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1

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Nonlinear observers synthesis based on strongly persistent inputs

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Design of a high gain observer is

- usually :
 - based on normal forms,
 - for non-linear system assumed to be observable independently on the input.
- Here :
 - also based on normal forms,
 - for nonlinear systems not observable independently on the input,
 - conditions and proof of convergence of this observer are given, base on $u(t)$.

Let us take the following system with normal form :

$$\begin{cases} \dot{z} = A(u)z + g(u, z) \\ y = Cz \end{cases} \quad (1)$$

where :

$$A(u) = \begin{pmatrix} 0 & a_1(u) & \dots & 0 \\ 0 & 0 & \ddots & \vdots \\ \vdots & \vdots & \ddots & a_{n-1}(u) \\ 0 & 0 & \dots & 0 \end{pmatrix} \quad (2)$$

$g_i(u, z) = g_i(u, z_1, \dots, z_i)$ and $C = [1, 0, \dots, 0]$
 $z \in \mathbb{R}^n, u \in \mathbb{R}, y \in \mathbb{R}$

Definition

System (1) is **uniformly observable (U.O.)** if every input $u(\cdot)$ defined on any $[t_0, t_0 + T]$ renders the system observable : $\forall x(t_0) \neq x'(t_0)$, the associated outputs $y(x(0), u, t)$, $y(x'(0), u, t)$ are not identically equal on $[t_0, t_0 + T]$.

If the system is **U.O.**, it was shown (Gauthier-Kupka, SIAM-J. 1994) that $a(u) \neq 0$, for every u , an observer for (1) is :

$$\dot{\hat{z}} = A(u)\hat{z} + g(u, \hat{z}) + K(C\hat{z} - y) \quad (3)$$

where K is a constant tuning gain.

Observer design problem tackled

- Here, systems (1) that are not **U.O.** are considered : the above observer synthesis cannot be applied.
- E.g., consider the following simple example :

$$\begin{cases} \dot{z}_1 = uz_2 \\ \dot{z}_2 = 0 \\ y = z_1 \end{cases} \quad (4)$$

- This system follows the form (1) considered.
- But the input $u = 0$ renders the system unobservable.
- Consequently, a constant gain observer (3) cannot be designed.

Grammian of observability G

- Consider the affine part of system (1) :

$$\begin{cases} \dot{z} = A(u)z \\ y = Cz \end{cases} \quad (5)$$

- with $\phi_u(t, s)$ its transition matrix :

- $\frac{d(\phi_u(t, s))}{dt} = A(u(t))\phi_u(t, s)$,
- $\phi_u(s, s) = I = \text{identity matrix}$.

Remark

- u renders system (1) observable on $[t_0, t_0 + T]$ iff it renders (5) observable on the same interval.
- u renders system (5) observable on $[t_0, t_0 + T]$ iff the grammian of observability is positive definite :

$$G(u, t_0, t_0 + T) = \int_{t_0}^{t_0+T} \phi_u^T(t_0, t) C^T C \phi_u(t_0, t) dt \geq \alpha I \quad (6)$$

In what follows, we will give a condition on $u(t)$ s.t. the following system :

$$\begin{cases} \dot{\hat{z}} = A(u)\hat{z} + g(u, \hat{z}) - S^{-1}C^T(C\hat{z} - y) \\ \dot{S} = -\theta S - A^T(u)S - SA(u) + C^T C \end{cases} \quad (7)$$

forms an observer for the non U.O. system (1), where $\theta > 0$ is the tuning parameter of the observer.

- The gain of the observer (7) is based on the inverse of $S(t)$:

$$S(t) = e^{-\theta t} \Phi_u^T(0, t) S(0) \Phi_u(0, t) + \int_0^t e^{-\theta(t-s)} \Phi_u^T(s, t) C^T C \Phi_u(s, t) ds \quad (8)$$

- To be inverted, $S(t)$ needs to be well conditioned.
- With bounded $u(\cdot)$ + large θ : first term of $S(t)$ exp. $\rightarrow 0$,
 $S(t)$ exp. $\rightarrow \Psi(t, \theta) = \int_0^t e^{-\theta(t-s)} \Phi_u^T(s, t) C^T C \Phi_u(s, t) ds$
- Given T , $t - s \geq T$ + large θ : $S(t)$ exp. $\rightarrow G(u, t - T, t)$
- Condition : to get a good conditioning of $S(t)$, the input u must render the system (1) observable on any $[t - T, t]$.

From an observer gain condition to an input condition

- This condition on u will now be described based on the diagonal coefficients Ψ_{ii} of $\Psi(t, \theta)$.
- Since $A(u)$ is a nilpotent matrix and $C = [1, 0, \dots, 0]$, then :

- $\Psi_{11}(t, \theta) = \int_0^t e^{-\theta(t-s)} ds$
- For $i \geq 2$, $\Psi_{ii}(t, \theta) = \int_0^t e^{-\theta(t-s)} \left(\int_s^t a_1(u(t_1)) \dots a_{i-1}(u(t_1)) dt_1 \dots dt_{i-1} \dots \left(\int_s^{t_1} \int_s^{t_1} \dots \int_s^{t_{i-2}} a_1(u(t_1)) \dots a_{i-1}(u(t_1)) dt_1 \dots dt_{i-1} \right) ds \right)$

- Let :

$$\Gamma(t, \theta) = \begin{pmatrix} \Psi_{11}(t, \theta) & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & \Psi_{nn}(t, \theta) \end{pmatrix} \quad (9)$$

Definition

A bounded input $u(\cdot)$ is said to be strongly persistent if :

$\exists \theta_0 > 0 ; \exists t_0 \geq 0 ; \forall \theta > \theta_0 ; \forall t \geq t_0 :$

i) $\Gamma(t, \theta)$ is invertible and satisfies :

$$\Gamma(t, \theta) \leq \alpha(\theta) \Psi(t, \theta), \text{ with } \lim_{\theta \rightarrow \infty} \frac{\alpha(\theta)}{\theta} = 0.$$

ii) $\exists \gamma > 0, \|\Gamma_{jj}^{-1}(t, \theta)\| \|\Gamma_{ii}(t, \theta)\| \leq \gamma, \text{ for } 1 \leq j \leq i.$

Theorem

Assume that the nonlinear term $g(u, z)$ in (1) is a global Lipschitz function with respect to z , locally to u .

Then for every strongly persistent inputs u , the following system forms an exponential observer for system (1) :

$$\begin{cases} \dot{\hat{z}} = A(u)\hat{z} + g(u, \hat{z}) - S^{-1}C^T(C\hat{z} - y) \\ \dot{S} = -\theta S - A^T(u)S - SA(u) + C^T C \end{cases} \quad (10)$$

Proof of convergence for the non U.O. system observer

- Set $e(t) = \widehat{z}(t) - z(t)$ and using (1)-(10) :

$$\begin{cases} \dot{e} = (A(u) - S^{-1}C^T C)e + \delta(g) \\ \dot{S} = -\theta S - A^T(u)S - SA(u) + C^T C \end{cases} \quad (11)$$

where $\delta(g) = g(u, \widehat{z}) - g(u, z)$

- Take equation 2 of (11)
- Apply the Cholesky decomposition : $S(t) = \Lambda(t)\Lambda^T(t)$, with $\Lambda(t)$ =lower triangular matrix with positive diagonal.
- Take the time variant linear change of coordinates $\epsilon(t) = \Lambda^T(t)e(t)$, with $\Lambda^{-T} = (\Lambda^T)^{-1}$
- Multiply both sides : to the left by Λ^{-1} , to the right by Λ^{-T} .
- Set $V(t) = \epsilon^T(t)\epsilon(t)$, then :

$$\begin{cases} \dot{V} = -\theta\|\epsilon\|^2 + 2\epsilon^T \Lambda^T \delta(g) + (\delta(g))^T \Lambda \Lambda^T \delta(g) - \epsilon^T \Lambda^{-1} C^T C \Lambda^{-T} \epsilon \dots \\ \dots \leq -\theta\|\epsilon\|^2 + 2\|\epsilon\| \|\Lambda^T \delta(g)\| + (\delta(g))^T \Lambda \Lambda^T \delta(g) \end{cases} \quad (12)$$

Proof of convergence for the non U.O. system observer

- Since there exists $\tau_0 > 0$; there exists $a > 0$, s.t. for $\forall t \geq \tau_0 : S(t) = \Lambda(t)\Lambda^T(t) \leq a\Psi(t, \theta)$.
- then, $\forall t \geq t_0$:

$$\dot{V} \leq -\theta\|\epsilon\|^2 + 2\sqrt{a}\|\epsilon\|\sqrt{(\delta(g))^T\Psi(t, \theta)\delta(g)} + a(\delta(g))^T\Psi(t, \theta)\delta(g) \quad (13)$$

- $\Psi(t, \theta)$ being a diagonal matrix, we obtain $(\delta(g))^T\Psi(t, \theta)\delta(g) = \sum_{i=1}^q (\delta(g))_i^T\Psi_{ii}(t, \theta)\delta(g)_i$.

Proof of convergence for the non U.O. system observer

- Using the mean value theorem : $\delta(g)_i = \sum_{j=1}^i a_j(\cdot) e_j$ where $e_j = \hat{z}_j - z_j$:

$$\begin{aligned} & (\delta(g))_i^T \Psi_{ii}(t, \theta) \delta(g)_i \\ &= \sum_{l,k=1}^i e_k^T a_j(\cdot) \Psi_{ii}(t, \theta) a_j(\cdot) e_l \\ &\leq \tilde{a} \|\Psi_{ii}(t, \theta)\| \sum_{l=1}^i e_l^T e_l, \text{ where } \tilde{a} > 0 \text{ is a constant} \end{aligned} \tag{14}$$

- Set $\Psi_{ii}(t, \theta) = \tilde{\Psi}_{ii} \tilde{\Psi}_{ii}^T$ where $\tilde{\Psi}_{ii}$ is a lower triangular matrix given by the Cholesky decomposition.
- Set $\tilde{\epsilon}_i = \tilde{\Psi}_{ii} e_i$ and $\tilde{\epsilon} = (\tilde{\epsilon}_1^T, \dots, \tilde{\epsilon}_q^T)^T$, from (14) and inequality ii) of definition 2 (S.P. (u), it follows :

$$\begin{aligned} & (\delta(g))_i^T \Psi_{ii}(t, \theta) \delta(g)_i \leq \tilde{a} \|\Psi_{ii}(t, \theta)\| \sum_{l=1}^i \|\Psi_{ii}^{-1}(t, \theta)\| \tilde{\epsilon}_l^T \tilde{\epsilon}_l \\ &\leq b \sum_{l=1}^i \tilde{\epsilon}_l^T \tilde{\epsilon}_l, \text{ where } b > 0 \text{ is a constant} \end{aligned} \tag{15}$$

Proof of convergence for the non U.O. system observer

- Combining (13) and (15), we can find two constants $\alpha_1 > 0$, $\alpha_2 > 0$ which not depend on θ , such that for every $t \geq \tau_0$ we have :

$$\dot{V} \leq -\theta \|\epsilon\|^2 + \alpha_1 \|\epsilon\| \sqrt{\tilde{\epsilon}^T \tilde{\epsilon}} + \alpha_2 \tilde{\epsilon}^T \tilde{\epsilon} \quad (16)$$

- By definition of $\tilde{\epsilon}$, $\tilde{\epsilon}^T \tilde{\epsilon} = e^T \Gamma(t, \theta) e$, where $\Gamma(t, \theta)$ is the diagonal matrix given by (9). From i) of definition 2, we have :

$$\exists \theta_0 > 0; \exists t_0 \geq 0; \forall \theta > \theta_0; \forall t \geq t_0, \quad \Gamma(t, \theta) \leq \alpha(\theta) \Psi(t, \theta), \quad \text{with } \lim_{\theta \rightarrow \infty} \frac{\alpha(\theta)}{\theta} = 0 \quad (17)$$

Proof of convergence for the non U.O. system observer

- Now, using the facts that $\epsilon^T \epsilon = e^T S(t)e$, and that $\Psi(t, \theta)$ behaves like $S(t)$, then there exist constants $\tilde{\alpha}_1 > 0$, $\tilde{\alpha}_2 > 0$ which not depend on θ such that :

$$\dot{V} \leq -\theta \|\epsilon\|^2 + \tilde{\alpha}_1 \sqrt{\alpha(\theta)} \|\epsilon\|^2 + \tilde{\alpha}_2 \alpha(\theta) \|\epsilon\|^2 \quad (18)$$

- Finally, since $\lim_{\theta \rightarrow \infty} \frac{\alpha(\theta)}{\theta} = 0$ (see i) of definition 2), terms 2 and 3 in (18) are negligible.
- Hence, the estimation error $\|e(t)\|$ between the state of the system (1) and the state of the observer (10) converges to 0.

Contacts and discussion

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Our research team 1 page resume in **chinese** :
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Time for discussion!!!