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# Modelling and Analysis of a Non-Synchronized Transport Network using Petri Nets and (Max, Plus) Algebra

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## Abstract

Modelling, performance analysis and control play a major role in the design of discrete event systems (DES). An adequate formal method to describe the behaviour of such systems is required. This paper deals with the modelling and the performance evaluation of a public transportation system. The behaviour of such system is studied in the framework of DES. Specifically, the study focuses on modelling of transportation system using the two complementary tools Petri nets (PN) and dioid algebra. In this study, we focus mainly on a logical modelling avoiding temporal synchronization phenomena of resources. The performance analysis of the considered network is made by solving and analysing the (max, +) state model which describes the analytical behaviour of the network. Journey times of passengers and calculation of shortest path from an origin to a destination will be considered in the stage of performance evaluation.

**Keywords:** Discrete Event systems, Transport Systems, Modelling, Petri Nets, Dioid algebra

## 1. INTRODUCTION

The conception and the planning (timetabling or resources assignment) of transport systems are essential tasks to ensure an optimized management of bus networks. Modelling, performance analysis and control of transportation networks are issues arousing an ever-increasing interest in many researches [7, 8, 9, 10]. These networks can be considered as complex large-scale systems which require dedicated tools for their study. In particular, the development of collective transportation mode is one key to improve urban quality of life, by significantly reducing various problems like traffic jams, pollution, and risks of accidents. To improve the performance of a public transportation mode, it is first necessary to evaluate and analyse the strength and weakness of the existing offer, so as to identify critical points of the network.

Transport systems can be considered as a discrete event system where the dynamic evolves in a space of discrete state, states changes are conditioned by events. The study of DES was the subject of several researches and developments [1, 2, 3, 4].

In this paper, we focus on modelling and evaluation analysis of a public transport network using two complementary tools Petri nets and (max, +) algebra. Our contribution is to propose a graphic-based model wherein temporal synchronization phenomena of buses in the connection stops are avoided. This hypothesis will not be considered in our modelling in order to avoid delays of buses on the lines and then enormous waiting times of passengers. By removing this hypothesis, the

buses don't have to wait for each other to exchange passengers. They can thus work independently. Another objective of our study is to propose a calculus method allowing the passengers to move from a given stop to a destination, with exchange of bus, with minimal waiting times and by using the shortest itinerary.

The strength and the feasibility of Petri nets, especially its class Timed Event Graph (TEG) [5, 6] for modelling and analysis of various class of DES (transportation systems, production systems, information systems,...) were proved in several researches [9, 11, 12] for planning bus operations in transportation network; and [13] to study a manufacturing system. To model and analyse thoroughly both the structural and analytical behaviour of DES, Petri nets are complemented by (max, +) equations in dioid algebra. The efficiency and the simplicity of this algebra in modelling, analysis, and performance evaluation were proved in several researches [1, 2, 7, 8]. For all these reasons our choice to study a bus network is based on these two complementary tools.

This paper is organised as follows: in section 2, we describe the studied transportation network. In section 3, we propose a graphic and (max, +)-based models of the network. An analytical study of network performances of this system by solving the (max, +) model is given in section 4. Last section gives some conclusions and suggestions for further researches.

## 2. DESCRIPTION OF STUDIED SYSTEM

First of all, we describe the system we will study in this paper. It is about a public transport network wherein several bus lines are connected. These connections, called *connection stops* allow every user of this means of transport to move from his departure stop to a desired destination.

In this study, we limit ourselves to a network consisting of four bus lines (Figure 1). We underline that the network considered in this paper is just an application example; the proposed modelling methodology can be generalized to other infrastructures and other transportation mode. In the considered infrastructure, each line contains several stops among which the connection and simple stops. On each line, buses circulate according to a defined timetable. Considering our objective which concerns the connections management of the network, we are only going to represent, in the modelling stage, the bus terminals (departure stops and terminuses) and connection stops. Other stops (simple stops) are integrated into the bus line (Figure 1). With each portion of itinerary, we associate a travelling time, which represents the necessary average time for a bus to move on the portion. For every line the following data are supposed to be fixed:

- The required time for a bus to perform a trip,
- The number of buses circulating on the line,
- Journey time between each two successive stops.

Furthermore, we suppose that the synchronization of buses is not considered. Every bus arriving in a stop leaves this last when passengers go up and/or get off the bus.

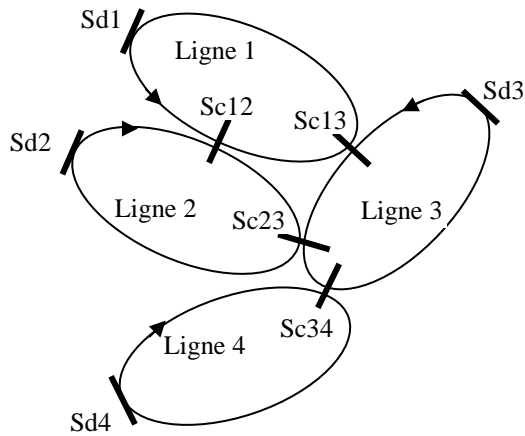


Figure 1. The considered network structure.

**Legend:**

Sdi ( $1 \leq i \leq 4$ ) : Departure stop of the line Li ;  
 Sc12 : connection stop between L1 and L2;  
 Sc13 : connection stop between L1 and L3;  
 Sc23 : connection stop between L2 and L3;  
 Sc34 : connection stop between L3 and L4.

### 3. GRAPHICAL AND MATHEMATICAL MODELING

#### 3.1 Motivation

To model the considered system, we choose a timed event graph which is a significant class of PN. As we already underlined, the choice of this graphical tool is justified by its powerful modelling of DES. It is also chosen for its helpful to facilitate the description of the behaviour of these systems by linear equations in dioid algebra.

In our study, we are interested in the management of passenger connections on the network. At each connection stop between two lines Li and Lj, we are interested at the same time in the passengers making the connection from Li to Lj and from Lj to Li. This situation can be expressed by a synchronization phenomenon in the stop. However this phenomenon represents a strong constraint in the modelling of the concrete functioning of buses on the network. In the railways networks, this problem is less present. Indeed, the trains are generally expected to be able to exchange the passengers. Their expectations are taken into account during the planning of their timetables. This situation produces the synchronization of trains at the railway station. Whereas, buses don't wait each other, at the connection stops to exchange passengers.

Two manners to model a connection stop in a bus network are given in figure 2. Let us note that in the graphic-based model that represents the network, transitions represent stops, places model portions of road between stops and tokens represent the buses

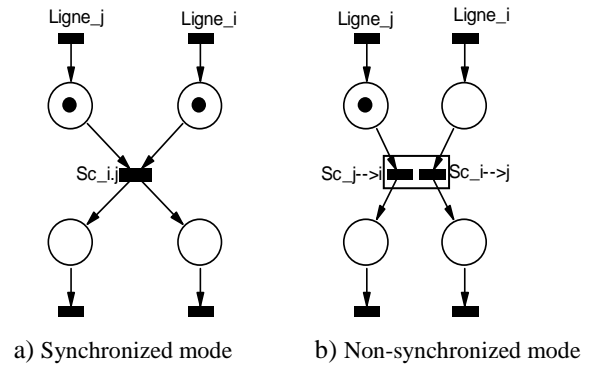


Figure 2: Modelling of a connection stop between two liens Li and Lj.

Our initiative is to manage the synchronization phenomenon of buses without imposing any undesirable waiting of vehicles. Figure 2 represents a portion of two lines related by a connection stop. In this figure, we represent, in two different ways, a connection stop between two lines.

Figure a) represents the connection stop renowned  $Sc_{i,j}$  between the line  $Li$  and the line  $Lj$ . According to the firing rules of Petri net transitions, the firing of the transition  $Sc_{i,j}$  cannot be done except when all its upstream places contain each one at least one token. Knowing that tokens represents buses, the departure of a bus from this stop can be done only if another bus circulating on the other line arrives at the same stop. This functioning mode is caused by the temporal synchronization phenomenon between buses.

In Figure b), we relax the synchronization constraint of buses. To avoid this constraint, the stop is represented by the couple of two transitions ( $Sc_{i \rightarrow j}$ ,  $Sc_{j \rightarrow i}$ ), called *composed transition*. The transition  $Sc_{i \rightarrow j}$  (resp.  $Sc_{j \rightarrow i}$ ) model the connection from the line  $Li$  to  $Lj$  (resp. the line  $Lj$  to  $Li$ ). In this situation, even if both transitions  $Sc_{i \rightarrow j}$  and  $Sc_{j \rightarrow i}$  represent the same stop, their firings are independent. If a bus of the line  $Li$  (resp. line  $Lj$ ) arrives at the stop  $Sc_{i \rightarrow j}$  (resp.  $Sc_{j \rightarrow i}$ ), it can leave this last one just when passengers go up and/or get off the bus without having to wait the arrival of a bus of the line  $Lj$  (resp.  $Li$ ). This modeling approach allows insuring a non-synchronized functioning of buses.

Using this approach, it is possible to model all connection stops of the network, avoiding the synchronization constraint of buses.

### 3.2 Graphic-based model

In this section, we give the event graph model that represents the behaviour network of the figure 1. Each line is represented by the departure stop, connection stop (s) and terminus.

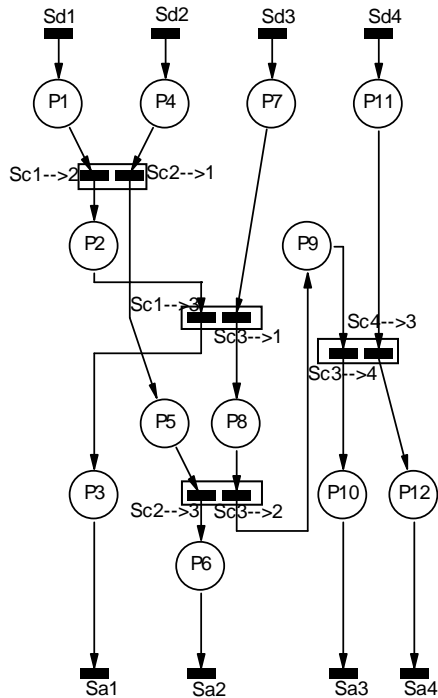


Figure 3: Event Graph Model of bus network.

As shown in figure 3, the four lines are represented as follows:

- The line  $L1$  :  $Sd1, Sc1 \rightarrow 2, Sc1 \rightarrow 3, Sa1$  ;
- The line  $L2$  :  $Sd2, Sc2 \rightarrow 1, Sc2 \rightarrow 3, Sa2$  ;
- The line  $L3$  :  $Sd3, Sc3 \rightarrow 1, Sc3 \rightarrow 2, Sc3 \rightarrow 4, Sa3$  ;
- The line  $L4$  :  $Sd4, Sc4 \rightarrow 3, Sa4$ .
- Different places  $P_i$  ( $1 \leq i \leq 12$ ) represent the portions of road between connection stops.

In the graphic-based model of figure 3, various connection stops are represented as follows:

$Sc12 = (Sc1 \rightarrow 2, Sc2 \rightarrow 1)$  ;  $Sc13 = (Sc1 \rightarrow 3, Sc3 \rightarrow 1)$  ;  
 $Sc23 = (Sc2 \rightarrow 3, Sc3 \rightarrow 2)$  ;  $Sc34 = (Sc3 \rightarrow 4, Sc4 \rightarrow 3)$ .

### 3.3 (Max, +)-based model

Let us start this section by giving some basic elements of  $(\max, +)$  algebra. Let us denote  $\mathbb{R}_{\max}$ , called *dioid*, the set  $\mathbb{R} \cup \{-\infty\}$  associated with the two laws  $\oplus$  and  $\otimes$ . These laws are defined by:  $\forall a, b \in \mathbb{R}_{\max}$  we have:

- $a \oplus b = \max(a, b)$
- $a \otimes b = a + b$  (often we write “ $ab$ ” instead of  $a \otimes b$ )

The law  $\oplus$  (resp.  $\otimes$ ) has a neutral element, denoted  $\varepsilon$  (resp.  $e$ ), for addition (resp. multiplication) :  $a \oplus \varepsilon = \varepsilon \oplus a = a$  (resp.  $a \otimes e = e \otimes a = a$ ). The element  $\varepsilon$  is absorbing for the law  $\otimes$  ( $a \otimes \varepsilon = \varepsilon \otimes a = \varepsilon$ ). The laws  $\oplus$  and  $\otimes$  have several properties and characteristics such as: associativity, distributivity, neutral elements, etc...

Solving  $(\max, +)$  equations is perfectly possible. Let us consider for example this type of equation:  $a \otimes x \oplus b = x$ , this implicit equation has as solution:  $x = a^* \otimes b$ , with  $a^* = e \oplus a \oplus a^2 \oplus \dots \oplus a^n \oplus \dots$  (Kleene star). When  $a$  and  $b$  are matrices of orders respectively  $(nxn)$  and  $(nx1)$  we come down to the same resolution but in the dioid matrix  $\mathbb{R}_{\max}^{nxn}$ , ( $a^* = Id \oplus a \oplus a^2 \oplus \dots \oplus a^n \oplus \dots$ ) with

$Id \in \mathbb{R}_{\max}^{nxn}$  is the identity matrix. More details about this algebra can be found in [1].

With assigning variables to various transitions and temporisations to various places of the model, we can describe its behaviour by mathematical equations in  $(\max, +)$  algebra. To do this, we assign to every transition one variable (Table 1).

The temporization  $t_i$  associated with the place  $P_i$  represents the time needed to move between two stops (these stops are represented by both upstream and downstream transition of the place  $P_i$ ).

In the stage of the mathematical modelling, the figure 3 is replaced by the figure 4 wherein different variables and temporisations are presented.

		Transitions	Variables
Input system		Sd1	u1
		Sd2	u2
		Sd3	u3
		Sd4	u4
Internal state of the system	connection L1 - L2	Sc1→2	x1
		Sc2→1	x2
	Connection L1 - L3	Sc1→3	x3
		Sc3→1	x4
	Connection L2 -L3	Sc2→3	x5
		Sc3→2	x6
	Connection L3 - L4	Sc3→4	x7
		Sc4→3	x8
Output system		Sc1	y1
		Sc2	y2
		Sc3	y3
		Sc4	y4

Table 1: Definition of model variables.

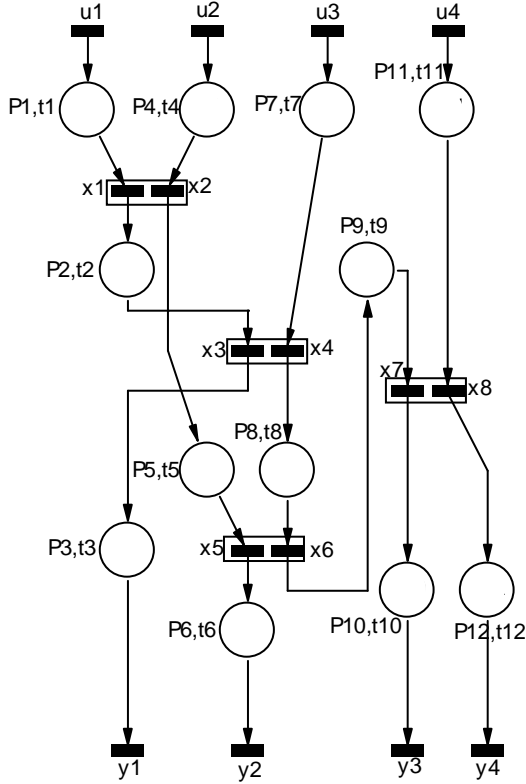


Figure 4: TEG model: definition of variables and temporization.

In order to describe the behaviour of the graphic-based model by (max, +) equations, we associate with each variable  $x_i$ , for  $1 \leq i \leq 8$  (resp.  $u_i$  and  $y_i$ , for  $1 \leq i \leq 4$ ), the parameter  $x_i(k)$  (resp.  $u_i(k)$  and  $y_i(k)$ ). These parameters, called *dates*, represent the dates of  $k^{\text{th}}$  firing of the associated transition.

The (max, +)-based model describing the analytical behaviour of the studied network, is given by (1).

$$\forall k \geq 1, \quad \begin{cases} x1(k) = t1 \otimes u1(k) \\ x2(k) = t4 \otimes u2(k) \\ x3(k) = t2 \otimes x1(k) \\ x4(k) = t7 \otimes u3(k) \\ x5(k) = t5 \otimes x2(k) \\ x6(k) = t8 \otimes x4(k) \\ x7(k) = t9 \otimes x6(k) \\ x8(k) = t11 \otimes u4(k) \\ y1(k) = t3 \otimes x3(k) \\ y2(k) = t6 \otimes x5(k) \\ y3(k) = t10 \otimes x7(k) \\ y4(k) = t12 \otimes x8(k) \end{cases} \quad (1)$$

By defining the input vectors  $U(k)$ , the state vectors  $X(k)$  and the output vectors  $Y(k)$  relative to the considered system, we put the system (1) in a matrix form.

Let us note:

$U(k) = [u1(k), u2(k), u3(k), u4(k)]^t$  : input vector;

$X(k) = [x1(k), x2(k), \dots, x8(k)]^t$  : state vector ;

$Y(k) = [y1(k), y2(k), y3(k), y4(k)]^t$  : output vector.

Using these three vectors, the system (1) can be written as follows:

$$\forall k \geq 1, \quad \begin{cases} X(k) = A \otimes X(k) \oplus B \otimes U(k) \\ Y(k) = C \otimes X(k) \end{cases} \quad (2) \quad (3)$$

The characteristic matrices  $A$ ,  $B$  and  $C$  ( $A \in \mathbb{R}_{\max}^{8 \times 8}$ ,  $B \in \mathbb{R}_{\max}^{8 \times 4}$  and  $C \in \mathbb{R}_{\max}^{4 \times 8}$ ) are expressed according to the system data.

From the equation (2) it is possible to calculate and evaluate all states of the system at any time. To do this, we proceed to solve this equation. So we replace by an iterative way the expression of  $X(k)$  by its value (right member of (2)). Thus, we obtain:

$$\begin{aligned} \forall k \geq 1, \\ X(k) &= A \otimes X(k) \oplus B \otimes U(k) \\ &= A \otimes (A \otimes X(k) \oplus B \otimes U(k)) \\ &= \dots \\ &= A^n X(k) \otimes (A^{n-1} \oplus A^{n-2} \oplus \dots \oplus A \oplus \text{Id}) \otimes B \otimes U(k) \\ &= A^* \otimes B \otimes U(k) \end{aligned} \quad (4)$$

Where:

- $A^* = \varepsilon$  because the TEG model of the figure 4 does not contain strong connex components.
- $A^* (= A^{n-1} \oplus A^{n-2} \oplus \dots \oplus A \oplus \text{Id})$  is the quasi-inverse of the matrix  $A$ , called also *star of Kleene*.

Using (4), the expression of the output (equation (3)) is given by:  $\forall k \geq 1$ ,

$$Y(k) = C \otimes A^* \otimes B \otimes U(k) \quad (5)$$

From the equation (5), we can evaluate different arrival times of buses at each terminus.

#### 4. EVALUATION AND ANALYSIS OF TRAVELING TIMES

From every couple of equations of the system (1) and by using the solution given in (4), it is possible to evaluate the journey times (connection times + travel times) of passengers on the network. From this evaluation, we try to define the shortest itinerary for a given passenger who has several possibilities to reach a desired destination.

Considering , for example, the connection stop Sc12, the waiting time for passengers who want to make a connection from L1 to L2 and L2 to L1 are given respectively by  $T_{1,2}$  and  $T_{2,1}$ . These times are expressed as:  $\forall k \geq 1$ ,

$$\begin{cases} T_{1,2}(k) = x_2(k_2) - x_1(k) \\ T_{2,1}(k) = x_1(k_1) - x_2(k) \end{cases} \quad (6)$$

With:

$$\begin{aligned} k_1 &= \inf_j \{j \in \mathbb{N} \text{ such as } x_1(j) \geq x_2(k)\} \\ k_2 &= \inf_j \{j \in \mathbb{N} \text{ such as } x_2(j) \geq x_1(k)\} \end{aligned}$$

In the same way, we calculate the other waiting times in different connection stops.

As we have already mentioned, we try to define the shortest itinerary for a passenger who travels on the network by making one or several connections. For example, we choose a passenger who wants to move from Sd1 (departure stop of L1) to Sa4 (terminus of L4). In this study, we are going to evaluate the travel times of this passenger on the various possible itineraries and for different departure times from Sd1. This evaluation is based on (max, +) solution given in (4) and by calculating waiting times of passengers in connections stops as in (6) and (7). By observing the network, the possible itineraries from Sd1 to Sa4 are:

$$1- Sd1 \rightarrow Sc12 \rightarrow Sc23 \rightarrow Sc34 \rightarrow Sa4$$

Or

$$2- Sd1 \rightarrow Sc13 \rightarrow Sc34 \rightarrow Sa4$$

Therefore, we are going to evaluate the travelling time for these two itineraries for a working period of buses.

If we consider a working period of buses which corresponds to the time interval  $[\alpha, \beta]$ . We suppose that the departure dates of buses from their departure stops are include in this interval.

- *First itinerary:*

The various connection times of a passenger arriving at Sd1 at the time “t” are given by:

$$\forall k \in \{j \in \mathbb{N} \text{ such as } u_1(j) \in [\alpha, \beta]\},$$

$$\begin{aligned} T_{1,2}^1(k, t) &= x_2(k_2) - x_1(k) \\ T_{2,3}^1(k, t) &= x_6(k_3) - x_5(k_2) \\ T_{3,4}^1(k, t) &= x_8(k_4) - x_7(k_3) \end{aligned}$$

With:

$$\begin{aligned} k_2 &= \inf_j \{j \in \mathbb{N} \text{ such as } x_2(j) \geq x_1(k)\} \\ k_3 &= \inf_j \{j \in \mathbb{N} \text{ such as } x_6(j) \geq x_5(k_2)\} \\ k_4 &= \inf_j \{j \in \mathbb{N} \text{ such as } x_8(j) \geq x_7(k_3)\} \end{aligned}$$

The sum of these waiting times gives the global waiting time of the passenger, noted  $T_{\text{waiting}}^1(k, t)$  to go from Sd1 to Sa4. So we obtain:

$$T_{\text{waiting}}^1(k, t) = T_{1,2}^1(k, t) + T_{2,3}^1(k, t) + T_{3,4}^1(k, t) \quad (8)$$

- *Second itinerary:*

The connection times in this case are given by :

$$\forall k \in \{j \in \mathbb{N} \text{ such as } u_1(j) \in [\alpha, \beta]\},$$

$$\begin{aligned} T_{1,3}^2(k, t) &= x_4(k'3) - x_3(k) \\ T_{3,4}^2(k, t) &= x_8(k'4) - x_7(k'3) \end{aligned}$$

With:

$$\begin{aligned} k'3 &= \inf_j \{j \in \mathbb{N} \text{ such as } x_4(j) \geq x_3(k)\} ; \\ k'4 &= \inf_j \{j \in \mathbb{N} \text{ such as } x_8(j) \geq x_7(k'3)\} ; \end{aligned}$$

As a result, the sum of waiting times for this second itinerary is:

$$T_{\text{waiting}}^2(k, t) = T_{1,3}^2(k, t) + T_{3,4}^2(k, t) \quad (9)$$

By comparing the two results in (8) and (9), we choose, for each date “t”, the shortest itinerary that must be taken by passengers.

#### 5. CONCLUSION

In this paper we described the modelling and the evaluation of a public transport network, in a first time, with a subclass of Petri Nets called dynamic timed event graph, and in a second time with the mathematical approach (max, +) algebra. Those tools enabled us to model our transportation network. A modelling approach is proposed to avoid synchronised working mode of buses. Without considering this working mode, the buses can be circulating without enormous waiting times at the connections stops. Buses are not constrained to wait each other at the connection stops.

By solving and analysing the (max, +) model, we evaluated some performance of bus network (in terms of connection and travelling times of passengers). We showed also, how to calculate a shortest itinerary on the network basing on these analytical evaluations. The prospect of this work is the extension of the obtained results to more complex transport systems, for example multimodal networks.

## REFERENCES

- [1] Baccelli F., Cohen G., Olsder G-J. et Quadrat J-P., 1992. *Synchronisation and linearity, Algebra for discrete Event Systems*. John Wiley et sons Ltd.
- [2] Cohen G, Dubois D, Quadrat J-P, Viot M. A linear-system-theoretic view of discrete-event process and its use for performance evaluation in manufacturing. *IEEE Transaction on Automatic Control* 1985 ; 30 ; 3; 210-220.
- [3] Gaubert S. Systèmes dynamiques à événements discrets. Lecture notes, ENSMP, Automation and Master degree ATS Orsay, INRIA. France: Rocquencourt ; 1999.
- [4] De Schutter B. Max-Algebra system theory for discrete event systems. Ph.D thesis of Katholieke Universiteit Leuven Heverlee 1996.
- [5] David R. et Alla H. , 1992. *Du grafset aux réseaux de pétri*. Paris: Hermes.
- [6] C.A. Petri, Kommunikation mit automaten, PhD thesis, Institut für instrumentelle Mathematik, Bonn, 1962.
- [7] Olsder G-J. , Subbiono et Gettrick M-M. ,1998. 26<sup>ème</sup> école de printemps d'informatique, Algèbre Max-Plus et applications en informatique: *On large scale max-plus algebra model in railway systems*, 4-7 mai 1998, p. 177-192, Ile de Noirmoutier, Vendée, France.
- [8] A. Nait-Sidi-Moh, M.-A. Manier and A. El Moudni. Spectral analysis for performance evaluation in a bus network. *European Journal of Operational Research* Volume 193, Issue 1, Pages 289-302. 16 February 2009.
- [9] Nait-Sidi-Moh A. Modélisation et commande de systèmes de transport avec l'algèbre des dioïdes. PhD thesis in Automation and Computer Science, University of Franche-Comté, Belfort, december 2003.
- [10] Bussieck, Michael R, Winter T, Zimmermann, Uwe T. Discrete optimization in public rail transport. *Mathematical Programming* 1997; 79; 3; 415-444.
- [11] Houssin L, Lahaye S, Boimond J-L. Timetable synthesis using (max, +) algebra. *Proceeding of the 12th IFAC Symposium Information Control Problems in Manufacturing*. France; St-Etienne; 2006; 375-380.
- [12] Karlaftis M, Kepaptsoglou K, Stathopoulos A. Decision Support Systems For Planning Bus Operations During Mega Events: The Athens 2004 Summer Olympics. *Proceeding of the 11th IFAC Symposium on Control in Transportation Systems*. The Netherlands; Delft; 2006. CD-ROM (189).
- [13] Proth, J.-M., Xie, X. . Les réseaux de Petri pour la conception et la gestion des systèmes de production. Masson 1995.