relations. From [5], we deduce that the \mathbf{P} -natural system $\Pi(\Sigma)$ is finitely generated if and only if the natural system $\Pi(\Sigma')$ is finitely generated.

From this fact, we get a necessary condition to reduce the coherence problem of 1.3.3 in a 2-pro \mathbf{P} , “every \mathbf{P} -diagram commutes”, to “if a specified finite set of \mathbf{P} -diagrams commute then every diagram commutes”.

2.2.2. Proposition. *Let \mathbf{P} be a 2-pro. If there is a finite presentation Σ of \mathbf{P} such that the abelian natural system $\Pi(\Sigma)$ on \mathbf{P} is not finitely generated, then there is no solution to the coherence problem with a finite set of \mathbf{P} -diagrams.*

2.3. Application: coherence for monoidal categories

2.3.1. Definition. A *monoidal category* is a data $(\mathcal{C}, \otimes, e, \alpha, \lambda, \rho)$ made of a category \mathcal{C} , two functors

$$\otimes : \mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C}, \quad e : * \rightarrow \mathcal{C}$$

and three natural isomorphisms

$$\alpha_{x,y,z} : (x \otimes y) \otimes z \xrightarrow{\sim} x \otimes (y \otimes z), \quad \lambda_x : e \otimes x \xrightarrow{\sim} x, \quad \rho_x : x \otimes e \xrightarrow{\sim} x,$$

such that the following two diagrams commute in \mathcal{C} :

$$\begin{array}{ccc} & (x \otimes (y \otimes z)) \otimes t & \xrightarrow{\alpha} x \otimes ((y \otimes z) \otimes t) \\ \alpha \nearrow & & \searrow \alpha \\ ((x \otimes y) \otimes z) \otimes t & \text{\textcircled{C}} & x \otimes (y \otimes (z \otimes t)) \\ \alpha \searrow & & \nearrow \alpha \\ & (x \otimes y) \otimes (z \otimes t) & \end{array} \quad \begin{array}{ccc} & x \otimes (e \otimes y) & \\ \alpha \nearrow & & \searrow \lambda \\ (x \otimes e) \otimes y & \text{\textcircled{C}} & x \otimes y \\ & \xrightarrow{\rho} & \end{array} \quad (2)$$

A *monoidal functor* from \mathcal{C} to \mathcal{D} is a triple (F, ϕ, ι) made of a functor $F : \mathcal{C} \rightarrow \mathcal{D}$, a natural isomorphism

$$\phi_{x,y} : Fx \otimes Fy \xrightarrow{\sim} F(x \otimes y),$$

2.3. Application: coherence for monoidal categories

and an isomorphism $\iota : e \xrightarrow{\sim} F(e)$ such that the following diagrams commute in \mathcal{D} :

$$\begin{array}{ccc}
 & Fx \otimes (Fy \otimes Fz) \xrightarrow{1 \otimes \phi} Fx \otimes F(y \otimes z) & \\
 \alpha \nearrow & & \searrow \phi \\
 (Fx \otimes Fy) \otimes Fz & \text{\textcircled{C}} & F(x \otimes (y \otimes z)) \\
 \phi \otimes 1 \searrow & & \nearrow F\alpha \\
 & F(x \otimes y) \otimes Fz \xrightarrow{\phi} F((x \otimes y) \otimes z) &
 \end{array} \tag{3}$$

$$\begin{array}{ccc}
 Fx \otimes e \xrightarrow{\rho} Fx & & e \otimes Fx \xrightarrow{\lambda} Fx \\
 1 \otimes \iota \downarrow \text{\textcircled{C}} \uparrow F\rho & & \iota \otimes 1 \downarrow \text{\textcircled{C}} \uparrow F\lambda \\
 Fx \otimes Fe \xrightarrow{\phi} F(x \otimes e) & & Fe \otimes Fx \xrightarrow{\phi} F(e \otimes x)
 \end{array} \tag{4}$$

2.3.2. The 2-pro of monoidal categories. Let \mathbf{Mon} be the 2-pro presented by the 4-polygraph $\mathbf{Mon} = (\mathbf{Mon}_3, \mathbf{Mon}_4)$ where \mathbf{Mon}_3 is the 3-polygraph with two 2-cells $\blacktriangledown, \blacklozenge$ and three 3-cells

$$\begin{array}{ccc}
 \blacktriangledown \Rightarrow \blacktriangledown & \blacklozenge \Rightarrow | & \blacklozenge \Rightarrow | \\
 \text{\textcircled{C}} & \text{\textcircled{C}} & \text{\textcircled{C}}
 \end{array} \tag{5}$$

and \mathbf{Mon}_4 is the cellular extension of \mathbf{Mon}_3^\top made of the following two 4-cells:

$$\begin{array}{ccc}
 \text{Diagram 1} & & \text{Diagram 2} \\
 \text{\textcircled{C}} & & \text{\textcircled{C}}
 \end{array} \tag{6}$$

2.3.3. Lemma. *The category of small monoidal categories and monoidal functors is isomorphic to the category $\mathcal{Alg}(\mathbf{Mon})$.*

Proof. For a monoidal category $(\mathcal{C}, \otimes, e, \alpha, \lambda, \rho)$, the correspondance with a \mathbf{Mon} -algebra A is given by:

$$A(|) = \mathcal{C}, \quad A(\blacktriangledown) = \otimes, \quad A(\blacklozenge) = e, \quad A(\blacktriangledown) = \alpha, \quad A(\blacklozenge) = \lambda, \quad A(\blacklozenge) = \rho. \tag{7}$$

The two commutative diagrams (2) correspond to commutative diagrams $A(\blacktriangledown)$ and $A(\blacklozenge)$ in \mathcal{Cat} . If F is a monoidal functor from \mathcal{C} to \mathcal{D} , the correspondance with a morphism Ψ of associated \mathbf{Mon} -algebras is given by:

$$\Psi_{|} = F, \quad \Psi_{\blacktriangledown} = \phi, \quad \Psi_{\blacklozenge} = \iota.$$

The relations (3) and (4) satisfied by ϕ and ι correspond exactly to the ones satisfied by Ψ . □

2. Coherence in monoidal categories

2.3.4. Proposition ([4]). *The cellular extension Mon_4 is a homotopy basis of the free track 3-category Mon_3^\top .*

Proof. First, we prove that the 3-polygraph polygraph Mon_3 is convergent. For termination, we use the methodology from [3], reformulated with the vocabulary of [4]. For that, we first consider the 2-functor $X : \text{Mon}_2 \rightarrow \mathbf{Ord}$ given on generators as follows, where \mathbf{Ord} is the monoidal category of ordered sets and monotone maps, seen as a 2-category with one 0-cell:

$$X(|) = \mathbb{N} \setminus \{0\}, \quad X(\nabla)(i, j) = i + j, \quad X(\bullet) = 1.$$

Then, we consider the following assignment of 2-cells of Mon_2 :

$$\partial(\nabla)(i, j) = i, \quad \partial(\bullet) = 0.$$

This assignment extends, in a unique way, to a derivation of Mon_2^* with values into the X , *i.e.*, a map ∂ that sends each 2-cell $f : m \Rightarrow n$ of Mon_2^* to a monotone map $\partial(f) : \mathbb{N}^m \rightarrow \mathbb{N}$ that satisfies the following relations [4]:

$$\partial(f \star_0 g)(i_1, \dots, i_{m+n}) = \partial(f)(i_1, \dots, i_m) + \partial(g)(i_{m+1}, \dots, i_{m+n})$$

and

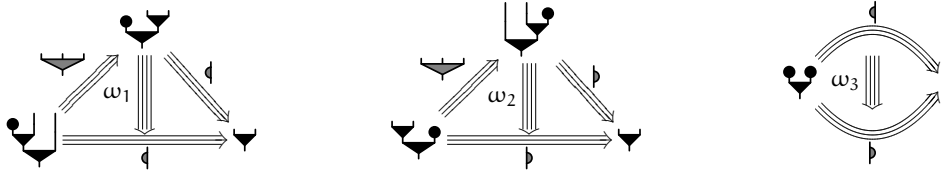
$$\partial(f \star_1 g)(i_1, \dots, i_m) = \partial(f)(i_1, \dots, i_m) + \partial(g) \circ X(f)(i_1, \dots, i_m).$$

We check that, for every 3-cell α of Mon_3 , we have:

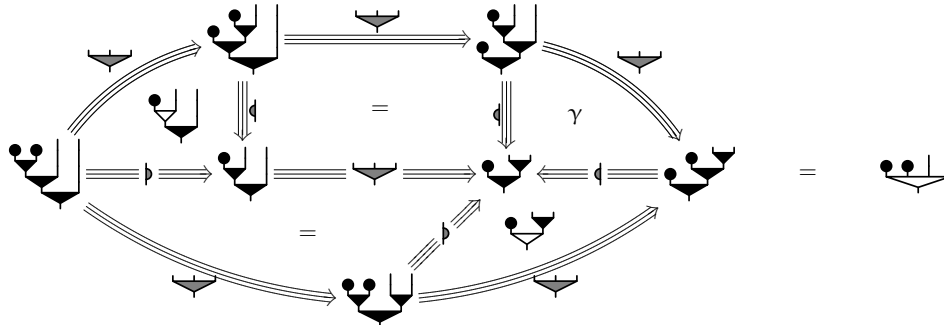
$$X(s\alpha) \geq X(t\alpha) \quad \text{and} \quad \partial(s\alpha) > \partial(t\alpha)$$

where monotone maps are compared pointwise. Following [3, 4], this gives termination of Mon_3 .

For confluence, we study the critical branchings of Mon_3 : it has five regular critical branchings. Each of them is confluent, yielding a cellular extension Γ of Mon_3^\top with five 4-cells, the ones of Mon_4 plus the following three 4-cells:

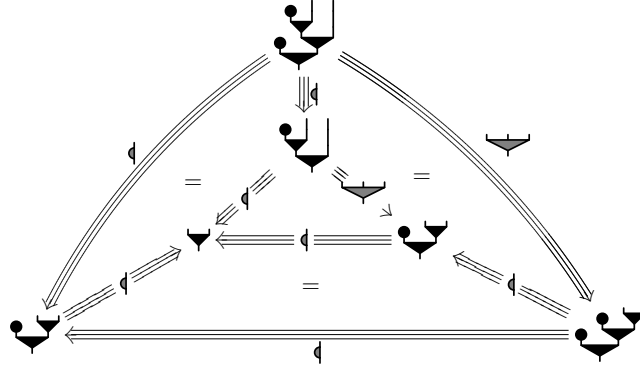


Hence Γ is a homotopy basis of Mon_3^\top . To prove that Mon_4 is a homotopy basis, we show that, for each 4-cell ω_i , we have $s(\omega_i) \approx_{\text{Mon}_4} t(\omega_i)$. For ω_1 , we define the 4-cell γ of Mon_3^\top (Mon_4) by the following relation, where we abusively denote 3-cells by the generating 3-cell of Mon_3 they contain:

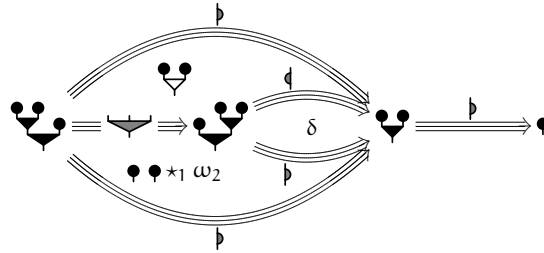


2.3. Application: coherence for monoidal categories

As a consequence of this construction, we have $s\gamma \approx_{\text{Mon}_4} t\gamma$. Then we build the following diagram, proving that $s(\omega_1) \approx_{\text{Mon}_4} t(\omega_1)$ also holds:



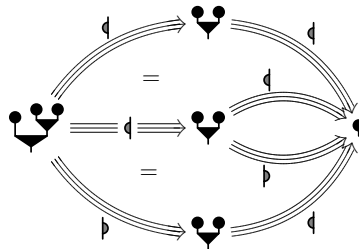
For the 4-cell ω_2 , one proceeds in a similar way, starting with the 4-cell $\downarrow \downarrow \bullet \bullet$. Finally, let us consider the case of the 4-cell ω_3 . First, we define the 4-cell δ of $\text{Mon}_3^{\text{T}}(\text{Mon}_4)$ by the following relation:



As a consequence, we have $s\delta \approx_{\text{Mon}_4} t\delta$, hence

$$s(\delta *_2 \bullet) = t(\delta *_2 \bullet).$$

The diagram



yields $s(\omega_3) \approx_{\Gamma} t(\omega_3)$, thus concluding the proof. □

We can deduce, from this result and Proposition 2.2.1, a generating set for the abelian natural system of identities among relations $\Pi(\text{Mon})$ on \mathbf{Mon} :

$$\widetilde{\text{Mon}}_4 = \left\{ \left[\downarrow \downarrow \bullet \bullet *_2 \downarrow \downarrow \bullet \bullet *_2 \downarrow \downarrow \bullet \bullet *_2 \left(\downarrow \downarrow \bullet \bullet \right)^{-} *_2 \left(\downarrow \downarrow \bullet \bullet \right)^{-} \right], \left[\downarrow \downarrow \bullet \bullet *_2 \downarrow \downarrow \bullet \bullet *_2 \left(\downarrow \downarrow \bullet \bullet \right)^{-} \right] \right\}$$

From Proposition 2.3.4, we have:

2.3.5. Corollary (Coherence theorem for monoidal categories, [7]). *The 2-pro \mathbf{Mon} is aspherical.*

3. Coherence in symmetric monoidal categories

3. COHERENCE IN SYMMETRIC MONOIDAL CATEGORIES

3.1. Presentations of 2-props

We recall from [2] the following characterization of 1-props, derived from a similar result for algebraic theories [1].

3.1.1. Proposition. *A 1-pro \mathbf{P} is a 1-prop if and only if it contains a 2-cell $\tau : 2 \Rightarrow 2$ such that, with the inductively defined notations*

$$\tau_{0,1} = \tau_{1,0} = 1_1, \quad \tau_{n+1,1} = (n \star_0 \tau) \star_1 (\tau_{n,1} \star_0 1), \quad \tau_{1,n+1} = (\tau \star_0 n) \star_1 (1 \star_0 \tau_{1,n}),$$

the following relations hold:

- *Symmetry relation:*

$$\tau \star_1 \tau = 1_2. \quad (8)$$

- *Yang-Baxter relation:*

$$(\tau \star_0 1) \star_1 (1 \star_0 \tau) \star_1 (\tau \star_0 1) = (1 \star_0 \tau) \star_1 (\tau \star_0 1) \star_1 (1 \star_0 \tau). \quad (9)$$

- *For every 2-cell $f : m \Rightarrow n$, naturality relations for f :*

$$(f \star_0 1) \star_1 \tau_{n,1} = \tau_{m,1} \star_1 (1 \star_0 f) \quad \text{and} \quad (1 \star_0 f) \star_1 \tau_{1,n} = \tau_{1,m} \star_1 (f \star_0 1). \quad (10)$$

3.1.2. Presentations of 1-props. Let Σ be a 2-polygraph with one 0-cell and one 1-cell. We denote by $S\Sigma$ the 3-polygraph obtained from Σ by adjoining a 2-cell $\tau : 2 \Rightarrow 2$ and 3-cells given by the symmetry (8), Yang-Baxter (9) and naturality (10) relations for every 2-cell of Σ , directed from left to right. The *free 1-prop generated by Σ* is the 2-category, denoted by Σ^S , presented by the 3-polygraph $S\Sigma$.

Let \mathbf{P} be a 1-prop. A *presentation of \mathbf{P}* is a pair (Σ_2, Σ_3) , made of a 2-polygraph Σ_2 with one 0-cell and one 1-cell and a cellular extension Σ_3 of the free 2-category on the 2-polygraph $\Sigma_2 \amalg \{ \bowtie \}$, such that

$$\mathbf{P} \simeq \Sigma_2^S / \Sigma_3.$$

3.1.3. The 1-prop of permutations. The free 1-prop on no 2-cell is the strict monoidal category **Perm** of permutations: it is presented by the 3-polygraph with one 2-cell \bowtie and two 3-cells

$$\begin{array}{c} \bowtie \\ \Rightarrow \\ | \quad | \end{array} \quad \text{and} \quad \begin{array}{c} \bowtie \quad \bowtie \\ \Rightarrow \\ \bowtie \quad \bowtie \end{array}$$

There exists an isomorphism between the category of small categories and functors and the category $\mathcal{Alg}(\mathbf{Perm})$. The correspondence is given, for a **Perm**-algebra $\mathcal{C} : \mathbf{Perm} \rightarrow \mathbf{Cat}$, by

$$\mathcal{C}(\bowtie) = T_{\mathcal{C}, \mathcal{C}},$$

where $T_{\mathcal{C}, \mathcal{C}}$ is the endofunctor of $\mathcal{C} \times \mathcal{C}$ sending (x, y) to (y, x) .

3.2. Convergent presentations of algebraic 2-props and asphericity

3.1.4. Proposition. A 2-pro \mathbf{P} is a 2-prop if and only if it contains a 2-cell $\tau : 2 \Rightarrow 2$ such that the following relations hold:

- Symmetry and Yang-Baxter relations.
- Naturality relations for every 2-cell of \mathbf{P} .
- For every 3-cell $A : f \Rightarrow g : m \Rightarrow n$, naturality relations for A :

$$(A \star_0 1) \star_1 \tau_{n,1} = \tau_{m,1} \star_1 (1 \star_0 A) \quad \text{and} \quad (1 \star_0 A) \star_1 \tau_{1,n} = \tau_{1,m} \star_1 (A \star_0 1). \quad (11)$$

Proof. This is an immediate extension of Proposition 3.1.1 for symmetric monoidal categories. \square

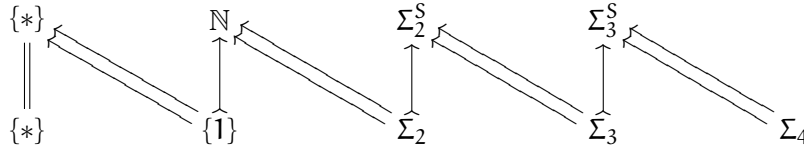
3.1.5. Presentations of 2-props. Let $\Sigma = (\Sigma_2, \Sigma_3)$ be a presentation of a 1-prop. We denote by $S\Sigma$ the 4-polygraph obtained from the 3-polygraph $S\Sigma_2$ by adjoining the 3-cells of Σ_3 and 4-cells given by the naturality relations (11) for every 3-cell A of Σ_3 , directed from left to right. The *free 2-prop generated by Σ* is the track 3-category, denoted by Σ^S , given by:

$$\Sigma^S = \Sigma_2^S(\Sigma_3)/\Sigma_4.$$

Let \mathbf{P} be a 2-prop. A *presentation of \mathbf{P}* is a pair (Σ_3, Σ_4) , made of a presentation of a 1-prop Σ_3 and a cellular extension Σ_4 of the free track 3-category Σ_3^\top , such that

$$\mathbf{P} \simeq \Sigma_3^S/\Sigma_4.$$

To summarize, a presentation of \mathbf{P} yields a diagram which is similar to the one corresponding to the inductive construction of a 4-polygraph, see [1]:



3.2. Convergent presentations of algebraic 2-props and asphericity

3.2.1. Convergent presentations of algebraic 2-props. A presentation Σ of a 2-prop is *convergent* when the 3-polygraph $S\Sigma$ is convergent. A presentation Σ of a 1-prop (resp. 2-prop) is *algebraic* when every 2-cell (resp. every 2-cell and every 3-cell) of Σ has 1-target equal to the generating 1-cell 1. A 2-prop is *algebraic* when it admits an algebraic presentation.

3.2.2. Classification of critical branchings. Let Σ be an algebraic presentation of a 1-prop \mathbf{P} . We recall from [2, 3] that the critical branchings of the 3-polygraph $S\Sigma$ are organized as follows:

1. Five critical branchings generated by the symmetry and Yang-Baxter 3-cells.
2. For every 2-cell φ of Σ , four critical branchings, generated, on the one hand, by the naturality 3-cells for φ and, on the other hand, by the symmetry and Yang-Baxter 3-cells.

3. Coherence in symmetric monoidal categories

3. For every pair (φ, ψ) of 2-cells of Σ , one critical branching generated by the first naturality 3-cell of φ and the second naturality 3-cell of ψ .
4. For every 3-cell $\alpha : f \Rightarrow g$ of Σ , two critical branchings generated by α and the naturality 3-cells for f .
5. The other critical branchings, generated by at least a 3-cell of Σ .

The critical branchings of the first four families are always confluent. The cellular extension corresponding to their confluence diagrams satisfies the following property: the source and target of each one of its 4-cells are identified through the canonical projection $\pi : S\Sigma^\top \rightarrow \mathbf{P}$. For example, the confluence diagrams of the fourth family correspond to the naturality 4-cells.

When Σ is convergent, we denote by Γ_Σ a (chosen) cellular extension corresponding to the confluence diagrams of the critical branchings of the last family.

3.2.3. Lemma. *Let Σ be an algebraic convergent presentation of a 1-prop \mathbf{P} . Then $\pi(\Gamma_\Sigma)$ is a homotopy basis of \mathbf{P} .*

3.2.4. Theorem. *If a 2-prop \mathbf{P} admits an algebraic convergent presentation (Σ_3, Σ_4) such that Σ_4 is Tietze-equivalent to $\pi(\Gamma_{\Sigma_3})$, then \mathbf{P} is aspherical.*

3.3. Application to symmetric monoidal categories

3.3.1. Definition. A (small) *symmetric monoidal category* is given by data $(\mathcal{C}, \otimes, e, \alpha, \lambda, \rho, \tau)$ made up of a monoidal category $(\mathcal{C}, \otimes, e, \alpha, \lambda, \rho)$ together with a natural isomorphism

$$\tau_{x,y} : x \otimes y \rightarrow y \otimes x,$$

called the *symmetry* and such that the following two diagrams commute in \mathcal{C} :

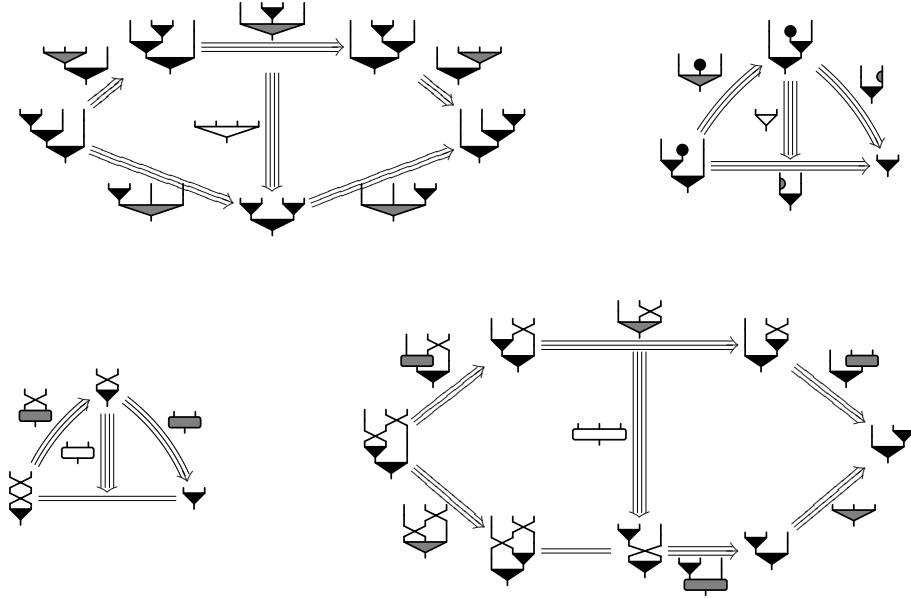
$$\begin{array}{ccc}
 & & x \otimes (y \otimes z) \xrightarrow{\tau} (y \otimes z) \otimes x \\
 & \alpha \nearrow & \\
 x \otimes (y \otimes z) & & \\
 \alpha \searrow & & \\
 (x \otimes y) \otimes z & \text{\textcircled{C}} & (y \otimes z) \otimes x \\
 \tau \searrow & & \alpha \searrow \\
 (y \otimes x) \otimes z & & y \otimes (z \otimes x) \\
 \alpha \searrow & & \tau \nearrow \\
 (y \otimes x) \otimes z & \xrightarrow{\alpha} & y \otimes (x \otimes z)
 \end{array} \tag{12}$$

3.3.2. The 2-prop of symmetric monoidal categories. Let \mathbf{Sym} be the 2-prop presented by \mathbf{Sym} given as follows:

- \mathbf{Sym}_2 is \mathbf{Mon}_2 , with two 2-cells \blacktriangledown and \bullet .
- \mathbf{Sym}_3 is \mathbf{Mon}_3 extended with a 3-cell:



- Sym_4 is Mon_4 extended with two 4-cells:



3.3.3. Lemma. *The category of small symmetric monoidal categories and symmetric monoidal functors is isomorphic to the category $\text{Alg}(\mathbf{Sym})$.*

Proof. Given a symmetric monoidal category $(\mathcal{C}, \otimes, e, \alpha, \lambda, \rho, \tau)$, the correspondence with a \mathbf{Sym} -algebra A is given by (7) for the monoidal underlying structure and by

$$A(\text{diag}) = \tau,$$

for the symmetry. The two commutative diagrams (2) of Lemma 2.3.3 correspond to commutative diagrams $A(\text{diag}_1)$ and $A(\text{diag}_2)$ in Cat and the commutative diagrams (12) correspond to commutative diagrams $A(\text{diag}_3)$ and $A(\text{diag}_4)$.

A symmetric monoidal functor from \mathcal{C} to \mathcal{D} is a monoidal functor (F, ϕ, ι) such that the following diagram commutes:

$$\begin{array}{ccc} Fx \otimes Fy & \xrightarrow{\tau} & Fy \otimes Fx \\ \phi \downarrow & \text{\textcircled{C}} & \downarrow \phi \\ F(x \otimes y) & \xrightarrow{F\tau} & F(y \otimes x) \end{array} \quad (13)$$

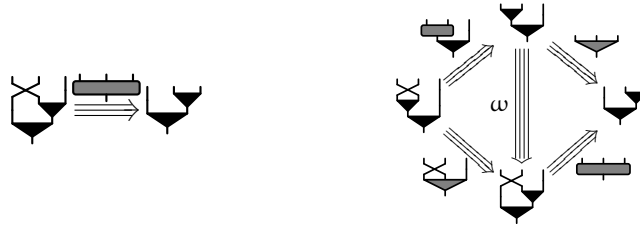
The correspondance with a morphism Ψ of associated \mathbf{Sym} -algebras is given by:

$$\Psi_{\downarrow} = F, \quad \Psi_{\blacktriangledown} = \phi, \quad \Psi_{\bullet} = \iota.$$

The relations (3), (4) and (13) correspond to the properties of the morphism Ψ . □

3. Coherence in symmetric monoidal categories

3.3.4. A convergent presentation. We define Sym' as the presentation Sym extended with one 3-cell and one 4-cell:



3.3.5. Lemma. *The 2-prop Sym is presented by Sym' .*

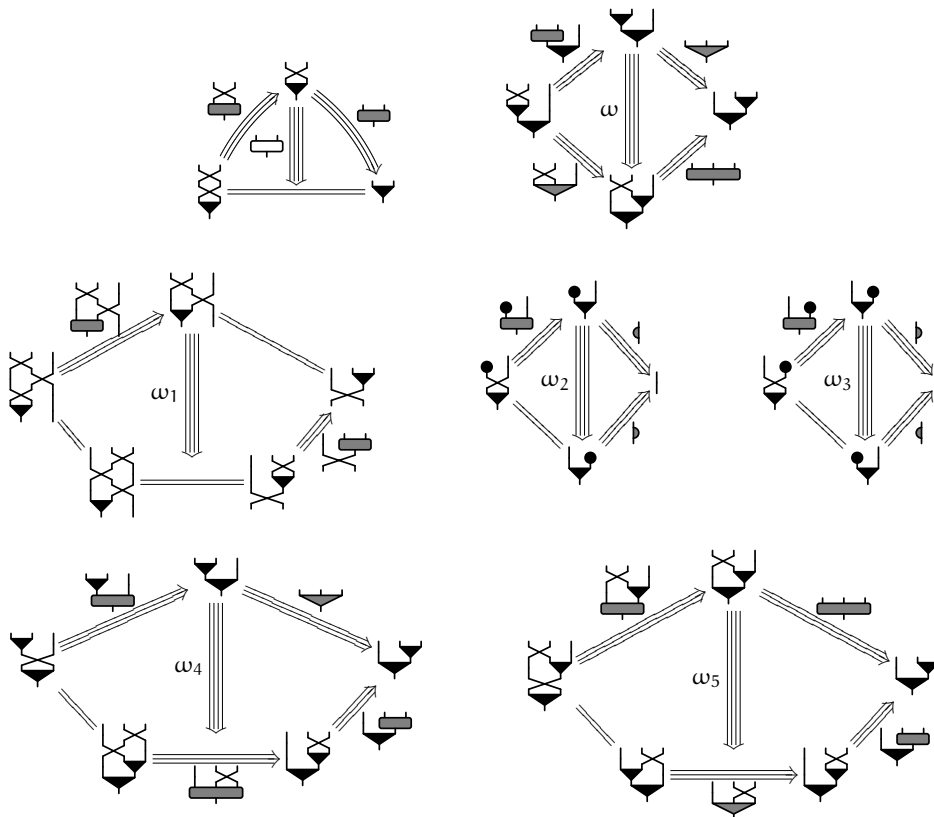
Proof. The 4-cell ω corresponds exactly to the following relation:

$$\text{[shaded box]} \approx_{\omega} \left(\text{[crossing]} \right)^{-1} *_2 \text{[shaded box]} *_2 \text{[crossing]}$$

This induces an isomorphism between the 2-props $(\text{Sym}'_3)^S / \text{Sym}'_4$ and $\text{Sym}_3^S / \text{Sym}_4 \simeq \mathbf{P}$. □

3.3.6. Proposition. *The 3-polygraph $S(\text{Sym}'_3)$ is convergent and the cellular extension Sym'_4 is Tietze-equivalent to $\pi(\Gamma_{S(\text{Sym}'_3)})$.*

Proof. The convergence of $S(\text{Sym}'_3)$ is proved in [3]. The image through π of $\Gamma_{S(\text{Sym}'_3)}$ has seven 4-cells:



In a similar way to the case of monoidal categories, we prove that, for every 4-cell ω_i , we have $s(\omega_i) \approx_{\text{Sym}_4'} t(\omega_i)$. The projection sends ω_1 to one of the naturality relations for \square . Then, for each 4-cell ω_i , with $2 \leq i \leq 5$, we consider a 4-cell W_i of $\text{Sym}_3^\top(\text{Sym}_4)$, built as an instance of the 4-cell \square composed with 2-cells:

$$W_2 = \begin{array}{c} \bullet \bullet \\ \hline \end{array}, \quad W_3 = \begin{array}{c} \bullet \bullet \\ \hline \end{array}, \quad W_4 = \begin{array}{c} \nabla \bullet \\ \hline \end{array}, \quad W_5 = \begin{array}{c} \nabla \nabla \\ \hline \end{array}.$$

On the one hand, the boundary of W_i satisfies $s(W_i) \approx_{\text{Sym}_4} t(W_i)$. On the other hand, we partially fill the boundary of W_i with 4-cells of Sym_4 , plus exchange and naturality relations, until reaching the boundary of ω_i (or of ω_i^-), thus yielding the result. \square

3.3.7. Corollary (Coherence theorem for symmetric monoidal categories, [7]). *The 2-prop Sym is aspherical.*

4. COHERENCE FOR BRAIDED MONOIDAL CATEGORIES

4.1. Generalized coherence problem

4.1.1. Definition. A (small) *braided monoidal category* is a monoidal category $(\mathcal{C}, \otimes, e, \alpha, \lambda, \rho)$ equipped with a natural isomorphism

$$\beta_{x,y} : x \otimes y \rightarrow y \otimes x,$$

called the *braiding* and such that the following diagrams commute in \mathcal{C} :

$$\begin{array}{ccc} & x \otimes (y \otimes z) & \xrightarrow{\beta} & (y \otimes z) \otimes x \\ & \alpha \nearrow & & \searrow \alpha \\ (x \otimes y) \otimes z & & \textcircled{C} & & y \otimes (z \otimes x) \\ & \beta \searrow & & \nearrow \beta \\ & (y \otimes x) \otimes z & \xrightarrow{\alpha} & y \otimes (x \otimes z) \end{array}$$

$$\begin{array}{ccc} & x \otimes (y \otimes z) & \xrightarrow{\beta^-} & (y \otimes z) \otimes x \\ & \alpha \nearrow & & \searrow \alpha \\ (x \otimes y) \otimes z & & \textcircled{C} & & y \otimes (z \otimes x) \\ & \beta^- \searrow & & \nearrow \beta^- \\ & (y \otimes x) \otimes z & \xrightarrow{\alpha} & y \otimes (x \otimes z) \end{array}$$

4. Coherence for braided monoidal categories

4.1.2. Generalized coherence theorem. Contrary to the case of monoidal and symmetric monoidal categories, we do not have that every diagram commutes in a braided monoidal category. For example, the morphisms $\beta_{x,y}$ and $\beta_{y,x}^{-1}$, from $x \otimes y$ to $y \otimes x$, have no reason to be equal. In fact, they are equal if and only if β is a symmetry, hence if and only if all diagrams commute.

As a consequence, the coherence problem for braided monoidal categories requires a generalized version of the coherence problem we have considered so far.

THE GENERALIZED COHERENCE PROBLEM: Given a 2-prop \mathbf{P} , decide, for any 3-sphere γ of \mathbf{P} , whether or not the diagram $\mathcal{C}(\gamma)$ commutes for every \mathbf{P} -algebra \mathcal{C} .

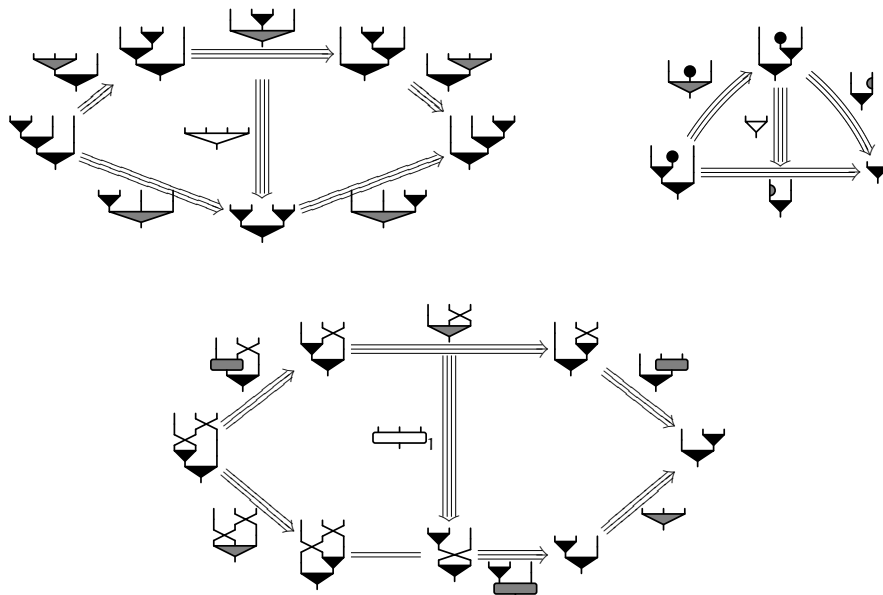
Hence, a solution for the generalized coherence problem is a decision procedure for the equality of 3-cells of \mathbf{P} . For the coherence problems considered so far, this decision procedure answers yes for every 3-sphere. We consider methods to study the generalized coherence theorem of 2-props and illustrate those methods on the 2-prop of braided monoidal categories.

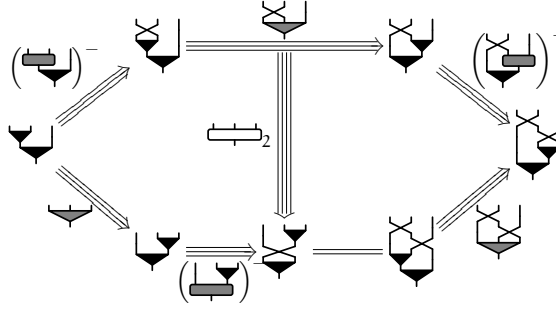
4.1.3. The 2-prop of braided monoidal categories. Let \mathbf{Br} be the 2-prop presented by Br , which is defined as follows:

- Br_2 is Mon_2 , made of two 2-cells ∇ and \blacklozenge .
- Br_3 is Sym_3 , made of four 3-cells:



- Br_4 is Mon_4 extended with two 4-cells:





The category of (small) braided monoidal categories is isomorphic to the category $\mathcal{Alg}(\mathbf{Br})$.

4.2. Preservation of coherence by equivalences

4.2.1. Equivalence of 2-props. Let \mathbf{P} and \mathbf{Q} be 2-props. A *morphism of 2-props* from \mathbf{P} to \mathbf{Q} is a 3-functor $F : \mathbf{P} \rightarrow \mathbf{Q}$ which is the identity on 1-cells, i.e., $F(n) = n$, for every 1-cell $n \in \mathbb{N}$. If $F, G : \mathbf{P} \rightarrow \mathbf{Q}$ are two morphisms of 2-props, a *natural transformation from F to G* is a family α of 3-cells of \mathbf{Q}

$$\alpha_f : F(f) \Rightarrow G(f)$$

indexed by the 2-cells of \mathbf{P} and such that, for every 3-cell $A : f \Rightarrow g$ of \mathbf{P} , the following diagram commutes in \mathbf{Q} :

$$\begin{array}{ccc} F(f) & \xRightarrow{\alpha_f} & G(f) \\ F(A) \Downarrow & \text{\textcircled{C}} & \Downarrow G(A) \\ F(g) & \xRightarrow{\alpha_g} & G(g). \end{array}$$

If $F : \mathbf{P} \rightarrow \mathbf{Q}$ is a morphism of 2-props, a *quasi-inverse for F* is a morphism of 2-props $G : \mathbf{Q} \rightarrow \mathbf{P}$ such that there exist natural isomorphisms

$$GF \simeq 1_{\mathbf{P}} \quad \text{and} \quad FG \simeq 1_{\mathbf{Q}}.$$

An *equivalence between \mathbf{P} and \mathbf{Q}* is a morphism of 2-props $F : \mathbf{P} \rightarrow \mathbf{Q}$ that admits a quasi-inverse.

4.2.2. Proposition. Let $F : \mathbf{P} \rightarrow \mathbf{Q}$ be an equivalence between 2-props \mathbf{P} and \mathbf{Q} and let (A, B) be a 3-sphere of \mathbf{P} . Then $A = B$ if and only if $F(A) = F(B)$.

Proof. Let $A, B : f \Rightarrow g$ be a 3-sphere of \mathbf{P} such that $F(A) = F(B)$. We denote by $G : \mathbf{Q} \rightarrow \mathbf{P}$ a quasi-inverse of F and by α the natural isomorphism from GF to $1_{\mathbf{P}}$. We have, by definition of α , commutative diagrams in \mathbf{P} :

$$\begin{array}{ccc} GF(f) & \xRightarrow{\alpha_f} & f \\ GF(A) \Downarrow & \text{\textcircled{C}} & \Downarrow A \\ GF(g) & \xRightarrow{\alpha_g} & g \end{array} \quad \begin{array}{ccc} GF(f) & \xRightarrow{\alpha_f} & f \\ GF(B) \Downarrow & \text{\textcircled{C}} & \Downarrow B \\ GF(g) & \xRightarrow{\alpha_g} & g \end{array}$$

4. Coherence for braided monoidal categories

By hypothesis, we have $\text{GF}(A) = \text{GF}(B)$. Thus:

$$A = \alpha_f^- \star_2 \text{GF}(A) \star_2 \alpha_g = \alpha_f^- \star_2 \text{GF}(B) \star_2 \alpha_g = B. \quad \square$$

4.2.3. Notation. If \mathbf{P} and \mathbf{Q} are 2-props with $\mathbf{Q} \subseteq \mathbf{P}$, we denote by \mathbf{P}/\mathbf{Q} the quotient of \mathbf{P} by the 3-cells of \mathbf{Q} .

4.2.4. Theorem. *Let \mathbf{P} and \mathbf{Q} be 2-props with \mathbf{Q} aspherical and $\mathbf{Q} \subseteq \mathbf{P}$. Then the canonical projection $\pi : \mathbf{P} \rightarrow \mathbf{P}/\mathbf{Q}$ is an equivalence of 2-props.*

Proof. Let f and g be parallel 2-cells of \mathbf{P} . Since \mathbf{Q} is aspherical, the 3-cells of \mathbf{P} from f to g are in bijective correspondence with the 3-cells of \mathbf{P}/\mathbf{Q} from $\pi(f)$ to $\pi(g)$. Since π is surjective, this yields a quasi-inverse of π . \square

By Corollary 2.3.5, the 2-pro \mathbf{Mon} is aspherical, we have

4.2.5. Corollary. *Let (A, B) be a 3-sphere of \mathbf{Br} . Then $A = B$ if and only if $\pi(A) = \pi(B)$ in \mathbf{Br}/\mathbf{Mon} .*

4.3. The initial algebra of an algebraic 2-prop

4.3.1. Definition. Let \mathbf{P} be an algebraic 2-prop, with an algebraic presentation Σ . A 2-cell f of \mathbf{P} is *algebraic* when it has target 1 and it is a 2-cell of the free 2-category Σ_2^* .

The *initial \mathbf{P} -algebra* is the \mathbf{P} -algebra \mathcal{P} defined as follows. The category $\mathcal{P}(1)$, or simply \mathcal{P} , is given by:

- Its objects are the algebraic 2-cells of \mathbf{P} .
- A morphism from $f : n \Rightarrow 1$ to $g : n \Rightarrow 1$ is a pair (σ, A) where σ is a permutation, seen as a 2-cell of \mathbf{P} generated by $\{\bowtie\}$, and $A : \sigma \star_1 f \Rightarrow g$ is a 3-cell of \mathbf{P} .
- The composite of (σ, A) and (τ, B) is $(\tau \star_1 \sigma, (\tau \star_1 A) \star_2 B)$.
- The identity of $f : n \Rightarrow 1$ is $(1_n, f)$.

If $\varphi : n \Rightarrow 1$ is a 2-cell of Σ , then the functor $\mathcal{P}(\varphi) : \mathcal{P}^n \rightarrow \mathcal{P}$ is given by

$$\mathcal{P}(\varphi)(f_1, \dots, f_n) = (f_1 \star_0 \cdots \star_0 f_n) \star_1 \varphi.$$

If $\alpha : f \Rightarrow g : n \Rightarrow 1$ is a 3-cell of Σ , then the component at (f_1, \dots, f_n) of the natural transformation $\mathcal{P}(\alpha)$ is the pair

$$\mathcal{P}(\alpha)_{(f_1, \dots, f_n)} = (1_n, (f_1 \star_0 \cdots \star_0 f_n) \star_1 \alpha).$$

4.3.2. Theorem. *Let \mathbf{P} be an algebraic 2-prop and let (A, B) be a 3-sphere of \mathbf{P} . Then $A = B$ if and only if $\mathcal{P}(A) = \mathcal{P}(B)$.*

Proof. Let us assume that $A, B : f \Rightarrow g : m \Rightarrow n$ are such that $\mathcal{P}(A) = \mathcal{P}(B)$. Then we have, by definition of \mathcal{P} , for every algebraic 2-cells f_1, \dots, f_m of \mathbf{P} :

$$(1_m, (f_1 \star_0 \cdots \star_0 f_m) \star_1 A) = (1_m, (f_1 \star_0 \cdots \star_0 f_m) \star_1 B).$$

In particular, we take $f_i = 1$, for every $i \in \{1, \dots, m\}$, to get $A = B$. \square

4.4. The main result

4.4.1. Theorem. *The (underlying category of the) initial algebra \mathcal{B} of \mathbf{Br}/\mathbf{Mon} is the monoidal category of braids.*

Proof. We note that, in \mathbf{Br}/\mathbf{Mon} , there is exactly one algebraic 2-cell for each natural number n . As a consequence, a \mathbf{Br}/\mathbf{Mon} -algebra is a strict monoidal category \mathcal{C} , with natural numbers as objects and monoidal product given by $m \star_0 n = m + n$, for every $m, n \in \mathbb{N}$ and $u \star_0 v = \mathcal{C}(\nabla)(u, v)$, for every morphisms u, v with target 1. In particular, in the initial algebra \mathcal{B} , the product of (σ, A) and (τ, B) , where $t_1(A) = t_1(B) = 1$, is given by:

$$(\sigma, A) \star_0 (\tau, B) = (\sigma \star_0 \tau, (A \star_0 B) \star_1 \nabla).$$

Thus, \mathcal{B} is generated, as a monoidal category, by the the following endomorphisms of 2:

$$\overline{\times} = (\overline{\times}, \overline{\times}) \quad \text{and} \quad \overline{\times} = \left(\overline{\times}, \left(\overline{\times} \right)^{-} \right).$$

We have:

$$\begin{aligned} \overline{\times} &= (\overline{\times}, \overline{\times}) \left(\overline{\times}, \left(\overline{\times} \right)^{-} \right) \\ &= \left(1_2, \overline{\times} \star_2 \left(\overline{\times} \right)^{-} \right) = (1_2, \mu). \end{aligned}$$

We note that $(1_2, \mu)$ is the identity of 2 in \mathcal{B} to get:

$$\overline{\times} = | |$$

We prove, in a similar way:

$$\overline{\times} = | |$$

We check:

$$\begin{aligned} \overline{\times} \overline{\times} &= (\overline{\times} \downarrow \overline{\times} \downarrow) (| \overline{\times} \downarrow) (\overline{\times} \downarrow \overline{\times} \downarrow) \\ &= \left(\overline{\times} \overline{\times}, \overline{\times} \star_2 \overline{\times} \star_2 \overline{\times} \right). \end{aligned}$$

Computing in \mathbf{Br}/\mathbf{Mon} , we get:

$$\overline{\times} \overline{\times} \star_2 \overline{\times} \star_2 \overline{\times} = \overline{\times} \star_2 \overline{\times} = \overline{\times} \overline{\times}.$$

Then, we consider:

$$\overline{\times} \overline{\times} = \left(\overline{\times} \overline{\times}, \overline{\times} \star_2 \overline{\times} \star_2 \overline{\times} \right).$$

REFERENCES

In **Br/Mon**, we have:

$$\text{Diagram 1} * 2 \text{Diagram 2} * 2 \text{Diagram 3} = \text{Diagram 4} * 2 \text{Diagram 5} = \text{Diagram 6}.$$

Thus, it follows:

$$\text{Diagram 7} = \text{Diagram 8}$$

Finally, one checks that the image through \mathcal{B} of the 4-cells Diagram 9 , Diagram 10_1 and Diagram 10_2 induce no other relation on endomorphisms Diagram 11 and Diagram 12 . \square

4.4.2. Corollary (Coherence theorem for braided monoidal categories, [6]). *Let (A, B) be a 3-sphere of **Br**. Then $A = B$ if and only if the braids $\mathcal{B}(A)$ and $\mathcal{B}(B)$ are equal.*

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YVES GUIRAUD

INRIA NANCY, LORIA, 615 rue du Jardin Botanique, CS 20101, F-54603 Villers-lès-Nancy Cedex, France

E-mail address: yves.guiraud@loria.fr

PHILIPPE MALBOS

UNIVERSITÉ DE LYON, UNIVERSITÉ LYON 1, CNRS, UMR 5208, INSTITUT CAMILLE JORDAN, Batiment du Doyen Jean Braconnier, 43, blvd du 11 novembre 1918, F-69622 Villeurbanne Cedex, France

E-mail address: malbos@math.univ-lyon1.fr