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Nonlinear Observers Synthesis Based on Strongly Persistent Inputs

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Abstract: In most of the literature, the design of a high gain observer is based on normal forms. In the case where a nonlinear system, which is characterized by a normal form, is observable independently on the input, it is shown here that the gain of this observer does not require any differential equation. In this paper, we also investigate the problem of the high gain observer of a class of nonlinear systems which are not observable independently on the input. It is shown that if the input satisfies some observability condition, then an exponential observer, whose gain derived from a Lyapunov differential equation, can be designed.
Key Words: Nonlinear System, Nonlinear Observer, Normal Form

1 INTRODUCTION

The problem of state estimation is an important issue for the control, the diagnosis and the process monitoring. Many methods have been developed for designing an observer for nonlinear systems. Among these methods, a rather natural approach consists in characterizing nonlinear systems which can be steered, by a change of coordinates, into state affine systems up to output injection. Indeed, an extended Luenberger (or Kalman) observer can be designed for these class of systems (see for instance [1, 2]). For stationary linear systems, the observability does not depend on the inputs. An extension of this result for a more general class of nonlinear systems consists in giving normal forms which characterize nonlinear systems that are observable independently on the inputs. This problem has been initiated by [3] in the single output case. This normal form has been used in [4] in order to design a high gain observer in the control affine case. An extension of this observer synthesis for non control affine is stated in the single output case in [5] and in [6] in the multi-output case. For several other subclasses of multiple input multiple output (MIMO) uniformly observable systems, observers have been designed (see eg. [7-10]).

Based on the previous works mentioned above, recently, in [11], the authors gave a sufficient condition permitting to characterize nonlinear systems which can be immersed into high dimension normal form. The proposed class of systems may admit inputs which render them unobservable. In order to design an observer for these systems, the authors assume that the inputs must render the system sufficiently observable in some sense. The gain of the proposed observer is then derived from a Lyapunov differential equation.

In this paper, we will consider the same class of normal form stated in [11].First, we will show that if an input satisfies the hypothesis which is stated in [11], then a classical high gain observer can be designed: i.e., the gain of the observer does not require any Lyapunov differential equation.Next, we will propose an observer design based on a class of inputs which satisfy a weaker hypothesis than those considered in [11].

This paper is organized as follows: in the next section, we give the problem statement and we discuss the observer synthesis based on the observability hypothesis stated in [11]. In section 3, we propose a new observer synthesis based on hypothesis weaker than the hypothesis stated in [11].

2 PROBLEM STATEMENT AND SOME PRELIMINARY RESULTS

In this section, in the field of high gain observers [12], we recall some normal forms that have been used for their design. Two cases are presented: the first one concerns normal forms which characterized nonlinear systems which are observable independently on the input. In this case, the observer gain does not require any differential equation. The second case concerns normal forms of nonlinear systems which admit inputs that render them unobservable. For these systems, the observer synthesis depends on the input excitation.More precisely, in [11], the authors gave an observability condition that an input must satisfy in order to design a high gain observer whose gain derived from a differential equation. In subsection 2.2, we show that if such condition holds, then a classical high observer can be designed, namely, the gain of the observer can be synthesized without the use of differential equation.

2.1 Problem statement

Consider the single output nonlinear system:

$$\begin{cases} \dot{x} = f(x,u), \\ y = h(x), \end{cases} \tag{1}$$

the state $x(t) \in \mathbb{R}^n$, the input $u(t) \in \mathbb{R}^m$ and the output $y(t) \in \mathbb{R}$.

System (1) is said to be uniformly observable if every input $u(.)$ distinguishes every two different initial states $x(0)$, $x'(0)$ on any interval $[0, T(x(0), x'(0))]$ in which $x(.)$ and $x'(.)$ are well defined.

In the control affine case, the authors in [3] (see also [4] for a simple proof), showed that if (1) is uniformly observable, then it can be transformed locally almost everywhere

by a change of coordinates into the following normal form:

$$\left\{ \begin{array}{l} \dot{z} = Az + g(u,z), \\ y = Cz, \end{array} \right. \quad (2)$$

where:

$$A = \begin{pmatrix} 0 & 1 & \dots & 0 \\ 0 & 0 & \ddots & \vdots \\ \vdots & \vdots & \ddots & 1 \\ 0 & 0 & \dots & 0 \end{pmatrix}, \; C = [1, 0, \dots, 0],$$

$$g(u,z) = \begin{pmatrix} g_1(u, z_1) \\ \vdots \\ g_{n-1}(u, z_1, \dots, z_{n-1}) \\ g_n(u,z) \end{pmatrix}, \; z = (z_1, ..., z_n)^T.$$

This normal form is then used in [4], in order to design an observer of the form:

$$\dot{\widehat{z}} = A\widehat{z} + g(u, \widehat{z}) + \Delta_\theta K(C\widehat{z} - y), \quad (3)$$

where Δ_θ is the diagonal $n \times n$ matrix $\Delta_\theta = \begin{pmatrix} \theta & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & \theta^n \end{pmatrix}$ and K is a constant n-column vector such that $(A + KC)$ is a Hurwitz matrix. This observer exponentially converges whenever g is a Global Lipschitz function and for θ sufficiently large.

This canonical form together with its associated observer synthesis has been extended to non affine control systems by [5] and [6] respectively in the single input and the multi-output cases. In the non affine control systems, the normal form takes the following form:

$$\left\{ \begin{array}{l} \dot{z_1} = F_1(u, z_1, z_2), \\ \vdots \\ \dot{z_i} = F_i(u, z_1, \dots, z_{i+1}), \\ \vdots \\ \dot{z_q} = F_q(u, z), \\ y = Cz, \end{array} \right. \quad (4)$$

where $z = \begin{pmatrix} z_1 \\ \vdots \\ z_q \end{pmatrix}$, $z_i \in \mathbb{R}^{n_i}$, $u \in U$ a connected and bounded set of $\mathbb{R}^m$. $n_1 \geqslant n_2 \geqslant \dots \geqslant n_q$; $n_1 + n_2 + \dots + n_q = n$, and where:

$$\text{Rank} \left(\frac{\partial F_i}{\partial z_{i+1}}(u,z) \right) = n_{i+1}; \; \forall z \in \mathbb{R}^n; \forall u \in \text{U}; \forall i, \quad (5)$$

In the single output case, we have $q = n$ and all the n_i's are equal to 1, and the condition (5) means that $\frac{\partial F_i}{\partial z_{i+1}}(u,z) \neq 0$, for every (u,z).

In the single output case, the observer synthesis for the class of systems (4) is based on the following lemma:

Lemma 1 *Consider a $n \times n$ matrix $A(t) = \begin{pmatrix} 0 & a_1(t) & \dots & 0 \\ 0 & 0 & \ddots & \vdots \\ \vdots & \vdots & \ddots & a_{n-1}(t) \\ 0 & 0 & \dots & 0 \end{pmatrix}$, and assume that $0 < a \leqslant a_i(t) \leqslant a'$ for some constants a, a'. Then there exist a constant vector K; a symmetric positive definite matrix P and a constant $\alpha > 0$ such that:*

$$(A(t) + KC)^T P + P(A(t) + KC) \leqslant -\alpha I \quad (6)$$

The following result has been stated in [5] for the single output systems. Its extension to multi-output systems is shown in [6].

Theorem 1 *Let $u(.)$ be an input such that $0 < a \leqslant |\frac{\partial F_i}{\partial z_{i+1}}(u(t), z)| \leqslant a'$, for every $t \geqslant 0$ and for every z, where a, a' are constants. Assume again that the sign of $\frac{\partial F_i}{\partial z_{i+1}}(u(t), z)$ does not change. Then there exists $\theta_0 > 0$, such that for every $\theta > \theta_0$, the following system forms an exponential observer for system (4):*

$$\dot{\widehat{z}} = F(u, \widehat{z}) + \Delta_\theta K(C\widehat{z} - y), \quad (7)$$

where Δ_θ is the diagonal matrix given above, K is the constant gain stated in lemma 1 and where $F_i(u, z_1, \dots, z_{i+1})$ is the i^{th} component of F.

We have seen that an observer for control affine system (2) is of the form (3), and that the gain K of this observer derived from the pair (C, A) (the matrix $A + KC$ is Hurwitz). This idea has been extended in [11] to systems of the form:

$$\left\{ \begin{array}{l} \dot{z} = A(u)z + g(u,z), \\ y = Cz, \end{array} \right. \quad (8)$$

where:

$$A(u) = \begin{pmatrix} 0 & a_1(u) & \dots & 0 \\ 0 & 0 & \ddots & \vdots \\ \vdots & \vdots & \ddots & a_{n-1}(u) \\ 0 & 0 & \dots & 0 \end{pmatrix}, \quad (9)$$

$g_i(u,z) = g_i(u, z_1, \dots, z_i)$ and $C = [1, 0 \dots, 0]$.

In order to design an observer for (8) based on the following state affine system:

$$\left\{ \begin{array}{l} \dot{z} = A(u)z, \\ y = Cz, \end{array} \right. \quad (10)$$

the authors assume that the inputs $u(.)$ that render the system (10) observable in a strong sense are given by the following definition:

Definition 1 *[11]*
A bounded input $u(.)$ is said be locally regular, if there exist $t_0 \geqslant 0$, $\theta_0 > 0$ and $\alpha > 0$, such that for every $\theta \geqslant \theta_0$; for every $t \geqslant \max\{\frac{1}{\theta_0}, t_0\}$ we have:

$$G(u, t - \frac{1}{\theta}, t) = \int_{t-\frac{1}{\theta}}^{t} \phi_u^T(s,t) C^T C \phi_u(s,t) ds \geqslant \alpha \theta \Delta_\theta^{-2}, \quad (11)$$

where $G(u, t-\frac{1}{\theta}, t)$ is the Grammian of observability defined on $[t-\frac{1}{\theta}, t]$, $\phi_u(t,s)$ is the transition matrix of system (10) ($\frac{d(\phi_u(t,s))}{dt} = A(u(t))\phi_u(t,s)$), with $\phi_u(s,s) = I$ is the identity matrix, and Δ_θ is the diagonal $n \times n$ matrix defined above.

Theorem 2 *[11]*
If $u(.)$ is locally regular, then an observer for (8) takes the following form:

$$\begin{cases} \dot{\widehat{z}} = A(u)\widehat{z} + g(u,\widehat{z}) - \Delta_\theta S^{-1} C^T (C\widehat{z} - y), \\ \dot{S} = -\theta(\gamma S + A^T(u)S + SA(u) - C^T C), \\ S(0) \text{ is a } n \times n \text{ symmetric positive definite matrix}, \end{cases} \tag{12}$$

where $\theta > 0$ is a constant parameter which may be chosen sufficiently large and $\gamma > 0$ is a constant parameter which guaranties the stability of the differential equation $\dot{S}_0 = -\gamma S_0 + A^T(u)S_0 + S_0 A(u) - C^T C$, in the sense that $\alpha_0 I \leqslant S_0(t) \leqslant \beta_0 I$, for every $t \geqslant 0$, $\alpha_0 > 0$, $\beta_0 > 0$ are positive constants.

This theorem therefore states that a Lyapunov differential equation needs to be solved in order to get the observer gain for the observer (12) of the system (8). In the next part of this section, we will prove that a simpler constant gain observer may be designed for the system (8).

2.2 Preliminary results

We give here a proposition which shows that for continuous locally regular inputs, a constant gain observer of the form (7) can be designed for system (8). This means that the system of differential equations $\dot{S} = -\theta(\gamma S + A^T(u)S + SA(u) - C^T C)$ is not necessary in order to design an observer for systems (8). To do so, it suffices to show that if $u(.)$ is a continuous locally regular input, then it satisfies the conditions of theorem 1.

Proposition 1 *If $u(.)$ is a continuous locally regular input, then there exists $\tau_0 \geqslant 0$; there exist constants a, a', $0 < a < a'$ such that for every $t \geqslant \tau_0$, $a \leqslant |a_i(u(t))| \leqslant a'$, for $1 \leqslant i \leqslant n-1$. Moreover, the sign of $a_i(u(t))$ does not change.*

Proof

Since $A(u)$ is a nilpotent matrix and $C = [1, 0, \ldots, 0]$, we deduce that:

$$\begin{array}{l} C\phi_u(s,t) = [1, \ldots \\ \ldots \int_t^s a_1(u(t_1))dt_1, \ldots \\ \ldots \int_t^s \int_t^{t_1} \ldots \int_t^{t_{n-2}} a_1(u(t_1)) \ldots \\ \ldots a_{n-1}(u(t_{n-1}))dt_1 \ldots dt_{n-1}]. \end{array} \tag{13}$$

If we set:

$$\begin{cases} \alpha_1(t,s) = 1, \\ \alpha_2(t,s) = \int_t^s a_1(u(t_1))dt_1, \\ \alpha_n(t,s) = \int_t^s \int_t^{t_1} \ldots \int_t^{t_{n-2}} a_1(u(t_1)) \ldots \\ \qquad \ldots a_{n-1}(u(t_{n-1}))dt_1 \ldots dt_{n-1}, \end{cases} \tag{14}$$

then we get:

$$\begin{array}{l} \int_{t-\frac{1}{\theta}}^t \phi_u^T(s,t) C^T C \phi_u(s,t) ds = \psi(t,\theta) \ldots \\ \ldots = \begin{pmatrix} \psi_{11}(t,\theta) & \psi_{12}(t,\theta) & \ldots & \psi_{1n}(t,\theta) \\ \psi_{12}(t,\theta) & \psi_{22}(t,\theta) & \ldots & \psi_{2n}(t,\theta) \\ \vdots & \vdots & \ddots & \vdots \\ \psi_{1n}(t,\theta) & \psi_{2n}(t,\theta) & \ldots & \psi_{nn}(t,\theta) \end{pmatrix}, \end{array} \tag{15}$$

where $\psi_{ij}(t,\theta) = \int_{t-\frac{1}{\theta}}^t \alpha_i(t,s)\alpha_j(t,s)ds$.
The condition (11) implies:

$$\begin{cases} \exists \theta_0 > 0; \exists t_0 \geqslant 0; \exists \beta > 0; \forall \theta \geqslant \theta_0, \text{ we have:} \\ \forall t \geqslant \max\{t_0, \theta\}; \forall i, 1 \leqslant i \leqslant n, \psi_{ii}(t,\theta) \geqslant \frac{\beta}{\theta^{2i-1}}. \end{cases} \tag{16}$$

Let us analyse condition (16) for $i = 2, \ldots, n$:

- For $i = 2$:

$$\begin{cases} \psi_{22}(t,\theta) = \int_{t-\frac{1}{\theta}}^t (\int_t^s a_1(u(t_1))dt_1)^2 ds ... \\ ... \leqslant \max\{a_1^2(u(\tau)), t-\frac{1}{\theta} \leqslant \tau \leqslant t\} \int_{t-\frac{1}{\theta}}^t (t-s)^2 ds \\ \leqslant \max\{a_1^2(u(\tau)), t-\frac{1}{\theta} \leqslant \tau \leqslant t\} \frac{1}{3\theta^3}. \end{cases} \tag{17}$$

- For $i > 2$, setting:
  $\gamma_i = \max\{a_1^2(u(\tau_1)) \ldots a_{i-1}^2(u(\tau_{i-1})), t-\frac{1}{\theta} \leqslant \tau_j \leqslant t, 1 \leqslant j \leqslant i-1\}$,
  we obtain:

$$\begin{cases} \psi_{ii}(t,\theta) = \int_{t-\frac{1}{\theta}}^t (\int_t^s \ldots \int_t^{t_{i-2}} a_1(u(t_1)) ... \\ \ldots a_{i-1}u((t_{i-1}))dt_1 \ldots dt_{i-1})^2 ds ... \\ \leqslant \gamma_i \int_{t-\frac{1}{\theta}}^t (\int_t^s \ldots \int_t^{t_{i-2}} dt_1 \ldots dt_{i-1})^2 ds ... \\ \leqslant \dfrac{\gamma_i}{(2i-1)((i-1)!)^2 \theta^{2i-1}}. \end{cases} \tag{18}$$

Combining (16), (17) and (18), we deduce that there exists a constant $\widetilde{\gamma} > 0$, such that for every $t \geqslant \max\{t_0, \theta\}$, $a_i^2(u(t)) \geqslant \widetilde{\gamma}$. Finally, using the continuity of $a(.)$ and $u(.)$, it follows that the sign of $a_i(u(t))$ does not change for every $t \geqslant \max\{t_0, \theta\}$.
On one hand, one has to notice that (8) is a particular form of (4): indeed, $F_i(u,z) = a_i(u)z_{i+1} + g_i(u, z_1, \ldots, z_i)$. Therefore $\dfrac{\partial F_i}{\partial z_{i+1}}(u(t), z) = a_i(u)$. On the other hand, we just proved with the proposition 1 that the sign of $a_i(u(t))$ does not change. Therefore, the conditions of the theorem 1 are satisfied: a constant gain observer of the form (7) may also be applied on system (8), hence without the need to solve any Lyapunov differential equation.

3 MAIN CONTRIBUTION: OBSERVER SYNTHESIS BASED ON STRONGLY PERSISTENT INPUTS

In this section, for a particular set of inputs, we give an observer for system (8) based on a weaker condition than this proposed in theorem 2. To do so, consider the system of differential equations defined on the manifold of symmetric positive definite matrices $\mathcal{S}^+$:

$$\begin{cases} \dot{S} = -\theta S - A^T(u)S - SA(u) + C^T C, \\ S(0) \in \mathcal{S}^+, \end{cases} \tag{19}$$

where $A(u)$ is the $n \times n$ matrix given in (8). A simple calculation gives:

$$S(t) = e^{-\theta t}\Phi_u^T(0,t)S(0)\Phi_u(0,t) + ... \\ ... \int_0^t e^{-\theta(t-s)}\Phi_u^T(s,t)C^TC\Phi_u(s,t)ds, \quad (20)$$

where $\Phi_u(0,t) = \Phi_u^{-1}(t,0)$, $\Phi_u(t,0)$ is the transition matrix of the system $\dot{x} = A(u)x$, with $\Phi_u(0,0) = I$.If $u(.)$ is bounded, then $\|\Phi_u(s,t)\| \leqslant e^{m(t-s)}$ where $m = \|A(u(.))\|_\infty$.

Remark 1 *1) For $\theta > 2m$, $e^{-\theta t}\Phi_u^T(0,t)S(0)\Phi_u(0,t)$ exponentially converges to 0. In particular, for t sufficiently large, the behavior of $S(t)$ becomes similar to the behavior of:*

$$\Psi(t,\theta) = \int_0^t e^{-\theta(t-s)}\Phi_u^T(s,t)C^TC\Phi_u(s,t)ds = \begin{pmatrix} \Psi_{11}(t,\theta) & \Psi_{12}(t,\theta) & \dots & \Psi_{1n}(t,\theta) \\ \Psi_{12}(t,\theta) & \Psi_{22}(t,\theta) & \dots & \Psi_{2n}(t,\theta) \\ \vdots & \vdots & \ddots & \vdots \\ \Psi_{1n}(t,\theta) & \Psi_{2n}(t,\theta) & \dots & \Psi_{nn}(t,\theta) \end{pmatrix}. \quad (21)$$

Namely, $\exists \omega > 0; \forall t \geqslant 0, \forall \theta > 2m$:

$$S(t) - \omega e^{-(\theta-2m)t}I \leqslant \Psi(t,\theta) \leqslant S(t) + \omega e^{-(\theta-2m)t}I. \quad (22)$$

2) $\Psi_{ij}(t,\theta) = \Psi_{ji}^T(t,\theta)$ is $n_i \times n_j$ matrix defined by:

- $\Psi_{11}(t,\theta) = \int_0^t e^{-\theta(t-s)}ds$;
- $\Psi_{12}(t,\theta) = \int_0^t e^{-\theta(t-s)} \int_s^t A_{12}(u(t_1))dt_1ds$;
- *For $i \geqslant 3$,* $\Psi_{1i}(t,\theta) = \int_0^t e^{-\theta(t-s)} \int_s^t \int_s^{t_1} \dots \int_s^{t_{i-2}} A_{12}(u(t_1)) \dots ...A_{1i}(u(t_1))dt_1 \dots dt_{i-1}ds$;
- *For $i = 2$,* $\Psi_{22}(t,\theta) = \int_0^t e^{-\theta(t-s)} \dots \dots (\int_s^t A_{12}^T(u(t_1))dt_1)(\int_s^t A_{12}(u(t_1))dt_1)ds$;
- *For $i \geqslant 3$,* $\Psi_{2i}(t,\theta) = \int_0^t e^{-\theta(t-s)}(\int_s^t A_{12}^T(u(t_1))dt_1)... ...(\int_s^t \int_s^{t_1} \dots \int_s^{t_{i-2}} A_{12}(u(t_1)) \dots A_{i-1,i}(u(t_1))... ...dt_1 \dots dt_{i-1})ds$;
- *For $i, j \geqslant 3$,* $\Psi_{ij}(t,\theta) = \int_0^t e^{-\theta(t-s)}(\int_s^t A_{12}^T(u(t_1)) \dots A_{i-1,i}^T(u(t_1))... ...dt_1 \dots dt_{i-1})(\int_s^t \int_s^{t_1} \dots \int_s^{t_{i-2}} A_{12}(u(t_1)) \dots ...A_{j-1,j}(u(t_1))dt_1 \dots dt_{j-1})ds$.

In the sequel we will denote by $\Gamma(t,\theta)$ the diagonal matrix:

$$\Gamma(t,\theta) = \begin{pmatrix} \Psi_{11}(t,\theta) & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & \Psi_{nn}(t,\theta) \end{pmatrix}. \quad (23)$$

By construction, this diagonal matrix is positive definite.

Definition 2 *A bounded input $u(.)$ is said to be strongly persistent if:*
$\exists \theta_0 > 0$; $\exists t_0 \geqslant 0$; $\forall \theta > \theta_0$; $\forall t \geqslant t_0$:

i) $\Gamma(t,\theta)$ is invertible and satisfies:
$\Gamma(t,\theta) \leqslant \alpha(\theta)\Psi(t,\theta)$, *with* $\lim_{\theta\to\infty} \frac{\alpha(\theta)}{\theta} = 0$.

ii) $\exists \gamma > 0$, $\|\Gamma_{jj}^{-1}(t,\theta)\|\|\Gamma_{ii}(t,\theta)\| \leqslant \gamma$, for $1 \leqslant j \leqslant i$.

The set of such strongly persistent inputs will be denoted by $\mathcal{C}_{SP}$.

Remark 2 *Denoting by $\mathcal{C}_{LR}$ the set of locally regularly inputs defined in definition 1, then $\mathcal{C}_{LR} \subset \mathcal{C}_{SP}$*

As the following example shows, generally we have $\mathcal{C}_{LR} \subsetneq \mathcal{C}_{SP}$.

Example 1

$$\begin{cases} \dot{x}(t) = \begin{pmatrix} 0 & u(t) \\ 0 & 0 \end{pmatrix} x(t), \\ y(t) = (1 \ \ 0)x(t), \end{cases} \quad (24)$$

$u(t) = \cos(t)$ *belongs to $\mathcal{C}_{SP}$, but it is not in $\mathcal{C}_{LR}$.*

Now, we can state our main result of this section:

Theorem 3 *Assume that the nonlinear term $g(u,z)$ in (8) is a global Lipschitz function with respect to z, locally to u. Then for every $u \in \mathcal{C}_{SP}$, the following system forms an exponential observer for system (8):*

$$\begin{cases} \dot{\widehat{z}} = A(u)\widehat{z} + g(u,\widehat{z}) - S^{-1}C^T(C\widehat{z} - y), \\ \dot{S} = -\theta S - A^T(u)S - SA(u) + C^TC, \end{cases} \quad (25)$$

Proof
Setting $e(t) = \widehat{z}(t) - z(t)$ and using (8)-(25), we obtain:

$$\begin{cases} \dot{e} = (A(u) - S^{-1}C^TC)e + \delta(g), \\ \dot{S} = -\theta S - A^T(u)S - SA(u) + C^TC, \end{cases} \quad (26)$$

where $\delta(g) = g(u,\widehat{z}) - g(u,z)$. Applying the Cholesky decomposition to $S(t)$, we obtain $S(t) = \Lambda(t)\Lambda^T(t)$ where $\Lambda(t)$ is a lower triangular matrix with a positive diagonal. Now consider the time variant linear change of coordinates $\epsilon(t) = \Lambda^T(t)e(t)$, we deduce:

$$\dot{\epsilon} = (\Lambda^TA(u)\Lambda^{-T} - \Lambda^{-1}C^TC\Lambda^{-T})\epsilon + \Lambda^T\delta(g) + \dot{\Lambda}^T\Lambda^{-T}\epsilon, \quad (27)$$

where Λ^{-T} denotes the inverse of Λ^T.
Using the relation $S(t) = \Lambda(t)\Lambda^T(t)$ and the fact that $S(t)$ is a solution of the second equation of (25), we obtain:

$$\dot{\Lambda}\Lambda^T + \Lambda\dot{\Lambda}^T = -\theta\Lambda\Lambda^T - A^T(u)\Lambda\Lambda^T - \Lambda\Lambda^TA(u) + C^TC. \quad (28)$$

Let us multiply both sides of (28): to the left by Λ^{-1}, and to the right by Λ^{-T}. Then, we get:

$$\Lambda^{-1}\dot{\Lambda} + \dot{\Lambda}^T\Lambda^{-T} = ... \\ ... - \theta I - \Lambda^{-1}A^T(u)\Lambda - \Lambda^TA(u)\Lambda^{-T} + \Lambda^{-1}C^TC\Lambda^{-T}. \quad (29)$$

Now setting $V(t) = \epsilon^T(t)\epsilon(t)$:

$$\begin{cases} \dot{V} = \dot{\epsilon}^T\epsilon + \epsilon^T\dot{\epsilon}... \\ ... = \epsilon^T[\Lambda^{-1}A^T(u)\Lambda - \Lambda^{-1}C^TC\Lambda^{-T}]\epsilon... \\ ... + \epsilon^T[\Lambda^TA(u)\Lambda^{-T} - \Lambda^{-1}C^TC\Lambda^{-T}]\epsilon... \\ ... + 2\epsilon^T\Lambda^T\delta(g) + (\delta(g))^T\Lambda\Lambda^T\delta(g)... \\ ... + \epsilon^T[\Lambda^{-1}\dot{\Lambda} + \dot{\Lambda}^T\Lambda^{-T}]\epsilon. \end{cases} \quad (30)$$

Combining (29) and (30), it follows:

$$\left\{\begin{array}{l}\dot{V} = -\theta\|\epsilon\|^2 + 2\epsilon^T\Lambda^T\delta(g) + (\delta(g))^T\Lambda\Lambda^T\delta(g) - ... \\ ...\epsilon^T\Lambda^{-1}C^TC\Lambda^{-T}\epsilon... \\ ... \leqslant -\theta\|\epsilon\|^2 + 2\|\epsilon\|\|\Lambda^T\delta(g)\| + (\delta(g))^T\Lambda\Lambda^T\delta(g),\end{array}\right. \tag{31}$$

where $\|\|$ denotes the euclidian norm.
Using 1) of remark 1, there exists $\tau_0 > 0$; there exists $a > 0$, such that for every $t \geqslant \tau_0$, we have $S(t) = \Lambda(t)\Lambda^T(t) \leqslant a\Psi(t,\theta)$. Combining this inequality together with (31), we deduce that for every $t \geqslant t_0$:

$$\begin{array}{l}\dot{V} \leqslant -\theta\|\epsilon\|^2 + 2\sqrt{a}\|\epsilon\|\sqrt{(\delta(g))^T\Psi(t,\theta)\delta(g)} + ... \\ ...a(\delta(g))^T\Psi(t,\theta)\delta(g).\end{array} \tag{32}$$

$\Psi(t,\theta)$ being a diagonal matrix, we obtain $(\delta(g))^T\Psi(t,\theta)\delta(g) = \sum_{i=1}^{q}(\delta(g))_i^T\Psi_{ii}(t,\theta)\delta(g)_i$. Now using the mean value theorem, we obtain: $\delta(g)_i = \sum_{j=1}^{i} Aij(.)e_j$ where $e_j = \widehat{z}_j - z_j$ and $A_{ij}(.)$ is a $n_i \times n_j$ matrix which is a bounded function of $(u, z, \widehat{z})$, and hence:

$$\begin{array}{l}(\delta(g))_i^T\Psi_{ii}(t,\theta)\delta(g)_i \\ = \sum_{l,k=1}^{i} e_k^T A_{ik}^T(.)\Psi_{ii}(t,\theta)A_{il}(.)e_l \\ \leqslant \widetilde{a}\|\Psi_{ii}(t,\theta)\|\sum_{l=1}^{i} e_l^T e_l, \text{ where } \widetilde{a} > 0 \text{ is a constant,}\end{array} \tag{33}$$

Now set, $\Psi_{ii}(t,\theta) = \widetilde{\Psi}_{ii}\widetilde{\Psi}_{ii}^T$ where $\widetilde{\Psi}_{ii}$ is a lower triangular matrix given by the Cholesky decomposition. Set $\widetilde{\epsilon}_i = \widetilde{\Psi}_{ii}e_i$ and $\widetilde{\epsilon} = (\widetilde{\epsilon}_1^T, \ldots, \widetilde{\epsilon}_q^T)^T$, from (33) and inequality ii) of definition 2, it follows:

$$\begin{array}{l}(\delta(g))_i^T\Psi_{ii}(t,\theta)\delta(g)_i \\ \leqslant \widetilde{a}\|\Psi_{ii}(t,\theta)\|\sum_{l=1}^{i}\|\Psi_{ll}^{-1}(t,\theta)\|\widetilde{\epsilon}_l^T\widetilde{\epsilon}_l \\ \leqslant b\sum_{l=1}^{i}\widetilde{\epsilon}_l^T\widetilde{\epsilon}_l, \text{ where } b > 0 \text{ is a constant,}\end{array} \tag{34}$$

Combining (32) and (34), we can find two constants $\alpha_1 > 0$, $\alpha_2 > 0$ which not depend on θ, such that for every $t \geqslant \tau_0$ we have:

$$\dot{V} \leqslant -\theta\|\epsilon\|^2 + \alpha_1\|\epsilon\|\sqrt{\widetilde{\epsilon}^T\widetilde{\epsilon}} + \alpha_2\widetilde{\epsilon}^T\widetilde{\epsilon}, \tag{35}$$

By definition of $\widetilde{\epsilon}$, $\widetilde{\epsilon}^T\widetilde{\epsilon} = e^T\Gamma(t,\theta)e$, where $\Gamma(t,\theta)$ is the diagonal matrix given by (23). From i) of definition 2, we have:

$$\begin{array}{c}\exists\theta_0 > 0; \exists t_0 \geqslant 0; \forall\theta > \theta_0; \forall t \geqslant t_0, \\ \Gamma(t,\theta) \leqslant \alpha(\theta)\Psi(t,\theta), \text{ with} \lim_{\theta\to\infty}\frac{\alpha(\theta)}{\theta} = 0,\end{array} \tag{36}$$

Now, using the fact that $\epsilon^T\epsilon = e^TS(t)e$ that $\Psi(t,\theta)$ have the same behavior as $S(t)$ (see inequalities (22) of remark 1), then there exist constants $\widetilde{\alpha}_1 > 0$, $\widetilde{\alpha}_2 > 0$ which not depend on θ such that:

$$\dot{V} \leqslant -\theta\|\epsilon\|^2 + \widetilde{\alpha}_1\sqrt{\alpha(\theta)}\|\epsilon\|^2 + \widetilde{\alpha}_2\alpha(\theta)\|\epsilon\|^2 \tag{37}$$

Finally, since $\lim_{\theta\to\infty}\frac{\alpha(\theta)}{\theta} = 0$ (see i) of definition 2), we can conclude that $V(t)$ exponentially converges to 0. Hence, it is the same for the estimation error $\|e(t)\|$ between the state of the system (8) and the state of the observer (25).

4 CONCLUSIONS

In this paper, we have recalled some normal forms which permit to design a high gain observer. In the case when the system is observable independently on the inputs, this normal forms have been used to design a high gain observer whose gain does not require any differential equation. In this paper, we have also investigated the case when the system admits inputs which render it unobservable. We have shown that if the input satisfies some observability condition (strong persistent input), then the observer synthesis requires a Lyapunov differential equation which permits to calculate the gain of the observer.

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