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# A simple proof that generic 3-RPR manipulators have two aspects

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## Introduction

Planar 3-RPR manipulators have been extensively studied because they meet several interesting features such as potential industrial applications, relative kinematic simplicity and nice mathematical properties [1-11]. Moreover, the study of the 3-RPR planar manipulator may help better understand the kinematic behaviour of its more complex spatial counterpart, the 6-dof octahedral manipulator, as reported in [3]. An important feature of these manipulators is their ability to change assembly-mode without encountering a singularity [1-6]. Since a parallel manipulator becomes uncontrollable on a singular configuration, this feature is interesting as it can enlarge its usable workspace. Knowing whether a parallel manipulator, or, more interestingly, a family of manipulators has this feature or not is of interest for both the designer and the end-user. Determining the number of aspects (singularity-free domain) may help answering this question.

Recently [6], M. Husty announced a proof that a generic 3-RPR manipulator has two aspects. Since, in the same time, there are up to 6 assembly-modes, this shows that a generic manipulator has more than one assembly mode in one of its aspects, thus showing that non-singular assembly-mode changing motions are possible. His proof is rather technical and involved, and the details of the proof are yet to appear in a joint paper with J. Schicho. We propose here a much simpler proof of Husty's result. This new proof is based on a parameterization of the singular surface different from the one used by M. Husty and on arguments of a simple topological nature.

The rest of this paper is organized as follows. Section 1 describes the new parameterization of the singular surfaces. Section 2 is devoted to the proof; we also discuss briefly the non-generic cases.

The author thanks Philippe Wenger for his useful advice.

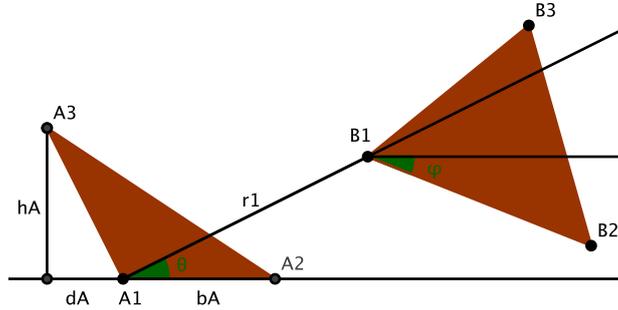


Figure 1: Parameters and coordinates

# 1 The parameterization of the singular surface

## 1.1 Notations and coordinates

In order to describe the manipulator, we use here the following notations:

- The base triangle is  $A_1A_2A_3$  (with the direct orientation). We take  $b_A = A_1A_2$  as the base of this triangle; the coordinates of the point  $A_3$  in the direct orthonormal frame  $\mathcal{F}$  with origin  $A_1$  and first coordinate axis directed and oriented by  $\overrightarrow{A_1A_2}$  are denoted by  $(h_A, d_A)$  ( $h$  for height and  $d$  for “déport” meaning offset).
- The platform triangle is denoted by  $B_1B_2B_3$  where  $B_i$  is linked to  $A_i$  by a leg of the manipulator ( $B_1B_2B_3$  may be in indirect orientation). We use the parameters  $(b_B, h_B, d_B)$ , analogous to the ones defined above for the base triangle, to encode the geometry of the platform triangle. Note that  $h_B$  may be negative.

We encode the position of the platform triangle with respect to the base triangle by means of the coordinates  $(\phi, \theta, r_1) \in [-\pi, \pi] \times [-\pi/2, \pi/2] \times \mathbb{R}$  which are defined in the following way:

- $\phi$  measures the angle  $(\overrightarrow{A_1A_2}, \overrightarrow{B_1B_2})$ ,
- the coordinates of  $B_1$  in the frame  $\mathcal{F}$  defined above are  $(r_1 \cos \theta, r_1 \sin \theta)$ .

Note that  $r_1$  may be negative; its absolute value is the length of the first leg of the manipulator, usually denoted by  $\rho_1$  in the literature. Of course, we make identifications:  $(\phi, -\pi/2, r_1)$  is identified with  $(\phi, \pi/2, -r_1)$  and  $(-\pi, r_1, \theta)$  with  $(\pi, r_1, \theta)$ . The angle  $\phi$  is an oriented angle of vectors and it is measured modulo  $2\pi$ . The angle  $\theta$  is an oriented angle of lines and it is measured modulo  $\pi$ .

It is in order here to comment the use of these coordinates for the group of plane motions. Usually, the group of plane motions would be described by using a rotation part (encoded by the angle  $\phi$ ) and a translation part (the vector  $\overrightarrow{A_1A_2}$ ), as  $S^1 \times \mathbb{R}^2$ . The coordinates we shall use have the peculiarity that all triples  $(\phi, \theta, 0)$  with  $\phi$  fixed and  $\theta$  varying correspond actually to the same position of the manipulator (in some sense, we fix arbitrarily a direction for

the first leg of the manipulator, although its length is null). From the algebro-geometric point of view, we have blown-up the origin  $0 \in \mathbb{R}^2$ , getting thus the Moebius strip  $[-\pi/2, \pi/2] \times \mathbb{R}$  where  $(-\pi/2, r_1)$  is identified with  $(\pi/2, -r_1)$ . The use of these coordinates may look awkward at first sight, but it will enable us to obtain a useful parameterization of the singular surface in our blown-up workspace, with  $r_1$  as a function of  $\phi$  and  $\theta$ .

We denote by  $W$  the blown-up workspace with coordinates  $(\phi, \theta, r_1)$  and by  $T$  the torus  $\mathbb{R}/2\pi\mathbb{Z} \times \mathbb{R}/\pi\mathbb{Z}$  with coordinates  $(\phi, \theta)$ . We denote by  $p : W \rightarrow T$  the projection defined by  $p(\phi, \theta, r_1) = (\phi, \theta)$ . Note that, for each point  $(\phi, \theta)$  of the torus  $T$ , the fiber  $p^{-1}(\phi, \theta)$  of the projection is a line, but  $W$  is not homeomorphic to the product  $T \times \mathbb{R}$ ; the blown-up workspace  $W$  is actually a nontrivial line bundle over  $T$ .

## 1.2 Parameterization of the singular surface

The equation of the singular locus in the blown-up workspace  $W$  is obtained by expressing the fact that the legs  $(A_1B_1)$ ,  $(A_2B_2)$  and  $(A_3B_3)$  are concurrent or parallel. There is of course a factor  $r_1$  in this equation (a result of the blowing-up of the origin of  $\mathbb{R}^2$ ). The other factor (which is actually the equation of the strict transform of the singular surface by the blowing up) has the form

$$N(\phi, \theta) - D(\phi, \theta) r_1 ,$$

where  $N(\phi, \theta)$  and  $D(\phi, \theta)$  are polynomials in the trigonometric functions of  $\phi$  and  $\theta$ . Precisely, we have:

$$\begin{aligned} N(\phi, \theta) = & b_B \left( (b_A d_B - h_A h_B - d_A d_B) (\sin \phi)^2 - b_A h_A \sin \phi \right. \\ & \left. + (h_A d_B + b_A h_B - d_A h_B) \sin \phi \cos \phi \right) \cos \theta \\ & + \left( b_A (h_A d_B - d_A h_B) \cos \phi + b_A (d_A b_B - d_A d_B - h_A h_B) \sin \phi \right. \\ & \left. + b_A b_B h_B + b_B (d_A h_B - b_A h_B - h_A d_B) (\cos \phi)^2 \right. \\ & \left. + b_B (h_A h_B + d_A d_B - b_A d_B) \cos \phi \sin \phi \right) \sin \theta , \end{aligned} \quad (1)$$

and

$$\begin{aligned} D(\phi, \theta) = & \left( (b_A d_B - d_A b_B) (\sin \theta)^2 + (h_A b_B - b_A h_B) \cos \theta \sin \theta \right) \cos \phi \\ & + \left( (d_A b_B - b_A d_B) \sin \theta \cos \theta - h_A b_B (\cos \theta)^2 - b_A h_B (\sin \theta)^2 \right) \sin \phi . \end{aligned} \quad (2)$$

In the following, we shall call *singular surface* the surface with equation

$$N(\phi, \theta) - D(\phi, \theta) r_1 = 0 \quad (3)$$

and *singular locus* the union of this surface with  $r_1 = 0$ .

Solving the equation (3) in  $r_1$  we get the parameterization  $r_1^{\text{Sing}}(\phi, \theta) = \frac{N(\phi, \theta)}{D(\phi, \theta)}$ . Observe that we have  $r_1^{\text{Sing}}(\phi, \theta + \pi) = -r_1^{\text{Sing}}(\phi, \theta)$ , which agrees with the identification made in the description the blown-up workspace.

The parameterization gives an infinite value for  $r_1^{\text{Sing}}$  when  $D(\phi, \theta) = 0$ , and the zero value when  $N(\phi, \theta) = 0$ . These two equations describe curves on the torus  $T$  which is the space of the angular coordinates  $(\phi, \theta)$ .

Considering the formula (1) for  $N(\phi, \theta)$ , it appears that  $N(\phi, \theta) = 0$  can be solved (in generic cases) in  $\tan \theta = \nu(\phi)$ , where  $\nu$  is a rational fraction in  $\cos \phi$  and  $\sin \phi$ . This can be explained from a geometric point of view : the angle  $\theta$  is the angle from the line  $(A_1A_2)$  to the line joining  $A_1$  to the intersection point of the legs  $(A_2B_2)$  and  $(A_3B_3)$

On the other hand, the formula (2) for  $D(\phi, \theta)$  shows that  $D(\phi, \theta) = 0$  can be solved in  $\tan \phi = \tau(\theta)$ , where  $\tau$  is a rational fraction in  $\tan \theta$ ; this gives two values of  $\phi$  which differ by  $\pi$  for each value of  $\theta$ . This is again clear from a geometric point of view, since the ‘‘asymptotic’’ singular situations (i.e., those with  $r_1^{\text{Sing}}$  infinite) are characterized by the fact that

$$\frac{\overline{\ell(A_1)\ell(A_2)}}{\overline{\ell(A_1)\ell(A_3)}} = \frac{\overline{\ell(B_1)\ell(B_2)}}{\overline{\ell(B_1)\ell(B_3)}},$$

where  $\ell$  is a projection parallel to the common direction of the infinite legs, which is given by  $\theta$ ; this is realized by two orientations of the platform with respect to the base (encoded by the angle  $\phi$ ) which differ by  $\pi$ .

Actually, the curve  $D(\phi, \theta) = 0$  has two branches (or connected components) one of which is obtained from the other by the translation  $\phi \mapsto \phi + \pi$ . This point, which will be of importance later, is not a priori clear, since there could be only one branch possessing the translation symmetry  $\phi \mapsto \phi + \pi$ . But indeed the equation  $\cos \phi = 0$ , together with  $D(\phi, \theta) = 0$ , is an equation of degree 2 in  $\tan \theta$ , precisely

$$b_A h_B (\tan \theta)^2 + (b_A d_B - b_B d_A) \tan \theta + b_B h_A = 0.$$

This equation has 0 or 2 solutions in  $\theta$ . The fact that this number is even implies that, if one follows continuously a determination of  $\phi$  along  $D(\phi, \theta) = 0$  when  $\theta$  varies from  $-\pi/2$  to  $\pi/2$ , one returns to the same determination of  $\phi$  modulo  $2\pi$ . Hence, there are two disjoint branches of the curve  $D(\phi, \theta) = 0$ .

### 1.3 Indetermination points

For a generic 3-RPR manipulator, the two curves  $N(\phi, \theta) = 0$  and  $D(\phi, \theta) = 0$  on the torus  $T$  have no common component and intersect transversally in finitely many points which are the indetermination points of the parameterization. Locally around an indetermination point, the two curves  $N(\phi, \theta) = 0$  and  $D(\phi, \theta) = 0$  cut out four ‘‘quadrants’’ on  $T$ .

The indetermination points can be computed using the resultant of  $N(\phi, \theta)$  and  $D(\phi, \theta)$  with respect to  $\tan \theta$ . This resultant has four factors

$$\begin{aligned} F_1(\phi) &= b_A b_B \sin \phi, \\ F_2(\phi) &= (d_A h_B - h_A d_B) \cos \phi + (h_A h_B + d_A d_B) \sin \phi \\ &= A_1 A_3 \times B_1 B_3 \times \sin(\phi - \alpha_1 + \beta_1), \\ F_3(\phi) &= (d_A h_B - h_A d_B + b_B h_A - b_A h_B) \cos \phi \\ &\quad + (d_A d_B + b_A b_B + h_A h_B - b_B d_A - b_A d_B) \sin \phi \\ &= A_2 A_3 \times B_2 B_3 \times \sin(\phi + \alpha_2 - \beta_2), \\ F_4(\phi) &= (-b_A h_B - b_B h_A) \cos \phi + (-b_A d_B + b_B d_A) \sin \phi + b_A h_A + b_B h_B, \end{aligned}$$

where we use the angles  $\alpha_1 = (\overrightarrow{A_1A_2}, \overrightarrow{A_1A_3})$ ,  $\beta_1 = (\overrightarrow{B_1B_2}, \overrightarrow{B_1B_3})$ ,  $\alpha_2 = (\overrightarrow{A_2A_3}, \overrightarrow{A_2A_1})$  and  $\beta_2 = (\overrightarrow{B_2B_3}, \overrightarrow{B_2B_1})$ .

The vanishing of the first three factors corresponds to the parallelism of the sides  $[A_1A_2]$  and  $[B_1B_2]$  (respectively  $[A_1A_3]$  and  $[B_1B_3]$ ,  $[A_2A_3]$  and  $[B_2B_3]$ ). The fact that this gives rise to situations where the singular value of  $r_1$  is not determined has an easy geometric interpretation: this is for instance the case if the two sides  $[A_1A_2]$  and  $[B_1B_2]$  are on the same line.

There is clearly another situation where the singular value of  $r_1$  is not determined : when the three legs of the manipulator are parallel. The conditions for the parallelism of the second and third legs with the first one are as follows:

$$\begin{aligned} b_A \sin \theta &= b_B \sin(\theta - \phi) \\ d_A \sin \theta - h_A \cos \theta &= d_B \sin(\theta - \phi) - h_B \cos(\theta - \phi) . \end{aligned}$$

Writing these equation in  $\tan \theta$  and taking the resultant with respect to  $\tan \theta$  gives indeed the fourth factor, up to sign.

Each of these first three factors has two solutions in  $\phi$  which differ by  $\pi$ . The last factor may have or not two solutions in  $\phi$ . The value of  $\theta$  for the indetermination points is given by  $\tan \theta = \nu(\phi)$ . Hence, for a generic 3-RPR manipulator, the parameterization has 6 or 8 indetermination points.

## 1.4 Examples

The first example corresponds to the manipulator with parameters  $b_A = 7$ ,  $h_A = 3$ ,  $d_A = -2$ ,  $b_B = 1$ ,  $h_B = 2$ ,  $d_B = 1$ . The figure 2 shows the curve  $N(\phi, \theta) = 0$  in red, the two branches of the curve  $D(\phi, \theta) = 0$  in blue and six indetermination points in green.

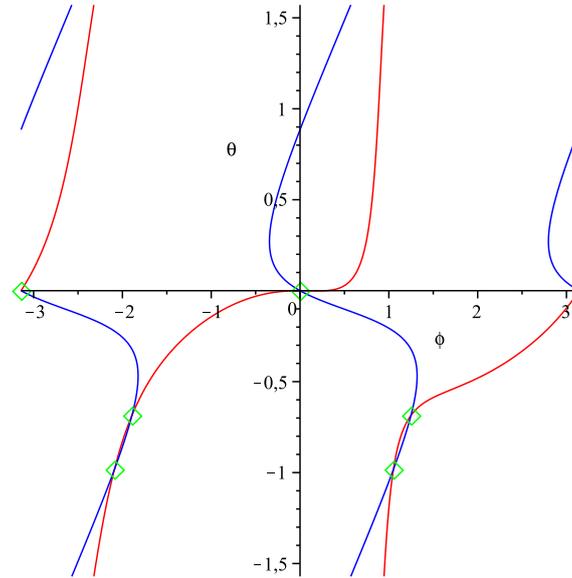


Figure 2: Example with 6 indetermination points

Recall that the space of  $(\phi, \theta)$  is actually a torus  $T$  obtained by gluing the left side of the picture with the right side, and the bottom side with the top side. The complement of the union of the curves in  $T$  has six connected components (two of which are very small).

The second example shown on figure 3 has parameters corresponding approximately to the Innocenti–Merlet manipulator:  $b_A = 15.9$ ,  $h_A = 10$ ,  $d_A = 0$ ,  $b_B = 17$ ,  $h_B = 16.1$ ,  $d_B = 13.2$ . It exhibits eight indetermination points. There are eight connected components in the complement of the union of the curves  $D(\phi, \theta) = 0$  and  $N(\phi, \theta) = 0$  in  $T$ , among which two represent almost all of  $T$ .

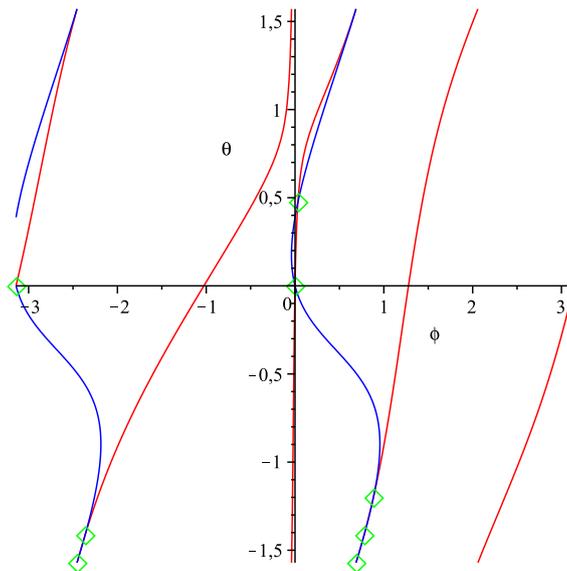


Figure 3: The Innocenti–Merlet manipulator with 8 indetermination points

The fact that there are respectively 6 and 8 connected components in the complement of the curves is not just a coincidence. Let  $I$  be the number of indetermination points. It can be checked that the curves  $N(\phi, \theta) = 0$  and  $D(\phi, \theta) = 0$  induce a cellular decomposition of the torus  $T$ , having  $I$  0-cells (the indetermination points) and  $2I$  1-cells which are intervals of the curves between indetermination points. Consequently, since the Euler characteristic of the torus is 0, there are  $I$  2-cells which are precisely the connected components of the complement of the curves.

## 2 Proof that there are generically two aspects

The blown-up workspace has been obtained by blowing up the locus  $r_1 = 0$  in the usual workspace, and this locus is contained in the singular locus. Hence, the complement of the singular locus in the usual workspace is the same as the complement of the singular locus in the blown-up workspace  $W$ . (Recall that this singular locus is the union of  $r_1 = 0$  and the singular surface  $N(\phi, \theta) = 0$  –

$D(\phi, \theta) r_1 = 0$ .) We now proceed to prove that a generic 3-RPR manipulator has two aspects, i.e. that the complement of the singular locus in  $W$  has two connected components.

Outside of the curves  $N(\phi, \theta) = 0$  and  $D(\phi, \theta) = 0$  on the torus  $T$ , the value of  $r_1^{\text{Sing}}(\phi, \theta)$  is different from 0 and infinity, and the line  $p^{-1}(\phi, \theta)$  above  $(\phi, \theta)$  is divided into three intervals by  $r_1^{\text{Sing}}(\phi, \theta)$  and 0, each one contained in a connected component of the complement of the singular locus in the blown-up workspace  $W$ .

The complement of the union of the curves  $N(\phi, \theta) = 0$  and  $D(\phi, \theta) = 0$  in the torus  $T$  has finitely many connected components (six or eight as we saw above, but the precise number does not matter here). If  $C$  is any of these connected components in  $T$ , the complement of the singular locus in  $p^{-1}(C) \subset W$  has three connected components: the one which is bounded by  $r_1 = 0$  and by the graph of  $r_1^{\text{Sing}}$ , the one which is bounded by  $r_1 = 0$  on one side and unbounded on the other side and the one which is bounded by the graph of  $r_1^{\text{Sing}}$  on one side and unbounded on the other side.

The topological proof that there are two aspects will consist in examining how these connected components glue together when crossing the curves  $N(\phi, \theta) = 0$  and  $D(\phi, \theta) = 0$ .

## 2.1 Turning around an indetermination point

The main step in the proof is to realize how the three intervals cut on the line  $p^{-1}(\phi, \theta)$  vary when one turns around an indetermination point of the parameterization. The two curves  $N(\phi, \theta) = 0$  and  $D(\phi, \theta) = 0$  intersect transversally in such a point. The value  $r_1^{\text{Sing}}(\phi, \theta)$  changes sign by going through 0 when one crosses the curve  $N(\phi, \theta) = 0$ , and it changes sign by going through infinity when one crosses the curve  $D(\phi, \theta) = 0$ . We label  $a, b, c$  the three intervals of the line  $p^{-1}(\phi, \theta)$  in a local quadrant (the upper-left one on figure 4).

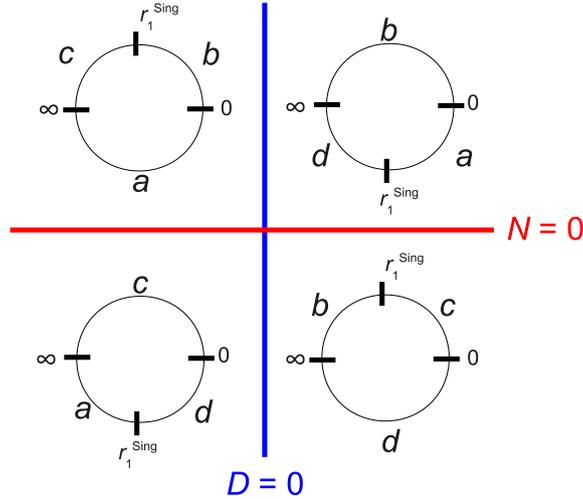


Figure 4: Connected components around an indetermination point

When we go to the lower-right quadrant counterclockwise, the interval  $b$  between  $r_1^{\text{Sing}}$  and 0 is squeezed when crossing  $N(\phi, \theta) = 0$ , and then the interval  $a$  is squeezed when crossing  $D(\phi, \theta) = 0$ . When we go clockwise, the interval  $c$  between  $r_1^{\text{Sing}}$  and  $\infty$  is squeezed when crossing  $D(\phi, \theta) = 0$ , and then the interval  $a$  is squeezed when crossing  $N(\phi, \theta) = 0$ .

We label  $d$  the third interval over the lower-right quadrant, i.e. the interval between 0 and  $\infty$  not containing  $r_1^{\text{Sing}}$ . This interval survives in the lower-left and upper-right quadrants.

Summarizing, we have proved that the singular locus cuts out four connected components in  $p^{-1}(U)$ , where  $U$  is a small neighborhood of an indetermination point.

## 2.2 Following a branch of $D(\phi, \theta) = 0$ , and conclusion

We follow now closely one of the two branches of the curve  $D(\phi, \theta) = 0$  by letting  $\theta$  vary from  $-\pi/2$  to  $\pi/2$ . The identification of  $(\phi, \theta, r_1)$  with  $(\phi, \theta + \pi, -r_1)$  shows that  $r_1^{\text{Sing}}$  has to change sign an odd number of times during this loop. Hence, the branch we follow will have an odd number of indetermination points, precisely 3 or 5 (we have seen before that there are at least three pairs of indetermination points, each stable under the translation  $\phi \mapsto \phi + \pi$ , plus possibly two other indetermination points).

So after one loop we have to identify the two pictures on the right hand side of the curve  $D(\phi, \theta) = 0$  in figure 4, modulo  $r_1 \mapsto -r_1$ . This identification yields  $a = c$  and  $b = d$ . (The same would be obtained by identifying the two pictures on the left hand side of the curve.) This shows that the singular locus cuts out only two connected components in  $p^{-1}(B)$ , where  $B$  is a small band around a branch of the curve  $D(\phi, \theta) = 0$  in the torus  $T$ .

Since the curve  $N(\phi, \theta) = 0$  and the two branches of  $D(\phi, \theta) = 0$  all contain indetermination points, all connected components of the complement of these curves in the torus  $T$  contain indetermination points in their boundaries. Moreover, there is an interval of the curve  $N(\phi, \theta) = 0$  joining indetermination points on different branches of  $D(\phi, \theta) = 0$ ; this is enough to glue together the two connected components above a band around one branch of  $D(\phi, \theta) = 0$  to the corresponding two connected components above a band around the other branch.

We completed the proof of the result of M. Husty:

**Theorem 1** *A generic planar 3-RPR manipulator has two aspects.*

## 2.3 Non-generic cases

We made in the proof genericity assumptions at several places. We now discuss briefly the non-generic cases. We shall only consider manipulators where neither the base triangle nor the platform triangle are flat (i.e.  $b_A h_A b_B h_b \neq 0$ ).

The most severe failure to genericity is the case when the two curves  $N(\phi, \theta) = 0$  and  $D(\phi, \theta) = 0$  have a common component. This case can be determined by computing resultants, and it occurs for the following peculiar geometries of the manipulator :

- “Similar” manipulators where the platform triangle and the base triangle are similar (i.e  $b_B = \lambda b_A$ ,  $h_B = \lambda h_A$  and  $d_B = \lambda d_A$  for some  $\lambda > 0$ ).

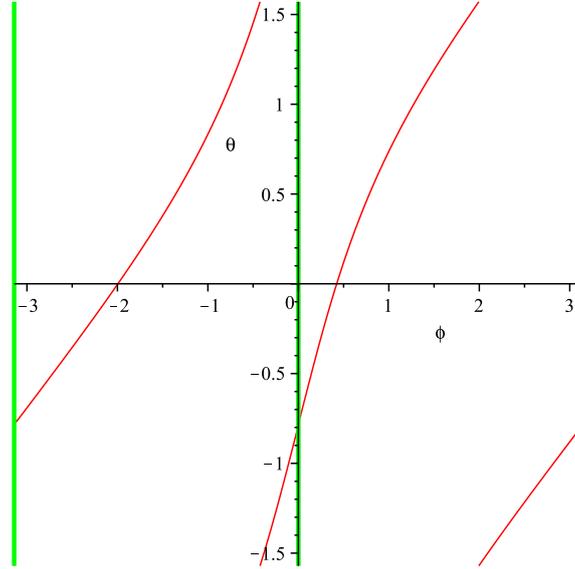


Figure 5: Example of a “similar” manipulator

An example is given in Figure 5, where the common components (two vertical lines at  $\phi = 0$  and  $\phi = \pm\pi$  are indicated in green. It is known there there are four aspects in this case, and this can be checked on the figure by easy topological arguments of the kind we used above.

- “Symmetric” manipulator where the platform triangle is the image of the base triangle by an indirect isometry of the plane (i.e  $b_B = b_A$ ,  $h_B = -h_A$  and  $d_B = d_A$ ).

An example is given in Figure 6. Here the common component indicated in green is described by  $\phi - 2\theta = \pm\pi$ . It is known [13] that there are two aspects in this case.

Other possibilities for non-genericity are :

- when the curve  $N(\phi, \theta) = 0$  has components which are vertical lines (on such a line we don’t have  $\theta$  as a function of  $\phi$ ). This happens when  $A_1A_2 = B_1B_2$  or  $A_1A_3 = B_1B_3$  or  $\angle(A_1A_2, A_1A_3) = \angle(B_1B_2, B_1B_3)$  or the height from vertex  $A_1$  in  $A_1A_2A_3$  is equal to the height from vertex  $B_1$  in  $B_1B_2B_3$ . In most cases (except of course the two mentioned above), the non-genericity is only apparent and disappears when one changes the choice of the privileged leg  $A_1B_1$ , but we have not investigated all possibilities.
- when the curves  $N(\phi, \theta) = 0$  and  $D(\phi, \theta) = 0$  have non transverse intersections. The topological arguments are a little more intricate in this case, but this is not a serious problem.

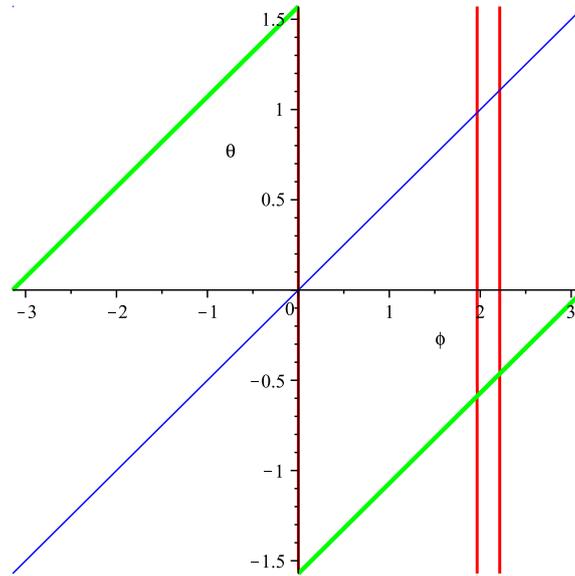


Figure 6: Example of a “symmetric” manipulator

In conclusion of this incomplete discussion, we conjecture that the only case where there are four aspects is the case of “similar” manipulators.

## Conclusion

The coordinates  $(\phi, \theta, r_1)$  we used for the workspace enabled us to express  $r_1$  as a function of the angular coordinates  $\phi$  and  $\theta$  on the singular surface. This description of the singular surface was then used in simple topological arguments to prove that, for a generic 3-RPR manipulator, the complement of the singular locus in the workspace has two connected components, i.e. the manipulator has two aspects. We also conjecture that the only case where there are four aspects instead of two is when the platform triangle is similar to the base triangle (assuming neither triangle is flat).

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