Derivatives and Asymptotics of Whittaker functions

Nadir Matringe

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Abstract

Let F be a p-adic field, and G_n one of the groups $GL(n,F)$, $GSO(2n-1,F)$, $GSp(2n,F)$, or $GSO(2(n-1),F)$. Using the mirabolic subgroup or analogues of it, and related "derivative" functors, we give an asymptotic expansion of functions in the Whittaker model of generic representations of G_n, with respect to a minimal set of characters of subgroups of the maximal torus. Denoting by Z_n the center of G_n, and by N_n the unipotent radical of its standard Borel subgroup, we characterize generic representations occurring in $L^2(Z_nN_n\backslash G_n)$ in terms of these characters.
This is related to a conjecture of Lapid and Mao for general split groups, asserting that the generic representations occurring in $L^2(Z_nN_n\backslash G_n)$ are the generic discrete series; we prove it for the group G_n.

Introduction

Let G_n be the points of one of the groups $GL(n)$, $GSO(2n-1)$, $GSp(2n)$, or $GSO(2(n-1))$ over a p-adic field K. The main result (Theorem 2.1) of this work describes the asymptotic behaviour of the restriction of Whittaker functions to the standard maximal torus, in terms of a family of characters which is minimal in some sense. From results of [L-M], this restriction can be described for split reductive groups in terms of cuspidal exponents.
Here, after having defined analogues of the mirabolic subgroup for the groups G_n, and the corresponding derivative functors, following [C-P] (where the case of completely reducible derivatives is treated for $GL(n)$), we choose to describe the restriction of Whittaker functions to the torus in terms of central exponents of the derivatives.
This description, inspired by [B], is better adapted to understanding when the Whittaker model of a unitary generic representation is a subspace of $L^2(Z_nN_n\backslash G_n)$ (these representations are conjectured to be generic discrete series by Lapid and Mao).
In the first section, we review the groups in question and define their mirabolic subgroups. We also give a decomposition of the unipotent radical of the standard Borel subgroup, and a description of how nondegenerate characters of this radical behave with respect to this decomposition.
In Section 2, we give properties of the derivative functors, and use them to prove our asymptotic expansion of Whittaker functions, which is Theorem 2.1.
In Section 3, we characterize generic representations with Whittaker model included in $L^2(Z_nN_n\backslash G_n)$ in terms of central exponents of the derivatives, in Corollary 3.1. We then prove in Theorem 3.2 the conjecture 3.5 of [L-M].

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1 Mirabolic subgroup and nondegenerate characters

Let F be nonarchimedean local field, we denote by $\mathfrak{O}_F$ its ring of integer, and by $\mathfrak{P}_F = \varpi_F \mathfrak{O}_F$ the maximal ideal of this ring, where ϖ_F is a uniformiser of F.
We give a list of groups, and describe some of their properties which will be used in the sequel:

- **Case A:**
  The group G_0 is trivial, and for $n \geq 1$, the group G_n is $GL(n, F)$.

  We consider the maximal torus of G_n consisting of diagonal matrices, it is isomorphic to $(F^*)^n$.
  For $n \geq 2$, the simple roots of this group can be chosen to be the characters

  $$\alpha_i(diag(x_1, \dots, x_n)) = x_i x_{i+1}^{-1}$$

  for i between 1 and $n-1$.
  The root subgroup U_{α_i} is given by matrices of the form $I_n + xE_{i,i+1}$ for x in F.

  Standard Levi subgroups of G_n are given by matrices of the form $diag(a_1, \dots, a_r)$ where a_i belongs to $GL(n_i, F)$, with $n_1 + \dots + n_r = n$. We denote the preceding group by $M_{(n_1,\dots,n_r)}$, and the corresponding standard parabolic subgroup is denoted by $P_{(n_1,\dots,n_r)}$, with unipotent radical $U_{(n_1,\dots,n_r)}$.

  For $n \geq 2$ we denote by U_n the group $U_{(n-1,1)}$, of matrices of the form $\begin{bmatrix} I_{n-1} & V \\ & 1 \end{bmatrix}$.
  It is isomorphic to F^{n-1}.

  For $n > k \geq 1$, the group G_k embeds naturally in G_n, and is given by matrices of the form $diag(g, I_{n-k})$; we denote by Z_k its center.
  We denote by P_n the mirabolic subgroup $G_{n-1} \ltimes U_n$.

- **Case B:**
  The group G_0 is trivial.
  For $n \geq 2$, the group $G_n = GO(2n-1, F)$ is the group of matrices g in $GL(2n-1, F)$ such that ${}^t g J g$ belongs to $F^* J$. We call the multiplier of an element g in G_n the scalar $\mu(g)$ such that ${}^t g J g$ is equal to $\mu(g)J$ (one checks that for this group, the multiplier actually belongs to $(F^*)^2$), where J is the antidiagonal matrix of $GL(2n-1, F)$ with ones on the second diagonal. It is the direct product of $SO(2n-1, F)$ with F^*, more precisely, of $SO(2n-1, F)$ and the group $I(F^*)$ of matrices $I(t) = tI_{2n-1}$ for t in F^*.

  The maximal torus of G_n is equal to the product of the torus of matrices of the form $diag(x_{n-1}^{-1}, \dots, x_1^{-1}, 1, x_1, \dots, x_{n-1})$ with $I(F^*)$ and is isomorphic to $(F^*)^n$.
  For $n \geq 3$, the simple roots of this group can be chosen to be the characters

  $$\alpha_{i+1}(diag(tx_{n-1}^{-1}, \dots, tx_1^{-1}, t, tx_1, \dots, tx_{n-1})) = x_i x_{i+1}^{-1}$$

  for i between 1 and $n-2$, and $\alpha_1(diag(tx_{n-1}^{-1}, \dots, tx_1^{-1}, t, tx_1, \dots, tx_{n-1})) = x_1^{-1}$.
  The root subgroups U_{α_i} are given by matrices of the form

  $$diag(u, \dots, 1, u, 1, \dots, 1, u^{-1}, 1, \dots, 1)$$

  for matrices u in the unipotent radical of the Borel of $GL(2, F)$.

For $n > k \geq 1$, the group G_k embeds naturally in G_n, and is given by matrices of the form $diag(\mu(g)I_k, g, I_{n-k})$, where $\mu(g)$ is the multiplier of the element g of G_k.
Its center Z_k is given by matrices of the form $z_k(t) = diag(t^2 I_{n-k}, tI_{2k-1}, I_{n-k})$ for t in K^*.

For $n \geq 1$, the standard Levi subgroups of G_n are given by matrices of the form

$$diag(\mu(g)^\tau a_{r-1}^{-1}, \dots, \mu(g)^\tau a_1^{-1}, g, a_1, \dots, a_r),$$

where a_i belongs to $GL(n_i, F)$, ${}^\tau a$ is the transpose of a with respect to the second diagonal, g belongs to G_m, with $2m-1+2n_1+\dots+2n_r = 2n-1$. We denote the preceding group by $M_{(m;n_1,\dots,n_r)}$, and the corresponding standard parabolic subgroup consisting of block upper triangular matrices is denoted by $P_{(m;n_1,\dots,n_r)}$, with unipotent radical $U_{(m;n_1,\dots,n_r)}$.

For $n \geq 2$ we denote by U_n the group $U_{(n-1;1)}$, of matrices of the form $\begin{bmatrix} 1 & -{}^\tau V & -{}^\tau V V/2 \\ & I_{2n-3} & V \\ & & 1 \end{bmatrix}$.

It is isomorphic to F^{2n-3}.

For $n \geq 2$, we denote by P_n the "mirabolic" subgroup $G_{n-1} \ltimes U_n$.

- **Case C:**
The group G_0 is trivial, the group G_1 is F^*.
For $n \geq 2$, the group $G_n = GSp(2(n-1), F)$, where $GSp(2(n-1), F)$ is the group of matrices g in $GL(2(n-1), F)$ such that ${}^t g J g$ belongs to $F^* J$, where $J = \begin{bmatrix} 0 & W \\ -W & 0 \end{bmatrix}$ and W is the antidiagonal matrix of $GL(n-1, F)$ with ones on the second diagonal. It is the semi-direct product of $Sp(2(n-1), F)$ with F^*, more precisely, of $Sp(2n, F)$ and the group $I(F^*)$ of matrices $I(t) = diag(tI_{n-1}, I_{n-1})$ for t in F^*.

The maximal torus of G_n is equal to the product of the torus of matrices of the form $diag(x_{n-1}^{-1}, \dots, x_1^{-1}, x_1, \dots, x_{n-1})$ with $I(F^*)$ and is isomorphic to $(F^*)^n$.
The simple roots of this group are the characters $\alpha_{i+1}(diag(tx_{n-1}^{-1}, \dots, tx_1^{-1}, x_1, \dots, x_{n-1})) = x_i x_{i+1}^{-1}$ for i between 1 and $n-2$, and $\alpha_1(diag(tx_{n-1}^{-1}, \dots, tx_1^{-1}, x_1, \dots, x_{n-1})) = tx_1^{-2}$.
For i less than n, the root subgroup U_{α_i} is given by matrices of the form

$$diag(1, \dots, 1, u, 1, \dots, 1, u^{-1}, 1, \dots, 1),$$

for matrices u in the unipotent radical of the Borel of $GL(2, F)$, whereas U_{α_n} is given by matrices $diag(1, \dots, 1, u, 1, \dots, 1)$, for matrices u in the unipotent radical of the Borel of $GL(2, F)$.

For $n > k \geq 2$, the group G_k embeds naturally in G_n, and is given by matrices of the form $diag(\mu(g)I_{n-k}, g, I_{n-k})$, where $\mu(g)$ is the multiplier of the element g of G_k. Its center Z_k is given by matrices of the form $z_k(t) = diag(t^2 I_{n-k}, tI_{2(k-1)}, I_{n-k})$ for t in F^*. The group G_1, which is equal to its center Z_1, embeds as $I(F^*)$.

For $n \geq 1$, the standard Levi subgroups of G_n are:

  - either matrices
$$diag(\mu(g)^\tau a_{r-1}^{-1}, \dots, \mu(g)^\tau a_1^{-1}, g, a_1, \dots, a_r),$$
where a_i belongs to $GL(n_i, F)$, g belongs to G_m with $m \geq 2$, with $2(m-1) + 2n_1 + \dots + 2n_r = 2(n-1)$. We denote the preceding group by $M_{(m;n_1,\dots,n_r)}$, and the corresponding standard parabolic subgroup consisting of block upper triangular matrices is

denoted by $P_{(m;n_1,\dots,n_r)}$, with unipotent radical $U_{(m;n_1,\dots,n_r)}$.

– or the matrices
$$z_1.diag(a_{r-1}^{-1},\dots,a_1^{-1},a_1,\dots,a_r),$$
with a_i in $GL(n_i,F)$, $2n_1+\cdots+2n_r=2(n-1)$, and z_1 in G_1.

For $n\geq 2$, we denote by U_n the group $U_{(n-1;1)}$, of matrices of the form $\begin{bmatrix}1 & -{}^{\tau}V_2 & {}^{\tau}V_1 & x\\ & I_{n-2} & & V_1\\ & & I_{n-2} & V_2\\ & & & 1\end{bmatrix}$.

For $n\geq 3$, it is an extension of $F^{2(n-2)}$ by F, which is the Heisenberg group corresponding to the alternating bilinear form on $F^{2(n-2)}$, given by $(W_1,W_2)\times(V_1,V_2)\mapsto -{}^{\tau}W_2.V_1+{}^{\tau}W_1.V_2$. It is a two steps nilpotent subgroup, with center equal to its derived subgroup, given by matrices with $V_1=V_2=0$, and the maximal abelian quotient U_n^{ab} of U_n is $F^{2(n-2)}$.
The group U_2 is the unipotent radical of the standard Borel of $G_2=GSp(2,F)=GL(2,F)$, and is isomorphic to $(F,+)$.

For $n\geq 2$, we denote by P_n the "mirabolic" subgroup $G_{n-1}\ltimes U_n$.

- **Case D:**
We denote by G_0 the trivial group. The group G_1 is F^*.

For $n\geq 2$, the group $G_n=GSO(2(n-1),F)$ is the group of matrices g in $GL(2n,F)$ satisfying that ${}^tgJg$ belongs to F^*J, where J is the antidiagonal matrix of $GL(2n,F)$ with ones an the second diagonal. It is the semi-direct product of $SO(2(n-1),F)$ with the group $I(F^*)$, of the matrices $I(t)=diag(tI_n,I_n)$ for t in F^*.

The maximal torus of G_n is equal to the product of the torus of matrices of the form $diag(x_{n-1}^{-1},\dots,x_1^{-1},x_1,\dots,x_{n-1})$ with $I(F^*)$ and is isomorphic to $(F^*)^n$.
If n is 2, then G_2 is the diagonal torus of $GL(2,F)$.
For $n\geq 3$, the simple roots of this group can be chosen to be the characters
$$\alpha_{i+1}(diag(tx_{n-1}^{-1},\dots,tx_1^{-1},x_1,\dots,x_{n-1}))=x_ix_{i+1}^{-1}$$
for i between 1 and $n-2$, and $\alpha_1(diag(tx_n^{-1},\dots,tx_1^{-1},x_1,\dots,x_n))=tx_1^{-1}x_2^{-1}$.
For $i\geq 1$, the root subgroup $U_{\alpha_{i+1}}$ is given by matrices of the form
$$diag(1,\dots,1,u,1,\dots,1,u^{-1},1,\dots,1)$$
for matrices u in the unipotent radical of the Borel of $GL(2,F)$, whereas U_{α_1} is given by matrices $diag(1,\dots,1,u,1,\dots,1)$ for matrices u of the form $\begin{bmatrix}1 & & y & \\ & 1 & & -y\\ & & 1 & \\ & & & 1\end{bmatrix}$, with y in F.

For $n>k\geq 2$, the group G_k embeds naturally in G_n, and is given by matrices of the form $diag(\mu(g)I_{n-k},g,I_{n-k})$, where $\mu(g)$ is the multiplier of the element g of G_k. For $k\geq 3$, its center Z_k is given by matrices of the form $z_k(t)=diag(t^2I_{n-k},tI_{2(k-1)},I_{n-k})$ for t in F^*.
We denote by $\mathbf{Z_2}$ the subgroup of the torus $\mathbf{G_2}$, given by matrices of the form $\mathbf{z_2(t)=diag(tI_{n-2},1,t,I_{n-2})}$ for t in $\mathbf{F^*}$.
The group G_1 which is equal to its center Z_1, embeds as $I(F^*)$.

The standard Levi subgroups of G_n are the following:

– The groups given by matrices of the form

$$diag(\mu(g)^{\tau}a_r^{-1},\dots,\mu(g)^{\tau}a_1^{-1},g,a_1,\dots,a_r),$$

where a_i belongs to $GL(n_i,F)$, g belongs to G_m with $m\geq 3$, with $2(m-1)+2n_1+\dots+2n_r=2(n-1)$. We denote the preceding group by $M_{(m;n_1,\dots,n_r)}$, and the corresponding standard parabolic subgroup is denoted by $P_{(m;n_1,\dots,n_r)}$, with unipotent radical $U_{(m;n_1,\dots,n_r)}$.

– The groups given by matrices of the form

$$g_2.diag({}^{\tau}a_r^{-1},\dots,{}^{\tau}a_1^{-1},1,1,a_1,\dots,a_r),$$

with a_i in $GL(n_i,F)$, $2n_1+\dots+2n_r=2(n-2)$, and g_2 in G_2.

– The groups given by matrices of the form

$$z_1.diag({}^{\tau}a_r^{-1},\dots,{}^{\tau}a_1^{-1},a_1,\dots,a_r),$$

with a_i in $GL(n_i,F)$, $2n_1+\dots+2n_r=2(n-1)$, and z_1 in G_1.

– The groups given by matrices of the form

$$z_2.diag({}^{\tau}a_r^{-1},\dots,{}^{\tau}a_1^{-1},g,a_1,\dots,a_r),$$

where a_i belongs to $GL(n_i,F)$, and g in $GL(2(m-1),F)$ is of the form

$$\begin{bmatrix} A & & V & \\ & t' & & L' \\ L & & t & \\ & V' & & A' \end{bmatrix},$$

with

$$\begin{bmatrix} A & V \\ L & t \end{bmatrix}\in GL(m-1,F),$$

A a $(m-2\times m-2)$-matrix,

$$\begin{bmatrix} {}^{\tau}A' & {}^{\tau}L' \\ {}^{\tau}V' & t' \end{bmatrix}=\begin{bmatrix} A & V \\ L & t \end{bmatrix}^{-1},$$

with $2m+2n_1+\dots+2n_r=2n$ and z_2 in Z_2.

For $n\geq 3$ we denote by U_n the subgroup $U_{(n-1;1)}$ of G_n, of matrices of the form

$$\begin{bmatrix} 1 & -{}^{\tau}V & -{}^{\tau}VV/2 \\ & I_{2(n-2)} & V \\ & & 1 \end{bmatrix}.$$

It is isomorphic to F^{2n-2}.

For $n\geq 3$ denote by P_n the "mirabolic" subgroup $G_{n-1}\ltimes U_n$.
We denote by U_2 the group U_{α_1}, and by P_2 the group $G_1\ltimes U_2$.

Lemma 1.1. *We denote by Z_i the center of G_i, except in case D when $n=2$, where we denote by Z_2 the subgroup $diag(1,t)$ with t in F^* of G_2. In all cases, one checks that the maximal torus A_n of G_n is the direct product $Z_1.Z_2\dots Z_{n-1}.Z_n$, and each Z_i is isomorphic to F^*. Moreover, the i-th root has the property that $\alpha_i(z_1\dots z_n)=z_i$, in other words these coordinates parametrize the torus A_n such that simple roots become canonical projections.*

The unipotent radical N_{n+1} of the standard Borel subgroup of G_{n+1} is equal to $U_2 \dots U_{n+1}$. Let θ be a nondegenerate character of N_{n+1} (i.e. that restricts non trivially to any of the simple root subgroups). We denote by θ_{i+1} the character $\theta_{|U_{i+1}}$, except in the case D, for $i = 2$. In this case $U_3 = U_{\alpha_1} \times U_{\alpha_2}$, and we denote by θ_3 the character $\theta_3(u_{\alpha_1} u_{\alpha_2}) = \theta(u_{\alpha_2})$.
Because θ is trivial on U_{der}, and according to the description of U_{der} in Theorem 4.1 of [B-H], the character θ_{i+1} must be trivial on every root subgroup U_α contained in U_{i+1} such that α is not simple, moreover for case D, $n = 2$, the character θ_3 is trivial on U_{α_1}.
Conversely, if a non trivial character θ_{i+1} of U_{i+1} is trivial on every $U_\alpha \subset U_{i+1}$ which is not simple, and if, in case D, $n = 2$, we impose in addition that θ_3 is trivial on U_{α_1}, then one checks that the normalizer of θ_{i+1} in the mirabolic subgroup P_{i+1} is $P_i U_{i+1}$. As the group $U_2 \dots U_i$ is a subgroup of P_i, a family of non trivial characters θ_{i+1} of U_{i+1}, trivial on every $U_\alpha \subset U_{i+1}$ except U_{α_i}, defines a nondegenerate character of $N_{n+1} = U_2 \dots U_{n+1}$ by $\theta(u_2 \dots u_{n+1}) = \prod_{i=1}^{n} \theta_{i+1}(u_{i+1})$.

Now we fix such a nondegenerate character θ, and write θ^k for the character $\theta_2 \dots \theta_k$ of N_k.

2 Derivatives and Whittaker functions

If G is an l-group, we denote by $Alg(G)$ the category of smooth complex G-modules. If (π, V) belongs to $Alg(G)$, H is a closed subgroup of G, and χ is a character of H, we denote by $V(H, \chi)$ the subspace of V generated by vectors of the form $\pi(h)v - \chi(h)v$ for h in H and v in V. This space is actually stable under the action of the subgroup $N_G(\chi)$ of the normalizer $N_G(H)$ of H in G, which fixes χ.
We denote by δ_H the positive character of $N_G(H)$ such that if μ is a right Haar measure on G, and λ is the left translation of smooth functions with compact support on G, then $\mu \circ \lambda(n^{-1}) = \delta_H(n)\mu$ for n in N.
This gives the spaces $V(H, \chi)$ and $V_{H,\chi} = V/V(H, \chi)$ (that we simply denote by V_H when χ is trivial) a structure of smooth $N_G(\chi)$-modules.

Notations being as in the first section, and for k be an integer between 2 and n we define the following functors:

- First we recall the definition of the Jacquet functors:
  Let P be a parabolic subgroup of G_n, with Levi subgroup M, and unipotent radical U.
  We denote by J_P the functor from $Alg(G_n)$ to $Alg(M)$ such that, if (π, V) is a smooth G_n-module, we have $J_P(V) = V_U$, and M acts on $J_P(V)$ by $J_P\pi(m)(v + V(U, 1)) = \delta_U(m)^{-1/2}\pi(m)v + V(U, 1)$.

- With the same notations, we denote by i_P^G the functor from $Alg(M)$ to $Alg(G_n)$ such that, if ρ is a smooth M-module, and $\bar{\rho}$ is the corresponding P-module obtained by inflation of ρ to P, then $i_P^G(\rho)$ is the G_n-module $ind_P^{G_n}(\bar{\rho})$ where ind is the usual compact induction.

- The functor $\Phi_{\theta_k}^-$ (denoted r_{U_k,θ_k} in section 1 of [B-Z.2]) from $Alg(P_k)$ to $Alg(P_{k-1})$ such that, if (π, V) is a smooth P_k-module, $\Phi_{\theta_k}^- V = V_{U_k,\theta_k}$, and P_{k-1} acts on $\Phi_{\theta_k}^-(V)$ by $\Phi_{\theta_k}^-\pi(p)(v + V(U_k, \theta_k) = \delta_{U_k}(p)^{-1/2}\pi(p)(v + V(U_k, \theta_k)$.

- The functor $\Phi_{\theta_k}^+$ (denoted i_{U_k,θ_k} in section 1 of [B-Z.2]) from $Alg(P_{k-1})$ to $Alg(P_k)$ such that, for π in $Alg(P_{k-1})$, one has $\Phi_{\theta_k}^+\pi = ind_{P_{k-1}U_k}^{P_k}(\delta_{U_k}^{1/2}\pi \otimes \theta_k)$, where ind is the usual compact induction.

- The functor $\hat{\Phi}_{\theta_k}^+$ (I_{U_k,θ_k} in section 1 of [B-Z.2]) from $Alg(P_{k-1})$ to $Alg(P_k)$ such that, for π in $Alg(P_{k-1})$, one has $\Phi_{\theta_k}^+\pi = Ind_{P_{k-1}U_k}^{P_k}(\delta_{U_k}^{1/2}\pi \otimes \theta_k)$, where Ind is the usual induction.

- The functor Ψ^- is the Jacquet functor J_{U_k}, (denoted $r_{U_k,1}$ in section 1 of [B-Z.2]) from $Alg(P_k)$ to $Alg(G_{k-1})$, such that if (π, V) is a smooth P_k-module, $\Psi^- V = V_{U_k,1}$, and G_{k-1} acts on $\Psi^-(V)$ by $\Psi^-\pi(g)(v) + V(U_k, \theta_k) = \delta_{U_k}(g)^{-1/2}\pi(p)(v + V(U_k, 1))$.
- The functor Ψ^+ (denoted $i_{U_k,1}$ in section 1 of [B-Z.2]) from $Alg(G_{k-1})$ to $Alg(P_k)$, such that for π in $Alg(G_{k-1})$, one has $\Psi^+\pi = ind_{G_{k-1}U_k}^{P_k}(\delta_{U_k}^{1/2}\pi \otimes 1) = \delta_{U_k}^{1/2}\pi \otimes 1$.

As we already fixed the character θ of N_n, we will most of the time forget the dependence in θ_k of $\Phi^-_{\theta_k}$ and $\Phi^+_{\theta_k}$, and we will write these functors Φ^- and Φ^+. These functors have the following properties which follow (except for c) an d) which are trivial) from Proposition 1.9 of [B-Z.2]:

Proposition 2.1. *a) The functors Φ^-, Φ^+, Ψ^-, and Ψ^+ are exact.*
b) Ψ^- is left adjoint to Ψ^+.
b') Φ^- is left adjoint to $\hat{\Phi}^+$.
c) $\Phi^-\Psi^+ = 0$
d) $\Psi^-\Psi^+ = Id$.

Now we want to know how these functors restrict to smooth P_k-modules which are submodules of the space $C^\infty(N_k\backslash P_k, \theta^k) = Ind_{N_k}^{P_k}(\theta^k)$ of functions on P_k, fixed by some open subgroup of P_k under right translation, and which transform by θ^k under left translation by elements of N_k. The next proposition shows the stability of this type of modules under Φ^- and Φ^+.

Proposition 2.2. *For any submodule τ of $C^\infty(N_k\backslash P_k, \theta^k)$, the P_{k-1}-module $\Phi^-\tau$ is a submodule of $C^\infty(N_{k-1}\backslash P_{k-1}, \theta^{k-1})$, with model given by restriction of functions $\delta_{U_k}^{-1/2}W$ in τ to P_{k-1}, and such that we have $\Psi^-\tau(p)W = \rho(p)W$ for p in P_{k-1}, where ρ is the action by right translation. Conversely, the P_{k+1}-module $\Phi^+\tau$ can be identified with a submodule of $C^\infty(N_{k+1}\backslash P_{k+1}, \theta^{k+1})$, with the natural action of P_{k+1} by right translation.*

Proof. The first property will hold if we show that $C^\infty(N_k\backslash P_k, \theta^k)(U_k, \theta_k)$ is the kernel of the restriction map to $C^\infty(N_{k-1}\backslash P_{k-1}, \theta^{k-1})$, this is a straightforward adaptation of the proof of Proposition 2.1 of [C-P].
The second property is a consequence of the following equalities and inclusions:

$$\begin{aligned} \Phi^+(C^\infty(N_k\backslash P_k, \theta^k)) &= ind_{P_kU_{k+1}}^{P_{k+1}}(\delta_{U_{k+1}}^{1/2}.Ind_{N_k}^{P_k}(\theta^k) \otimes \theta_{k+1}) \\ &\subset Ind_{P_kU_{k+1}}^{P_{k+1}}(\delta_{U_{k+1}}^{1/2}.Ind_{N_k}^{P_k}(\theta^k) \otimes \theta_{k+1}) \end{aligned}.$$

Then

$$\delta_{U_{k+1}}^{1/2}.Ind_{N_k}^{P_k}(\theta^k) \simeq Ind_{N_k}^{P_k}(\theta^k)$$

because the character $\delta_{U_{k+1}}^{1/2}$ of P_k is trivial on N_k.
Finally

$$Ind_{P_kU_{k+1}}^{P_{k+1}}(Ind_{N_k}^{P_k}(\theta^k) \otimes \theta_{k+1}) \simeq Ind_{N_{k+1}}^{P_{k+1}}(\theta^{k+1})$$

□

More can be said about smooth P_k-submodules of the space $C^\infty(N_k\backslash P_k, \theta^k) = Ind_{N_k}^{P_k}(\theta^k)$. If τ is a P_k-submodule of $C^\infty(N_k\backslash P_k, \theta^k)$, then the derived subgroup of U_k (which is trivial except in case D) acts trivially.
To see this, take W in $C^\infty(N_k\backslash P_k, \theta^k)$, we claim that if u belongs to the derived subgroup U_k^{der} of U_k, then $\tau(u)W$ and W are equal. So let p belong to P_k; one has $\tau(u)W(p) = W(pu) = W(pup^{-1}p) = \theta^k(pup^{-1})W(p)$. But P_k normalizes U_k (so $\theta^k(pup^{-1}) = \theta_k(pup^{-1})$), so that it normalizes its derived subgroup as well; as θ_k is trivial on this subgroup, this proves our claim.

For such modules P_k-modules, there is a nice interpretation of $V(U_k, 1)$ in terms of the analytic behaviour of Whittaker functions. First, we make the following observation.

Remark 2.1. For $k \geq 3$, as a consequence of the Iwasawa decomposition, any element g of G_{k-1} can be written in the form pzc with p in P_{k-1}, z in Z_{k-1}, and k in $K = G_{k-1}(\mathfrak{O}_F)$, and the absolute value of z depends only on g, so we can write it $|z(g)|_F$.
If a function W is in the space of $C^\infty(N_k \backslash P_k, \theta^k)$, then for g in G_{k-1}, we show that $W(g)$ vanishes whenever $|z(g)|_F$ is large enough.
Indeed if we take the "natural" group isomorphism u from $(F^m, +)$ to U_k^{ab}, for some positive integer m, and recalling that it is in fact U_k^{ab} that acts on V, then $u(x)$ will fix W for x near zero in F^m.
But then, for g in G_{k-1} of the form pzk, one has $W(g) = W(gu(x)) = \theta_k(gu(x)g^{-1})W(g)$, which is equal $\theta_k(zku(x)(cz)^{-1})W(g)$ because P_{k-1} normalizes θ_k. This implies the equality $[\theta_k(zku(x)(kz)^{-1}) - 1]W(g) = [\theta_k(u(zkx)) - 1]W(g) = 0$ for any x in a neighbourhood of zero depending only on W. The assertion follows easily.

Proposition 2.3. *Let (τ, V) be a P_k-submodule of $C^\infty(N_k \backslash P_k, \theta^k)$. Then the space $V(U_k, 1)$ is the subspace of V, of functions W such that there exists an integer N_W with $W(g) = 0$, for any g satisfying $|z(g)|_F \leq q_F^{-N_W}$.*

Proof. Suppose first that a function W is in $V(U_k, 1)$, so we can write it $\pi(u)W' - W'$ for some u in U_k^{ab} and some W' in V. Then, writing g as pzk, and u as $u(x)$ for x in F^m, we have $[\pi(u)W' - W'](g) = [\theta_k(u(zkx)) - 1]W'(g)$, which will be zero to 0 when $|z|_F$ is close to zero.
Conversely, we use the characterization of Jacquet and Langlands asserting that the elements W of $V(U_k, 1) = V(U_k^{ab}, 1)$ are those such that $\int_U \tau(u)Wdu$ is zero as soon as the open compact subgroup U of U_k^{ab} contains some compact open subgroup U_W. So suppose W is in V and that it vanishes on elements g of $G_{n-1}(F)$ satisfying $|z(g)|_F \leq q_F^{-N_W}$.
Let U be any open compact subgroup of U_k^{ab}, that we identify with a subgroup of F^m. The integral $\int_U \tau(u)Wdu$ evaluated at $g = pzk$, is equal to $\int_{x \in U} \theta_k(zkx)W(g)dx$. Hence this integral is always zero for $|z|_F \leq q_F^{-N_W}$ because $W(g)$ is.
We now recall that as θ_k is a non trivial character of U_k^{ab}, there exists a compact open ball U_0 of $U_k^{ab} \simeq F^n$ such that, the integral $\int_{x \in U} \theta_k(x)dx$ is zero whenever the compact open subgroup U of U_k^{ab} contains U_0. But then for $|z|_F \geq q_F^{-N_W}$, if t_W is an element of F^* of absolute value $q_F^{N_W}$, the integral $\int_{x \in U} \theta_k(zkx)W(g)dx$ is also zero as soon as U contains $t_W U_0$. Hence $\int_U \tau(u)Wdu$ is zero when U is a compact open subgroup of U_k^{ab} containing $U_W = t_W U_0$, and W belongs to $V(U_k, 1)$.

□

For any smooth P_n-module τ, and any integer $k \geq 1$, we denote by $\tau_{(k)}$ the representation of P_{n-k+1} equal to $\Phi^{k-1}\tau$, and by $\tau^{(k)}$ the representation of G_{n-k} equal to $\Psi^- \Phi^{k-1}\tau = \Psi^- \tau_{(k)}$.
We say that a smooth irreducible representation π of G_n is θ^n-generic if it is isomorphic to a submodule of the induced representation $Ind_{N_n}^{G_n}(\pi)$. If it is the case, the submodule of $Ind_{N_n}^{G_n}(\pi)$ isomorphic to π is unique, it is called the Whittaker model of π and denoted by $W(\pi, \theta^n)$.
Now we let (π, V) be a θ^n-generic representation of G_n (hence a smooth P_n-module as well), we denote by (π', V') the representation of P_n obtained on the space of restrictions of functions in $W(\pi, \theta)$ to P_n, it is a quotient of π as a P_n-module, and restriction to P_n is known to be an isomorphism in case A.

The following proposition follows from applying repeatedly Proposition 2.2, and from Proposition 2.3.

Proposition 2.4. *Let τ be a smooth P_n-submodule of $C^\infty(N_n \backslash P_n, \theta^n)$, and $k \geq 0$ be an integer, then the P_{k+1}-module $\tau_{(n-k-1)}$ is a submodule of $C^\infty(N_{k+1} \backslash P_{k+1}, \theta^{k+1})$, with model given by restriction of functions $[\delta_{U_{k+2}} \ldots \delta_{U_n}]^{-1/2}W$ in τ to P_{k+1}. In this realisation, one has $\tau_{(n-k-1)}(p)W = \rho(p)W$ for p in P_{k+1}, where ρ is the action by right translation.*

The next proposition asserts amongst other things that for every $k \geq 1$, the G_{n-k}-module $\pi^{(k)}$ has finite length.

Proposition 2.5. *If (π, V) is a smooth representation of G_n of finite length, then for k between 1 and $n-1$, the G_k-module $\pi^{(n-k)}$ (hence its quotient $\pi'^{(n-k)}$) has finite length.*

Proof. For $k \geq 1$, except in case D, $k = 2$, we denote by $U_{k,n-k}$ the unique standard unipotent radical (denoted by $U_{(k;n-k)}$ in the previous section) containing U_{α_k} as only simple root subgroup. In case D, for $k = 2$, we be denote $U_{2,n-2}$ the unique standard unipotent radical containing U_{α_1} and U_{α_2} as only simple root subgroups.
In all cases, the corresponding Levi $M_{k,n-k}$ is the direct product of G_k with $GL(n-k, F)$.
Now the module G_k-module $\pi^{(n-k)}$ is a quotient of the Jacquet $G_k \times GL(n-k)$-module

$$(\pi_{U_{k,n-k}}, V/V(U_{k,n-k}, 1)),$$

as the kernel of the surjective map $\pi \twoheadrightarrow \pi^{(n-k)}$ contains $V(U_{k,n-k}, 1)$. More precisely, let $N_{n-k,A}$ be the unipotent radical of the standard Borel subgroup of $GL(n-k, F)$, the group $U_{k+1} \dots U_n$ is the semidirect product $N_{n-k,A} \ltimes U_{k,n-k}$, so that the space $V^{(n-k)}$ of $\pi^{(n-k)}$ is equal to the quotient

$$V/V(N_{n-k,A} \ltimes U_{k,n-k}, \theta^n_{|N_{n-k,A}} \otimes 1_{U_{k,n-k}})$$

where V is the space of π.
We denote by I_k the surjection obtained by facorisation from $V_{U_{k,n-k}}$ onto $V^{(n-k)}$. From Lemma 2.32 of [B-Z], the map I_k identifies with the projection

$$V_{U_{k,n-k}} \twoheadrightarrow (V_{U_{k,n-k}})_{N_{n-k,A}, \theta^n_{|N_{n-k,A}}} = V_{U_{k,n-k}}/V_{U_{k,n-k}}(N_{n-k,A}, \theta^n_{|N_{n-k,A}}).$$

The map I_k is in fact a G_k-modules morphism, because of the equality of modulus characters

$$(\delta_{U_{k,n-k}})_{|G_k} = (\delta_{U_{k+1}} \dots \delta_{U_n})_{|G_k}$$

which is itself a consequence of the decomposition

$$U_{k,n-k} = \prod_{i=k+1}^{n} (U_{k,n-k} \cap U_i).$$

The group $N_{A,n-k}$ being a union of compact subgroups, the map I_k preserves exact sequences. As the Jacquet module functor preserves finite length, the $G_k \times GL(n-k, F)$-module $\pi_{U_{k,n-k}}$ has a finite composition series $0 \subset (\pi_{U_{k,n-k}})_1 \subset \dots \subset (\pi_{U_{k,n-k}})_{r_k} = \pi_{U_{k,n-k}}$. We put $\pi_i^{(n-k)} = I_k[(\pi_{U_{k,n-k}})_i]$.
Hence $\pi_i^{(n-k)}/\pi_{i-1}^{(n-k)}$ is equal to $[(\pi_{U_{k,n-k}})_i/(\pi_{U_{k,n-k}})_{i-1}]_{N_{n-k,A}, \theta_{|N_{n-k,A}}}$, but as a $G_k \times GL(n-k, F)$-module, the quotient $(\pi_{U_{k,n-k}})_i/(\pi_{U_{k,n-k}})_{i-1}$ isomorphic to $\rho_1 \otimes \rho_2$ for irreducible representations ρ_1 and ρ_2 of G_k and $GL(n-k, F)$ respectively. Because the character θ^n restricts to $N_{n-k,A}$ as a nondegenerate character, the quotient $\pi_i^{(n-k)}/\pi_{i-1}^{(n-k)}$ is equal to $\rho_1 \otimes (\rho_2)_{N_{n-k,A}, \theta^n_{|N_{n-k,A}}}$, thus it is zero unless ρ_2 is generic, in which case it is equal to the irreducible representation ρ_1.
So we proved that $\pi^{(n-k)}$ has finite length as G_k-module, smaller than the length of the Jacquet module $\pi_{U_{k,n-k}}$ as a $G_k \times GL(n-k, F)$-module.

□

There is another property of the maps I_k defined in the proof of the preceding proposition that is worth mentioning, which is that their restriction to generalised characteristic subspaces is nonzero. More formally, let G be an l-group, and T be a closed abelian subgroup of G. If V is a smooth G-module, following [C], we define for each character χ of T, the T-submodule

$$V_{\chi,\infty} = \{v \in V | \ \exists n \in \mathbb{N}, \forall t \in T, \ (\tau(t) - \chi(t)Id)^n(v) = 0\}.$$

If V is T-finite (i.e. every vector in V generates a finite dimensional T-module), then it is the finite direct sum of its (nonzero by definition) generalised characteristic subspaces, and every such (nonzero) $V_{\chi,\infty}$ contains the nonzero generalised eigenspace

$$V_{\chi} = \{v \in V |\ \exists n \in \mathbb{N}, \forall t \in T,\ (\tau(t) - \chi(t)Id)(v) = 0\}.$$

First recall that smooth $(F^*)^r$-modules E, with a filtration $0 = E_0 \subset E_1 \subset \cdots \subset E_{r-1} \subset E_r = E$ such that $(F^*)^r$ acts by a character on each quotient are $(F^*)^r$-finite.

Lemma 2.1. *Let E be a smooth $(F^*)^r$-module E, with a filtration $0 = E_0 \subset E_1 \subset \cdots \subset E_{r-1} \subset E_r = E$ such that $(F^*)^r$ acts by a character c_{i+1} on each quotient E_{i+1}/E_i, then any vector of E lies in a finite dimensional $(F^*)^r$-submodule.*

Proof. One proves this by induction on the smallest i such that E_i contains v. If this i is 1, the group $(F^*)^r$ only multiplies v by a scalar, and we are done.
Suppose that the result is known for E_i, and take v in E_{i+1} but not in E_i. Then for every t in $(F)^*$, the vector $\tau(t)v - c_{i+1}(t)v$ belongs to E_i. By smoothness, the set $\{\tau(u)v\ |t \in (U_F)^r\}$ is actually equal to $\{\tau(u)v\ |u \in P\}$ for P a finite set of $(U_F)^r$. The vector space generated by this set is stabilized by $(U_F)^r$, and has a finite basis $v_1, \dots, v_m$. Now the vectors

$$\tau(1, \dots, 1, \varpi_F, 1, \dots, 1)v_l - c_{i+1}(1, \dots, 1, \varpi_F, 1, \dots, 1)v_l$$

belong to E_i, hence by induction hypothesis, to a finite dimensional $(F^*)^r$-submodule V_l of E_i. Finally the finite dimensional space $Vect(v_1, \dots, v_m) + V_1 + \cdots + V_m$ is stable under $(U_F)^r$ and the elements $(1, \dots, 1, \varpi_F, 1, \dots, 1)$, hence $(F^*)^r$, and contains v. □

This in particular applies to the Z_kZ_n-module $V_{U_{k,n-k}}$ and the Z_k-module V described in the proof of Proposition 2.5, as both are respectively $G_k \times GL(n-k, F)$ and G_k-modules of finite length.

Now we can prove the following property of the maps I_k:

Proposition 2.6. *Let (π, V) be a θ^n-generic representation of G_n, and for $k \geq 1$, let $U_{k,n-k}$ and $M_{k,n-k} \simeq G_k \times GL(n-k, F)$ the subgroups of G_n defined in the proof of Proposition 2.5. Let χ be a character of the central subgroup Z_kZ_n of $M_{k,n-k}$, and denote by the same letter its restriction to the central subgroup Z_k of G_k. If the generalised characteristic subspace $(V_{U_{k,n-k}})_{\chi,\infty}$ is nonzero, then the map I_k restricts non trivially to $(V_{U_{k,n-k}})_{\chi}$. In particular the space $V_{\chi}^{(n-k)}$ is nonzero.*

Proof. Suppose that the subspace $(V_{U_{k,n-k}})_{\chi,\infty}$ of $V_{U_{k,n-k}}$ is nonzero, hence $(V_{U_{k,n-k}})_{\chi}$ is nonzero. The space $(V_{U_{k,n-k}})_{\chi}$ is $M_{k,n-k}$-submodule of $V_{U_{k,n-k}}$, so it has finite length, hence it contains some irreducible representation $\rho_1 \otimes \rho_2$ of $M_{k,n-k}$. Hence $Hom_{M_{k,n-k}}(\rho_1 \otimes \rho_2, V_{U_{k,n-k}})$ is nonzero, but then from Bernstein's second adjointness theorem (see [Bu], Theorem 3), we deduce that V is a quotient of the representation $\rho_1 \times \rho_2$ parabolically induced from $\rho_1 \otimes \rho_2$. As V admits a nonzero Whittaker form, so does $\rho_1 \times \rho_2$, and from a classical result of Rodier (Theorem 7 of [R]), this implies that ρ_1 and ρ_2 are generic with respect to some nondegenrate character. As genericiticity doesn't depend on the character for $GL(n-k, F)$, we deuce that $I_k(\rho_1 \otimes \rho_2) = \rho_1$. Hence I_k restricts non trivially to $(V_{U_{k,n-k}})_{\chi}$, and the image $I_k[(V_{U_{k,n-k}})_{\chi}]$ contains ρ_1 which is a nonzero submodule of $V_{\chi}^{(n-k)}$. □

We will also need to know that, if k is an integer between 1 and $n-1$ and χ is a character of Z_k, then the Z_k-modules $(V^{(n-k)})_{\chi}$ and $(V'^{(n-k)})_{\chi}$ are nonzero at the same time. We already know from the previous proposition that this is equivalent to the fact that the Z_k-modules $(V'^{(n-k)})_{\chi}$ and $(V_{U_{k,n-k}})_{\chi}$ are nonzero at the same time, and that $(V'^{(n-k)})_{\chi}$ nonzero implies that $(V_{U_{k,n-k}})_{\chi}$ is nonzero.

Proposition 2.7. *If (π, V) is a θ^n-generic representation of G_n, and for $k \geq 1$, let $U_{k,n-k}$ and $M_{k,n-k} \simeq G_k \times GL(n-k,F)$ be the subgroups of G_n defined in the proof of Proposition 2.5. Let χ be a character of the central subgroup $Z_k Z_n$ of $M_{k,n-k}$, and denote by the same letter its restriction to the central subgroup Z_k of G_k, then the space $(V_{U_{k,n-k}})_\chi$ is nonzero if and only if the space $(V'^{(n-k)})_\chi$ is nonzero.*

Proof. We only need to prove that if $(V_{U_{k,n-k}})_\chi$ is nonzero, then the space $(V'^{(n-k)})_\chi$ is nonzero. So suppose that the $M_{k,n-k}$-module $(V_{U_{k,n-k}})_\chi$ is nonzero, it is of finite length, hence it contains an irreducible $M_{k,n-k}$-submodule ρ. Call $P_{k,n-k}$ the parabolic subgroup $M_{k,n-k}U_{k,n-k}$, and $P^-_{k,n-k}$ its opposite parabolic subgroup (with unipotent radical $(U_{k,n-k})^-$). We already saw that by Bernstein's second adjointness theorem, the induced representation $i_{P^-_{k,n-k}}^{G_n}(\rho)$ has π as a quotient, and therefore $i_{P^-_{k,n-k}}^{G_n}(\rho)$ is θ^n-generic. Then from Theorem 7 of [R], the $M_{k,n-k}$-module ρ is θ-generic, where θ is the restriction of θ^n to the unipotent radical of the Borel of $M_{k,n-k}$. Both have the same Whittaker model $W(\pi, \theta^n)$, i.e. the (unique up to scalar) Whittaker form on the space of $i_{P^-_{k,n-k}}^{G_n}(\rho)$ facorises through the projection from $i_{P^-_{k,n-k}}^{G_n}(\rho)$ to π. Let L^- be a nonzero θ-Whittaker form on the space of ρ, by Theorems 1.4 and 1.6 of [C-S], there is a nonzero Whittaker form L on the space of $i_{P^-_{k,n-k}}^{G_n}(\rho)$ whose restriction to the subspace $C_c^\infty(P^-_{k,n-k}\backslash P^-_{k,n-k}U_{k,n-k}, (\delta_{U_k^-})^{1/2}\rho)$ of functions with support in $P^-_{k,n-k}U_{k,n-k}$ is given by

$$f \mapsto \int_{U_{k,n-k}} L^-(f(u))\theta^{-1}(u)du.$$

We denote by $\bar{L}$ the Whittaker form on the space of π which lifts to L. In particular, for any $\bar{f}$ in the space of π, which is the image of f in $C_c^\infty(P^-_{k,n-k}\backslash P^-_{k,n-k}U_{k,n-k})$, one has

$$\bar{L}(\bar{f}) = \int_{U_{k,n-k}} L^-(f(u))\theta^{-1}(u)du.$$

Let v be a vector in the space of ρ, such that $L^-(v)$ is nonzero. Let K' be a compact subgroup of G_n, with Iwahori decomposition with respect to $P_{k,n-k}$, and such that $K' \cap M_{k,n-k}$ fixes v, then the function α equal to $u^- m u \mapsto \rho(m)v$ on $(U_{k,n-k})^- M_{k,n-k}(U_{k,n-k} \cap K')$, and zero outside, is well defined and belongs to the space $C_c^\infty(P^-_{k,n-k}\backslash P^-_{k,n-k}U_{k,n-k}, \rho)$. We denote by W_α the corresponding Whittaker function $g \mapsto \bar{L}(\pi(g)\bar{\alpha})$. If a belongs to the group Z_k, one has

$$\begin{aligned} W_\alpha(a) &= \int_{U_{k,n-k}} L^-(\alpha(ua))\theta^{-1}(u)du \\ &= \chi(a)\delta_{U_{k,n-k}}^{-1/2}(a) \int_{U_{k,n-k}} L^-(\alpha(a^{-1}ua))\theta^{-1}(u)du \\ &= \chi(a)\delta_{U_{k,n-k}}^{1/2}(a) \int_{U_{k,n-k}} L^-(\alpha(u))\theta^{-1}(aua^{-1})du \\ &= \chi(a)\delta_{U_{k,n-k}}^{1/2}(a)L^-(v) \int_{U_{k,n-k}} \theta^{-1}(aua^{-1})du \end{aligned}$$

In this last integral, u stays in the compact set $U_{k,n-k} \cap K'$, hence there is a (punctured) neighbourhood of zero in $Lie(Z_k) = F$, such that $a(U_{k,n-k} \cap K')a^{-1}$ is a subset of $Ker(\theta)$ when a belongs to this neighbourhood. Finally, up to multiplication of the function α by a scalar, one has

$$W_\alpha(a) = \chi(a)\delta_{U_{k,n-k}}^{1/2}(a)L^-(v)$$

whenever a is this neighbourhood of zero.
A similar computation gives the equality

$$W_\alpha(zg) = \chi(z)\delta_{U_{k,n-k}}(z)^{1/2}c_g$$

for z in Z_k in a neighbourhood of zero and g in G_k, where c_g is the constant $\int_{U_{k,n-k}} L^-(\alpha(ug))du$. Hence from Propositions 2.3, 2.4, and the equality $(\delta_{U_{k,n-k}})_{|G_k} = (\delta_{U_{k+1}} \dots \delta_{U_n})_{|G_k}$, we deduce that the vector $(\delta_{U_{k+2}} \dots \delta_{U_n})^{-1/2} W_\alpha$ in the space of $\pi'_{(n-k+1)}$ is such that its image in $\pi'^{(n-k)}$ is nonzero and belongs to the space $(\pi'^{(n-k)})_\chi$. This proves the proposition. □

A straightforward generalisation of the proof of the preceding proposition gives the following corollary.

Corollary 2.1. *Let G be the F-points of a quasi-split reductive group defined over F. Let P be a parabolic subgroup of G with a Levi subgroup M, and P^- its opposite subgroup with $P \cap P^- = M$. Let (π, V) be a smooth θ-generic representation of G, for some nondegenerate character θ of the unipotent radical U of a Borel subgroup of G contained in P. We denote by j_{P^-} be the map defined in Theorem 3.4 of [D] from $(V^*)^{U_\theta}$ to $(J_P(V)^*)^{M\cap U_\theta}$. If L is a nonzero vector of the line $(V^*)^{U_\theta}$, then the linear form $j_{P^-}(L)$ restricts non trivially to any irreducible M-submodule of $J_P(V)$ whenever the Jacquet module $J_P(V)$ is nonzero.*

We now come back to the study of F^*-modules E with finite factor series such that F^* acts by a character on each quotient. From Lemma 2.1, any vector of E will belong to a finite dimensional F^*-submodule E' as in:

Proposition 2.8. *If E' is a non zero finite dimensional F^*-submodule of E, then E' has a basis B in which the action of F^*r is given by a block diagonal matrix $Mat_B(\tau(t))$ with each block of the form:*

$$\begin{pmatrix} c(t) & c(t)P_{1,2}(v_F(t)) & c(t)P_{1,3}(v_F(t)) & \dots & c(t)P_{1,q}(v_F(t)) \\ & c(t) & c(t)P_{2,3}(v_F(t)) & \dots & c(t)P_{2,q}(v_F(t)) \\ & & \ddots & & \vdots \\ & & & c(t) & c(t)P_{q-1,q}(v_F(t)) \\ & & & & c(t) \end{pmatrix},$$

for c one of the c_i's, q a positive integer depending on the block, and the $P_{i,j}$'s being polynomials with no constant term of degree at most $j-i$.

Proof. First we decompose E' as a direct sum under the action of the compact abelian group U_F. Because E' has a filtration by the spaces $E' \cap E_i$, and that F^* acts on each sub factor as one of the c_i's, the group U_F acts on each weight space as the restriction of one of the c_i's. Now each weight space is stable under F^* by commutativity, and so we can restrict ourselves to the case where E' is a weight-space of U_F.
Again E' has a filtration, such that F^* acts on each sub factor as one of the c_i's (with all these characters having the same restriction to U_F), say $c_{i_1}, \dots, c_{i_k}$, in particular, we deduce that the endomorphism $\tau(\varpi_F)$ has a triangular matrix in a basis adapted to this filtration, with eigenvalues $c_{i_1}(\varpi_F), \dots, c_{i_k}(\varpi_F)$. As $\tau(\varpi_F)$ is trigonalisable, the space E' is the direct sum its characteristic subspaces, and again these characteristic subspaces are stable under F^*.
So finally one can assume that E' is a characteristic subspace for some eigenvalue $c(\pi)$ of $\tau(\varpi_F)$, on which U_F acts as the character c, where c is one of the c_i's.

Hence there is a basis B of E' such that

$$Mat_B(c^{-1}(t)\tau(t)) = \begin{pmatrix} 1 & A_{1,2}(t) & A_{1,3}(t) & \dots & A_{1,q}(t) \\ & 1 & A_{2,3}(t) & \dots & A_{2,q}(t) \\ & & \ddots & & \vdots \\ & & & 1 & A_{q-1,q}(t) \\ & & & & 1 \end{pmatrix}$$

for any t in F^*, where the $A_{i,j}$'s are smooth functions on F^*. So we only have to prove that the $A_{i,j}$'s are polynomials of the valuation of F with no constant term.

We do this by induction on q.
It is obvious when $q=1$. Suppose the statement holds for $q-1$, and suppose that E' is of dimension q, with basis $B=(v_1,\dots,v_q)$. Considering the two $c^{-1}\tau(F^*)$-modules $Vect(v_1,\dots,v_{q-1})$ and $Vect(v_1,\dots,v_q)/Vect(v_1)$ of dimension $q-1$, we deduce that for every couple (i,j) different from $(1,q)$, there is a polynomial with no constant term $P_{i,j}$ of degree at most $j-i$, such that $A_{i,j}=P_{i,j}\circ v_F$. Now because $c^{-1}\tau$ is a representation of F^*, and because the $P_{i,j}\circ v_F$'s vanish on U_F for $(i,j)\neq(1,q)$, we deduce that $A_{1,q}$ is a smooth morphism from $(U_F,\times)$ to $(\mathbb{C},+)$, which must be zero because $(\mathbb{C},+)$ has no nontrivial compact subgroups. From this we deduce that $A_{1,q}$ is invariant under translation by elements of U_F (i.e. $A_{1,q}(\varpi_F^k u)=A_{1,q}(\varpi_F^k)$ for every U in U_F).
Denote by $M(k)$ the matrix $Mat_B(c^{-1}\tau(\varpi_F^k))$ for k in $\mathbb{Z}$. One has $M(k)=M(1)M(k-1)$ for $k\geq 1$, which in implies $A_{1,q}(\varpi_F^k)=\sum_{j=2}^{q-1}P_{1,j}(1)P_{j,q}(k-1)+A_{1,q}(\varpi_F^{k-1})+A_{1,q}(\varpi_F)=Q(k)+A_{1,q}(\varpi_F^{k-1})+A_{1,q}(\varpi_F)$ for Q a polynomial of degree at most $q-2$. This in turn implies that $A_{1,q}(\varpi_F^k)=\sum_{l=1}^{k-1}Q(l)+kA_{1,q}(\varpi_F)=R(k)$ for R a polynomial of degree at most $q-1$, according to the theory of Bernoulli polynomials, for any $k\geq 0$. The same reasoning for $k\leq 0$, implies $A_{1,q}(\varpi_F^k)=R'(k)$ for R' a polynomial of degree at most $q-1$, for any $k\leq 0$. We need to show that $R=R'$ to conclude.
We know that $M(k)$ is a matrix whose coefficients are polynomials in k for $k>0$ of degree at most $q-1$, we denote it by $P(k)$. The matrix $M(k)$ has the same property for $k<0$, we denote it by $P'(k)$. Moreover for any $k\geq 0$ and $k'\leq 0$, with $k+k'\geq 0$, one has $P(k+k')=P(k)P'(k')$. Fix $k>q-1$, then the matrices $P(k+k')$ and $P(k)P'(k')$ are equal for k' in $[1-q,0]$, as their coefficients are polynomials in k' with degree at most $q-1$, the equality $P(k+z')=P(k)P'(z')$ holds for any complex number z'. Now fix such a complex number z', the equality $P(k+z')$ and $P(k)P'(z')$ holds for any integer $k>q-1$, and as both matrices have coefficients which are polynomials in k, this equality actually holds for any complex number z, so that $P(z+z')$ equals $P(z)P'(z')$ for any complex numbers z and z'.
As $P(0)=I_q$, we deduce that P and P' are equal on $\mathbb{C}$, and this implies that R is equal to R'. □

From this we deduce the following theorem, giving an expansion at infinity of Whittaker functions of generic representations of G_n, for $GL(n)$, the statement holds for ant smooth P_n-submodule of finite length of $C^\infty(N_n\backslash P_n)$:

Theorem 2.1. *Let θ be a nondegenerate character of the group N_n, let π be a θ-generic representation of G_n, and let $c_{1,n-k},\dots,c_{r_k,n-k}$ be the characters of Z_k appearing in a composition series of $\tau=\pi^{(n-k)}$. Then, for any function W in the space of π, the function*

$$W(z_1,z_2,...,z_{n-1})=W(z_1z_2,...z_{n-1})$$

is a linear combination of functions of the form

$$\prod_{k=1}^{n-1}[c_{i_k,k}\delta_{U_{k+1}}^{1/2}\dots\delta_{U_n}^{1/2}](z_k)v_F(z_k)^{m_k}\phi_k(z_k)$$

for i_k between 1 and r_k, positive integers m_k, and functions ϕ_k in $C_c^\infty(F)$.

Proof. Actually we prove the following stronger statement, which is satisfied by $\pi_{(0)}$ according to Proposition 2.5:

. *Let π be a submodule of $C^\infty(N_n\backslash P_n,\theta)$, such that for every k between 1 and $n-1$, the G_k-module $\tau=\pi^{(n-k)}=\Psi^-(\Phi^-)^{n-k-1}(\pi)$ has a composition series such that on each respective quotient, the central subgroup Z_k acts by the characters $c_{1,n-k},\dots,c_{r_k,n-k}$.*
Then, for any function W in the space of π, the function $W(z_1,z_2,...,z_{n-1})=W(z_1z_2,...z_{n-1})$ is a linear combination of functions of the form

$$\prod_{k=1}^{n-1}[c_{i_k,k}\delta_{U_{k+1}}^{1/2}\dots\delta_{U_n}^{1/2}](z_k)v_F(z_k)^{m_k}\phi_k(z_k)$$

for i_k between 1 and r_k, positive integers m_k, and functions ϕ_k in $C_c^\infty(F)$.

The proof is by induction on n.
Let W belong to the space of π. We denote by v its image in the space E of $\pi^{(1)}$. The vector v belongs to a finite dimensional Z_{n-1}-submodule E' of E, on which Z_{n-1} acts by a matrix of the form determined in Proposition 2.8. We fix a basis $B=(e_1,\dots,e_q)$ of E', and denote by $M(a)$ the matrix $M_B(\tau(a))$ (with a in Z_{n-1} and $\tau(a)=\pi^{(1)}(a)$), hence we have $\tau(a)e_l=\sum_{k=1}^q M(a)_{k,l}e_k$ for each l between 1 and q.

Taking preimages $\tilde{E}_1,\dots,\tilde{E}_q$ of $e_1,\dots,e_q$ in $\pi_{(0)}$, we denote by $\tilde{E}$ the function vector $\begin{pmatrix}\tilde{E}_1\\ \vdots\\ \tilde{E}_q\end{pmatrix}$.

If the image v of W in $\pi^{(1)}$ is equal to $x_1e_1+\dots+x_qe_q$, there is an integer M, such that for every $(z_1,\dots,z_{n-2})$ in $Z_1\times\dots\times Z_{n-2}$, the function

$$W(z_1,\dots,z_{n-1})-(x_1,\dots,x_q)\tilde{E}(z_1,\dots,z_{n-1})$$

vanishes for $|z_{n-1}|_F\le q_F^{-M}$. We denote by S the function $(x_1,\dots,x_q)\tilde{E}$.
Because of Remark 2.1, there is an integer M', such that for any $(z_1,\dots,z_{n-2})$ in $Z_1\times\dots\times Z_{n-2}$, and any z_{n-1} in Z_{n-1} of absolute value greater than $q_F^{M'}$, both $W(z_1,\dots,z_{n-1})$ and $S(z_1,\dots,z_{n-1})$ are zero, so that the difference $D(z_1,\dots,z_{n-1})$ of the two functions is a smooth function which vanishes whenever z_{n-1} has absolute value outside $[q_F^{-M},q_F^{M'}]$. Moreover there is a compact subgroup U of $Z_{n-1}(\mathfrak{O}_F)$ independent of $(z_1,\dots,z_{n-1})$ such that both functions (hence D) are invariant when z_{n-1} is multiplied by an element of U. Denoting by $(z_\alpha)_{\alpha\,\in\,A}$ a finite set of representatives for

$$\{z\;|q_F^{-M}\le|z_{n-1}|_F\le q_F^{M'}\}/U,$$

this implies that $D(z_1,\dots,z_{n-1})$ is equal to $\sum_{\alpha\,\in\,A}D(z_1,\dots,z_{n-2},z_\alpha)\mathbf{1}_{z_\alpha U}(z_{n-1})$, which we can always write as $\sum_{\alpha\,\in\,A}D(z_1,\dots,z_{n-2},z_\alpha)\delta_{U_n}^{1/2}(z_{n-1})D_\alpha(z_{n-1})$ with $D_\alpha=\delta_{U_n}^{-1/2}\mathbf{1}_{z_\alpha U}$ in $C_c^\infty(Lie(Z_{n-1}))$.
Each function $D(z_1,\dots,z_{n-2},z_\alpha)$ is equal to $W(z_1,\dots,z_\alpha)-S(z_1,\dots,z_\alpha)$, and the restrictions to P_{n-1} of the functions $\delta_{U_n}^{-1/2}[\pi(z_\alpha)D]$ belong to the smooth submodule $\Phi^-(\pi)$ of $C^\infty(N_{n-1}\backslash P_{n-1},\theta)$, which still satisfies the hypothesis of the statement.
Hence, by induction hypothesis, the function D is a sum of functions of the form

$$\prod_{k=1}^{n-1}[c_{i_k,k}\delta_{U_{k+1}}^{1/2}\dots\delta_{U_n}^{1/2}](z_k)v_F(z_k)^{m'_k}\phi'_k(z_k)$$

for i_k between 1 and r_k, null or positive integers or integer vectors m'_k, and functions ϕ'_k in $C_c^\infty(Lie(Z_k))$.

Now, call p the projection $W'\mapsto(\delta_{U_n}^{1/2}W')_{|P_{n-1}}$ from $\pi_{(0)}$ to $\pi^{(1)}$, then for any a in Z_{n-1}, one has $\rho(a)p(\tilde{E}_l)=\sum_{k=1}^q M(a)_{k,l}p(\tilde{E}_k)$. Hence as $\rho(a)p(\tilde{E}_l)$ equals $\delta_{U_n}^{-1/2}(a)\pi_{(0)}(a)\tilde{E}_l$, we deduce that there is a punctured neighbourhood of zero in Z_{n-1}, such that for each l, the function $\delta_{U_n}^{-1/2}(a)\pi_{(0)}(a)\tilde{E}_l-\sum_{k=1}^q M(a)_{k,l}\tilde{E}_k$ vanishes on elements $g=pac$ of G_{n-1} (p in P_{n-1}, a in Z_{n-1}, c in $G_{n-1}(\mathfrak{O}_F)$) such that a is in this neighbourhood.
In particular, there exists N_a such that for every $(z_1,\dots,z_{n-1})$, the vector function

$$\delta_{U_n}^{-1/2}(a)\pi_{(0)}(a)\tilde{E}(z_1,\dots,z_{n-1})-{}^tM(a)\tilde{E}(z_1,\dots,z_{n-1})$$

vanishes when we have $|z_{n-1}|_F\le q_F^{-N_a}$.

This implies, as in the proof of Proposition 2.6. of [C-P], the following claim:

Claim. *There is actually an M'', such that for every z in Z_{n-1}, with $|z_{n-1}|_F \leq q_F^{-M''}$, and every a in Z_{n-1}, with $|a|_F \leq 1$, the function $\tilde{E}(z_1, \dots, z_{n-1}a)$ is equal to $\delta_{U_n}^{1/2}(a)^t M(a)\tilde{E}((z_1, \dots, z_{n-1})$.*

Proof of the claim. We denote $(z_1, \dots, z_{n-2})$ by x, and z_{n-1} by z.
If U is an open compact subgroup of $Z_{n-1}(\mathfrak{O}_F)$, such that $\tilde{E}$ and the homomorphism $a \in Z_{n-1} \mapsto M(a) \in G_q(\mathbb{C})$ are U invariant, we denote by $u_1, \dots, u_s$ the representatives of $Z_{n-1}(\mathfrak{O}_F)/U$, and by ω, the canonical generator of $Z_{n-1}/Z_{n-1}(\mathfrak{O}_F)$. We put $M'' = \max_{i,j}(N_{u_i}, N_\omega)$.
Then for z in $\{z \in Z_{n-1}, |z|_F \leq q_F^{-M''}\}$, and $a = \omega^r u_i u$ in $\{z \in Z_{n-1}, |z|_F \leq 1\}$ (with u in U, and $r \in \mathbb{N}$), we have

$$\tilde{E}(x, za) = \tilde{E}(x, z\omega^r u_i) = \delta_{U_n}^{1/2}(u_i)^t M(u_i)\tilde{E}(x, z\omega^r)$$

because $z\omega^r$ belongs to $\{z \in Z_{n-1}, |z|_F \leq q_F^{-M''}\} \subset \{z \in Z_{n-1}, |z|_F \leq q_F^{-N_{u_i}}\}$. But if $r \geq 1$, again one has

$$\tilde{E}(x, z\omega^r) = \delta_{U_n}^{1/2}(\omega)^t M(\omega)(\omega_i)\tilde{E}(x, z\omega^{r-1}),$$

and $z\omega^{r-1}$ belongs to

$$\{z \in Z_{n-1}, |z|_F \leq q_F^{-N_2}\} \subset \{z \in Z_{n-1}, |z|_F \leq q_F^{-N_\omega}\},$$

and repeating this step, we deduce the equality $\tilde{E}(x, za) = \delta_{U_n}^{1/2}(a)^t M(a)\tilde{E}(x, z)$. ☐

Hence there is an element z_0 in Z_{n-1} with $|z_0|_F = q_F^{-M''}$, such that for every $(z_1, \dots, z_{n-2})$ in $Z_1 \times \dots \times Z_{n-2}$, the vector $\tilde{E}(z_1, \dots, z_{n-1})$ is equal to

$$\delta_{U_n}^{1/2}(z_{n-1})^t M(z_{n-1})(z_{n-1})[\delta_{U_n}^{-1/2}(z_0)^t M(z_0^{-1})]\tilde{E}(z_1, \dots, z_{n-2}, z_0)$$

for any z_{n-1} with $|z_{n-1}|_F \leq 1$.
Hence the function $\mathbf{1}_{\{|z_{n-1}|\leq 1\}}S(z_1, \dots, z_{n-1})$ is equal to

$$(x_1, \dots, x_q)^t M(z_{n-1})(z_{n-1})[\delta_{U_n}^{-1/2}(z_0)^t M(z_0^{-1})]\tilde{E}(z_1, \dots, z_{n-2}, z_0)\delta_{U_n}^{1/2}(z_{n-1})\mathbf{1}_{\{|z_{n-1}|\leq 1\}}.$$

One proves as for the function D, that function $\mathbf{1}_{\{|z_{n-1}|>1\}}(z_{n-1})S(z_1, \dots, z_{n-1})$ is of the form

$$\sum_{\beta \in B} S(z_1, \dots, z_{n-2}, z_\beta)\delta_{U_n}^{1/2}(z_{n-1})S_\beta(z_{n-1})$$

with S_β in $C_c^\infty(F)$ for some finite set B.
By induction hypothesis again, applied to the function $(\delta_{U_n}^{-1/2}\tilde{E}_i)(z_1, \dots, z_{n-2}, z_0)$ and the function $(\delta_{U_n}^{-1/2}S)(z_1, \dots, z_{n-2}, z_\beta)$, we deduce that the function $S = \mathbf{1}_{\{|z_{n-1}|\leq 1\}}S + \mathbf{1}_{\{|z_{n-1}|>1\}}S$ is a sum of functions of the form $\prod_{k=1}^{n-1} c_{i_k,n-k}(z_k)\delta_{U_{k+1}}^{1/2} \dots \delta_{U_n}^{1/2}(z_k)v_F(z_k)^{m_j''}\phi_k''(z_k)$ for i_k between 1 and r_k, null or positive integers or integer vectors m_k'', and functions ϕ_k'' in $C_c^\infty(F)$.
The statement follows as the function W equals $D + S$. ☐

3 $L^2(Z_nN_n\backslash G_n)$ and discrete series

First we characterise the Whittaker functions which belong to $\int_{N_n\backslash P_n} |W(p)|^2 dp$ in terms of exponents of the "shifted derivatives" (see [B], 7.2.). This result has been used in [M].
We say that a character of a F^* is positive if its (complex) absolute value, is of the form $|\ |_F^r$ for some positive real r.

Theorem 3.1. *Let θ be a nondegenerate character of the group N_n, and π be a θ-generic representation of G_n, let the $c_{1,n-k},\dots,c_{r_k,n-k}$ be the characters of Z_k appearing in a composition series of $\tau=\pi^{(n-k)}$. Then the integral*

$$\int_{N_n\backslash P_n} |W(p)|^2 dp$$

converges for any W in π if and only if all the characters $c_{i_k,k}\delta_{U_{k+1}}^{1/2}$ are positive for k between 1 and $n-1$.

Proof. Again we prove the stronger statement:

. *Let (π,V) be a P_n-submodule of $C^\infty(N_n\backslash P_n,\theta)$, such that for every k between 1 and $n-1$, the G_k-module $\tau=\pi^{(n-k)}=\Psi^-(\Phi^-)^{n-k-1}(\pi)$ has a composition series such that, on each respective quotient, the central subgroup Z_k acts by the characters $c_{1,n-k},\dots,c_{r_k,n-k}$. Then the integral*

$$\int_{N_n\backslash P_n} |W(p)|^2 dp$$

converges for any W in π if and only if all the characters $c_{i_k,k}\delta_{U_{k+1}}^{1/2}$ are positive for k between 1 and $n-1$.

Suppose first that all the characters $c_{i_k,k}\delta_{U_{k+1}}^{1/2}$ are positive. Let W belong to the space of π, first we notice the equality

$$\int_{N_n\backslash P_n} |W(p)|^2 dp = \int_{N_{n-1}\backslash G_{n-1}} |W(g)|^2 dg.$$

Now the Iwasawa decomposition reduces the convergence of this integral to that of

$$\int_{A_{n-1}} |W(a)|^2 \delta_{N_{n-1}}^{-1}(a) d^*a$$

Using coordinates $(z_1,\dots,z_{n-1})$ (see Lemma 1.1) of A_{n-1}, the function $\delta_{N_{n-1}}^{-1}(z_1,\dots,z_{n-1})$ is equal to $\prod_{k=1}^{n-2}(\delta_{U_{k+1}}\dots\delta_{U_{n-1}})^{-1}(z_k)$.
According to Theorem 2.1 the function $|W(z_1,\dots,z_{n-1})|^2$ is bounded by a sum of functions of the form

$$\prod_{k=1}^{n-1} |c_{i_k,k}|(z_k)|c_{l_k,k}|(z_j)(\delta_{U_k+1}\dots\delta_{U_n})(z_k)v_F(z_k)^{m_k}\phi_k(z_k).$$

Hence our integral will converge if the same is true of the integrals

$$\int_{A_{n-1}} \prod_{k=1}^{n-1} |c_{i_k,k}|(z_k)|c_{l_k,k}|\delta_{U_n}(z_k)v_F(t_k)^{m_k}\phi_k(z_k)dz_1\dots dz_n,$$

i.e. if the integrals $\int_{Z_k}|c_{i_k,k}|(z_k)|c_{l_k,k}|(z_k)\delta_{U_n}(z_k)v_F(t_k)^{m_k}\phi_k(t_k)dz_k$ converge for any k between 1 and $n-1$.
But the restriction of δ_{U_n} to Z_k is equal to $\delta_{U_{k+1}}$, so the convergence follows from our assertion on the characters $c_{i_k,k}\delta_{U_n}^{1/2}$.
Conversely, suppose that every W in $\pi_{(0)}$ belongs to the space $L^2(N_n\backslash P_n)$ corresponding to a right invariant measure on $N_n\backslash P_n$.
By Iwasawa decomposition, one gets that $\int_{N_n\backslash P_n}|W(p)|^2dp=\int_{N_{n-1}\backslash G_{n-1}}|W(g)|^2dg$ is equal to $\int_{A_{n-1}\times F_n}|W(ak)|^2\delta_{N_{n-1}}^{-1}(a)d^*adk$ which is greater than $dk(U)\int_{A_{n-1}}|W(a)|^2\delta_{N_{n-1}}^{-1}(a)d^*a$ for some compact open subgroup U fixing W. In particular the integral $\int_{A_{n-1}}|W(a)|^2\delta_{N_{n-1}}^{-1}(a)d^*a$ converges for any W in π.

This by Fubini's theorem and smoothness of W, implies that $\int_{A_{n-2}} |W(a)|^2 \delta_{N_{n-1}}^{-1}(a)d^*a$ is finite for any W in π. But the restriction of $\delta_{N_{n-1}}$ to A_{n-1} is equal to $\delta_{N_{n-2}}\delta_{U_{n-1}}$, so that the integral $\int_{A_{n-2}} |W(a)|^2 \delta_{U_{n-1}}^{-1}\delta_{N_{n-2}}^{-1}(a)d^*a$ is finite for W in π, which by Iwasawa decomposition again, implies that $\int_{N_{n-1}\backslash P_{n-1}} |\delta_{U_{n-1}}^{-1/2} W(p)|^2 dp$ is finite.
The functions $\delta_{U_{n-1}}^{-1/2} W$ belong to the space of $\phi^-(\pi)$, hence by induction, all the characters $c_{i_k,k}\delta_{U_{k+1}}^{1/2}$ are positive for $k \leq n-2$. So we only need to check that the characters $c_{i_{n-1},n-1}\delta_{U_n}^{1/2}$ are positive. Suppose that one of them, $c_{1,n-1}\delta_{U_n}^{1/2}$ for instance, wasn't.
Then, taking v nonzero in $\Psi^-(\pi)$ such that Z_{n-1} multiplies v by $c_{1,n-1}$, according to Proposition 2.3 and taking W a preimage of v in π, there is a positive integer N_a, such that $\delta_{U_n}^{-1/2}(a)\pi(a)W(g) - c_{1,n-1}(a)W(g)$ is zero whenever for any g in G_{n-1} with $|z(g)|_F \leq q_F^{-N_a}$. As in Claim 2, this implies that there is a positive integer N, such that $W(ag)$ is equal to $\delta_{U_n}^{1/2}(a)c_{1,n-1}(a)W(g)$ whenever $|z(g)|_F \leq q_F^{-N}$ and $|a|_F \leq 1$. We recall that W doesn't belong to $V(U_n, 1)$ (otherwise v would be zero), hence according to Proposition 2.3, there is g_0 in G_{n-1} with $|z(g_0)|_F \leq q_F^{-N}$, such that $W(g_0)$ is nonzero. We denote by W_0 the function $\pi(g_0)W$, and we recall that the integral

$$\int_{A_{n-1}} |W_0(a)|^2 \delta_{N_{n-1}}^{-1}(a)d^*a = \int_{Z_1\times\cdots\times Z_{n-1}} |W_0(z_1\ldots z_{n-1})|^2 \delta_{N_{n-1}}^{-1}(z_1\ldots z_{n-1})dz_1\ldots dz_{n-1},$$

is finite. Hence the smoothness of W_0 and Fubini's theorem imply that the integral

$$\int_{Z_{n-1}} |W_0(z_{n-1})|^2 \delta_{N_{n-1}}^{-1}(z_{n-1})dz_{n-1} = \int_{Z_{n-1}} |W_0(z_{n-1})|^2 dz_{n-1}$$

is finite. But for $|z_{n-1}|_F \leq 1$, the function $W_0(z_{n-1})$ is equal to $\delta_{U_n}^{1/2}(z_{n-1})c_{1,n-1}(z_{n-1})W(g_0)$ with $W(g_0)$ nonzero, hence it is square integrable at zero if and only if $\delta_{U_n}c_{1,n-1}^2$, thus $\delta_{U_n}^{1/2}c_{1,n-1}$ is positive.

□

Remark 3.1. The last proof more or less contains the following fact (which is more precisely a consequence of an induction, and the last step of the proof):
For every character $c_{i_k,n-k}$ appearing in a factor series of $\pi^{(n-k)}$, there is W in V, such that $W(z_k)$ is equal to $[c_{i_k,n-k}\delta_{U_{k+1}}^{1/2}\ldots\delta_{U_n}^{1/2}](z_k)$ near zero. Hence this family of characters is minimal in the sense that each of them must occur in the expansion given in Proposition 2.1 of some W in V.

From this we deduce a characterization of the Whittaker functions in $L^2(Z_nN_n\backslash G_n)$.

Corollary 3.1. *Let θ be a nondegenerate character of the group N_n, and π be a θ-generic representation of G_n with unitary central character, let the $c_{1,n-k},\ldots,c_{r_k,n-k}$ be the central characters appearing in the factor series of $\tau = \pi^{(n-k)}$. Then the integral*

$$\int_{Z_nN_n\backslash G_n} |W(g)|^2 dg$$

converges for any W in π if and only if all the characters $c_{i_k,k}$ are positive for k between 1 and $n-1$.

Proof. By the Iwasawa decomposition, the integral $\int_{Z_nN_n\backslash G_n} |W(g)|^2 dg$ converges for every W in $W(\pi,\theta)$ if and only if the $\int_{A_{n-1}} |W(a)|^2 \delta_{N_n}^{-1}(a)dg$ converges for every W in $W(\pi,\theta)$.
As the character δ_{N_n} restricts to G_{n-1} as $\delta_{N_{n-1}}\delta_{U_n}$, this integral is equal to

$$\int_{A_{n-1}} |\delta_{U_n}^{-1/2} W(a)|^2 \delta_{N_{n-1}}^{-1}(a)dg.$$

But this integral converges for any W in $W(\pi,\theta)$ if and only if so does the integral

$$\int_{N_n \backslash P_n} |\delta_{U_n}^{-1/2} W(p)|^2 dp$$

for any W in $W(\pi,\theta)$.
By the statement in the proof of theorem 3.1, applied to $\delta_{U_n}^{-1/2} \otimes \pi'$, this is the case if and only all the characters $c_{i_k,k}$ are positive for k between 1 and $n-1$.

□

Let P be a standard proper parabolic subgroup of G_n, U its unipotent radical, and M its standard Levi subgroup. If (π, V) is a smooth irreducible representation of G_n, one calls cuspidal exponent of π with respect to P, a character χ of the center of M such that the characteristic space of the Jacquet module $(V_U)_{\chi,\infty}$ is nonzero. Denoting by Δ the set of simple roots $\{\alpha_1, \dots, \alpha_n\}$ of G_n, We denote by $P^{\{i_1,\dots,i_t\}}$ the standard parabolic subgroup associated with the set of positive roots $\Delta - \{\alpha_{i_1}, \dots, \alpha_{i_t}\}$, by $U^{\{i_1,\dots,i_t\}}$ its unipotent radical, by $M^{\{i_1,\dots,i_t\}}$ its standard Levi subgroup, which admits as a central subgroup the product $Z_{i_1} \dots Z_{i_t}$.
Notice that except for case D, for $\{i_1, \dots, i_t\} = \{2\}$, where we used the notation $U_{2,n-2}$ for $U^{\{1,2\}}$, the group $U^{\{k\}}$ is what we already denoted by $U_{k,n-k}$ before.
We denote by $A^-_{i_1,\dots,i_t}$ the set

$$\{z_{i_1} \dots z_{i_t} \in Z_{i_1} \dots Z_{i_t},\ |z_{i_k}|_F \leq 1, \text{ and } |z_{i_1} \dots z_{i_t}|_F < 1\}.$$

Theorem 4.4.6 of [C] then asserts that π with unitary central character is a discrete series representation if and only if, for every standard parabolic subgroup $P^{\{i_1,\dots,i_t\}}$, if χ is a cuspidal exponent of π with respect to $P^{\{i_1,\dots,i_t\}}$, the restriction of χ to $A^-_{i_1,\dots,i_t}$ is less than 1, or equivalently if χ restricted to $Z_{i_1} \dots Z_{i_t}$ is positive.
We also notice that for any k, the Jacquet module $V_{U^{\{i_k\}}}$ surjects onto $V_{U^{\{i_1,\dots,i_t\}}}$, and that the character $\delta_{U^{\{i_1,\dots,i_t\}}}$ restricts to Z_{i_k} as $\delta_{U^{\{i_k\}}}$, hence if χ is a cuspidal exponent of π with respect to $P^{\{i_1,\dots,i_t\}}$, then $\chi_{|Z_{i_k}}$ is the restriction to Z_{i_k} of a cuspidal exponent of π with respect to $P^{\{i_k\}}$. This implies that π irreducible with unitary central character is a discrete series representation if and only if the cuspidal exponents of π with respect to maximal parabolic subgroups $P^{\{i_k\}}$ have positive restriction to Z_{i_k}.

We call a character χ of Z_k such that $(V^{(n-k)})_\chi$ (or equivalently $(V^{(n-k)})_{\chi,\infty}$) is nonzero an exponent of the derivative $(\pi^{(n-k)}, V^{(n-k)})$. Now we recall that we showed in Proposition 2.7, that the Z_k modules $V_{U_{k,n-k}}$ and $V^{(n-k)}$ have the same nonzero weight subspaces. This allows to prove in our four cases the following conjecture of Lapid and Mao ([L-M], conjecture 3.5).

Theorem 3.2. *Let π be a generic representation of G_n with unitary central character and with Whittaker model $W(\pi,\theta)$, then the following statements are equivalent:*

i) The integral

$$\int_{N_n Z_n \backslash G_n} |W(g)|^2 dg$$

converges for any W in $W(\pi,\theta)$.

ii) All the exponents of the derivatives of π are positive.

iii) the representation π is square-integrable.

Proof. By assumption, the exponents of the derivatives of π are the characters $c_{i_k,k}$ of corollary 3.1, hence i) $\Leftrightarrow$ ii) is corollary 3.1.
ii) $\Leftrightarrow$ iii): we treat the case D separately, so assume first that G_n is not $GSO(2(n-1), F)$.
By Proposition 2.6, every cuspidal exponent of π corresponding to $V_{U^{\{k\}}}$ is positive if and only if every exponent of the derivative $\pi^{(n-k)}$ is positive. But we have already seen that this implies

that π is a discrete series representation.
For the case D, we could have reversed the roles of the roots α_1 and α_2 (which correspond to the two symmetric roots at the end of the Dynkin diagram). The only effect it would have is to change the definition of the derivative functors $\pi^{(n-2)}$ and $\pi^{(n-1)}$. Indeed U_2 would become U_{α_2}, Z_1 and Z_2 would be exchanged. The character θ_3 would have to be trivial on U_{α_2} instead of being trivial on U_{α_1}. But i) and ii) would still be equivalent in this case, and i) is independent of these choices.
In both cases, the maps I_k from $V_{U_{k,n-k}}$ to $V^{(n-k)}$ take nonzero weight subspaces to nonzero weight subspaces. For $n \geq 3$, the space $V_{U_{k,n-k}}$ is equal to $V_{U\{k\}}$. In the first case, $V_{U_{1,n-1}}$ is equal to $V_{U\{1\}} = V_{U_{\alpha_1}}$, and it is equal to $V_{U\{2\}} = V_{U_{\alpha_2}}$ in the second case. This implies that all the exponents of the derivatives of π are positive if and only if all cuspidal exponents of π with respect to maximal parabolic subgroups are positive. Again this proves ii) $\Leftrightarrow$ iii). □

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