Coupled damage-plasticity constitutive model and direct stress interpolation

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Computational Mechanics manuscript No.
(will be inserted by the editor)

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Received: date / Accepted: date

Abstract In this paper we develop the governing equations of the coupled damage-plasticity model, which is capable of representing the main mechanisms of inelastic behavior including irreversible plastic deformation, change of elastic response and the localized failure. We show in particular how such model should be implemented within the stress-based variational formulation, providing an important advantage for local computation of the internal variables, which thus remains very robust and even non-iterative for the case of linear hardening model. Several simple examples are presented in order to illustrate the kind of response the model can represent.

Keywords coupled damage-plasticity · stress interpolation · cyclic loading

1 Introduction

It is often the case that the basic phenomenological models of inelastic behavior, on one side plasticity and on another damage, cannot represent in a reliable manner all the salient phenomena observed in inelastic behavior of real materials. In other words, for a number of applications one needs not only a reliable representation of irreversible deformation upon unloading as provided by the plasticity model [2] [11] [18] but also the elastic response modification upon unloading such as provided by the damage model [8] [10]. The case in point concerns the inelastic behavior of porous materials [1] [13] or more general class of the inelastic materials under cyclic loading. For any such case, the minimum requirement we need for representing with irreversible deformation and change of elastic response leads to a coupled damage-plasticity model [7] [9] [12] [16]. The class of coupled damage-plasticity models studied in this work is even more general from the initial models of this kind proposing plasticity criterion in terms of damage-modified effective stresses [7] [9] [16] in that it accommodates the independent criteria, the first for triggering the evolution of the irreversible deformation as opposed to the second governing the evolution of the elastic response modification [4] [6]. The main objective of this work is to discuss the theoretical formulation of such a coupled damage-plasticity model, as well as the numerical implementation. We show how to build the corresponding strain energy for such model and how to compute the evolution of its internal variables over a typical time step of the incremental / iterative scheme. We show in particular that the stress-based interpolation can lead to a very robust numerical implementation where the internal variables computation is guaranteed to converge. The latter is thus an additional advantage to what has been illustrated previously [17] about a superior accuracy of stress-based finite element approximations.

The outline of the paper is as follows. In the next section we briefly recall the governing equations of the coupled damage-plasticity model [4] [6] and its novel stress-based variational formulation. In section three, we present the local computation of the internal variables, which turns out to be very robust and even non-iterative for the case of linear hardening model. A couple of illustrative examples are presented in section four and concluding remarks are stated in section five.

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2 Variational formulation of coupled damage-plasticity model

2.1 Governing equations of the constitutive model

In this section we first present the governing equations for this constitutive model of coupled damage-plasticity. We show in particular that all these equations can be derived from three main ingredients: additive split of the total deformation field, the strain energy and yield / damage criteria, along with the principle of maximum dissipations for damage and plasticity. More precisely, we first assume that the total deformation ε can be split additively into elastic part ε^e, plastic part ε^p and damage part ε^d to write:

$$\varepsilon = \varepsilon^e + \varepsilon^p + \varepsilon^d \tag{1}$$

Contrary to the plastic deformation which is the main internal variable for plasticity, the damage deformation is just the vehicle for connecting two models and the main internal damage variable still remains the damage compliance D through the result $\varepsilon^d = D\sigma$ which will be proved shortly afterwards.

We can postulate the strain energy of the coupled damage-plasticity model according to:

$$\begin{aligned}\psi(u,\sigma,\varepsilon^p,D,\xi^p,\xi^d) = \\ \psi^e(\varepsilon^e) + \psi^d(\varepsilon^d,D) + \Xi^p(\xi^p) + \Xi^d(\xi^d)\end{aligned} \tag{2}$$

where, for generality, we accounted for eventual hardening effects with ξ^p and ξ^d as hardening variables for plasticity and damage.

The plasticity and damage mechanisms of inelastic behavior are activated, respectively, for a zero-value of plasticity and damage criterion:

$$\phi^p(\sigma,q^p) = 0 \ , \ \phi^d(\sigma,q^d) = 0 \tag{3}$$

In equation (3) above, q^p and q^d are the stress-like variables which control the evolution of the plasticity and damage thresholds as a function of hardening variables ξ^p and ξ^d, respectively.

Model problem in 1D setting which we choose in order to clearly illustrate the developments to follow considers a simple quadratic form of strain energy and a linear isotropic hardening with:

$$\begin{aligned}
&\psi^e(\varepsilon^e) = \sigma\varepsilon^e - \chi^e(\sigma)\\
&\text{with} \quad \chi^e(\sigma) = \frac{1}{2}\sigma E^{-1}\sigma\\
&\psi^d(\varepsilon^d,D) = \sigma\varepsilon^d - \chi^d(\sigma,D)\\
&\text{with} \quad \chi^d(\sigma,D) = \frac{1}{2}\sigma D\sigma\\
&\Xi^p(\xi^p) = \frac{1}{2}\xi^p K^p \xi^p\\
&\Xi^d(\xi^d) = \frac{1}{2}\xi^d K^d \xi^d\\
&\phi^p(\sigma,q^p) = |\sigma| - (\sigma_y - q^p)\\
&\phi^d(\sigma,q^d) = |\sigma| - (\sigma_f - q^d)
\end{aligned} \tag{4}$$

In equations (4) above, E is Young's modulus, K^p and K^d are plastic and damage hardening moduli, whereas σ_y and σ_f are yield and fracture limits, respectively. We note in passing that no essential restrictions are introduced with 1D case, and very much the same development is followed for 2D or 3D cases.

The total dissipation produced by this coupled damage-plasticity model, which must remain non-negative, can be written by appealing to the second principle of the thermodynamics [11]:

$$\begin{aligned}
0 \le \dot{\mathscr{D}} =& \sigma\dot{\varepsilon} - \dot{\psi}\\
=& \dot{\sigma}\Big(\frac{\partial\chi^e}{\partial\sigma} - \varepsilon^e\Big) + \dot{\sigma}\Big(\frac{\partial\chi^d}{\partial\sigma} - \varepsilon^d\Big) +\\
& \underbrace{\sigma\dot{\varepsilon}^p - \frac{\partial\Xi^p}{\partial\xi^p}\dot{\xi}^p}_{\dot{\mathscr{D}}^p} + \underbrace{\frac{\partial\chi^d}{\partial D}\dot{D} - \frac{\partial\Xi^d}{\partial\xi^d}\dot{\xi}^d}_{\dot{\mathscr{D}}^d}
\end{aligned} \tag{5}$$

The last statement leads to two possible interpretations:

1. Elastic process which is characterized by the frozen values of the internal variables with $\dot{\varepsilon}^p = 0$, $\dot{\xi}^p = 0$, $\dot{D} = 0$ and $\dot{\xi}^d = 0$, which also implies that plastic and damage dissipations $\dot{\mathscr{D}}^p = 0$ and $\dot{\mathscr{D}}^d = 0$. We thus obtain from equations (5) the constitutive equations for the stress, the definition of the damage strain as well as the hardening variables q^p and q^d according to:

$$\begin{aligned}
\dot{\mathscr{D}} = 0 \Rightarrow & \varepsilon^e = \frac{\partial\chi^e}{\partial\sigma} = E^{-1}\sigma\\
& \varepsilon^d = \frac{\partial\chi^d}{\partial\sigma} = D\sigma\\
& q^p = -\frac{\partial\Xi^p}{\partial\xi^p} = -K^p\xi^p\\
& q^d = -\frac{\partial\Xi^d}{\partial\xi^d} = -K^d\xi^d
\end{aligned} \tag{6}$$

2. By assuming the last results to remain valid for the inelastic process, we can obtain the corresponding interpretation of the inelastic dissipation for an inelastic process where internal variables evolution takes place:

$$0 < \dot{\mathscr{D}} = \underbrace{\sigma\dot{\varepsilon}^p + q^p\dot{\xi}^p}_{\dot{\mathscr{D}}^p} + \underbrace{\sigma\dot{\varepsilon}^d + q^d\dot{\xi}^d}_{\dot{\mathscr{D}}^d} \tag{7}$$

The only remaining hypothesis which is needed is the one based on maximizing the dissipation in any such inelastic process. The latter can be set as the corresponding constrained minimization problem and handled by the Lagrange multiplier method [6] [19] according to:

$$\begin{aligned}
& \min_{\phi^p(\sigma,q^p)=0;\phi^d(\sigma,q^d)=0} \quad [-\mathscr{D}^p(\sigma,q^p) - \mathscr{D}^d(\sigma,q^d)]\\
& \Rightarrow \max_{\dot{\gamma}^p,\dot{\gamma}^d} \min_{\forall\sigma,q^p,q^d} [\mathscr{L}^p(\sigma,q^p) + \mathscr{L}^d(\sigma,q^d)]\\
& \mathscr{L}^p(\sigma,q^p) = -\mathscr{D}^p(\sigma,q^p) + \dot{\gamma}^p \cdot \phi^p(\sigma,q^p)\\
& \mathscr{L}^d(\sigma,q^d) = -\mathscr{D}^d(\sigma,q^d) + \dot{\gamma}^d \cdot \phi^d(\sigma,q^d)
\end{aligned} \tag{8}$$

The Kuhn-Tucker optimality conditions for those kind of minimization problem can then be written providing the evolution equations for all the internal variables along with the loading / unloading conditions for plasticity and damage components:

$$
\begin{aligned}
&0=\frac{\partial \mathscr{L}^p}{\partial \sigma}=-\dot{\varepsilon}^p+\dot{\gamma}^p\frac{\partial \phi^p}{\partial \sigma}\Rightarrow \dot{\varepsilon}^p=\dot{\gamma}^p\frac{\partial \phi^p}{\partial \sigma}\\
&0=\frac{\partial \mathscr{L}^p}{\partial q^p}=-\dot{\xi}^p+\dot{\gamma}^p\frac{\partial \phi^p}{\partial q^p}\Rightarrow \dot{\xi}^p=\dot{\gamma}^p\frac{\partial \phi^p}{\partial q^p}\\
&0=\frac{\partial \mathscr{L}^p}{\partial \dot{\gamma}^p}=\phi^p\\
&0=\frac{\partial \mathscr{L}^d}{\partial \sigma}=-\dot{D}\sigma+\dot{\gamma}^d\frac{\partial \phi^d}{\partial \sigma}\Rightarrow \dot{D}\sigma=\dot{\gamma}^d\frac{\partial \phi^d}{\partial \sigma}\\
&0=\frac{\partial \mathscr{L}^d}{\partial q^d}=-\dot{\xi}^d+\dot{\gamma}^d\frac{\partial \phi^d}{\partial q^d}\Rightarrow \dot{\xi}^d=\dot{\gamma}^d\frac{\partial \phi^d}{\partial q^d}\\
&0=\frac{\partial \mathscr{L}^d}{\partial \dot{\gamma}^d}=\phi^d
\end{aligned}
\tag{9}
$$

By admitting that a negative value of yield and damage criteria corresponds to the elastic process, we can write a generalized form of the loading / unloading conditions with:

$$
\begin{aligned}
&\dot{\gamma}^p\geq 0 \quad ; \quad \phi^p(\sigma,q^p)\leq 0 \quad ; \quad \dot{\gamma}^p\cdot\phi^p(\sigma,q^p)=0\\
&\dot{\gamma}^d\geq 0 \quad ; \quad \phi^d(\sigma,q^d)\leq 0 \quad ; \quad \dot{\gamma}^d\cdot\phi^d(\sigma,q^d)=0
\end{aligned}
\tag{10}
$$

The plastic and damage multipliers remain equal to zero in any elastic process and we easily show from equations (9) that there is no change of internal variables. On the other hand, the plastic and damage multipliers take positive values in an inelastic process with the corresponding change of internal variables computed from equations (9). The values of multipliers can be computed from the consistency conditions imposing that the stress field remains admissible with respect to the chosen yield and damage criteria:

$$
\begin{aligned}
&\dot{\gamma}^p>0 \quad ; \quad \phi^p=0 \quad ; \quad \dot{\phi}^p=0\\
&\Rightarrow \frac{\partial \phi^p}{\partial \sigma}\dot{\sigma}+\frac{\partial \phi^p}{\partial q^p}\dot{q}^p=0\\
&\Rightarrow \dot{\gamma}^p=\frac{\frac{\partial \phi^p}{\partial \sigma}E(\dot{\varepsilon}-\dot{\varepsilon}^d)}{\frac{\partial \phi^p}{\partial \sigma}E\frac{\partial \phi^p}{\partial \sigma}-\frac{\partial \phi^p}{\partial q^p}\frac{dq^p}{d\xi^p}\frac{\partial \phi^p}{\partial q^p}}\\
&\dot{\gamma}^d>0 \quad ; \quad \phi^d=0 \quad ; \quad \dot{\phi}^d=0\\
&\Rightarrow \frac{\partial \phi^d}{\partial \sigma}\dot{\sigma}+\frac{\partial \phi^d}{\partial q^d}\dot{q}^d=0\\
&\Rightarrow \dot{\gamma}^d=\frac{\frac{\partial \phi^d}{\partial \sigma}D^{-1}\dot{\varepsilon}^d}{\frac{\partial \phi^d}{\partial \sigma}D^{-1}\frac{\partial \phi^d}{\partial \sigma}-\frac{\partial \phi^d}{\partial q^d}\frac{dq^d}{d\xi^d}\frac{\partial \phi^d}{\partial q^d}}
\end{aligned}
\tag{11}
$$

These values of plastic and damage multipliers can be exploited to obtain the stress rate constitutive equations:

$$
\begin{aligned}
&\dot{\sigma}=C^{ep}(\dot{\varepsilon}-\dot{\varepsilon}^d)\\
&C^{ep}=E-\frac{\frac{\partial \phi^p}{\partial \sigma}EE\frac{\partial \phi^p}{\partial \sigma}}{\frac{\partial \phi^p}{\partial \sigma}E\frac{\partial \phi^p}{\partial \sigma}-\frac{\partial \phi^p}{\partial q^p}\frac{dq^p}{d\xi^p}\frac{\partial \phi^p}{\partial q^p}}=\frac{EK^p}{E+K^p}\\
&\dot{\sigma}=C^{ed}\dot{\varepsilon}^d\\
&C^{ed}=D^{-1}-\frac{\frac{\partial \phi^d}{\partial \sigma}D^{-1}D^{-1}\frac{\partial \phi^d}{\partial \sigma}}{\frac{\partial \phi^d}{\partial \sigma}D^{-1}\frac{\partial \phi^d}{\partial \sigma}-\frac{\partial \phi^d}{\partial q^d}\frac{dq^d}{d\xi^d}\frac{\partial \phi^d}{\partial q^d}}=\frac{D^{-1}K^d}{D^{-1}+K^d}
\end{aligned}
\tag{12}
$$

These two equations can be combined in order to obtain the elasto-plastic-damage tangent modulus leading to:

$$
\begin{aligned}
&\dot{\sigma}=C^{epd}\dot{\varepsilon} \quad ; \quad C^{epd}=\frac{C^{ep}C^{ed}}{C^{ep}+C^{ed}}\\
&C^{epd}=\frac{ED^{-1}K^pK^d}{ED^{-1}K^p+ED^{-1}K^d+EK^pK^d+D^{-1}K^pK^d}
\end{aligned}
\tag{13}
$$

where the explicit form for linear hardening case is also recorded.

2.2 Hellinger-Reissner type of variational principle

With all the equations governing the evolution of internal variables listed in previous section, we can obtain the governing equations for other state variables by appealing to the Hellinger-Reissner type of variational principle; namely, for the fixed values of the internal variables, we can seek the stationarity condition for the energy functional:

$$
\begin{aligned}
&\Pi_{HR}(u,\sigma,\varepsilon^p,D,\xi^p,\xi^d)=\\
&\int_\Omega[-\chi^e(\sigma)-\chi^d(\sigma,D)+\sigma(\frac{du}{dx}-\varepsilon^p)]dV-\int_\Gamma u\cdot\bar{t}dA
\end{aligned}
\tag{14}
$$

where the last term corresponds to the external energy produced by the boundary traction forces.

At the fixed values of internal variables, one can obtain from relation (14) above the corresponding variational equations:

$$
\begin{aligned}
0&=G_u(u,\sigma,\varepsilon^p,D,\xi^p,\xi^d,w)\\
&=\int_\Omega\frac{dw}{dx}\sigma dV-\int_\Gamma w\cdot\bar{t}dA\\
0&=G_\sigma(u,\sigma,\varepsilon^p,D,\xi^p,\xi^d,\tau)\\
&=\int_\Omega\tau(\frac{du}{dx}-\varepsilon^p-D\sigma-E^{-1}\sigma)dV
\end{aligned}
\tag{15}
$$

where w and τ are, respectively, the virtual displacement and virtual stress fields. It is easy to see that the Euler-Lagrange equations corresponding to $(15)_1$ is nothing else but the local equilibrium, $\frac{d\sigma}{dx}=0$, accompanied by the natural boundary condition $\sigma\cdot n|_\Gamma$, whereas the same kind of equation for

$(15)_2$ leads to the additive decomposition of the total deformation into elastic, plastic and damage component, $\frac{du}{dx} = E^{-1}\sigma + \varepsilon^p + D\sigma$.

We seek an approximate solution to equations (15) by using the finite element method [21]. To that end, we choose the standard isoparametric interpolations for the displacement field along with the stress field representation discontinuous from element to element:

$$u^h = \bigcup_{\Omega^e} N_a(x) d_a \quad ; \quad \sigma^h|_{\Omega^e} = S_b \beta_b$$
$$w^h = \bigcup_{\Omega^e} N_a(x) c_a \quad ; \quad \tau^h|_{\Omega^e} = S_b \gamma_b \tag{16}$$

We indicate in (16) that the same kind of interpolations are chosen for real and virtual fields as the optimal choice for this kind of problems [6] [21]. With these approximations on hand we can write the discrete form of the variational equations in (15):

$$0 = G_u^h = \sum_a c_a \Big(\sum_b \int_{\Omega^e} B_a^T S_b dV \cdot \beta_b - \int_{\Gamma^e} N_a \bar{t} dA \Big)$$
$$0 = G_\sigma^h = \sum_b \gamma_b \Big(\sum_a \int_{\Omega^e} S_b^T B_a dV \cdot d_a - \sum_c \int_{\Omega^e} S_b^T E^{-1} S_c dV \cdot \beta_c - \sum_c \int_{\Omega^e} S_b^T D S_c dV \cdot \beta_c - \int_{\Omega^e} S_b^T \varepsilon^p dV \Big)$$

By considering that the virtual fields interpolation parameters can be picked arbitrarily we obtain from (15) the discrete form of the equilibrium equations which we constructed by the finite element assembly $A_{e=1}^{n^{el}}$ of the elements contributions:

$$A_{e=1}^{n^{el}} \begin{pmatrix} 0 & \mathbf{F}^{eT} \\ \mathbf{F}^e & -(\mathbf{H}^{E,e} + \mathbf{H}_{n+1}^{D,e}) \end{pmatrix} \begin{Bmatrix} \mathbf{d}_{n+1}^e \\ \beta_{n+1}^e \end{Bmatrix} = A_{e=1}^{n^{el}} \begin{Bmatrix} \mathbf{f}_{n+1}^e \\ \mathbf{e}_{n+1}^{p,e} \end{Bmatrix} \tag{18}$$

where

$$\mathbf{F}^e = \int_{\Omega^e} \mathbf{S}^T \mathbf{B} dV \quad ; \quad \mathbf{f}_{n+1}^e = \int_{\Gamma^e} \mathbf{N}^T \bar{t}_{n+1} dA$$
$$\mathbf{H}^{E,e} = \int_{\Omega^e} \mathbf{S}^T E^{-1} \mathbf{S} dV \quad ; \quad \mathbf{H}_{n+1}^{D,e} = \int_{\Omega^e} \mathbf{S}^T D_{n+1} \mathbf{S} dV \tag{19}$$
$$\mathbf{e}_{n+1}^{p,e} = \int_{\Omega^e} \mathbf{S}^T \varepsilon_{n+1}^p dV$$

We have indicated in (18) above that the solution to the equilibrium problem is sought at a given pseudo-time value t_{n+1} of the imposed loading program with $d_{n+1} = d(t_{n+1})$ and $\beta_{n+1} = \beta(t_{n+1})$. The main source of nonlinearity in this set of equations pertains to the corresponding value of the internal variables for plasticity and damage defined by ε_{n+1}^p and D_{n+1} respectively. As shown in the next section, the latter can be computed by incremental analysis with no need to iterate in each increment.

3 Noniterative solution to local problem

3.1 Coupled damage-plasticity internal variable computations

The local problem of plastic and damage flow computation, yet referred as the central problem of computational inelasticity, is solved by an incremental procedure. In each increment we employ the implicit Euler scheme to integrate the model evolution equations in (9) to obtain:

Local problem of internal variables computation

$$\begin{array}{ll} \text{Given:} & \beta_{n+1}^{trial,(i)}, \varepsilon_n^p, D_n, \xi_n^p, \xi_n^d \\ \text{Find :} & \varepsilon_{n+1}^p, D_{n+1}, \xi_{n+1}^p, \xi_{n+1}^d \\ \text{Such that :} & \dot{\gamma}_{n+1}^p \cdot \phi_{n+1}^p = 0 \quad , \quad \dot{\gamma}_{n+1}^d \cdot \phi_{n+1}^d = 0 \end{array} \tag{20}$$

with the last condition which is needed in order to guarantee the admissibility of the computed stress in the sense of the chosen criteria. More precisely, we first start by integrating the evolution equations in (9) by the implicit Euler scheme, which leads to:

$$\varepsilon_{n+1}^p = \varepsilon_n^p + \gamma_{n+1}^p \frac{\partial \phi_{n+1}^p}{\partial \sigma_{n+1}}$$
$$\xi_{n+1}^p = \xi_n^p + \gamma_{n+1}^p \frac{\partial \phi_{n+1}^p}{\partial q_{n+1}^p}$$
$$D_{n+1}\sigma_{n+1} = D_n \sigma_{n+1} + \gamma_{n+1}^d \frac{\partial \phi_{n+1}^d}{\partial \sigma_{n+1}} \tag{21}$$
$$\xi_{n+1}^d = \xi_n^d + \gamma_{n+1}^d \frac{\partial \phi_{n+1}^d}{\partial q_{n+1}^d}$$

In any of those two equations we do not know the value of γ_{n+1}^p nor γ_{n+1}^d and hence we start with the elastic trial state, which considers the trial values of plasticity $\gamma_{n+1}^{p,trial} = 0$ and damage multipliers $\gamma_{n+1}^{d,trial} = 0$, which implies that the internal variables will not change from the previous increment. This allows us to compute the trial values of stress or rather its interpolation parameters $\beta_{n+1}^{trial,(i)}$ according to:

$$\beta_{n+1}^{e,trial,(i)} = (\mathbf{H}^{E,e} + \mathbf{H}_n^{D,e})^{-1} (\mathbf{F}^e \mathbf{d}_{n+1}^{e,(i)} - \mathbf{e}_n^{p,e})$$
$$\Rightarrow \sigma_{n+1}^{trial,(i)} = \mathbf{S}\beta_{n+1}^{trial,(i)} \tag{22}$$

The corresponding trial values of the damage and yield criteria can then be written:

$$\phi_{n+1}^{p,trial} = |\sigma_{n+1}^{trial,(i)}| - (\sigma_y - q_n^p)$$
$$\phi_{n+1}^{d,trial} = |\sigma_{n+1}^{trial,(i)}| - (\sigma_f - q_n^d) \tag{23}$$

If both of these trial values are negative, the elastic trial step is confirmed as the good guess; if only one of them is positive, the problem remains of standard form [6] [18], and the most interesting is the case where both trial values are positive. The latter one will further be elaborated upon. To that end, we first make use of the auxiliary result:

$$\begin{aligned}
\sigma_{n+1}^{trial} &= (E^{-1}+D_n)^{-1}(\varepsilon_{n+1}^{(i)}-\varepsilon_n^p)\\
\sigma_{n+1} &= (E^{-1}+D_{n+1})^{-1}(\varepsilon_{n+1}^{(i)}-\varepsilon_{n+1}^p)\\
&= \sigma_{n+1}^{trial}-(E^{-1}+D_n)^{-1}(\gamma_{n+1}^p\frac{\partial\phi_{n+1}^p}{\partial\sigma_{n+1}}+\gamma_{n+1}^d\frac{\partial\phi_{n+1}^d}{\partial\sigma_{n+1}})
\end{aligned} \tag{24}$$

The last result provides the justification for the stress parameters update with:

$$\begin{aligned}
\beta_{n+1} &= \beta_{n+1}^{e,trial}-(\mathbf{H}^{E,e}+\mathbf{H}_n^{D,e})^{-1}(\mathbf{g}_{n+1}^{p,e}+\mathbf{g}_{n+1}^{d,e})\\
\mathbf{g}_{n+1}^{p,e} &= \int_{\Omega^e}\mathbf{S}^T\gamma_{n+1}^p\frac{\partial\phi_{n+1}^p}{\partial\sigma_{n+1}}dV\\
\mathbf{g}_{n+1}^{d,e} &= \int_{\Omega^e}\mathbf{S}^T\gamma_{n+1}^d\frac{\partial\phi_{n+1}^d}{\partial\sigma_{n+1}}dV
\end{aligned} \tag{25}$$

Moreover, the same auxiliary result can be exploited to obtain the values of the multipliers by enforcing the yield and damage criteria locally at each Gauss quadrature point, which allows us to write:

$$\begin{aligned}
&0=\phi_{n+1}^p=\\
&\phi_{n+1}^{p,trial}-[(E^{-1}+D_n)^{-1}+K^p]\gamma_{n+1}^p-(E^{-1}+D_n)^{-1}\gamma_{n+1}^p\\
&0=\phi_{n+1}^d=\\
&\phi_{n+1}^{d,trial}-(E^{-1}+D_n)^{-1}\gamma_{n+1}^p-[(E^{-1}+D_n)^{-1}+K^d]\gamma_{n+1}^d
\end{aligned} \tag{26}$$

The latter reduces to a set of two equations with the multipliers as unknowns, which can be solved in a close form:

$$\begin{Bmatrix}\gamma_{n+1}^p\\ \gamma_{n+1}^d\end{Bmatrix}=
\begin{pmatrix}(E^{-1}+D_n)^{-1}+K^p & (E^{-1}+D_n)^{-1}\\ (E^{-1}+D_n)^{-1} & (E^{-1}+D_n)^{-1}+K^d\end{pmatrix}^{-1}\begin{Bmatrix}\phi_{n+1}^{p,trial}\\ \phi_{n+1}^{d,trial}\end{Bmatrix} \tag{27}$$

With this computation of multipliers we can easily carry out the corresponding updates of internal variables in (21).

3.2 Softening response computation

The proposed damage-plasticity model can be further enhanced in order to handle the softening response without any mesh dependency. In that aspect we will follow the development presented in [5] for plasticity model. The key ingredient pertains to a modification of the strain field which allows a correct representation of the total strain field:

$$\varepsilon=\underbrace{\frac{d\bar{u}}{dx}+\tilde{G}\alpha}_{\bar{\varepsilon}}+\underbrace{\delta_{\bar{x}}\alpha}_{\bar{\bar{\varepsilon}}} \tag{28}$$

In (28) above, $\bar{\bar{\varepsilon}}=\delta_{\bar{x}}\alpha$ is the corresponding localized strain representation by Dirac function positioned at $\bar{x}$, α is the localized strain parameter, whereas $\tilde{G}$ is the function which defines the influence zone of the discontinuity typically limited to a single finite element. The latter allows to write:

$$t=-\int_{l^e}\tilde{G}\sigma dx \tag{29}$$

where t is the traction at discontinuity.

We also ought to modify the strain energy in (2) in order to account properly for the fracture energy which is needed to completely break the bond between two parts of the body:

$$\psi(\cdot)=\bar{\psi}(\cdot)+\delta_{\bar{x}}\bar{\bar{\psi}}(\bar{\bar{\xi}}) \tag{30}$$

where $\bar{\psi}(\cdot)$ is already defined in (2). The final modification concerns the corresponding criterion defining the stress value σ_u at which the bond starts breaking, as well as the softening law:

$$\begin{aligned}
&\bar{\bar{\phi}}(t,\bar{\bar{q}})=|t|-(\sigma_u-\bar{\bar{q}}(\bar{\bar{\xi}}))\leq 0\\
&\bar{\bar{q}}=-\bar{\bar{K}}\bar{\bar{\xi}}\quad;\quad G_f=\frac{\sigma_u^2}{2\bar{\bar{K}}}
\end{aligned} \tag{31}$$

With the remaining ingredients of the softening model obtained by the principle of maximum dissipation, we can carry out the computation in the very much same manner as already presented for the coupled model. The only difference concerns computing the trial value of the driving traction at discontinuity:

$$t_{n+1}^{trial}=-\int_{l^e}\tilde{G}\underbrace{S\beta_{n+1}^{(i)}}_{\sigma_{n+1}^{(i)}}dx \tag{32}$$

where the corresponding stress parameter values are furnished by the coupled damage-plasticity model as described in this section. A more detailed description of dealing with softening phenomena for 1D and 2D cases is presented previously in [5].

4 Finite element interpolations

At this stage we can turn to the global computation phase, which should provide the new iterative value of the trial stress, or rather the stress parameters $\beta_{n+1}^{trial,(n+1)}$. The corresponding set of equations to be solved can be written as:

$$\mathbf{r}=\begin{pmatrix}
\mathbf{F}^e\mathbf{d}_{n+1}^{e,(i+1)}-(\mathbf{H}^{E,e}+\mathbf{H}_n^{D,e})\beta_{n+1}^e-\mathbf{e}_n^{p,e}-\mathbf{g}_{n+1}^{p,e}-\mathbf{g}_{n+1}^{d,e}\\
\phi_{n+1}^p\\
-\xi_{n+1}^p+\xi_n^p+\gamma_{n+1}^p\frac{\partial\phi_{n+1}^p}{\partial q_{n+1}^p}\\
\phi_{n+1}^d\\
-\xi_{n+1}^d+\xi_n^d+\gamma_{n+1}^d\frac{\partial\phi_{n+1}^d}{\partial q_{n+1}^d}
\end{pmatrix}=\mathbf{0}$$

(33)

By taking into account that the solution of (33) is sought at the known values of multipliers and internal variables, we can further linearize this system and perform the static condensation [3]. The linearized form of the remaining first equation can be written as:

$$0 = \mathrm{Lin}[G_\sigma] = G_\sigma|_{n+1}^{(i)} + \mathbf{D}G_\sigma \cdot \Delta\beta_{n+1}^{(i)}$$
$$\beta_{n+1}^{(i+1)} = \beta_{n+1}^{(i)} + \Delta\beta_{n+1}^{(i)} \quad (34)$$

with

$$\mathbf{D}G_\sigma = -[\mathbf{H}^{E,e} + \mathbf{H}_n^{D,e} + \mathbf{G}_{n+1}^{p,e} + \mathbf{G}_{n+1}^{d,e}]$$
$$\mathbf{G}_{n+1}^{p,e} = \frac{\partial \mathbf{g}_{n+1}^{p,e}}{\partial \beta_{n+1}} = \int_{\Omega^e} \mathbf{S}^T [\gamma_{n+1}^p \frac{\partial^2 \phi_{n+1}^p}{\partial \sigma_{n+1}^2} - \frac{\partial \phi_{n+1}^p}{\partial \sigma_{n+1}} (\frac{\partial \phi_{n+1}^p}{\partial q_{n+1}^p} \frac{d q_{n+1}^p}{d \xi_{n+1}^p} \frac{\partial \phi_{n+1}^p}{\partial q_{n+1}^p})^{-1} \frac{\partial \phi_{n+1}^p}{\partial \sigma_{n+1}}] \mathbf{S} dV \quad (35)$$
$$\mathbf{G}_{n+1}^{d,e} = \frac{\partial \mathbf{g}_{n+1}^{d,e}}{\partial \beta_{n+1}} = \int_{\Omega^e} \mathbf{S}^T [\gamma_{n+1}^d \frac{\partial^2 \phi_{n+1}^d}{\partial \sigma_{n+1}^2} - \frac{\partial \phi_{n+1}^d}{\partial \sigma_{n+1}} (\frac{\partial \phi_{n+1}^d}{\partial q_{n+1}^d} \frac{d q_{n+1}^d}{d \xi_{n+1}^d} \frac{\partial \phi_{n+1}^d}{\partial q_{n+1}^d})^{-1} \frac{\partial \phi_{n+1}^d}{\partial \sigma_{n+1}}] \mathbf{S} dV$$

We note that all the results in (35) are computed with the corresponding admissible values of internal variables, which are obtained for the given best iterative guess on displacement and stress parameters. The improved parameters values can be obtained, if needed, by solving the linearized form of equilibrium equations in (18):

$$A_{e=1}^{n_{el}} \begin{pmatrix} \mathbf{0} & \mathbf{F}^{eT} \\ \mathbf{F}^e & -(\mathbf{H}^{E,e} + \mathbf{H}_n^{D,e} + \mathbf{G}_{n+1}^{p,e} + \mathbf{G}_{n+1}^{d,e}) \end{pmatrix} \begin{Bmatrix} \Delta \mathbf{d}_{n+1}^{e,(i)} \\ \Delta \beta_{n+1}^{e,(i)} \end{Bmatrix} = \begin{Bmatrix} \mathbf{f}_{n+1}^{e,(i)} \\ -\mathbf{g}_{n+1}^{e,(i)} \end{Bmatrix}$$
$$\mathbf{d}_{n+1}^{e,(i+1)} = \mathbf{d}_{n+1}^{e,(i)} + \Delta \mathbf{d}_{n+1}^{e,(i)} \Rightarrow$$
$$\mathbf{g}_{n+1}^{e,(i)} = \mathbf{F}^e \mathbf{d}_{n+1}^{e,(i)} - (\mathbf{H}^{E,e} + \mathbf{H}_n^{D,e}) \beta_{n+1}^{e,(i)} - \mathbf{e}_n^{e,p} - \mathbf{g}_{n+1}^{p,e} - \mathbf{g}_{n+1}^{d,e} \quad (36)$$

The last equation in (36) above can be solved at the element level which allows to reduce the system to the standard form and to obtain the element tangent stiffness matrix:

$$A_{e=1}^{n_{el}} \{ \underbrace{\mathbf{F}^{eT} (\mathbf{H}^{E,e} + \mathbf{H}_n^{D,e} + \mathbf{G}_{n+1}^{p,e} + \mathbf{G}_{n+1}^{d,e})^{-1} \mathbf{F}^e}_{\mathbf{K}_{n+1}^{e,(i)}} \Delta \mathbf{d}_{n+1}^{e,(i)} = \mathbf{f}_{n+1}^e - \underbrace{\mathbf{F}^{eT} (\mathbf{H}^{E,e} + \mathbf{H}_n^{D,e} + \mathbf{G}_{n+1}^{p,e} + \mathbf{G}_{n+1}^{d,e})^{-1} \mathbf{g}_{n+1}^{e,(i)}}_{\hat{\mathbf{f}}_{n+1}^{e,(i)}} \} \quad (37)$$

It is important to note that the tangent stiffness matrix of this kind has been computed without any local iteration (which is in sharp contrast with displacement-type formulation which requires a local iterative procedure to compute the stress).

4.1 1D case with 2-node truss-bar finite element interpolations

A very clear illustration of the result in (37) can be given for the simplest choice of hybrid stress interpolations for a 2-nodes truss bar element where displacement field is a linear polynomial and the stress field a constant:

$$\mathbf{u}^h(x)|_{\Omega^e} = N_1(x) d_1^e + N_2(x) d_2^e$$
$$N_1(x) = 1 - \frac{x}{l} \quad ; \quad N_2(x) = \frac{x}{l} \quad (38)$$
$$\beta^h(x)|_{\Omega^e} = S(x) \beta^e \quad ; \quad S(x) = 1$$

In this case we obtain the following results (valid for unit cross-section $A = 1$):

$$\mathbf{F}^e = \int_{l^e} (1) [-\frac{1}{l} \quad \frac{1}{l}] dx = [-1 \quad 1]$$
$$\mathbf{H}^{E,e} = \int_{l^e} (1) \frac{1}{E} (1) dx = \frac{l^e}{E}$$
$$\mathbf{H}_n^{D,e} = \int_{l^e} (1) D_n (1) dx = \frac{l^e}{\frac{1}{D_n}}$$
$$\mathbf{G}_{n+1}^{p,e} = \int_{l^e} (1) \frac{1}{K^p} (1) dx = \frac{l^e}{K^p}$$
$$\mathbf{G}_{n+1}^{d,e} = \int_{l^e} (1) \frac{1}{K^d} (1) dx = \frac{l^e}{K^d} \quad (39)$$
$$\mathbf{e}_n^{p,e} = \int_{l^e} (1) \varepsilon_n^p dx$$
$$\mathbf{g}_{n+1}^{p,e} = \int_{l^e} (1) \gamma_{n+1}^p \frac{\phi_{n+1}^p}{\sigma_{n+1}} dx$$
$$\mathbf{g}_{n+1}^{d,e} = \int_{l^e} (1) \gamma_{n+1}^d \frac{\phi_{n+1}^d}{\sigma_{n+1}} dx$$

With these results on hand the tangent stiffness matrix in (37) can be explicitly written as:

$$\mathbf{K}_{n+1}^e = \frac{C_{n+1}^{epd}}{l^e} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$$
$$C_{n+1}^{epd} = (\frac{1}{E} + \frac{1}{D_n^{-1}} + \frac{1}{K^p} + \frac{1}{K^d})^{-1} \quad (40)$$
$$= \frac{E D_n^{-1} K^p K^d}{E K^p K^d + E D_n^{-1} K^p + E D_n^{-1} K^d + D_n^{-1} K^p K^d}$$

We note that the tangent elasto-plastic-damage modulus in discrete problem computed in (40) is the same as the corresponding one for the continuum problem in (13). This kind of conclusion holds only for 1D case.

4.2 2D case with 4-node hybrid stress Pian-Sumihara finite element interpolations

The described formulation was implemented according to displacement and stress interpolation proposed by Pian and

Sumihara (e.g. see [14]). In this case, the displacement field interpolation is written in terms of natural coordinates η_1 and η_2 as a bilinear polynomial expression, which is identical to the standard 4-node isoparametric element (e.g. see [21]), i.e.

$$N_a(\eta_1,\eta_2) = \frac{1}{4}(1+\eta_{a1}\eta_1)(1+\eta_{a2}\eta_2), \tag{41}$$

where η_{a1},η_{a2} are the corresponding nodal value of natural coordinates (equal to ± 1). On the other hand, the stress field interpolation in natural coordinates no longer corresponds to the one provided by a 4-node isoparametric element, and can be written as:

$$\begin{aligned}\sigma_{11} &= \lambda_1 + \lambda_2\ \eta_2\\ \sigma_{22} &= \lambda_3 + \lambda_4\ \eta_1\\ \sigma_{12} &= \lambda_5.\end{aligned} \tag{42}$$

This kind of stress interpolation provides the element with nearly optimal performance in bending dominated problems. Moreover, the choice of the shape functions in (42) implies that equilibrium equations are directly verified in the parent domain, i.e. $div_\eta\ \sigma = 0$ for each set of parameters λ_α.

Next we transform the interpolation into the global coordinate system, (x_1,x_2). In order to preserve the statical admissibility ($div_\eta\ \sigma = 0 \to div_x\ \sigma = 0$), the transformation has to be of the form (e.g. see [21]),

$$\sigma(x_1,x_2) = T(\eta_1,\eta_2)\sigma(\eta_1,\eta_2)T^T(\eta_1,\eta_2). \tag{43}$$

If, in top of that, we demand that the constant stress field is properly transformed, the transformation tensor, T, has to be constant, $T(\eta_1,\eta_2) \equiv T$. The most efficient choice has proven to be

$$T_{ij} = J_{ij}(\eta_1 = 0, \eta_2 = 0) \tag{44}$$

where

$$J_{ij}(\eta_1,\eta_2) := \frac{\partial x_i}{\partial \eta_j} \tag{45}$$

is the jacobian tensor. Therefore, the stress can finally be expressed as,

$$\sigma = S\,\lambda, \tag{46}$$

where

$$S = T\begin{pmatrix}1\ 0\ 1\ 0\ 0\\ 0\ 1\ 0\ 1\ 0\\ 0\ 0\ 0\ 0\ 1\end{pmatrix}T^T. \tag{47}$$

The remaining part of computations follows closely the one presented for 1D case, with consistent matrix which ought to be computed by inversion.

5 Numerical Examples

In this section we present several illustrative numerical simulations, which consider the typical response curves for concrete in compression and in tension, as well as the response of porous metals. The implementation of the proposed model and all the computations are carried out with the general purpose finite element program FEAP [20].

5.1 Localization of the strain in a simple traction test

The first example presents the response of a bar computed during a single cycle of loading and unloading, as described in Figure 1a. The finite element for the bar is composed of three 2-nodes truss-bar elements with unit cross-section and unit length. The material model for the bar is the coupled damage-plasticity proposed in this work. The model can handle both the strain hardening and the softening response phase. The latter starts at the ultimate stress value of $\sigma_u = 30MPa$, chosen for all the elements except the one in the middle where a slightly reduced ultimate stress value is set to $\sigma_u = 29.9MPa$. This choice is made in order to control the section where the strain will localize, here attributed to the element in the middle, and turn this kind of bifurcation problem into a limit load problem.

As shown in Figure 1b, the plasticity component is activated first at point A, with the damage model following shortly and activated at point B. Since then until ultimate stress both plasticity and damage components are active and coupled as described previously. At point C, the ultimate stress is reached and yet another inelastic mechanism of the strain softening process is activated within the weakened section Γ in the middle of the bar. Following that point the localized deformation $\bar{\bar{\varepsilon}}$ will occur only in the middle of the bar. The localized strain leads to stress reduction and forces the rest of the bar to unload with inelastic strains ε^p and ε^d remaining fixed (see Figure 1c).

We assume herein that the localized strain is equivalent to a plasticity-like mechanism, which allows us to compute the irreversible localized strain value at section Γ, when the stress fully unloads to zero; namely, we obtain $\varepsilon = \varepsilon^p + \bar{\bar{\varepsilon}}$ in section Γ and $\varepsilon = \varepsilon^p$ in any other section. This plasticity-like mechanism for strain softening can be replaced by a damage-like model, which would apply the localized strain disappearance upon the stress unloading. Whatever is our choice made for the softening mechanism, the computed result will not depend on the mesh grading.

5.2 Cyclic behavior of concrete under compression

The constitutive model presented herein can also represent the behavior of concrete under cyclic loading. Indeed, in Figure 1b, the portion AB of the curve represent the plastic behavior of the material before the appearance of micro cracks

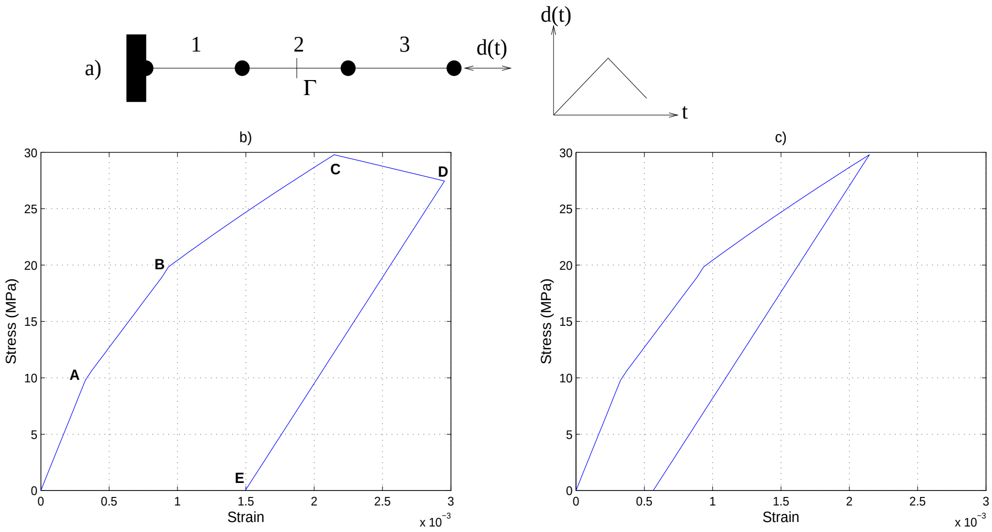

Fig. 1 a) Description of the bar; b) Response of element 2; c) Response of elements 1 and 3

at point B. The portion BC represents the fracture process zone (FPZ) which is assumed to develop in the whole bar. In the FPZ, $\varepsilon = \varepsilon^e + \varepsilon^p + \varepsilon^d$ with ε^p the irreversible part of the deformation due to the fact that micro cracks do not necessarily close entirely upon unloading because some particles may have penetrate in them. When the ultimate stress σ_u is reached, strain diminishes in the whole bar except in sections where macro cracks appear. Under tension, the localized strain parameter α clearly is the size of the crack opening perpendicular to the loading direction. Under compression, macro cracks develop parallel to the loading direction and the physical interpretation of α is somewhat less straightforward to provide than for the localized failure in tension.

To represent local hysteretic phenomena of concrete behavior under cyclic loading in compression (see Figure 2a), we enrich the continuum plasticity model presented in the previous sections with a linear kinematic hardening law. For that purpose, we introduce the internal variable κ^p, along with its conjugate state variable τ^p. The strain energy can then be written in a slightly generalized form with respect to (2):

$$\begin{aligned}&\psi(u,\sigma,\varepsilon^p,D,\xi^p,\xi^d,\kappa^p) = \\ &\psi^e(\varepsilon^e) + \psi^d(\varepsilon^d,D) + \Xi^p(\xi^p) + \Xi^d(\xi^d) + \lambda^p(\kappa^p) \\ &\Rightarrow \tau^p = -\frac{d\Lambda^p}{d\kappa^p} = -H^p\kappa^p\end{aligned} \quad (48)$$

The yield criterion is also modified from (4) in order to account for the elastic domain translation:

$$\phi^p(\sigma,q^p,\tau^p) = |\sigma + \tau^p| - (\sigma_y - q^p) \le 0 \quad (49)$$

If the imposed loading program considers only a compression or a tensile loading (without unloading), the formulation and implementation of such a modified model remain analogous to those of the proposed model without kinematic hardening.

By introducing the kinematic hardening law in the plasticity model, we can better represent some local hysteretic phenomena that occur in concrete and generally associated to sliding in cracks as shown in Figure 2b plotted with the following parameters: $E = 30GPa$, $\sigma_y = 8MPa$, $\sigma_f = 21MPa$, $\sigma_u = 28MPa$, $H^p = 40GPa$, $K^d = 8GPa$ and $\overline{\overline{K}} = -1.6GPa$.

5.3 Criteria for porous metals in tension

The porous metal coupled model was built along the lines of the pioneering work of Gurson ([1]), however with important difference regarding the present model, which has the ability to describe the closing of pores at unloading. Postulating that it is only spherical part of stress which determines the porosity, the damage criterion is given as

$$\phi^d(\sigma,q^d) = \langle tr(\sigma)\rangle - (\sigma_f^d - q^d), \quad (50)$$

where $tr(\sigma)$ denotes the trace of the tensor σ and $< \cdot >$ the Macauley brackets:

$$<x> = \begin{cases} x\ ; x \ge 0 \\ 0\ ; x < 0 \end{cases}. \quad (51)$$

Here we neglect the possibility that the material can be damaged in compression. To model the plasticity of metal matrix, we used the von Mises criterion:

$$\phi^p(\sigma,q^p) = \sqrt{dev(\sigma) : dev(\sigma)} - (\sigma_y^p - q^p), \quad (52)$$

where $dev(\sigma)$ denotes the deviatoric part of the tensor σ, $dev(\sigma) \equiv \sigma - \frac{1}{3}tr(\sigma)$.

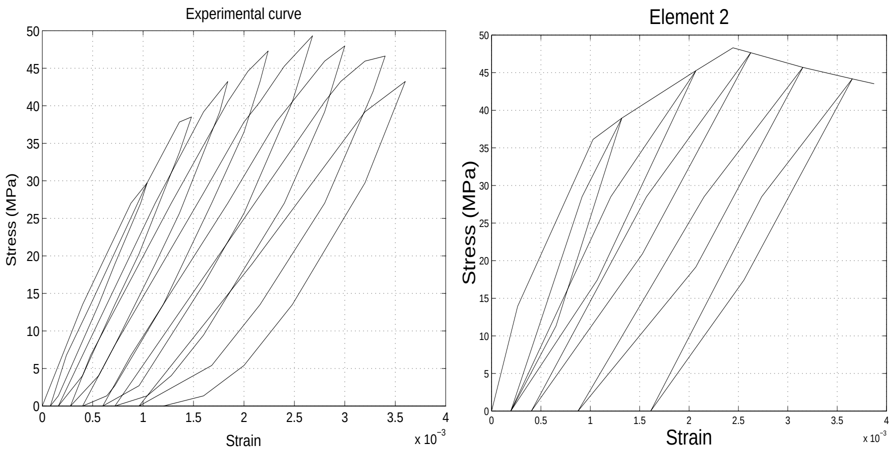

Fig. 2 a) Experimental result for cyclic behavior of concrete in compression (from [15]); b) Numerical result for cyclic behavior of concrete in compression by using the proposed coupled damage-plasticity model

From the choice of the criteria it follows that the evolution of damage variables depends only upon the spherical part of the stress tensor and the evolution of plastic variables upon its deviatoric part. Hence, the two nonlinear phenomena appear uncoupled in strain space. This is the direct consequence of the initial physical presumption that the opening of the micro-cracks is due to positive spherical part of the stress and sliding of crystal planes due to the stress deviator. The former corresponding to damage and the latter to plasticity.

Finally, we use an exponential hardening law for either phenomenon, plasticity and damage,

$$q^p(\xi^p) = (\sigma_y^p - \sigma_\infty^p)(1 - e^{-b^p\xi^p})$$
$$q^d(\xi^d) = (\sigma_f^d - \sigma_\infty^d)(1 - e^{-b^d\xi^d}), \tag{53}$$

where σ_∞^p and σ_∞^d are saturation values of stress, whereas b^p and b^d are the material parameters governing the rate of saturation.

The model is illustrated on an example of a rectangular plate with a circular hole in the middle, submitted to a simple tension test. By exploiting symmetry conditions, only one quarter of the model is used in the analysis; See Figure 3.

The material properties taken in the calculation were the following; (i) for elasticity: Young's modulus, $E = 240GPa$ and the shear modulus, $\mu = 92GPa$; (ii) for plasticity: yield stress, $\sigma_y = 170MPa$, hardening limit stress, $\sigma_\infty^p = 210MPa$ and saturation parameter, $b^p = 50$; (iii) for damage: fracture stress, $\sigma_f = 170MPa$, hardening limit stress, $\sigma_\infty^d = 210MPa$ and saturation parameter, $b^d = 50$.

In Figures 3 and 4 we show how the spreading of plastified and damaged regions will change with the other phenomenon being activated. Different stages of activation of either plasticity or damage models are illustrated by contours of hardening variables ξ^p and ξ^d, respectively.

We observe the complete disappearance of shear band (Figure 4), a typical response of metals or alloys with von Mises criterion, when damage is also taken into account. Besides, we notice that in the case where both phenomena are activated either region is reduced to a smaller volume, but the differences between the maximum and minimum value of ξ^p and ξ^d is larger. With other words, the phenomena are, when activated simultaneously, more localized.

6 Conclusion

The coupled damage-plasticity model proposed in this work goes beyond the minimum requirement we need for any such model of representing the irreversible deformation and change of elastic response, in that it also includes the strain localization softening phase. Any of the basic mechanisms of inelastic behavior is governed by an independent criterion, which specifies at what stage the corresponding evolution would start.

We have presented the governing equations for such model, which can be of interest for number of problems dealing with cyclic constitutive behavior. We have also shown that the best manner to provide the robust numerical implementation for such a model relies upon the direct stress interpolation. The latter provides the possibility to avoid any local iterative loop and much improves the model robustness.

It is clear that a constitutive model of this kind can be very useful for representing a number of experimentally observed inelastic phenomena including failure. However, the choice of model parameters, or rather the sequence of activation of each mechanism, ought to identified with care. This question is currently studied for a simpler constitutive model of anisotropic damage, with respect to modern testing procedures under heterogeneous stress field (see [22]); how to generalized these developments to current model will be examined in our future work.

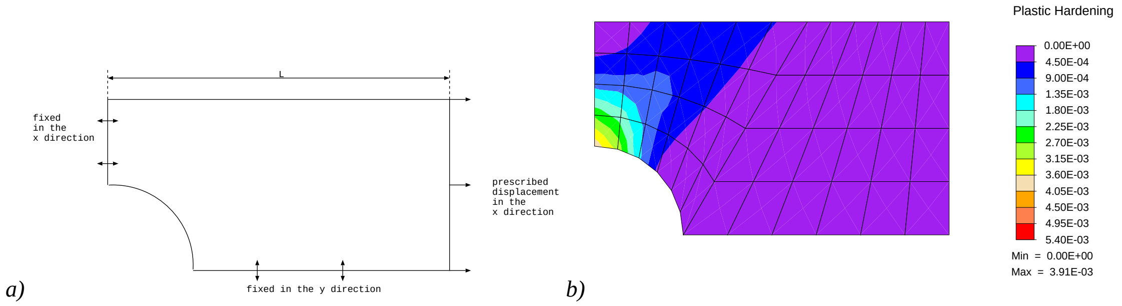

Fig. 3 *a) One quarter of the specimen: geometry and boundary conditions; b) Plasticity model: contours of the plastic hardening variable* ξ^p *showing typical shear bands.*

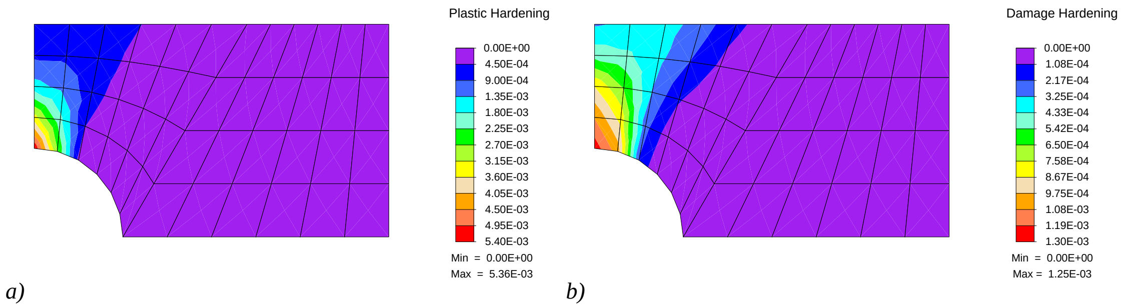

Fig. 4 *Coupled damage-plasticity model: a) contours of the plastic hardening variable* ξ^p*, and b) contours of the damage hardening variable* ξ^d.

Acknowledgements The financial support provided by the French Ministry of Research is gratefully acknowledged.

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