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# SPECTRAL SHIFT FUNCTION FOR OPERATORS WITH CROSSED MAGNETIC AND ELECTRIC FIELDS

MOUEZ DIMASSI AND VESSELIN PETKOV

ABSTRACT. We obtain a representation formula for the derivative of the spectral shift function  $\xi(\lambda; B, \epsilon)$  related to the operators  $H_0(B, \epsilon) = (D_x - By)^2 + D_y^2 + \epsilon x$  and  $H(B, \epsilon) = H_0(B, \epsilon) + V(x, y)$ ,  $B > 0, \epsilon > 0$ . We establish a limiting absorption principle for  $H(B, \epsilon)$  and an estimate  $\mathcal{O}(\epsilon^{n-2})$  for  $\xi'(\lambda; B, \epsilon)$ , provided  $\lambda \notin \sigma(Q)$ , where  $Q = (D_x - By)^2 + D_y^2 + V(x, y)$ .

## 1. INTRODUCTION

Consider the two-dimensional Schrödinger operator with homogeneous magnetic and electric fields

$$H = H(B, \epsilon) = H_0(B, \epsilon) + V(x, y), \quad D_x = -i\partial_x, \quad D_y = -i\partial_y,$$

where

$$H_0 = H_0(B, \epsilon) = (D_x - By)^2 + D_y^2 + \epsilon x.$$

Here  $B > 0$  and  $\epsilon > 0$  are proportional to the strength of the homogeneous magnetic and electric fields. We assume that  $V, \partial_x V \in C^0(\mathbb{R}^2; \mathbb{R}) \cap L^\infty(\mathbb{R}^2; \mathbb{R})$  and  $V(x, y)$  satisfies the estimate

$$|V(x, y)| \leq C(1 + |x|)^{-2-\delta}(1 + |y|)^{-1-\delta}, \quad \delta > 0. \quad (1.1)$$

For  $\epsilon \neq 0$  we have  $\sigma_{\text{ess}}(H_0(B, \epsilon)) = \sigma_{\text{ess}}(H(B, \epsilon)) = \mathbb{R}$ . On the other hand, for decreasing potentials  $V$  we may have embedded eigenvalues  $\lambda \in \mathbb{R}$  and this situation is completely different from that with  $\epsilon = 0$  when the spectrum of  $H(B, 0)$  is formed by eigenvalues with finite multiplicities which may accumulate only to Landau levels  $\lambda_n = (2n + 1)B$ ,  $n \in \mathbb{N}$  (see [9], [13], [15] and the references cited there). The spectral properties of  $H$  and the existence of resonances have been studied in [7], [8], [5] under the assumption that  $V(x, y)$  admits a holomorphic extension in the  $x$ -variable into a domain

$$\Gamma_{\delta_0} = \{z \in \mathbb{C} : 0 \leq |\text{Im } z| \leq \delta_0\}.$$

Moreover, without any assumption on the analyticity of  $V(x, y)$  we show in Proposition 2 below that the operator  $(H - z)^{-1} - (H_0 - z)^{-1}$  for  $z \in \mathbb{C}$ ,  $\text{Im } z \neq 0$ , is trace class and following the general setup [11], [20], we define the spectral shift function  $\xi(\lambda) = \xi(\lambda; B, \epsilon)$  related to  $H_0(B, \epsilon)$  and  $H(B, \epsilon)$  by

$$\langle \xi', f \rangle = \text{tr} \left( f(H) - f(H_0) \right), \quad f \in C_0^\infty(\mathbb{R}).$$

By this formula  $\xi(\lambda)$  is defined modulo a constant but for the analysis of the derivative  $\xi'(\lambda)$  this is not important. Moreover, the above property of the resolvents and Birman-Kuroda theorem imply  $\sigma_{\text{ac}}(H_0(B, \epsilon)) = \sigma_{\text{ac}}(H(B, \epsilon)) = \mathbb{R}$ . A representation of the derivative  $\xi'(\lambda; B, \epsilon)$  has been obtained in [5] for strong magnetic fields  $B \rightarrow +\infty$  under the assumption that  $V(x, y)$  admits an analytic

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continuation in  $x$ -direction. Moreover, the distribution of the resonances  $z_j$  of the perturbed operator  $H(B, \epsilon)$  has been examined in [5] and a Breit-Wigner representation of  $\xi'(\lambda; B, \epsilon)$  involving the resonances  $z_j$  was established.

In the literature there are a lot of works concerning Schrödinger operators with magnetic fields ( $\epsilon = 0$ ) but there are only few ones dealing with magnetic and Stark potentials ( $\epsilon \neq 0$ ) (see [7], [8], [5] and the references given there). It should be mentioned that the tools in [7], [8] and [5] are related to the resonances of the perturbed problem and to define the resonances one supposes that the potential  $V(x, y)$  has an analytic continuation in  $x$  variable. In this paper we consider the operator  $H$  without *any assumption* on the analytic continuation of  $V(x, y)$  and without the *restriction*  $B \rightarrow +\infty$ . Our purpose is to study  $\xi'(\lambda; B, \epsilon)$  and the existence of embedded eigenvalues of  $H$ . To examine the behavior of the spectral shift function we need a representation of the derivative  $\xi'(\lambda; B, \epsilon)$ . The key point in this direction is the following

**Theorem 1.** *Let  $V, \partial_x V \in C^0(\mathbb{R}^2; \mathbb{R}) \cap L^\infty(\mathbb{R}^2; \mathbb{R})$  and let (1.1) hold for  $V$  and  $\partial_x V$ . Then for every  $f \in C_0^\infty(\mathbb{R})$  and  $\epsilon \neq 0$  we have*

$$\operatorname{tr} \left( f(H) - f(H_0) \right) = -\frac{1}{\epsilon} \operatorname{tr} \left( \partial_x V f(H) \right). \quad (1.2)$$

The formula (1.2) has been proved by D. Robert and X.P.Wang [18] for Stark Hamiltonians in absence of magnetic field ( $B = 0$ ). In fact, the result in [18] says that

$$\xi'(\lambda; 0, \epsilon) = -\frac{1}{\epsilon} \int_{\mathbb{R}^2} \partial_x V \frac{\partial e}{\partial \lambda}(x, y, x, y; \lambda, 0, \epsilon) dx dy, \quad (1.3)$$

where  $e(\cdot, \cdot; \lambda, 0, \epsilon)$  is the spectral function of  $H(0, \epsilon)$ . The presence of magnetic field  $B \neq 0$  and Stark potential lead to some serious difficulties. The operator  $H$  is not elliptic for  $|x| + |y| \rightarrow \infty$  and we have double characteristics. On the other hand, the commutator  $[H, x]$  involves the term  $(D_x - By)$  and it creates additional difficulties. The proof of Theorem 1 is long and technical. We are going to study the trace class properties of the operators  $\psi(H \pm \mathbf{i})^{-N}$ ,  $\partial_x \circ \psi(H \pm \mathbf{i})^{-N-1}$ ,  $(H \pm \mathbf{i})\partial_x \circ \psi(H \pm \mathbf{i})^{-N-2}$  etc. for  $N \geq 2$  and  $\psi \in C_0^\infty(\mathbb{R}^2)$  (see Lemmas 1 and 2). Moreover, by an argument similar to that in Proposition 2.1 in [5], we obtain estimates for the trace norms of the operators

$$(z - H)^{-1}V(z' - H)^{-1}, V(z - H)^{-1}(z' - H)^{-1}, z \notin \mathbb{R}, z' \notin \mathbb{R}$$

and we apply an approximation argument. Notice that in [18] the spectral shift function is related to the trace of the *time delay* operator  $T(\lambda)$  defined via the corresponding scattering matrix  $S(\lambda)$  (see [17]). In contrast to [18], our proof is direct and neither  $T(\lambda)$  nor  $S(\lambda)$  corresponding to the operator  $H(B, \epsilon)$  are used.

The second question examined in this work is the existence of embedded real eigenvalues and the limiting absorption principle for  $H$ . In the physical literature one conjectures that for  $\epsilon \neq 0$  there are no embedded eigenvalues. We establish in Section 3 a weaker result saying that in any interval  $[a, b]$  we may have at most a finite number embedded eigenvalues with finite multiplicities. Under the assumption for analytic continuation of  $V$  it was proved in [7] that for some finite interval  $[\alpha(B, \epsilon), \beta(B, \epsilon)]$  there are no resonances  $z$  of  $H(B, \epsilon)$  with  $\operatorname{Re} z \notin [\alpha(B, \epsilon), \beta(B, \epsilon)]$ . Since the real resonances  $z$  coincide with the eigenvalues of  $H(B, \epsilon)$ , we obtain some information for the embedded eigenvalues. On the other hand, exploiting the analytic continuation and the resonances we proved in [5] that for  $B \rightarrow +\infty$  the real parts  $\operatorname{Re} z_j$  of the resonances  $z_j$  lie outside some

neighborhoods of the Landau levels. Thus the Landau levels play a role in the distribution of the resonances. It is known that the spectrum of the operator  $Q = (D_x - By)^2 + D_y^2 + V(x, y)$  with decreasing potential  $V$  is formed by eigenvalues (see [9], [13], [15]). In this paper we establish a limiting absorption principle for  $\lambda \notin \sigma(Q)$ . In particular, we show that there are no embedded eigenvalues outside  $\sigma(Q)$ . This agrees with the result in [5] obtained under the restrictions on the behavior of  $V$  and  $B \rightarrow +\infty$ . On the other hand, the result of Proposition 3 and the estimates (4.3) have been established by X. P. Wang [19] for Stark operators with  $B = 0$ .

Following the results in Section 4 and the representation of  $\xi'(\lambda; B, \epsilon)$  given in [5], it is natural to expect that for  $\lambda \notin \sigma(Q)$  the derivative of the spectral shift function  $\xi'(\lambda; B, \epsilon)$  must be bounded. In fact, we prove the following stronger result.

**Theorem 2.** *Let the potential  $V \in C^\infty(\mathbb{R}^2; \mathbb{R})$  satisfy with some  $\delta > 0$  and  $n \in \mathbb{N}$ ,  $n \geq 2$  the estimates*

$$|\partial_x^\alpha \partial_y^\beta V(x, y)| \leq C_{\alpha, \beta} (1 + |x|)^{-n-\delta-|\alpha|} (1 + |y|)^{-2-\delta-|\beta|}, \quad \forall \alpha, \forall \beta. \quad (1.4)$$

Then for  $\lambda_0 \notin \sigma(Q)$  we have

$$\xi'(\lambda; B, \epsilon) = \mathcal{O}(\epsilon^{n-2}) \quad (1.5)$$

uniformly for  $\lambda$  in a small neighborhood  $\Xi \subset \mathbb{R}$  of  $\lambda_0$ .

The estimate (1.5) has been obtained in [18] in the case of absence of magnetic field  $B = 0$  (for a Breit-Wigner formula see [10], [4] for Stark Hamiltonians and [5] for the operator  $H(B, \epsilon)$ ). Our approach is quite different from that in [18]. Our proof is going without an application of a representation similar to (1.3) which leads to complications connected with the behavior of the spectral function  $e(\cdot, \cdot; \lambda, B, \epsilon)$  corresponding to  $H(B, \epsilon)$ . The formula (1.2) plays a crucial role and our analysis is based on a complex analysis argument combined with a representation of  $f(H)$  involving the almost analytic continuation of  $f \in C_0^\infty(\mathbb{R})$ . In this direction our argument is similar to that developed in [4] and [5].

The plan of this paper is as follows. In Sect. 2 we establish Theorem 1. The embedded eigenvalues and Mourre estimates are examined in Sect. 3. In Sect. 4 we prove Proposition 3 concerning the limiting absorption principle for  $H(B, \epsilon)$ . Finally, in Sect. 5 we establish Theorem 2.

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## 2. REPRESENTATION OF THE SPECTRAL SHIFT FUNCTION

Throughout this work we will use the notations of [3] for symbols and pseudodifferential operators. In particular, if  $m : \mathbb{R}^4 \rightarrow [0, +\infty[$  is an order function (see [3], Definition 7.4), we say that  $a(z, \zeta) \in S^0(m)$  if for every  $\alpha \in \mathbb{N}^4$  there exists  $C_\alpha > 0$  such that

$$|\partial_{z, \zeta}^\alpha a(z, \zeta)| \leq C_\alpha m(z, \zeta).$$

In the special case when  $m = 1$ , we will write  $S^0$  instead of  $S^0(1)$ . We will use the standard Weyl quantization of symbols. More precisely, if  $p(z, \zeta)$ ,  $(z, \zeta) \in \mathbb{R}^4$ , is a symbol in  $S^0(m)$ , then

$P^w(z, D_z)$  is the operator defined by

$$P^w(z, D_z)u(z) = (2\pi)^{-2} \iint e^{i(z-z')\cdot\zeta} p\left(\frac{z+z'}{2}, \zeta\right) u(z') dz' d\zeta, \quad \text{for } u \in \mathcal{S}(\mathbb{R}^2).$$

We denote by  $P^w(z, hD_z)$  the semiclassical quantization obtained as above by quantizing  $p(z, h\zeta)$ .

Our goal in this section is to prove Theorem 1. For this purpose we need some Lemmas. We set

$$Q_0 = H_0 - \epsilon x = (D_x - By)^2 + D_y^2, \quad Q = Q_0 + V,$$

and in Lemma 1 we will use the notation  $H_1 = H$ . For the simplicity we assume that  $\epsilon = B = 1$ . The general case can be covered by the same argument.

**Lemma 1.** *Assume that  $V, \partial_x V \in C^0(\mathbb{R}^2; \mathbb{R}) \cap L^\infty(\mathbb{R}^2; \mathbb{R})$  and let  $\psi \in C_0^\infty(\mathbb{R}^2)$ . Then for  $N \geq 2$ ,  $j = 0, 1$  and for  $\text{Im } z \neq 0$ , the following operators are trace class:*

- i)  $\psi(H_j \pm \mathbf{i})^{-N}$ ,  $\partial_x \circ \psi(H_j \pm \mathbf{i})^{-N-1}$ ,  $(H_j \pm \mathbf{i})\partial_x \circ \psi(H_j \pm \mathbf{i})^{-N-2}$ .
- ii)  $(H_j \pm \mathbf{i})^{-N}\psi$ ,  $(H_j \pm \mathbf{i})^{-N-1}\psi \cdot \partial_x$ .
- iii)  $\psi \circ \partial_x(H_j \pm \mathbf{i})^{-N-1}$ ,  $(H_j \pm \mathbf{i})\psi \circ \partial_x(H_j \pm \mathbf{i})^{-N-2}$ .
- iv)  $(H_j \pm \mathbf{i})\partial_x(H_j \pm \mathbf{i})^{-N-2}\psi$ .
- v)  $(H_1 + \mathbf{i})\partial_x(H_1 + \mathbf{i})^{-N-1}(H_1 - z)^{-1}\psi$ .

Moreover,

$$\|(H_1 + \mathbf{i})\partial_x(H_1 + \mathbf{i})^{-N-1}(H_1 - z)^{-1}\psi\|_{\text{tr}} = \mathcal{O}\left(\frac{|z| + 1}{|\text{Im } z|^2}\right). \quad (2.1)$$

*Proof.* We will prove the lemma only for  $(H_1 + \mathbf{i})$ , the case concerning  $(H_1 - \mathbf{i})$  is similar. On the other hand, the statements for  $(H_0 + \mathbf{i})$  follow from those for  $(H_1 + \mathbf{i})$  when  $V = 0$ .

From the first resolvent equation, we obtain

$$\begin{aligned} (H_1 + z)^{-1} &= (Q_0 + z)^{-1} - (Q_0 + z)^{-1}(x + V)(H_1 + z)^{-1} \\ &= (Q_0 + z)^{-1} + \sum_{j=1}^{N+2} (-1)^j (Q_0 + z)^{-1} \left( (x + V)(Q_0 + z)^{-1} \right)^j \\ &\quad + (-1)^{N+3} \left( (Q_0 + z)^{-1}(x + V) \right)^{N+3} (H_1 + z)^{-1}. \end{aligned} \quad (2.2)$$

Taking  $(N - 1)$  derivatives with respect to  $z$  in the above identity and setting  $z = \mathbf{i}$ , we see that  $(H_1 + \mathbf{i})^{-N}$  is a linear combination of terms

$$\mathcal{K}_N := (Q_0 + \mathbf{i})^{-j_1} W(Q_0 + \mathbf{i})^{-j_2} W \dots (Q_0 + \mathbf{i})^{-j_r} W (H_1 + \mathbf{i})^{-p},$$

with  $j_1 + \dots + j_r \geq N$ ,  $j_1 \geq 1$ ,  $p \geq 0$  and  $W(x) = x + V(x)$ .

Recall that if  $P \in S^0(m)$  with  $m \in L^1(\mathbb{R}^4)$ , (resp.  $m \in L^2(\mathbb{R}^4)$ ) then the corresponding operator is trace class (resp. Hilbert-Schmidt). By using this and the fact that the symbol of  $(Q_0 + \mathbf{i})^{-1}$  is in  $S^0(\langle \xi - y, \eta \rangle^{-2})$ , we deduce that the operator

$$K_{l,p,l',p'}^j := \langle x \rangle^{-l} \langle y \rangle^{-p} (Q_0 + \mathbf{i})^{-j} \langle x \rangle^{l'} \langle y \rangle^{p'}$$

is trace class one for  $l - l', p - p' > 1, j \geq 2$  and Hilbert-Schmidt one for  $l - l', p - p' > 1/2, j \geq 1$ . Next, we write  $\psi \mathcal{K}_N$  as follows

$$\begin{aligned} \psi \mathcal{K}_N &= \psi \langle x \rangle^{3r} \langle y \rangle^{2r} K_{3r, 2r, 3r-2, 2r-2}^{j_1} W \langle x \rangle^{-1} K_{3r-3, 2r-2, 3r-1, 2r-4}^{j_2} W \langle x \rangle^{-1} \\ &\quad \dots W \langle x \rangle^{-1} \mathcal{K}_{3, 2, 1, 0}^{j_r} W \langle x \rangle^{-1} (H_1 + \mathbf{i})^{-p} \end{aligned} \quad (2.3)$$

Since  $j_1 + j_2 + \dots + j_r \geq N \geq 2$ , in the above decomposition, there are at least two Hilbert-Schmidt operators or one of trace class. Combining this with the fact  $\psi \langle x \rangle^{3r} \langle y \rangle^{2r}$ ,  $W \langle x \rangle^{-1}$  and  $(H_1 + \mathbf{i})^{-p}$  are bounded from  $L^2(\mathbb{R}^2)$  into  $L^2(\mathbb{R}^2)$ , we conclude that  $\psi \mathcal{K}_N$  is trace class operator. Thus  $\psi(H_1 + \mathbf{i})^{-N}$  is also a trace class operator. Repeating the same arguments, we obtain the proof for  $\partial_x \circ \psi(H_j \pm \mathbf{i})^{-N-1}$ .

As above to treat  $(H_j \pm \mathbf{i}) \partial_x \circ \psi(H_j \pm \mathbf{i})^{-N-2}$ , it suffices to show that  $(H_j \pm \mathbf{i}) \partial_x \circ \psi \mathcal{K}_N$  is trace class. If we have  $j_1 \geq 2$  the proof is completely similar to that of  $\psi(H_1 + \mathbf{i})^{-N}$ . In the case where  $j_1 = 1$  since  $(H_1 + \mathbf{i}) \partial_x (Q_0 + \mathbf{i})^{-1}$  is not bounded, we have to exploit the following representation

$$\begin{aligned} (H_1 + \mathbf{i}) \partial_x \circ \psi \mathcal{K}_N &= (H_1 + \mathbf{i}) (\partial_x \psi) \mathcal{K}_N \\ &+ (H_1 + \mathbf{i}) \psi (Q_0 + \mathbf{i})^{-1} \partial_x \circ W (Q_0 + \mathbf{i})^{-j_2} W \dots (Q_0 + \mathbf{i})^{-j_r} W (H_1 + \mathbf{i})^{-p}. \end{aligned}$$

Next use the fact that  $\partial_x W \in L^\infty$  and repeat the argument of the proof above.

Recall that  $A$  is trace class if and only if the adjoint operator  $A^*$  is trace class. Consequently, (i) implies (ii). Since  $\psi \cdot \partial_x = \partial_x \cdot \psi - (\partial_x \psi)$ , the assertion (iii) follows from (i).

To deal with (iv), we apply the following obvious identity with  $z = -\mathbf{i}$ ,

$$\partial_x (H - z)^{-1} = (H - z)^{-1} \partial_x + (H - z)^{-1} (1 + \partial_x V) (H - z)^{-1}, \quad (2.4)$$

and obtain

$$(H_1 + \mathbf{i}) \partial_x (H_1 + \mathbf{i})^{-N} \psi = (H_1 + \mathbf{i})^{-N} \partial_x \psi + \sum_{j=0}^{N-1} (H_1 + \mathbf{i})^{-j} (1 + \partial_x V) (H_1 + \mathbf{i})^{-N+j} \psi. \quad (2.5)$$

Applying (i) and (ii) to each term on the right hand side of (2.5), we get (iv).

Now we pass to the proof of (v). Applying (2.4), we obtain

$$\begin{aligned} (H_1 + \mathbf{i}) \partial_x (H_1 + \mathbf{i})^{-N-1} (H_1 - z)^{-1} \psi &= (H_1 + \mathbf{i}) (H_1 - z)^{-1} \partial_x (H_1 + \mathbf{i})^{-N-1} \psi \\ &+ (H_1 + \mathbf{i}) (H_1 - z)^{-1} (1 + \partial_x V) (H_1 - z)^{-1} (H_1 + \mathbf{i})^{-N} \psi. \end{aligned}$$

Combining the above equation with (i), (ii), (iv) and using the estimate

$$\|(H_1 + \mathbf{i}) (H_1 - z)^{-1}\| = \mathcal{O}\left(\frac{|z| + 1}{|\operatorname{Im} z|}\right),$$

we get (2.1). □

**Lemma 2.** *Assume that  $V(x, y) = \phi(x, y)W(x, y)$ , where  $\phi \in C_0^\infty(\mathbb{R}^2; \mathbb{R})$  and  $W, \partial_x W \in C^0(\mathbb{R}^2; \mathbb{R}) \cap L^\infty(\mathbb{R}^2; \mathbb{R})$ . Then for  $N \geq 4$  the operator*

$$(H + \mathbf{i}) \partial_x \left[ (H + \mathbf{i})^{-N} - (H_0 + \mathbf{i})^{-N} \right],$$

*is trace class.*

*Proof.* Taking  $(N - 1)$  derivatives with respect to  $z$  in the resolvent identity

$$(H + z)^{-1} - (H_0 + z)^{-1} = -(H + z)^{-1} V (H_0 + z)^{-1}$$

and setting  $z = \mathbf{i}$ , we see that  $(H + \mathbf{i})^{-N} - (H_0 + \mathbf{i})^{-N}$  is a linear combination of terms

$$(H + \mathbf{i})^{-j} V (H_0 + \mathbf{i})^{-(N+1+j)}$$

with  $1 \leq j \leq N$ . Composing the above terms by  $(H + \mathbf{i}) \partial_x$  and applying Lemma 1, we complete the proof. □

**Lemma 3.** *Assume that  $V$  satisfies the assumptions of Lemma 1. Let  $f \in C_0^\infty(\mathbb{R})$  and  $\psi \in C_0^\infty(\mathbb{R}^2)$ . Then the operators*

$$\psi f(H_i), \quad H_i \psi \partial_x f(H_i), \quad \psi \partial_x H_i f(H_i)$$

are trace class and we have

$$\mathrm{tr} \left( H_i \psi \partial_x f(H_i) \right) = \mathrm{tr} \left( \psi \partial_x H_i f(H_i) \right).$$

*Proof.* Set  $g(x) = (x + \mathbf{i})^4 f(x)$ . Since  $g(H_i)$  is bounded, it follows from Lemma 1 that the operators

$$\psi(H_i + \mathbf{i})^{-4} g(H_i), \quad H_i \psi \partial_x (H_i + \mathbf{i})^{-4} g(H_i), \quad \psi \partial_x (H_i + \mathbf{i})^{-4} H_i g(H_i),$$

are trace class, and the cyclicity of the trace yields

$$\begin{aligned} \mathrm{tr} \left( H_i \psi \partial_x f(H_i) \right) &= \mathrm{tr} \left( H_i \psi \partial_x (H_i + \mathbf{i})^{-4} g(H_i) \right) = \mathrm{tr} \left( H_i g(H_i) \psi \partial_x (H_i + \mathbf{i})^{-4} \right) \\ &= \mathrm{tr} \left( \psi \partial_x (H_i + \mathbf{i})^{-4} g(H_i) H_i \right) = \mathrm{tr} \left( \psi \partial_x H_i f(H_i) \right). \end{aligned}$$

Notice that in the above equalities we have used the fact that the operators  $g(H_i)$ ,  $H_i$  and  $(H_i + \mathbf{i})^{-4}$  commute.  $\square$

**Lemma 4.** *Let  $V$  be as in Lemma 2. Then for every  $f \in C_0^\infty(\mathbb{R})$  the operators*

$$f(H) - f(H_0), \quad \partial_x \left( f(H) - f(H_0) \right) \quad \text{and} \quad (H \pm \mathbf{i}) \partial_x \left( f(H) - f(H_0) \right)$$

are trace class.

*Proof.* Let  $g(x) = (x + \mathbf{i})^4 f(x)$  be as above. We decompose

$$\begin{aligned} (H + \mathbf{i}) \partial_x \left( f(H) - f(H_0) \right) &= (H + \mathbf{i}) \partial_x \left( (H + \mathbf{i})^{-4} - (H_0 + \mathbf{i})^{-4} \right) g(H_0) + \\ &\quad (H + \mathbf{i}) \partial_x (H + \mathbf{i})^{-4} \left( g(H) - g(H_0) \right) = I + II. \end{aligned}$$

According to Lemma 2, the operator  $I$  is trace class. To treat  $II$ , we use the Helffer-Sjöstrand formula

$$\begin{aligned} (II) &= -\frac{1}{\pi} \int \bar{\partial} \tilde{g}(z) (H + \mathbf{i}) \partial_x (H + \mathbf{i})^{-4} \left( (z - H)^{-1} - (z - H_0)^{-1} \right) L(dz) \\ &= -\frac{1}{\pi} \int \bar{\partial} \tilde{g}(z) (H + \mathbf{i}) \partial_x (H + \mathbf{i})^{-4} (z - H)^{-1} V (z - H_0)^{-1} L(dz), \end{aligned}$$

where  $\tilde{g}(z) \in C_0^\infty(\mathbb{C})$  is an almost analytic continuation of  $g$  such that  $\bar{\partial} \tilde{g}(z) = \mathcal{O}(|\mathrm{Im} z|^\infty)$ , while  $L(dz)$  is the Lebesgue measure on  $\mathbb{C}$ . Now applying Lemma 1, (v), we see that the operator

$$(H + \mathbf{i}) \partial_x (H + \mathbf{i})^{-4} (z - H)^{-1} V$$

is trace class. Since  $|z|$  is bounded on  $\mathrm{supp} \tilde{g}$ , we can apply (2.1) to the right hand part of the above equation and combining this with  $\bar{\partial} \tilde{g}(z) = \mathcal{O}(|\mathrm{Im} z|^\infty)$ , we deduce that  $II$  is trace class. Summing up, we conclude that  $(H + \mathbf{i}) \partial_x \left( f(H) - f(H_0) \right)$  is trace class. The same argument works for  $(H - \mathbf{i}) \partial_x \left( f(H) - f(H_0) \right)$ . The proof concerning  $f(H) - f(H_0)$  and  $\partial_x \left( f(H) - f(H_0) \right)$  are similar and simpler.  $\square$

To establish Theorem 1, we also need the following abstract result. For the reader convenience we present a proof.

**Proposition 1.** *Let  $A$  be an operator of trace class on some Hilbert space  $H$  and let  $\{K_n\}$  be sequences of bounded linear operator which converges strongly to  $K \in \mathcal{L}(H)$ . Then*

$$\lim_{n \rightarrow \infty} \|K_n A - K A\|_{\text{tr}} = 0.$$

*Proof.* First assume that  $A$  is a finite rank operator having the form  $A = \sum_{k=1}^m \langle \cdot, \psi_k \rangle \phi_k$ , where  $\psi_k, \phi_k \in H$ . Since

$$\|A\|_{\text{tr}} \leq \sum_{k=1}^m \|\phi_k\| \|\psi_k\|,$$

we have

$$\|(K_n - K)A\|_{\text{tr}} \leq \sum_{k=1}^m \|(K_n - K)\phi_k\| \|\psi_k\| \rightarrow 0, \quad n \rightarrow \infty. \quad (2.6)$$

The general case can be covered by an approximation. Since  $K_n$  converges strongly, it follows from the Banach-Streinhaus theorem that  $\mu = \sup_n \|K_n\| < \infty$ . Let  $\eta$  be an arbitrary positive constant and let  $A_\eta$  be a finite rank operator such that  $\|A - A_\eta\|_{\text{tr}} \leq \frac{\eta}{2\mu}$ . We have

$$\|(K_n - K)A\|_{\text{tr}} \leq \|(K_n - K)(A - A_\eta)\|_{\text{tr}} + \|(K_n - K)A_\eta\|_{\text{tr}} \leq \eta + \|(K_n - K)A_\eta\|_{\text{tr}}.$$

Next we apply (2.6) for the finite rank operator  $A_\eta$  and obtain

$$\lim_{n \rightarrow \infty} \|(K_n - K)A\|_{\text{tr}} \leq \eta,$$

which implies Proposition 1, since  $\eta$  is arbitrary.  $\square$

**Proof of Theorem 1.** Assume first that  $V = \phi W$  where  $\phi \in C_0^\infty(\mathbb{R}^2; \mathbb{R})$  and  $W, \partial_x W \in C^0(\mathbb{R}^2; \mathbb{R}) \cap L^\infty(\mathbb{R}^2; \mathbb{R})$ . Choose a function  $\chi \in C_0^\infty(\mathbb{R}^2)$  such that  $\chi = 1$  for  $|(x, y)| \leq 1$ . For  $R > 0$  set

$$\chi_R(x, y) = \chi\left(\frac{x}{R}, \frac{y}{R}\right),$$

and introduce

$$B_R := [\chi_R \partial_x, H]f(H) - [\chi_R \partial_x, H_0]f(H_0).$$

Here  $[A, B] = AB - BA$  denotes the commutator of  $A$  and  $B$ . According to Lemma 3, we have

$$\text{tr}\left([\chi_R \partial_x, H]f(H)\right) = \text{tr}\left([\chi_R \partial_x, H_0]f(H_0)\right) = 0.$$

Thus

$$\text{tr}(B_R) = 0. \quad (2.7)$$

On the other hand, a simple calculus shows that

$$B_R = \chi_R\left([\partial_x, H]f(H) - [\partial_x, H_0]f(H_0)\right) + [\chi_R, H_0]\partial_x\left(f(H) - f(H_0)\right) := B_R^1 + B_R^2, \quad (2.8)$$

where we have used that  $[\chi_R, H] = [\chi_R, H_0]$ .

Since  $[\partial_x, H] = 1 + \partial_x V$  and  $[\partial_x, H_0] = 1$ , it follows from Lemma 3, Lemma 4 and Proposition 1 that

$$\lim_{R \rightarrow \infty} \text{tr}(B_R^1) = \text{tr}\left(f(H) - f(H_0)\right) + \text{tr}\left(\partial_x V f(H)\right). \quad (2.9)$$

Next we claim that

$$\lim_{R \rightarrow \infty} B_R^2 = 0. \quad (2.10)$$



Using that  $[\chi_R, H_0] = \frac{2}{R}(D_x\chi_R)(D_x - y) - \frac{2}{R}(D_y\chi_R)D_y + \frac{1}{R^2}(\Delta\chi_R)$ , we decompose  $B_R^2$  as a sum of three terms  $B_R^2 = I_R^1 + I_R^2 + I_R^3$ , where

$$\begin{aligned} I_R^1 &= -\frac{2}{R}(D_x\chi_R)(D_x - y)\partial_x\left(f(H) - f(H_0)\right), \\ I_R^2 &= -\frac{2}{R}(D_y\chi_R)D_y\partial_x\left(f(H) - f(H_0)\right), \\ I_R^3 &= \frac{1}{R^2}(\Delta\chi_R)\partial_x\left(f(H) - f(H_0)\right). \end{aligned}$$

To treat  $I_R^1$ , we set  $Q = H - x$  and write

$$\begin{aligned} I_R^1 &= -\frac{2}{R}(D_x\chi_R)(D_x - y)(Q_0 - \mathbf{i})^{-1}(H - \mathbf{i})\partial_x\left(f(H) - f(H_0)\right) \\ &\quad + \frac{2}{R}(D_x\chi_R)[(D_x - y)(Q - \mathbf{i})^{-1}, x]\partial_x\left(f(H) - f(H_0)\right) \\ &\quad + \frac{2}{R}x(D_x\chi_R)(D_x - y)(Q - \mathbf{i})^{-1}\partial_x\left(f(H) - f(H_0)\right). \end{aligned}$$

The operators  $[(D_x - y)(Q - \mathbf{i})^{-1}, x]$  and  $(D_x - y)(Q - \mathbf{i})^{-1}$  are bounded, while  $\partial_x\left(f(H) - f(H_0)\right)$  and  $(H - \mathbf{i})\partial_x\left(f(H) - f(H_0)\right)$  are trace class operators (see Lemma 4). On the other hand,  $\frac{2}{R}(D_x\chi_R)$ ,  $\frac{2}{R}x(D_x\chi_R)$  converges strongly to zero. Indeed, since  $\chi(x, y) = 1$  for  $|(x, y)| \leq 1$ , we get

$$\int \left| \frac{x}{R}(D_x\chi_R)u \right|^2 dx dy \leq \sup_{(x,y) \in \mathbb{R}^2} |xD_x\chi(x, y)| \int_{\{|(x,y)| \geq R\}} |u|^2 dx dy \rightarrow 0, \quad R \rightarrow \infty,$$

for all  $u \in L^2(\mathbb{R}^2)$ . Applying Proposition 1, we conclude that

$$\lim_{R \rightarrow \infty} I_R^1 = 0. \quad (2.11)$$

To deal with  $I_R^2, I_R^3$ , notice that the operators  $D_y(Q - \mathbf{i})^{-1}$  and  $[D_y(Q - \mathbf{i})^{-1}, x]$  are bounded and we repeat the above argument. Thus we deduce

$$\lim_{R \rightarrow \infty} I_R^j = 0, \quad j = 2, 3. \quad (2.12)$$

Consequently, (2.11) and (2.12) imply (2.10) and the claim is proved. Now, combining (2.7), (2.8), (2.9) and (2.10), we obtain Theorem 1 in the case where  $V$  satisfies the assumption of Lemma 2 and  $\epsilon = 1$ .

**Proposition 2.** *Assume that  $V \in L^\infty(\mathbb{R}^2; \mathbb{R})$  satisfies (1.1). Then for  $z \notin \mathbb{R}$ ,  $z' \notin \mathbb{R}$  the operators  $(z - H)^{-1}V(z' - H)^{-1}$ ,  $V(z - H)^{-1}(z' - H)^{-1}$ ,  $(H - z)^{-1} - (H_0 - z)^{-1}$  are trace class and*

$$\begin{aligned} \|(z - H)^{-1}V(z' - H)^{-1}\|_{\text{tr}} &\leq C_1 |\text{Im } z|^{-1} |\text{Im } z'|^{-1}, \\ \|V(z - H)^{-1}(z' - H)^{-1}\|_{\text{tr}} &\leq C_1 |\text{Im } z|^{-1} |\text{Im } z'|^{-1}. \end{aligned} \quad (2.13)$$

Moreover, if  $g \in C_0^\infty(\mathbb{R})$ , then the operator  $Vg(H)$  is trace class.

*Proof.* Set  $g_\delta(x, y) = \langle x \rangle^{-1-\frac{\delta}{2}} \langle y \rangle^{-\frac{1+\delta}{2}}$  and  $f_\delta(x, y) = \langle x \rangle^{-2-\delta} \langle y \rangle^{-1-\delta}$ , where  $\delta$  is the constant in (1.1). According to Lemma 8 in the Appendix,  $g_\delta(H_0 + \mathbf{i})^{-1}$ ,  $(H_0 + \mathbf{i})^{-1}g_\delta$  are Hilbert-Schmidt operators and  $f_\delta(H_0 + \mathbf{i})^{-2}$  is a trace one. Since  $g_\delta^{-1}Vg_\delta^{-1}, Vf_\delta^{-1} \in L^\infty$ , it follows that

$$(H_0 + \mathbf{i})^{-1}V(H_0 + \mathbf{i})^{-1} = (H_0 + \mathbf{i})^{-1}g_\delta[g_\delta^{-1}Vg_\delta^{-1}]g_\delta(H_0 + \mathbf{i})^{-1}$$

and  $V(H_0 + \mathbf{i})^{-2}$  are trace class operators. Next we write

$$(H + \mathbf{i})^{-1} - (H_0 + \mathbf{i})^{-1} = -(H_0 + \mathbf{i})^{-1}V(H_0 + \mathbf{i})^{-1} + (H + \mathbf{i})^{-1}V(H_0 + \mathbf{i})^{-1}V(H_0 + \mathbf{i})^{-1}$$

and conclude that  $(H + \mathbf{i})^{-1} - (H_0 + \mathbf{i})^{-1} = -(H + \mathbf{i})^{-1}V(H_0 + \mathbf{i})^{-1}$  is trace class. Now consider the following equalities

$$\begin{aligned} (\mathbf{i} + H)^{-1}V(\mathbf{i} + H)^{-1} &= (\mathbf{i} + H_0)^{-1}V(\mathbf{i} + H_0)^{-1} + (\mathbf{i} + H)^{-1}V(\mathbf{i} + H_0)^{-1}V(\mathbf{i} + H_0)^{-1} + \\ &(\mathbf{i} + H_0)^{-1}V(\mathbf{i} + H_0)^{-1}V(\mathbf{i} + H)^{-1} + (\mathbf{i} + H)^{-1}V(\mathbf{i} + H_0)^{-1}V(\mathbf{i} + H_0)^{-1}V(\mathbf{i} + H)^{-1} \end{aligned}$$

and

$$V(H + \mathbf{i})^{-2} = V(H_0 + \mathbf{i})^{-2} - V(H_0 + \mathbf{i})^{-1}(H + \mathbf{i})^{-1}V(H_0 + \mathbf{i})^{-1} - V(H + \mathbf{i})^{-1}V(H_0 + \mathbf{i})^{-1}(H + \mathbf{i})^{-1}.$$

By using the trace class properties established above, we get (2.13) for  $z = z' = -i$ . By applying the first resolvent equation

$$(H - z)^{-1} = (H + \mathbf{i})^{-1} + (\mathbf{i} - z)(H + \mathbf{i})^{-1}(H - z)^{-1},$$

we obtain the general case.

To examine  $Vg(H)$ , consider the function  $h(x) = (x + \mathbf{i})^2g(x)$ . Then  $Vg(H) = V(H + \mathbf{i})^{-2}h(H)$  and since  $V(H + \mathbf{i})^{-2}$  is trace class, we obtain the result.  $\square$

For  $R > 0$  introduce

$$H_R := H_0 + \chi_R(x, y)V(x, y),$$

where  $\chi_R(x, y) = \chi(\frac{x}{R}, \frac{y}{R})$  with  $\chi \in C_0^\infty(\mathbb{R}^2)$  such that  $\chi = 1$  in a neighborhood of  $|(x, y)| \leq 1$ .

**Remark 1.** *The result of Proposition 2 concerning the trace class property of  $(H - z)^{-1} - (H_0 - z)^{-1}$ ,  $\text{Im } z \neq 0$ , improves considerably Proposition 2 in [5], where much more regular potentials have been examined. On the other hand, if the potential  $V$  satisfies (1.1) and  $V, \partial_x V \in C^0(\mathbb{R}^2; \mathbb{R}) \cap L^\infty(\mathbb{R}^2; \mathbb{R})$ , then the statements of Proposition 2 hold for the operators  $(z - H_R)^{-1}V(z' - H)^{-1}$ ,  $z \notin \mathbb{R}, z' \notin \mathbb{R}$ .*

The proof of Theorem 1 in the general case will be a simple consequence of the following

**Lemma 5.** *Let  $V(x, y)$  be as in Theorem 1. Then for  $f \in C_0^\infty(\mathbb{R})$  we have*

$$\lim_{R \rightarrow \infty} \text{tr} \left( f(H_R) - f(H) \right) = 0, \quad (2.14)$$

$$\lim_{R \rightarrow \infty} \text{tr} \left( \partial_x(\chi_R V)f(H_R) \right) = \text{tr} \left( \partial_x V f(H) \right). \quad (2.15)$$

*Proof.* Let  $g(x) = (x + \mathbf{i})f(x)$  be as above. We decompose

$$f(H_R) - f(H) = \left( (H_R + \mathbf{i})^{-1} - (H + \mathbf{i})^{-1} \right) g(H) + (H_R + \mathbf{i})^{-1} \left( g(H_R) - g(H) \right) = J_R + K_R.$$

From the first resolvent identity, we obtain

$$J_R = (H_R - \mathbf{i})^{-1}(1 - \chi_R)V(H + \mathbf{i})^{-1}g(H) = (H_R - \mathbf{i})^{-1}(1 - \chi_R)Vf(H).$$

According to Proposition 2, the operator  $Vf(H)$  is trace class and  $(H_R - \mathbf{i})^{-1}(1 - \chi_R)$  converges strongly to zero. Then from Proposition 1 it follows that

$$\lim_{R \rightarrow \infty} \text{tr } J_R = 0. \quad (2.16)$$

To treat  $\text{tr } K_R$ , as in the proof of Lemma 4, we use the Helffer-Sjöstrand formula and write

$$\begin{aligned} \text{tr } K_R &= -\frac{1}{\pi} \int \bar{\partial} \tilde{g}(z) \text{tr} \left( (H_R + \mathbf{i})^{-1} \left( (z - H_R)^{-1} - (z - H)^{-1} \right) \right) L(dz) \\ &= \frac{1}{\pi} \int \bar{\partial} \tilde{g}(z) \text{tr} \left( (H_R + \mathbf{i})^{-1} (z - H_R)^{-1} (1 - \chi_R) V (z - H)^{-1} \right) L(dz). \end{aligned}$$

By cyclicity of the traces we obtain

$$\begin{aligned} \text{tr} \left( (H_R + \mathbf{i})^{-1} (z - H_R)^{-1} (1 - \chi_R) V (z - H)^{-1} \right) &= \text{tr} \left( (z - H_R)^{-1} (1 - \chi_R) V (z - H)^{-1} (H_R + \mathbf{i})^{-1} \right) \\ &= \text{tr} \left( (z - H_R)^{-1} (1 - \chi_R) V (z - H)^{-1} (H + \mathbf{i})^{-1} \right) \\ &\quad + \text{tr} \left( (1 - \chi_R) V (H_R + \mathbf{i})^{-1} (z - H_R)^{-1} (1 - \chi_R) V (z - H)^{-1} (H + \mathbf{i})^{-1} \right). \end{aligned}$$

Now notice that for  $z \notin \mathbb{R}$  the operators  $(1 - \chi_R) V (H_R + \mathbf{i})^{-1} (z - H_R)^{-1} (1 - \chi_R)$  and  $(z - H_R)^{-1} (1 - \chi_R)$  converge strongly to zero. On the other hand, from Proposition 2 we deduce that the operator  $V (z - H)^{-1} (\mathbf{i} + H)^{-1}$  is trace class. Thus for  $z \notin \mathbb{R}$ , we conclude that the integrand converge to 0 as  $R \rightarrow \infty$ . An application of the Lebesgue convergence domination theorem combined with the estimates (2.13) yield

$$\lim_{R \rightarrow \infty} \text{tr } K_R = 0. \quad (2.17)$$

Putting together (2.16) and (2.17), we obtain (2.14).

Next, we pass to the proof of (2.15). A simple calculus shows that

$$\partial_x (\chi_R V) f(H_R) = \partial_x (\chi_R V) (f(H_R) - f(H)) + \frac{1}{R} (\partial_x \chi)_R V f(H) + (\chi_R \partial_x V) f(H). \quad (2.18)$$

Repeating the same arguments as in the proof of (2.14), we show that

$$\lim_{R \rightarrow \infty} \text{tr} \left( \partial_x (\chi_R V) (f(H_R) - f(H)) \right) = 0. \quad (2.19)$$

On the other hand, since  $\frac{1}{R} (\partial_x \chi)_R$  (resp.  $\chi_R$ ) converges strongly to zero (resp. 1), it follows from Proposition 1 that

$$\lim_{R \rightarrow \infty} \text{tr} \left( \frac{1}{R} (\partial_x \chi)_R V f(H) \right) = 0, \quad \lim_{R \rightarrow \infty} \text{tr} \left( \chi_R \partial_x V f(H) \right) = \text{tr} \left( \partial_x V f(H) \right),$$

which together with (2.18) and (2.19) yield (2.15).  $\square$

**End of the proof of Theorem 1.** Applying Theorem 1 to  $H_R$ , we obtain :

$$\text{tr} \left[ f(H_R) - f(H) \right] + \text{tr} \left[ f(H) - f(H_0) \right] = \text{tr} \left[ f(H_R) - f(H_0) \right] = -\text{tr} \left( \partial_x (\chi_R V) f(H) \right),$$

and an application of Lemma 5 implies Theorem 1.

## 3. MOURRE ESTIMATE AND EMBEDDED EIGENVALUES

Consider the operator

$$Q = (D_x - By)^2 + D_y^2 + V(x, y),$$

and set  $\langle x \rangle = (1 + |x|^2)^{1/2}$ ,  $\langle D_x \rangle = (1 + D_x^2)^{1/2}$ .

**Lemma 6.** *Assume that  $V, \partial_x V \in C^0(\mathbb{R}^2; \mathbb{R}) \cap L^\infty(\mathbb{R}^2; \mathbb{R})$  and let  $\|\mathbb{I}_{\{|x|+|y|>R\}}(x, y)\partial_x V\|_{L^\infty} \rightarrow 0$  for  $R \rightarrow +\infty$ . Then for all  $f \in C_0^\infty(\mathbb{R})$ , the operator  $f(H)\partial_x V f(H)$  is compact.*

*Proof.* Let  $\varphi(x, y) \in C_0^\infty(\mathbb{R}^2)$  be equal to one near zero. Set  $\varphi_n(x, y) = \varphi(\frac{x}{n}, \frac{y}{n})$ . According to Lemma 3, the operator  $f(H)\varphi_n\partial_x V f(H)$  is trace class. The set of compact operators is closed with respect to the norm  $\|\cdot\|_{\mathcal{L}(L^2)}$  and the lemma follows from the obvious estimate

$$\|f(H)(1 - \varphi_n)\partial_x V f(H)\|_{\mathcal{L}(L^2)} \leq \|f^2(H)\|_{\mathcal{L}(L^2)} \|(1 - \varphi_n)\partial_x V\|_\infty.$$

□

**Theorem 3.** *Let  $[a, b] \subset \mathbb{R}$ . Under the assumptions of Lemma 6, there exists a compact operator  $K$  such that*

$$\mathbb{I}_{[a,b]}(H)[\partial_x, H]\mathbb{I}_{[a,b]}(H) \geq \epsilon\mathbb{I}_{[a,b]}(H) + \mathbb{I}_{[a,b]}(H)K\mathbb{I}_{[a,b]}(H). \quad (3.1)$$

*Proof.* Since the operator  $\partial_x$  commutes with  $(D_x - By)$  and  $D_y^2$ , we have  $[\partial_x, H] = \epsilon + \partial_x V$ . Consequently,

$$\begin{aligned} \mathbb{I}_{[a,b]}(H)[\partial_x, H]\mathbb{I}_{[a,b]}(H) &= \epsilon\mathbb{I}_{[a,b]}(H) + \mathbb{I}_{[a,b]}(H)\partial_x V\mathbb{I}_{[a,b]}(H) \\ &= \epsilon\mathbb{I}_{[a,b]}(H) + \mathbb{I}_{[a,b]}f(H)\partial_x V f(H)\mathbb{I}_{[a,b]}(H), \end{aligned} \quad (3.2)$$

where  $f \in C_0^\infty(\mathbb{R})$  is a cut-off function such that  $f = 1$  on  $[a, b]$ . Thus, Theorem 3 follows from Lemma 6. □

The use of commutators with the operator  $\partial_x$  is well known for the analysis of the operator without magnetic field ( $B = 0$ ) (see the pioneering work [2] and [1] for a more complete list of references). On the other hand, to treat crossed magnetic and electric fields we need Lemma 1 and Lemma 3.

**Corollary 1.** *In addition to the assumptions of Theorem 3 assume that  $\partial_x^2 V \in C^0(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$ . Then the point spectrum of  $H$  in  $[a, b]$  is finite and with finite multiplicity. Moreover, the singular continuous spectrum of  $H$  is empty.*

*Proof.* Set  $A = D_x$  and let  $\alpha \in \mathbb{R}$ . The explicit formula

$$e^{i\alpha A}(H + \mathbf{i})^{-1} = (e^{i\alpha A}H e^{-i\alpha A} + \mathbf{i})^{-1} e^{i\alpha A} = (H + \epsilon\alpha + V(x + \alpha, y) - V(x, y) + \mathbf{i})^{-1} e^{i\alpha A}$$

shows that  $e^{i\alpha A}$  leaves  $D(H)$  invariant. On the other hand, since

$$\begin{aligned} \|He^{i\alpha A}(H + \mathbf{i})^{-1}\psi\| &= \|e^{-i\alpha A}H e^{i\alpha A}(H + \mathbf{i})^{-1}\psi\| \\ &= \left\| \left( H - \epsilon\alpha + V(x - \alpha, y) - V(x, y) \right) (H + \mathbf{i})^{-1}\psi \right\|, \end{aligned}$$

we deduce that for each  $\varphi \in D(H)$

$$\sup_{|\alpha| < 1} \|He^{i\alpha A}\varphi\| < \infty.$$

Combining this with the fact  $\mathbf{i}[A, H] = \epsilon + \partial_x V$ ,  $[A, [A, H]] = -\partial_x^2 V$  and using (3.1), we conclude that the self-adjoint operator  $A$  is a conjugate operator for  $H$  at every  $E \in \mathbb{R}$  in the sense of [14]. Consequently, Corollary 1 follows from the main result in [14] (see also [1], [6]).

□

**Remark 2.** For any sign-definite and bounded potential  $V(x, y)$  such that  $|V(x, y)| \rightarrow 0$  as  $|x| + |y| \rightarrow \infty$  sufficiently fast in [15] and [13] it was established that for  $\epsilon = 0$  the potential  $V$  creates an infinite number of eigenvalues of  $Q$  which accumulate to Landau levels. The above corollary shows that only a finite number of these eigenvalues may survive in the presence of a non vanishing constant electric field. In general, the problem of absence of embedded eigenvalues when  $\epsilon \neq 0$  remains open and this is an interesting conjecture.

For a fixed value of  $\epsilon \neq 0$ , the following result shows that there are potentials for which  $H$  has absolutely continuous spectrum without embedded eigenvalues.

**Corollary 2.** Fix  $\epsilon > 0$ . Assume that  $\partial_x^\alpha V \in C^0(\mathbb{R}^2; \mathbb{R}) \cap L^\infty(\mathbb{R}^2; \mathbb{R})$ ,  $\alpha = 0, 1, 2$  and

$$\epsilon + \partial_x V(x, y) > c > 0, \quad (3.3)$$

uniformly on  $(x, y) \in \mathbb{R}^2$ . Then  $H$  has no eigenvalues. Moreover, for  $s > 1/2$ , the following estimates holds uniformly on  $\lambda$  in a compact interval

$$\|\langle D_x \rangle^{-s} (H - \lambda \pm i0)^{-1} \langle D_x \rangle^{-s}\| = \mathcal{O}_\epsilon(1). \quad (3.4)$$

*Proof.* Let  $[a, b]$  be a compact interval in  $\mathbb{R}$ . From (3.1) and (3.3), we have

$$\mathbb{I}_{[a,b]}(H) [\partial_x, H] \mathbb{I}_{[a,b]}(H) \geq c \mathbb{I}_{[a,b]}(H). \quad (3.5)$$

According to the proof of Corollary 1,  $A = D_x$  is a conjugate operator in the sense of [14]. Combining this with (3.5) we deduce from [14] that  $H$  has no eigenvalue in  $\mathbb{R}$ . Applying once more Mourre theorem (see [14], [1], [6]), we obtain the estimate (3.4). □

#### 4. LIMITING ABSORPTION PRINCIPLE

In this section we treat the case when  $\epsilon$  is small enough. Notice that when  $\epsilon$  tends to zero in general the assumption  $\epsilon + \partial_x V > c > 0$  is not satisfied and we cannot apply Corollary 2. Our goal is to study the behavior of the resolvent  $(H - \lambda \pm i\delta)^{-1}$  as  $\delta \rightarrow 0$  for  $\lambda \notin \sigma(Q)$ . For such  $\lambda$  we could have eigenvalues of  $H$  and a direct application of Mourre argument is not possible. We will obtain the result assuming that  $\epsilon$  is small and for this purpose we need the following

**Lemma 7.** Assume that  $V \in L^\infty(\mathbb{R}^2; \mathbb{R})$  and let  $\lambda \notin \sigma(Q)$ . Let  $\chi \in C_0^\infty(\mathbb{R}; \mathbb{R})$  be equal to 1 near  $\lambda$  and let  $\text{supp } \chi \cap \sigma(Q) = \emptyset$ . Then

$$\|\chi(H) \langle x \rangle^{-2}\| \leq C\epsilon^2. \quad (4.1)$$

*Proof.* Since  $\text{supp } \chi \cap \sigma(Q) = \emptyset$ , the operators  $(z - Q)^{-1}$  and  $(z - Q)^{-1} x (z - Q)^{-1}$  are analytic operator valued functions for  $z$  in a complex neighborhood of  $\text{supp } \chi$ . Let  $\tilde{\chi}(z) \in C_0^\infty(\mathbb{C})$  be an almost analytic continuation of  $\chi(x)$  such that

$$\bar{\partial} \tilde{\chi}(z) = \mathcal{O}(|\text{Im } z|^\infty)$$

and  $\text{supp } \tilde{\chi}(z) \cap \sigma(Q) = \emptyset$ . We have the representation

$$\chi(H) = -\frac{1}{\pi} \int \bar{\partial} \tilde{\chi}(z) (z - H)^{-1} L(dz),$$

where  $L(dz)$  is the Lebesgue measure in  $\mathbb{C}$ . By using the resolvent identity, we get

$$(z - H)^{-1} = (z - Q)^{-1} + \epsilon (z - Q)^{-1} x (z - Q)^{-1} + \epsilon^2 (z - H)^{-1} x (z - Q)^{-1} x (z - Q)^{-1},$$

and we obtain

$$\begin{aligned}\chi(H) &= \chi(Q) - \frac{\epsilon}{\pi} \int \bar{\partial} \tilde{\chi}(z)(z-Q)^{-1}x(z-Q)^{-1}L(dz) \\ &\quad - \frac{\epsilon^2}{\pi} \int \bar{\partial} \tilde{\chi}(z)(z-H)^{-1}x(z-Q)^{-1}x(z-Q)^{-1}L(dz).\end{aligned}$$

Since  $\text{supp } \tilde{\chi}(z) \cap \sigma(Q) = \emptyset$ , the first two terms on the right hand side vanish. Consequently,

$$\chi(H) = -\frac{\epsilon^2}{\pi} \int \bar{\partial} \tilde{\chi}(z)(z-H)^{-1}x(z-Q)^{-1}x(z-Q)^{-1}L(dz). \quad (4.2)$$

Next we observe that

$$x(z-Q)^{-1} = (z-Q)^{-1}x + (z-Q)^{-1}[x, Q](z-Q)^{-1} = (z-Q)^{-1}x + L_1.$$

We have  $[x, Q] = 2(D_x - By)$ . Thus it is easy to see that for  $z \notin \sigma(Q)$ ,  $L_1 = (z-Q)^{-1}[x, Q](z-Q)^{-1}$  is a bounded operator since  $(D_x - By)(\mathbf{i} - Q)^{-1}$  is bounded and  $(z-Q)^{-1} = (\mathbf{i} - Q)^{-1} + (\mathbf{i} - Q)^{-1}(\mathbf{i} - z)(z-Q)^{-1}$ . We write

$$\begin{aligned}x(z-Q)^{-1}x(z-Q)^{-1} &= (z-Q)^{-1}x(z-Q)^{-1}x \\ &\quad + (z-Q)^{-1}xL_1 + L_1(z-Q)^{-1}x + L_1^2 = \sum_{j=1}^4 I_j.\end{aligned}$$

The operators  $I_4 = L_1^2$  and  $I_3 = L_1(z-Q)^{-1}x\langle x \rangle^{-2}$  are bounded. To see that  $I_1\langle x \rangle^{-2}$  is bounded, note that

$$I_1\langle x \rangle^{-2} = (z-Q)^{-2}x^2\langle x \rangle^{-2} + (z-Q)^{-1}L_1x\langle x \rangle^{-2}.$$

Finally,

$$I_2\langle x \rangle^{-2} = (z-Q)^{-2}x[x, Q](z-Q)^{-1}\langle x \rangle^{-2} + (z-Q)^{-1}L_1[x, Q](z-Q)^{-1}\langle x \rangle^{-2}$$

and since the second term on the right hand side is bounded, it remains to examine the operator

$$x[x, Q](z-Q)^{-1}\langle x \rangle^{-2} = [x, Q]x(z-Q)^{-1}\langle x \rangle^{-2} + 2\mathbf{i}(z-Q)^{-1}\langle x \rangle^{-2}.$$

Applying the above argument, we see that the last operator is bounded. Consequently, the operator under integration in (4.2) is bounded by  $\mathcal{O}(|\text{Im } z|^{-1})$  and this proves the statement.  $\square$

**Proposition 3.** *Assume that  $\partial_x^\alpha V \in C^0(\mathbb{R}^2; \mathbb{R}) \cap L^\infty(\mathbb{R}^2; \mathbb{R})$  for  $\alpha = 0, 1, 2$  and let  $\langle x \rangle^2 \partial_x V \in L^\infty(\mathbb{R}^2)$ . Let  $[a, b]$  be a compact interval such that  $[a, b] \cap \sigma(Q) = \emptyset$ . Then for  $s > 1/2$  and sufficiently small  $\epsilon_0 > 0$  we have the following estimate uniformly with respect to  $\lambda \in [a, b]$  and  $\epsilon \in ]0, \epsilon_0]$*

$$\|\langle D_x \rangle^{-s}(H - \lambda \pm \mathbf{i}0)^{-1}\langle D_x \rangle^{-s}\| \leq C\epsilon^{-1}. \quad (4.3)$$

Moreover,  $H$  has no embedded eigenvalues and singular continuous spectrum in  $[a, b]$ .

*Proof.* Let  $[a - \delta, b + \delta] \cap \sigma(Q) = \emptyset$  for  $0 < \delta \ll 1$ . Choose a function  $\chi(t) \in C_0^\infty(\mathbb{R}; \mathbb{R})$  such that  $\text{supp } \chi \subset [a - \delta, b + \delta]$  and  $\chi(t) = 1$  for  $a_1 = a - \delta/2 \leq t \leq b + \delta/2 = b_1$ . Then

$$\begin{aligned}\mathbb{I}_{[a_1, b_1]}(H)[\partial_x, H]\mathbb{I}_{[a_1, b_1]}(H) &= \epsilon \mathbb{I}_{[a_1, b_1]}(H) + \mathbb{I}_{[a_1, b_1]}(H)\partial_x V \mathbb{I}_{[a_1, b_1]}(H) \\ &= \epsilon \mathbb{I}_{[a_1, b_1]}(H) + \mathbb{I}_{[a_1, b_1]}(H)\left(\chi(H)\langle x \rangle^{-2}\right)\left(\langle x \rangle^2 \partial_x V\right)\mathbb{I}_{[a_1, b_1]}(H)\end{aligned}$$

Our assumption implies that the multiplication operator  $\langle x \rangle^2 \partial_x V \in L^\infty$ , while Lemma 7 says that

$$\|\chi(H)\langle x \rangle^{-2}\| \leq C\epsilon^2.$$

Thus

$$\mathbb{I}_{[a_1, b_1]}(H) \left( \chi(H) \langle x \rangle^{-2} \right) \left( \langle x \rangle^2 \partial_x V \right) \mathbb{I}_{[a_1, b_1]}(H) \leq C_1 \epsilon^2 \mathbb{I}_{[a_1, b_1]}(H)$$

and with a constant  $c_0 > 0$  we deduce

$$\mathbb{I}_{[a_1, b_1]}(H) [\partial_x, H] \mathbb{I}_{[a_1, b_1]}(H) \geq c_0 \epsilon \mathbb{I}_{[a_1, b_1]}(H).$$

Then it is well known (see for instance [14], [1], [6]) that for  $\lambda \in [a, b]$  we get (4.3) and  $H$  has no eigenvalues and singular continuous spectrum in  $[a, b]$ .  $\square$

**Remark 3.** *As we mentioned in Remark 2 for sign-definite rapidly decreasing potentials the spectrum of the operator  $Q$  is formed by infinite number eigenvalues having as points of accumulation the Landau levels  $\mu_n = (2n+1)B$ ,  $n \in \mathbb{N}$ . For such potentials Proposition 3 shows that the embedded eigenvalues of  $H$  could appear only in small neighborhoods of the eigenvalues of  $Q$ . Since in every interval we may have only a finite number of eigenvalues of  $H$ , it is clear that for some eigenvalues  $\nu$  of  $Q$  there are no eigenvalues of  $H$  in their neighborhoods. Moreover, it was proved in [12] that for potentials  $V \in C_0^\infty(\mathbb{R}^2)$  we have  $\sigma(Q) \cap ]\mu_n - B, \mu_n + B[ \subset (\mu_n - Cn^{-1/2}, \mu_n + Cn^{-1/2})$ ,  $n \geq N$  with  $C > 0$  and  $N$  depending only on  $\sup |V|$  and the diameter of the support of  $V$ . Thus for  $M$  large the embedded eigenvalues  $\lambda \geq M$  of  $H$  are sufficiently close to Landau levels  $\Lambda_n$ .*

## 5. ESTIMATES FOR THE DERIVATIVE OF THE SPECTRAL SHIFT FUNCTION

First we notice that the assumption (1.4) makes possible to define the spectral shift function  $\xi(\lambda, \epsilon)$  related to operators  $H_0(\epsilon) = H_0(B, \epsilon)$  and  $H(\epsilon) = H_0(B, \epsilon) + V(x, y)$  by the equality

$$\langle \xi', f \rangle = \text{tr} \left( f(H(\epsilon)) - f(H_0(\epsilon)) \right), \quad f \in C_0^\infty(\mathbb{R}).$$

Here and below we omit the dependence of  $B$  in the notations. Our purpose in this section is to establish Theorem 2. For the proof we need the following

**Proposition 4.** *Under the assumptions of Theorem 2, for  $\lambda_0 \notin \sigma(Q)$  and  $1/2 < s < \min(1/2 + \delta/4, 1)$  the operator*

$$\langle D_x \rangle^s \partial_x V \left[ (Q - z)^{-1} x \right]^n \langle D_x \rangle^s$$

*is trace class for  $z$  in a small complex neighborhood  $\Xi \subset \mathbb{C}$  of  $\lambda_0$ .*

*Proof.* Before starting the proof, notice that it is easy to establish the statement for  $z \ll 0$  since in this case the operator  $(Q - z)^{-1}$  is a pseudodifferential one and we can apply the calculus of pseudodifferential operators and the criteria which guarantees that a pseudodifferential operator is trace class (see for instance, [3], Theorem 9.4). For  $z \in \mathbb{R}^+ \setminus \sigma(Q)$  this is not the case and  $(Q - z)^{-1}$  is a bounded operator but not a pseudodifferential one. We may replace  $(Q - z)^{-1}$  by the pseudodifferential operator  $(Q - \mathbf{i})^{-1}$  modulo bounded operators but therefore it is difficult to examine the product involving many bounded operators and factors  $x^k$ . To overcome this difficulty, we are going to apply a convenient decomposition by product of operators having in mind that the operator on the left of a such product must be trace class one.

First we treat the case  $n = 2$ , the general case will be covered by a recurrence. We start with the analysis of the operator

$$\langle D_x \rangle^{2s} \partial_x V [(Q - z)^{-1} x]^2. \tag{5.1}$$

Our goal is to show that (5.1) is a trace class operator. Write

$$\langle D_x \rangle^{2s} \partial_x V \langle x \rangle^2 \langle x \rangle^{-2} (Q - z)^{-1} x (Q - z)^{-1} x = \langle D_x \rangle^{2s} (\partial_x V) \langle x \rangle^2 (Q - z)^{-1} \langle x \rangle^{-2} x (Q - z)^{-1} x$$

$$\begin{aligned}
& + \langle D_x \rangle^{2s} \partial_x V \langle x \rangle^2 (Q - z)^{-1} [Q, \langle x \rangle^{-2}] (Q - z)^{-1} x (Q - z)^{-1} x \\
& = \langle D_x \rangle^{2s} \partial_x V \langle x \rangle^2 (Q - z)^{-2} \left[ \langle x \rangle^{-2} x^2 + [Q, \langle x \rangle^{-2} x] (Q - z)^{-1} x \right] \\
& + \langle D_x \rangle^{2s} \partial_x V \langle x \rangle^2 (Q - z)^{-1} [Q, \langle x \rangle^{-2}] (Q - z)^{-1} x (Q - z)^{-1} x = T_1 + T_2.
\end{aligned}$$

To deal with  $T_1$ , we use the representation

$$T_1 = \langle D_x \rangle^{2s} \partial_x V \langle x \rangle^2 (Q - z)^{-2} W_1$$

and we will show that the operator

$$\begin{aligned}
W_1 & = \langle x \rangle^{-2} x^2 + [Q, \langle x \rangle^{-2} x] (Q - z)^{-1} x \\
& = \langle x \rangle^{-2} x^2 - \mathbf{i} \left[ (D_x - By) \frac{1 - x^2}{(1 + x^2)^2} + \frac{1 - x^2}{(1 + x^2)^2} (D_x - By) \right] (Q - z)^{-1} x
\end{aligned}$$

is bounded. Consider the operator

$$\begin{aligned}
(D_x - By) \frac{(1 - x^2)}{(1 + x^2)^2} (Q - z)^{-1} x & = (D_x - By) \frac{(1 - x^2)x}{(1 + x^2)^2} (Q - \mathbf{i})^{-1} \left[ 1 + (z - \mathbf{i})(Q - z)^{-1} \right] \\
& + (D_x - By) \frac{1 - x^2}{(1 + x^2)^2} (Q - z)^{-1} [Q, x] (Q - z)^{-1}.
\end{aligned}$$

The pseudodifferential operator

$$(D_x - By) \frac{(1 - x^2)x}{(1 + x^2)^2} (Q - \mathbf{i})^{-1}$$

is bounded and the product of this operator with  $\left[ 1 + (\mathbf{i} - z)(Q - z)^{-1} \right]$  is bounded, too. As in the proof of Lemma 7, we see that  $[Q, x](Q - z)^{-1}$  is bounded and with the same argument we treat the other terms. Thus we conclude that  $W_1$  is a bounded operator. Next we write

$$T_2 = \langle D_x \rangle^{2s} \partial_x V \langle x \rangle^2 (Q - z)^{-2} W_2,$$

where

$$W_2 = [Q, \langle x \rangle^{-2}] x (Q - z)^{-1} x + \left[ Q, [Q, \langle x \rangle^{-2}] \right] (Q - z)^{-1} x (Q - z)^{-1} x = W_{21} + W_{22}.$$

We have

$$W_{21} = 2\mathbf{i} \left[ (D_x - By) \frac{x^2}{(1 + x^2)^2} (Q - z)^{-1} x + \frac{x}{(1 + x^2)^2} (D_x - By) x (Q - z)^{-1} x \right]$$

and as above we deduce that  $W_{21}$  is a bounded operator. For the analysis of  $W_{22}$ , we write

$$\begin{aligned}
W_{22} & = \left\{ \frac{1 - 3x^2}{(1 + x^2)^3} 4(D_x - By)^2 + R_1(x)(D_x - By) + R_2(x) + \frac{x}{(1 + x^2)^2} (4\partial_x V + 8BD_y) \right\} \\
& \quad \times (Q - z)^{-1} x (Q - z)^{-1} x.
\end{aligned}$$

A simple calculus gives

$$\begin{aligned}
(Q - z)^{-1} x (Q - z)^{-1} x & = (Q - z)^{-1} x^2 (Q - z)^{-1} + (Q - z)^{-1} x M_1 \\
& = x^2 (Q - z)^{-2} + 4(Q - z)^{-1} x (D_x - By) (Q - z)^{-2} + x (Q - z)^{-1} M_1 + (Q - z)^{-1} M_2 \\
& = x^2 (Q - z)^{-2} + 4x (Q - z)^{-1} M_3 + (Q - z)^{-1} M_4 \\
& = x^2 (Q - \mathbf{i})^{-2} M_5 + 4x (Q - \mathbf{i})^{-1} M_6 + (Q - \mathbf{i})^{-1} M_7,
\end{aligned}$$



where  $M_k, k = 1, 2, \dots$ , denote bounded operators. The pseudodifferential calculus implies that the product of the term in the brackets  $\{\dots\}$  with  $x^j(Q - \mathbf{i})^{-j}$ ,  $j = 1, 2$  is a bounded operator. Combining this with the above equality, we conclude that  $W_{22}$  is bounded.

Now it remains to see that the operator

$$\mathcal{T} = \langle D_x \rangle^{2s} \partial_x V \langle x \rangle^2 (Q - z)^{-2}$$

is trace class. For this purpose we replace  $(Q - z)^2$  by

$$(Q - \mathbf{i})^{-2} \left[ I + (z - \mathbf{i})(Q - z)^{-1} \right]^2$$

and consider the pseudodifferential operator

$$\langle D_x \rangle^{2s} \partial_x V \langle x \rangle^2 (Q - \mathbf{i})^{-2} \tag{5.2}$$

with principal symbol

$$g_s(x, y, \xi, \eta) = \frac{\xi^{2s} (\partial_x V)(x, y) (1 + x^2)}{\left( (\xi - By)^2 + \eta^2 + V(x, y) - \mathbf{i} \right)^2}.$$

We use the estimate  $\langle \xi \rangle^{2s} \leq C \langle \xi - By \rangle^{2s} \langle y \rangle^{2s}$  and we apply Theorem 9.4 in [3] to deduce that (5.2) is a trace class operator. In fact we have

$$\sum_{|\alpha| \leq 5} \|\partial_{x,y,\xi,\eta}^\alpha g_s\|_{L^1(\mathbb{R}^4)} < \infty$$

since  $2s < 2$  guarantees that the integral with respect to  $\xi$  is convergent, while  $2s < 1 + \delta/2$  and the estimate (1.4) imply that integral with respect to  $y$  is convergent. Consequently,  $\mathcal{T}$  is a trace class operator and this completes the analysis of (5.1). Notice also that the same argument implies that the operator

$$\langle D_x \rangle^s \partial_x V \left[ (Q - z)^{-1} x \right]^2$$

is trace class.

To prove that the operator  $\langle D_x \rangle^s \partial_x V \left[ (Q - z)^{-1} x \right]^2 \langle D_x \rangle^s$  is trace class, we commute the operator  $\langle D_x \rangle^s$  with  $(Q - z)^{-1} x$  and  $\partial_x V$  in order to reduce the proof to that of (5.1). The commutators  $[x, \langle D_x \rangle^s]$  and  $[V, \langle D_x \rangle^s] x$  are bounded since  $s < 1$ . Next

$$\begin{aligned} [(Q - z)^{-1}, \langle D_x \rangle^s] x &= (Q - z)^{-1} [V, \langle D_x \rangle^s] (Q - z)^{-1} x \\ &= (Q - z)^{-1} [V, \langle D_x \rangle^s] \left( x(Q - z)^{-1} + (Q - z)^{-1} M_1 \right) = (Q - z)^{-1} M_2 \end{aligned}$$

and we obtain operators which can be handled by the above argument. Thus the assertion is proved for  $n = 2$ .

Passing to the general case  $n > 2$ , assume that the assertion holds for  $n = 2, \dots, k - 1$ , and suppose that  $V$  satisfy the estimate (1.4) with  $n = k$ . The idea is to replace the operator

$$\langle D_x \rangle^s \partial_x V \left[ (Q - z)^{-1} x \right]^k \langle D_x \rangle^s$$

by the trace class operator  $\langle D_x \rangle^s (\partial_x V) x^k (Q - z)^{-2} \langle D_x \rangle^s$  plus a sum of several operators which are trace class according to the recurrence assumption. Notice that if  $M_j$  is bounded operator obtained

as a product of  $(D_x - By)$  and  $(Q - z)^{-j}$ ,  $j \geq 1$ , the operator  $\langle D_x \rangle^{-s} M_j \langle D_x \rangle^s$  becomes a bounded operators and this makes possible to exploit the representation

$$\langle D_x \rangle^s \partial_x V (Q - z)^{-1} x \dots M_j \langle D_x \rangle^s = \left[ \langle D_x \rangle^s \partial_x V (Q - z)^{-1} x \dots \langle D_x \rangle^s \right] \left( \langle D_x \rangle^{-s} M_j \langle D_x \rangle^s \right)$$

Thus we reduce the analysis to the trace class property of  $\langle D_x \rangle^s \partial_x V (Q - z)^{-1} x \dots \langle D_x \rangle^s$ . For simplicity of the notations we will write  $A \sim_t B$  if the difference  $A - B$  is a trace class operator.

We start with the observation that

$$\langle D_x \rangle^s \partial_x V [(Q - z)^{-1} x]^k \langle D_x \rangle^s \sim_t \langle D_x \rangle^s \partial_x V [(Q - z)^{-1} x]^{k-2} (Q - z)^{-1} x^2 (Q - z)^{-1} \langle D_x \rangle^s.$$

We can establish this by a recurrence. For  $k - 1$  we apply the equality

$$\begin{aligned} \langle D_x \rangle^s \partial_x V [(Q - z)^{-1} x]^{k-1} \langle D_x \rangle^s &= \langle D_x \rangle^s \partial_x V [(Q - z)^{-1} x]^{k-3} (Q - z)^{-1} x^2 (Q - z)^{-1} \langle D_x \rangle^s \\ &\quad \langle D_x \rangle^s \partial_x V [(Q - z)^{-1} x]^{k-2} (Q - z)^{-1} [Q, x] (Q - z)^{-1} \langle D_x \rangle^s \\ &\sim_t \langle D_x \rangle^s \partial_x V [(Q - z)^{-1} x]^{k-3} (Q - z)^{-1} x^2 (Q - z)^{-1} \langle D_x \rangle^s. \end{aligned}$$

Commuting  $(Q - z)^{-1}$  and  $x^2$ , we obtain the result for  $k - 1$  and in the same way we continue for  $p \leq k - 1$ .

Next we commute  $(Q - z)^{-1}$  and  $x^2$  and get

$$\begin{aligned} \langle D_x \rangle^s \partial_x V [(Q - z)^{-1} x]^{k-2} (Q - z)^{-1} x^2 (Q - z)^{-1} \langle D_x \rangle^s \\ \sim_t \langle D_x \rangle^s \partial_x V [(Q - z)^{-1} x]^{k-3} (Q - z)^{-1} x^3 (Q - z)^{-2} \langle D_x \rangle^s. \end{aligned}$$

Indeed,  $[Q, x^2] = 4(D_x - By)x = -4\mathbf{i}x(D_x - By) - 2$  yields

$$(Q - z)^{-1} x^2 (Q - z)^{-1} = x^2 (Q - z)^{-2} - 4\mathbf{i} (Q - z)^{-1} x (D_x - By) (Q - z)^{-1} - 2(Q - z)^{-2}$$

and for the term

$$\langle D_x \rangle^s \partial_x V [(Q - z)^{-1} x]^{k-1} (D_x - By) (Q - z)^{-1} \langle D_x \rangle^s$$

we use the recurrence assumption and the fact that  $M_2 = (D_x - By)(Q - z)^{-1}$  is a bounded operator. In the same way for  $1 \leq j \leq k - 1$  we show that

$$\begin{aligned} \langle D_x \rangle^s \partial_x V [(Q - z)^{-1} x]^{k-j} (Q - z)^{-1} x^j (Q - z)^{-2} \langle D_x \rangle^s \\ \sim_t \langle D_x \rangle^s \partial_x V [(Q - z)^{-1} x]^{k-j-1} (Q - z)^{-1} x^{j+1} (Q - z)^{-2} \langle D_x \rangle^s, \end{aligned}$$

taking into account the equality

$$[Q, x^j] = 2j(D_x - By)x^{j-1} = 2jx^{j-1}(D_x - By) - 2\mathbf{i}j(j-1)x^{j-1}$$

and the recurrence assumption. Finally, we prove that

$$\langle D_x \rangle^s \partial_x V [(Q - z)^{-1} x]^k \langle D_x \rangle^s \sim_t \langle D_x \rangle^s (\partial_x V) x^k (Q - z)^{-2} \langle D_x \rangle^s$$

and, as in the proof in the case  $n = 2$ , we conclude that the operator on the right hand side is trace class one. □

After this preparation we pass to the proof of Theorem 2.

**Proof of Theorem 2.** Let  $\Xi \subset \mathbb{R}$  be a small neighborhood of  $\lambda_0$  such that  $\Xi \cap \sigma(Q) = \emptyset$ . For the simplicity of the notations we will write  $H(\epsilon)$ ,  $\xi(\lambda, \epsilon)$  instead of  $H(B, \epsilon)$ ,  $\xi(\lambda; B, \epsilon)$ . Given  $f \in C_0^\infty(\Xi)$ , introduce an almost analytic continuation  $\tilde{f} \in C_0^\infty(\mathbb{C})$  of  $f$  so that  $\bar{\partial}\tilde{f}(z) = \mathcal{O}(|\text{Im } z|^\infty)$  and  $\text{supp } \tilde{f}(z) \cap \sigma(Q) = \emptyset$ . Since  $(z-Q)^{-1}$  is analytic over the support of  $\tilde{f}(z)$ , applying the resolvent equality, we get

$$\begin{aligned} \partial_x V f(H(\epsilon)) &= -\frac{1}{\pi} \int \bar{\partial}\tilde{f}(z) \partial_x V (z - H(\epsilon))^{-1} L(dz) \\ &= (-1)^{n+1} \frac{\epsilon^n}{\pi} \int \bar{\partial}\tilde{f}(z) \partial_x V [(z - Q)^{-1} x]^n (z - H(\epsilon))^{-1} L(dz). \end{aligned} \quad (5.3)$$

Taking into account Proposition 4 and the cyclicity of the trace, we get

$$\begin{aligned} &\text{tr} \int \bar{\partial}\tilde{f}(z) \langle D_x \rangle^{-s} \left[ \langle D_x \rangle^s \partial_x V [(z - Q)^{-1} x]^n \langle D_x \rangle^s \right] \langle D_x \rangle^{-s} (z - H(\epsilon))^{-1} L(dz) \\ &= \text{tr} \int \bar{\partial}\tilde{f}(z) \left[ \langle D_x \rangle^s \partial_x V [(z - Q)^{-1} x]^n \langle D_x \rangle^s \right] \langle D_x \rangle^{-s} (z - H(\epsilon))^{-1} \langle D_x \rangle^{-s} L(dz). \end{aligned}$$

Set  $W(z) = \langle D_x \rangle^s \partial_x V [(z - Q)^{-1} x]^n \langle D_x \rangle^s$  and note that for  $z \in \text{supp } \tilde{f}$  this operator is trace class and  $W(z)$  is analytic. We write

$$\begin{aligned} &-\frac{1}{\pi} \int \bar{\partial}\tilde{f}(z) \text{tr} \left( \partial_x V [(z - Q)^{-1} x]^n (z - H(\epsilon))^{-1} \right) L(dz) \\ &= \frac{1}{\pi} \lim_{\eta \searrow 0} \left[ \int_{\text{Im } z > 0} \bar{\partial}\tilde{f}(z + \mathbf{i}\eta) \text{tr} \left[ \left( W(z + \mathbf{i}\eta) \langle D_x \rangle^{-s} (H(\epsilon) - (z + \mathbf{i}\eta))^{-1} \langle D_x \rangle^{-s} \right) \right] L(dz) \right. \\ &\quad \left. + \int_{\text{Im } z < 0} \bar{\partial}\tilde{f}(z - \mathbf{i}\eta) \text{tr} \left( W(z - \mathbf{i}\eta) \langle D_x \rangle^{-s} (H(\epsilon) - (z - \mathbf{i}\eta))^{-1} \langle D_x \rangle^{-s} \right) L(dz) \right]. \end{aligned}$$

Notice that the functions

$$\text{tr} \left( W(z \pm \mathbf{i}\eta) \langle D_x \rangle^{-s} (H(\epsilon) - (z \pm \mathbf{i}\eta))^{-1} \langle D_x \rangle^{-s} \right)$$

are analytic in  $\pm \text{Im } z > 0$ . Applying Green formula, as in Lemma 1 in [4], we deduce

$$\begin{aligned} \langle \xi'(\lambda, \epsilon), f \rangle &= \text{tr} \left( f(H(\epsilon) - f(H_0)) \right) = -\frac{1}{\epsilon} \text{tr} \left( \partial_x V f(H(\epsilon)) \right) \\ &= \lim_{\eta \searrow 0} \frac{(-1)^n \epsilon^{n-1}}{2\pi \mathbf{i}} \int f(\lambda) \text{tr} \left( W(\lambda) \left[ \langle D_x \rangle^{-s} \left( (H(\epsilon) - (\lambda + \mathbf{i}\eta))^{-1} - (H(\epsilon) - (\lambda - \mathbf{i}\eta))^{-1} \right) \langle D_x \rangle^{-s} \right] \right) d\lambda, \end{aligned}$$

where the integral is taken in the sense of distributions. On the other hand, Proposition 4 combined with (4.3) show that the right hand side of the above representation is finite and has order  $\mathcal{O}(\epsilon^{n-2})$ . Thus for  $\forall f \in C_0^\infty(\Xi)$  we obtain

$$\langle \xi'(\lambda, \epsilon), f \rangle = \int f(\lambda) T_\epsilon(\lambda) d\lambda$$

with  $T_\epsilon(\lambda) = \mathcal{O}(\epsilon^{n-2})$  and this completes the proof.

## 6. APPENDIX

The proof of the following Lemma is similar to the proof of Proposition 2.1 in [5] and for the reader convenience we give it.

**Lemma 8.** *Let  $\delta > 0$  and let  $k_j(x, y) = \langle x \rangle^{-j(1+\delta)} \langle y \rangle^{-j(\frac{1}{2}+\delta)}$ ,  $j = 1, 2$ . The operators  $G_2 := k_2(H_0 + \mathbf{i})^{-2}$ ,  $G_2^*$ , (resp.  $G_1 := k_1(H_0 + \mathbf{i})^{-1}$ ,  $G_1^*$ ), are trace class (resp. Hilbert-Schmidt).*

*Proof.* Without loss of the generality we may assume that  $B = \epsilon = 1$ . Introduce the unitary operator  $U : L^2(\mathbb{R}^2) \rightarrow L^2(\mathbb{R}^2)$  by

$$(Uu)(x, y) = \frac{2}{\pi} \iint_{\mathbb{R}^2} e^{i\varphi(x, y, x', y')} u(x', y') dx' dy',$$

where  $\varphi(x, y, x', y') = xy - xy' - x'y + x'y' - \frac{1}{2}y'$ . A simple calculus shows that

$$\tilde{H}_0 = U^{-1}H_0U = (D_y^2 + y^2) + x - \frac{1}{4},$$

$$\tilde{k}_j^\omega = U^{-1}k_jU = k_j^\omega \left( x - D_y - \frac{1}{2}, y + D_x \right).$$

Since  $U$  is unitary, it suffices to prove the lemma for  $\tilde{G}_j := UG_jU^{-1} = \tilde{k}_j^\omega(\tilde{H}_0 + \mathbf{i})^{-j}$ .

Let  $\chi(t) \in C_0^\infty(\mathbb{R}; [0, 1])$  be a cut-off function such that  $\chi(t) = 1$  for  $|t| \leq 1$  and  $\chi(t) = 0$  for  $|t| \geq 2$ . Fix a number  $k$ ,  $\max\{1, \frac{2}{1+2\delta}\} < k < 2$ , and introduce the symbol

$$q(x, y, \eta) = \chi \left( \frac{\langle y, \eta \rangle^k}{|\eta^2 + y^2 + (x + i)|} \right),$$

where  $\langle y, \eta \rangle = (1 + y^2 + \eta^2)^{1/2}$ . It clear that  $q(x, y, \eta) \in S^0(\mathbb{R}^4_{(x, \xi, y, \eta)})$  and we set  $A = q^\omega(x, y, D_y)$ . We decompose

$$\tilde{k}_j^\omega(\tilde{H}_0 + \mathbf{i})^{-j} = A\tilde{k}_j^\omega(\tilde{H}_0 + \mathbf{i})^{-j} + (I - A)\tilde{k}_j^\omega(\tilde{H}_0 + \mathbf{i})^{-j} = L_j + M_j. \quad (6.1)$$

To treat  $L_j$ , notice that on the support of  $q(x, y, \eta)$  we have

$$(\eta^2 + y^2 + x + \mathbf{i})^{-1} \in S^0(\mathbb{R}^4; \langle y, \eta \rangle^{-k}).$$

In fact, on the support of  $q$  we obtain

$$\langle y, \eta \rangle^k \leq 2|\eta^2 + y^2 + x + \mathbf{i}|,$$

and it is easy to estimate the derivatives of  $(\eta^2 + y^2 + x + \mathbf{i})^{-1}$ . According to the calculus of pseudodifferential operators,  $L_j$  becomes a pseudodifferential operator with symbol in

$$S^0(\mathbb{R}^4; \langle y, \eta \rangle^{-k} \langle x - \eta \rangle^{-j(1+\delta)} \langle y + \xi \rangle^{-j(\frac{1}{2}+\delta)}),$$

and the trace norm (resp. Hilbert-Schmidt norm) of  $L_2$  (resp.  $L_1$ ) can be estimated (see for instance, Proposition 9.2 and Theorem 9.4 in [3]) by

$$\begin{aligned} \|L_1\|_{\text{HS}}^2 + \|L_2\|_{\text{tr}} &\leq C_0 \iiint \langle y, \eta \rangle^{-2k} \langle x - \eta \rangle^{-2-2\delta} \langle y + \xi \rangle^{-1-2\delta} dx d\xi dy d\eta \\ &\leq C'_0 \iint \langle y, \eta \rangle^{-2k} dy d\eta \leq C''_0. \end{aligned} \quad (6.2)$$

To deal with  $M_j$ ,  $j = 1, 2$ , we will show that  $(I - A)\tilde{k}_2^\omega$  is trace class operator and  $(I - A)\tilde{k}_1^\omega$  is Hilbert-Schmidt one.

Notice that on the support of the symbol of  $(I - A)$  we have

$$\langle y, \eta \rangle^k \geq |\eta^2 + y^2 + x + \mathbf{i}|.$$

Taking into account the estimate  $\partial_x^l \partial_y^m k_j(x, y) = \mathcal{O}_{l,m}(\langle x \rangle^{-j(1+\delta)} \langle y \rangle^{-j(\frac{1}{2}+\delta)})$ , we get

$$\begin{aligned} \|(I - A)k_1^\omega\|_{\text{HS}}^2 + \|(I - A)k_2^\omega\|_{\text{tr}} &\leq C_1 \iiint_{\langle y, \eta \rangle^k \geq |\eta^2 + y^2 + x + \mathbf{i}|} \langle x - \eta \rangle^{-2-2\delta} \langle y + \xi \rangle^{-1-2\delta} dx d\xi dy d\eta \quad (6.3) \\ &\leq C_2 \iiint_{\langle y, \eta \rangle^k \geq |\eta^2 + y^2 + x + \mathbf{i}|} \langle x - \eta \rangle^{-2-2\delta} dx dy d\eta \leq C_2 \iiint_{\langle y, \eta \rangle^k \geq |\eta^2 + y^2 + \eta + u + \mathbf{i}|} \langle u \rangle^{-2-2\delta} du dy d\eta \\ &\leq C'_2 \iiint_{\substack{\langle y, \eta \rangle^k \geq |\eta^2 + y^2 + \eta + u|, \\ |u| \leq \frac{1}{2} \langle y, \eta \rangle^k}} \langle u \rangle^{-2-2\delta} du dy d\eta + C'_2 \iiint_{\substack{\langle y, \eta \rangle^k \geq |\eta^2 + y^2 + \eta + u|, \\ |u| \geq \frac{1}{2} \langle y, \eta \rangle^k}} \langle u \rangle^{-2-2\delta} du dy d\eta \\ &\leq C'_2 \left( \iiint_{|u| \leq C_3, |y| \leq C_3, |\eta| \leq C_3} \langle u \rangle^{-2-2\delta} du dy d\eta + \iiint_{|u| \geq \frac{1}{2} \langle y, \eta \rangle^k} \langle u \rangle^{-2-2\delta} du dy d\eta \right) \\ &\leq C_4 + C_5 \int \langle u \rangle^{-2-2\delta} \left( \int_0^{(2|u|)^{\frac{1}{k}}} r dr \right) du \leq C_4 + C_6 \int \langle u \rangle^{-2-2\delta+2/k} du \leq C_7, \end{aligned}$$

since  $-2 - 2\delta + 2/k < -1$ .

Using (6.1), (6.2), (6.3) and the fact that  $M$  is trace class (resp. Hilbert-Schmidt) operator if and only if  $M^*$  is trace class (resp. Hilbert-Schmidt) operator, we complete the proof of the lemma.  $\square$

#### REFERENCES

- [1] W. O. Amrein, A. M. Boutet de Monvel, V. Georgescu,  *$C_0$ -groups, commutator methods and spectral theory of  $N$ -body Hamiltonians*, Progress in Mathematics, **135**, Birkhäuser-Verlag, Basel, 1996.
- [2] F. Bentosela, R. Carmona, P. Duclos, B. Simon, B. Souillard, R. Weder, *Schrödinger operators with an electric field and random or deterministic potentials*, Comm. Math. Phys. **88** (1983), 387-397.
- [3] M. Dimassi and J. Sjöstrand, *Spectral asymptotics in semiclassical limit*, London Mathematical Society, Lecture Notes Series, **268**, Cambridge University Press, 1999.
- [4] M. Dimassi and V. Petkov, *Spectral shift function and resonances for non-semibounded and Stark hamiltonians*, J. Math. Pures et Appl. **82** (2003), 1303-1342.
- [5] M. Dimassi and V. Petkov, *Resonances for magnetic Stark hamiltonians in two dimensional case*, IMRN, **77** (2004), 4147-4179.
- [6] C. Gerard, *A proof of the abstract limiting absorption principle by energy estimates*, J. Funct. Anal. **254** (2008), 2707-2724.
- [7] C. Ferrari and H. Kovarik, *Resonances width in crossed electric and magnetic fields*, J. Phys. A: Math. Gen. **37** (2004), 7671-7697.
- [8] C. Ferrari and H. Kovarik, *On the exponential decay of magnetic Stark resonances*, Rep. Math. Phys. **56** (2005), 197-207.
- [9] V. Ivrii, *Analysis and Precise Spectral Asymptotics*, Springer Monographs in Mathematics, Springer, Berlin, 1998.
- [10] M. Klein, D. Robert, X. P. Wang, *Breit-Wigner formula for the scattering phase in the Stark effect*, Comm. Math. Phys. **131** (1990), no. 1, 109-124.
- [11] M. G. Krein, *On the ntrace formula in perturbation theory*, Mat. Sb. **33** (1953), 597-626 (in Russian).
- [12] E. Korotyaev, A. Pushnitski, *A trace formula and high energy spectral asymptotics for the perturbed Landau Hamiltonian*, J. Funct. Anal. **217** (2004), 221-248.
- [13] M. Melgaard and G. Rosenblum, *Eigenvalue asymptotics for weakly perturbed Dirac and Schrödinger operators with constant magnetic fields of full rank*, Commun. Partial Differential Equations, **28** (2003), 697-736.
- [14] E. Mourre, *Absence of singular continuous spectrum for certain self-adjoint operators*, Commun. Math. Physics, **78**, no. 3 (1981), 391-408.

- [15] G. Raikov and S. Warzel, *Quasi-classical versus non-classical spectral asymptotics for magnetic Schrödinger operators with decreasing electric potentials*, Rev. Math. Phys. **14** (2002), 1051-1072.
- [16] M. Reed and B. Simon, *Methods of Modern Mathematical Physics, IV, Analysis of Operators*, Academic Press, New York, 1978.
- [17] D. Robert and X. P. Wang, *Existence of time-delay operators for Stark Hamiltonians*, Commun. Partial Differential Equations, **14** (1989), 63-98.
- [18] D. Robert and X. P. Wang, *Time-delay and spectral density for Stark Hamiltonians. II. Asymptotics of trace formulae*, Chinese Ann. Math. Ser. B, **12** no. 3 (1991), 358–383.
- [19] X. P. Wang, *Weak coupling asymptotics of Schrödinger operators with Stark effect*, in: Harmonic Analysis, pp. 185–195, Lecture Notes in Math., 1494, Springer, Berlin, 1991.
- [20] D. Yafaev, *Mathematical Scattering Theory*, Amer. Math. Society, Providence, RI, 1992.

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