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THE POSSIBLE VALUES OF CRITICAL POINTS BETWEEN VARIETIES OF LATTICES

PIERRE GILLIBERT

ABSTRACT. We denote by $\text{Con}_c L$ the $(\vee, 0)$ -semilattice of all finitely generated congruences of a lattice L . For varieties (i.e., equational classes) \mathcal{V} and \mathcal{W} of lattices such that \mathcal{V} is contained neither in \mathcal{W} nor its dual, and such that every simple member of \mathcal{W} contains a prime interval, we prove that there exists a bounded lattice $A \in \mathcal{V}$ with at most \aleph_2 elements such that $\text{Con}_c A$ is not isomorphic to $\text{Con}_c B$ for any $B \in \mathcal{W}$. The bound \aleph_2 is optimal. As a corollary of our results, there are continuously many congruence classes of locally finite varieties of (bounded) modular lattices.

A large part of our work involves a categorical theory of partial algebras endowed with a partial subalgebra together with a semilattice-valued distance, that we call *gamps*. This part of the theory is formulated in any variety of (universal) algebras.

1. INTRODUCTION

1.1. **Background.** An *algebra* (in the sense of universal algebra) is a nonempty set A endowed with a collection of maps (“operations”) from finite powers of A to A . One of the most fundamental invariants associated with an algebra A is the lattice $\text{Con } A$ of all *congruences* of A , that is, the equivalence relations on A compatible with all the operations of A . It is often more convenient to work with the $(\vee, 0)$ -semilattice $\text{Con}_c A$ of all *finitely generated congruences* of A (for precise definitions we refer the reader to Section 2). We set

$$\text{Con}_c \mathcal{V} = \{S \mid (\exists A \in \mathcal{V})(S \cong \text{Con}_c A)\}, \quad \text{for every class } \mathcal{V} \text{ of algebras,}$$

and we call $\text{Con}_c \mathcal{V}$ the *compact congruence class* of \mathcal{V} . This object is especially studied for \mathcal{V} a *variety* (or *equational class*) of *algebras*, that is, the class of all algebras that satisfy a given set of *identities* (in a given similarity type).

For a variety \mathcal{V} , two fundamental questions arise:

- (Q1) Is $\text{Con}_c \mathcal{V}$ determined by a reasonably small fragment of itself—for example, is it determined by its *finite* members?
- (Q2) In what extent does $\text{Con}_c \mathcal{V}$ characterize \mathcal{V} ?

While both questions remain largely mysterious in full generality, partial answers are known in a number of situations. Here is an example that illustrates the difficulties underlying question (Q1) above. We are working with *varieties of lattices*. Denote by \mathcal{D} the variety of all distributive lattices and by \mathcal{M}_3 the variety of lattices

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generated by the five-element modular nondistributive lattice M_3 (cf. Figure 1, page 42). The finite members of both $\text{Con}_c \mathcal{D}$ and $\text{Con}_c \mathcal{M}_3$ are exactly the finite Boolean lattices. It is harder to prove that there exists a *countable* lattice $K \in \mathcal{M}_3$ such that $\text{Con}_c K \not\cong \text{Con}_c D$ for any $D \in \mathcal{D}$. This can be obtained easily from the results of [3], but a direct construction is also possible, as follows: denote by S a copy of the two-atom Boolean lattice in M_3 ; let K be the lattice of all eventually constant sequences of elements of M_3 such that the limit belongs to S . (This construction is a precursor of the *condensates* introduced in [3].)

This suggests a way to *measure* the “containment defect” of $\text{Con}_c \mathcal{V}$ into $\text{Con}_c \mathcal{W}$, for varieties (not necessarily in the same similarity type) \mathcal{V} and \mathcal{W} . The *critical point* between \mathcal{V} and \mathcal{W} is defined as

$$\text{crit}(\mathcal{V}; \mathcal{W}) = \begin{cases} \min\{\text{card } S \mid S \in (\text{Con}_c \mathcal{V}) - (\text{Con}_c \mathcal{W})\}, & \text{if } \text{Con}_c \mathcal{V} \not\subseteq \text{Con}_c \mathcal{W}, \\ \infty, & \text{if } \text{Con}_c \mathcal{V} \subseteq \text{Con}_c \mathcal{W}. \end{cases}$$

The example above shows that

$$\text{crit}(\mathcal{M}_3; \mathcal{D}) = \aleph_0.$$

Using varieties generated by finite non-modular lattices it is easy to construct *finite* critical points. By using techniques from *infinite combinatorics*, introduced in [15], Ploščica found in [10, 11] varieties of bounded lattices with critical point \aleph_2 ; the bounds of those examples were subsequently removed, by the author of the present paper, in [4]. For example, if we denote by \mathcal{M}_n the variety generated by the lattice of length 2 with n atoms, then

$$\text{crit}(\mathcal{M}_m; \mathcal{M}_n) = \aleph_2 \quad \text{if } m > n \geq 3.$$

In [3] we find a pair of finitely generated modular lattice varieties with critical point \aleph_1 , thus answering a 2002 question of Tůma and Wehrung from [14].

In “crossover” contexts (varieties on different similarity types), the situation between lattices and congruence-permutable varieties (say, *groups* or *modules*) is quite instructive. The five-element lattice M_3 is isomorphic to the congruence lattice of a group (take the Klein group) but not to the congruence lattice of any lattice (for it fails distributivity), and it is easily seen to be the smallest such example. Thus, if we denote by \mathcal{L} the variety of all lattices and by \mathcal{G} the variety of all groups, then

$$\text{crit}(\mathcal{G}; \mathcal{L}) = 5.$$

Much harder techniques, also originating from [15], are applied in [13], yielding the following result:

$$\text{crit}(\mathcal{L}; \mathcal{G}) = \aleph_2.$$

A general “crossover result”, proved in [3], is the following *Dichotomy Theorem*:

Dichotomy Theorem for varieties of algebras. *Let \mathcal{V} and \mathcal{W} be varieties of algebras, with \mathcal{V} locally finite and \mathcal{W} finitely generated congruence-distributive (for example, any variety of lattices). If $\text{Con}_c \mathcal{V} \not\subseteq \text{Con}_c \mathcal{W}$, then $\text{crit}(\mathcal{V}; \mathcal{W}) < \aleph_\omega$.*

In [6], we use techniques of *category theory* to extend this Dichotomy Theorem to much wider contexts, in particular to *congruence-modular* (instead of congruence-distributive) varieties and even to relative congruence classes of *quasivarieties*. Examples of congruence-modular varieties are varieties of *groups* (or even *loops*) or *modules*.

Now comes another mystery. We do not know any critical point, between varieties of algebras with (at most) countable similarity types, equal to \aleph_3 , or \aleph_4 , and so on: all known critical points are either below \aleph_2 or equal to ∞ . How general is that phenomenon?

One half of our main result, Theorem 11.6, is the following:

Dichotomy Theorem for varieties of lattices. *Let \mathcal{V} and \mathcal{W} be lattice varieties such that every simple member of \mathcal{W} contains a prime interval. If $\text{Con}_c \mathcal{V} \not\subseteq \text{Con}_c \mathcal{W}$, then $\text{crit}(\mathcal{V}; \mathcal{W}) \leq \aleph_2$.*

In particular, this result gives a solution to Question (Q1) for varieties of lattices where every simple member contains a prime interval. It turns out that the other half of Theorem 11.6 also solves Question (Q2) for those varieties, by proving that $\text{Con}_c \mathcal{V} \subseteq \text{Con}_c \mathcal{W}$ occurs only in the trivial cases, namely \mathcal{V} is contained in either \mathcal{W} or its dual.

1.2. Contents of the paper. Our main idea is the following. It is well-known that the assignment $A \mapsto \text{Con}_c A$ can be extended, in a standard way, to a *functor*, from all algebras of a given similarity type with their homomorphisms, to the category of all $(\vee, 0)$ -semilattices and $(\vee, 0)$ -homomorphisms: for a homomorphism $f: A \rightarrow B$ of algebras, $\text{Con}_c f$ sends any compact congruence α of A to the congruence generated by all pairs $(f(x), f(y))$ where $(x, y) \in \alpha$. Given a finite bounded lattice L , we construct a diagram of $(\vee, 0)$ -semilattices liftable in a variety of lattices (or bounded lattices) \mathcal{V} if and only if either L or its dual belongs to \mathcal{V} . Then, using a *condensate* [6, Section 3-1], we construct a $(\vee, 0)$ -semilattice liftable in \mathcal{V} if and only if either $L \in \mathcal{V}$ or $L^d \in \mathcal{V}$.

The main idea is to start with the *chain diagram* \vec{A} of L (cf. Definition 10.8). A precursor of that diagram can be found in [4, Section 4]. The chain diagram can be described in the following way. The arrows are the inclusion maps. At the bottom of the diagram we put $\{0, 1\}$, the sublattice of L consisting of the bounds of L . On the next level we put all three-element and four-element chains of L with extremities 0 and 1, over two chains we put the sublattice of L generated by those two chains, finally at the top we put the lattice L itself.

The diagram $\text{Con}_c \circ \vec{A}$ is liftable in any variety that contains either L or its dual; we do not know any counterexample of the converse yet. However, given a lifting \vec{B} of $\text{Con}_c \circ \vec{A}$, with all morphisms of \vec{B} being inclusion maps, if all lattices of \vec{B} that correspond to chains in \vec{A} contain “congruence chains” (cf. Definition 10.5) with the same extremities, we also require those chains be “direct” (cf. Definition 10.5). For example, if $A_i = \{0, x, 1\}$ is a chain, then $B_i = \{u, y, v\}$ is also a chain (with $u < y < v$), the congruence $\Theta_{A_i}(0, x)$ corresponds to the congruence $\Theta_{B_i}(u, y)$, and the congruence $\Theta_{A_i}(x, 1)$ corresponds to the congruence $\Theta_{B_i}(y, v)$. Under these assumptions we construct a sublattice of the top member of \vec{B} isomorphic to L .

The second step of our construction is to expand the chain diagram in order to force congruence chains to be direct (cf. Lemma 10.7). By gluing the chain diagram of L and copies of the diagram constructed in Lemma 10.7, we obtain a diagram \vec{A}' such that whenever \vec{B} is a lifting of $\text{Con}_c \circ \vec{A}'$ with enough congruence chains, then L embeds into a quotient of some lattice of \vec{B} or its dual.

The third step (cf. Lemma 11.2) is to ensure the existence of enough congruence chains. We can construct a diagram \vec{A}'' such that if $\text{Con}_c \circ \vec{A}''$ is liftable in some

variety \mathcal{W} , then either $L \in \mathcal{W}$ or $L \in \mathcal{W}^d$. For this step, we need a variety \mathcal{W} where every simple lattice has a prime interval (i.e., elements u and v such that $u \prec v$).

The last step is to use a *condensate* construction [6, Section 3-1] on \vec{A}'' to obtain a lattice B of cardinality \aleph_2 such that $\text{Con}_c B$ is liftable in \mathcal{W} if and only if $\text{Con}_c \circ \vec{A}''$ has a “partial lifting” in \mathcal{W} (cf. Theorem 9.3). Hence $\text{Con}_c B$ is liftable in \mathcal{W} if and only if either $L \in \mathcal{W}$ or $L \in \mathcal{W}^d$.

Theorem 9.3 is a generalization of [3, Theorem 6.9]. However, the construction used for the latter would give an upper bound \aleph_3 for critical points (or \aleph_2 in case the variety \mathcal{W} is finitely generated). A classical example illustrating the problem is that the three-element chain is the congruence lattice of a modular lattice, but not the congruence lattice of any finite modular lattice. In order to work around this problem, we apply the *Armature Lemma* from [6] to new objects called *gamps*, which are certain partial structures endowed with semilattice-valued distances (cf. Sections 4, 5, and 6). A *partial lifting* (cf. Definition 6.13) is a diagram of gamps that satisfies just enough properties to make the constructions in Sections 10 and 11 possible. The largest part of the paper studies algebraic properties, with a categorical slant, of these gamps; this part of the paper is not lattice-specific.

The chain diagram of a lattice and its uses (cf. Sections 10 and 11) can be illustrated by the following correspondences:

$$\begin{aligned} \mathbf{B} \text{ is a gamp of } \mathcal{W} &\iff \mathbf{B} = \mathbf{B}^* \text{ is a lattice in } \mathcal{W}, \\ \tilde{\mathbf{B}} &\iff \text{Con}_c \mathbf{B}, \\ \delta_{\mathbf{B}}(x, y) &\iff \Theta_{\mathbf{B}}(x, y), \\ \mathbf{g} \text{ is a morphism of gamps} &\iff \mathbf{g} \text{ is a morphism of lattices,} \\ \tilde{\mathbf{g}} &\iff \text{Con}_c \mathbf{g}, \\ \vec{\mathbf{B}} \text{ is partial lifting of } \vec{\mathbf{A}} \text{ in } \mathcal{W} &\iff \vec{\mathbf{B}} \text{ is a lifting of } \vec{\mathbf{A}} \text{ in } \mathcal{W}. \end{aligned}$$

The chain diagram also makes it possible to prove (cf. Theorem 12.2) that if there is a functor $\Psi: \mathcal{V} \rightarrow \mathcal{W}$ such that $\text{Con}_c \circ \Psi$ is equivalent to Con_c , then either $\mathcal{V} \subseteq \mathcal{W}$ or $\mathcal{V} \subseteq \mathcal{W}^d$. The functor Ψ itself does not need to be equivalent to either inclusion or dualization, however we prove in Theorem 12.2 that this holds “up to congruence-preserving extensions”.

The largest part of the paper, in particular the theory of *pregamps* and *gamps*, applies to any variety of algebras. Only the parts involving the *chain diagram* (essentially, from Section 10 on) are lattice-specific. It is conceivable that analogues of the chain diagram for other varieties of algebras could be worked out, thus making it possible to extend Theorem 11.6 to such varieties.

2. BASIC CONCEPTS

We denote by $\mathbf{2}$ the two-element lattice, or poset (i.e., partially ordered set), or $(\vee, 0)$ -semilattice, depending of the context.

We denote the *range* of a function $f: X \rightarrow Y$ by $\text{rng } f = \{f(x) \mid x \in X\}$. We use basic set-theoretical notation, for example ω is the first infinite ordinal, and also the set of all nonnegative integers; furthermore, $n = \{0, 1, \dots, n-1\}$ for every nonnegative integer n . By “countable” we will always mean “at most countable”.

Let X, I be sets, we often denote $\vec{x} = (x_i)_{i \in I}$ an element of X^I . In particular, for $n < \omega$, we denote by $\vec{x} = (x_0, \dots, x_{n-1})$ an n -tuple of X . If $f: Y \rightarrow Z$

is a function, where $Y \subseteq X$, we denote $f(\vec{x}) = (f(x_0), \dots, f(x_{n-1}))$ whenever it is defined. Similarly, if $f: Y \rightarrow Z$ is a function, where $Y \subseteq X^n$, we denote $f(\vec{x}) = f(x_0, \dots, x_{n-1})$ whenever it is defined. We also write $f(\vec{x}, \vec{y}) = f(x_0, \dots, x_{m-1}, y_0, \dots, y_{n-1})$ in case $\vec{x} = (x_0, \dots, x_{m-1})$ and $\vec{y} = (y_0, \dots, y_{n-1})$, and so on.

For example, let A and B be algebras of the same similarity type. Let ℓ be an n -ary operation. Let $f: A \rightarrow B$ a map. The map f is *compatible* with ℓ if $f(\ell(\vec{x})) = \ell(f(\vec{x}))$ for every n -tuple \vec{x} of X . Let $m \leq n \leq \omega$. Let \vec{x} be an n -tuple of X , we denote by $\vec{x} \upharpoonright m$ the m -tuple $(x_k)_{k < m}$.

If X is a set and θ is an equivalence relation on X , we denote by X/θ the set of all equivalence classes of θ . Given $x \in X$ we denote by x/θ the equivalence class of θ containing x . Given an n -tuple \vec{x} of X , we denote $\vec{x}/\theta = (x_0/\theta, \dots, x_{n-1}/\theta)$. Given $Y \subseteq X$, we set $Y/\theta = \{x/\theta \mid x \in Y\}$.

Given an algebra A , we denote by $\mathbf{0}_A$ (resp., $\mathbf{1}_A$) the smallest (resp. largest) congruence of A . Given x, y in A , we denote by $\Theta_A(x, y)$ the smallest congruence that identifies x and y , and we call it a *principal congruence*. A congruence of A is *finitely generated* if it is a finite join of principal congruences. The finitely generated congruences of A are exactly the *compact* elements of the algebraic lattice $\text{Con } A$. The set $\text{Con}_c A$ of all compact congruences of A is a $(\vee, 0)$ -subsemilattice of $\text{Con } A$. Given $X \subseteq A$ we denote by $\text{Con}_c^X A = \{\bigvee_{k < n} \Theta_A(x_k, y_k) \mid n < \omega, \text{ and } \vec{x}, \vec{y} \text{ are } n\text{-tuples of } X\}$.

The *kernel* $\ker f = \{(x, y) \in A^2 \mid f(x) = f(y)\}$ is a congruence of A , for any morphism of algebras $f: A \rightarrow B$. A *congruence-preserving extension* of an algebra A is an algebra B that contains A such that any congruence of A has a unique extension to B . Equivalently, $\text{Con}_c f$ is an isomorphism, where $f: A \rightarrow B$ denotes the inclusion map.

We denote by $\text{At } B$ the set of *atoms* (that is, the elements that cover 0) of a Boolean lattice B .

We denote by L^d the *dual* of a lattice L , that is, the lattice with the same universe as L and reverse ordering. Obviously, $\text{Con}_c L^d = \text{Con}_c L$. If \mathcal{V} is a variety of lattices, we denote by \mathcal{V}^d the class of all duals of lattices in \mathcal{V} . It is also the variety that satisfies all dual identities of those satisfied in \mathcal{V} . The assignment $L \mapsto L^d$ is functorial. We denote by \vec{L}^d the dual of a diagram \vec{L} of lattices (that is, we change each lattice of \vec{L} to its dual while keeping the same transition maps).

Let X be set, let κ be a cardinal. We denote by $\mathfrak{P}(X)$ the powerset of X and we set:

$$[X]^{<\kappa} = \{Y \in \mathfrak{P}(X) \mid \text{card } Y < \kappa\}.$$

The set X is κ -*small* if $\text{card } X < \kappa$.

We denote by **Set** the category of all sets with maps, and by **Sem** $_{\vee, 0}$ the category of $(\vee, 0)$ -semilattices with $(\vee, 0)$ -homomorphisms. We identify a class \mathcal{K} of algebras with the category of algebras of \mathcal{K} together with homomorphisms of algebras.

Let \mathcal{K} be a class of algebras on the same similarity type. The *variety generated by* \mathcal{K} , denoted by **Var** \mathcal{K} , is the smallest variety containing \mathcal{K} . If K is an algebra, we simply denote **Var** K instead of **Var** $\{K\}$. A variety \mathcal{V} is *finitely generated* if there exists a finite class \mathcal{K} of finite algebras such that $\mathcal{V} = \mathbf{Var } \mathcal{K}$. Equivalently, $\mathcal{V} = \mathbf{Var } K$ for a finite algebra K .

Given a poset P and subsets X and Q of P we set:

$$Q \downarrow X = \{q \in Q \mid (\exists x \in X)(q \leq x)\}$$

In case $X = \{a\}$ is a singleton, then we shall write $Q \downarrow a$ instead of $Q \downarrow \{a\}$. In case $P = Q$ we shall write $\downarrow X$ instead of $P \downarrow X$.

Given a poset P , we denote by $\text{Max } P$ the set of all maximal elements of P , and we set $P^\# = P - \text{Max } P$. Given $p, q \in P$, we say that q *covers* p , in notation $p \prec q$, if $p < q$ and there is no element $r \in P$ with $p < r < q$.

A poset P is *directed* if for all $x, y \in P$ there exists $z \in P$ such that $z \geq x, y$. A subset Q of a poset P is a *lower subset* if $Q = P \downarrow Q$, *spanning* if $P = P \downarrow Q = P \uparrow Q$. In case P has a least element 0 and a largest element 1 , Q is spanning if and only if $\{0, 1\} \subseteq Q$. For posets P and Q , a map $f: P \rightarrow Q$ is *isotone* if $x \leq y$ implies that $f(x) \leq f(y)$ for all $x, y \in P$.

Let $n \geq 2$ an integer. An algebra A is *congruence n -permutable* if the following equality holds:

$$\underbrace{\alpha \circ \beta \circ \alpha \circ \dots}_{n \text{ times}} = \underbrace{\beta \circ \alpha \circ \beta \circ \dots}_{n \text{ times}}, \text{ for all } \alpha, \beta \in \text{Con } A.$$

If $n = 2$ we say that A is *congruence-permutable* instead of *congruence 2-permutable*.

The following statement is folklore.

Proposition 2.1. *Let A be an algebra, let $n \geq 2$ an integer. The following conditions are equivalent:*

- (1) *The algebra A is congruence n -permutable.*
- (2) *For all $x_0, x_1, \dots, x_n \in A$, there are $y_0, y_1, \dots, y_n \in A$ such that the following containments hold:*

$$\begin{aligned} \Theta_A(y_k, y_{k+1}) &\subseteq \bigvee (\Theta_A(x_i, x_{i+1}) \mid i < n \text{ even}), & \text{for all } k < n \text{ odd,} \\ \Theta_A(y_k, y_{k+1}) &\subseteq \bigvee (\Theta_A(x_i, x_{i+1}) \mid i < n \text{ odd}), & \text{for all } k < n \text{ even.} \end{aligned}$$

Let $n \geq 2$. The class of all congruence n -permutable algebras of a given similarity type is closed under directed colimits and quotients. Moreover the class of congruence n -permutable algebras of a congruence-distributive variety is also closed under finite products (the latter statement is known not to extend to arbitrary algebras).

3. SEMILATTICES

In this section we give some well-known facts about $(\vee, 0)$ -semilattices. Most notions and results will have later a generalization with pregamps and gamps.

Proposition 3.1. *Let S, T be $(\vee, 0)$ -semilattices, let X be a set, and let $f: X \rightarrow S$ and $g: X \rightarrow T$ be maps. Assume that for every $x \in X$, for every positive integer n , and for every n -tuple \vec{y} of X the following implication holds:*

$$f(x) \leq \bigvee_{k < n} f(y_k) \implies g(x) \leq \bigvee_{k < n} g(y_k). \quad (3.1)$$

If S is join-generated by $f(X)$, then there exists a unique $(\vee, 0)$ -homomorphism $\phi: S \rightarrow T$ such that $\phi(f(x)) = g(x)$ for each $x \in X$.

If the converse of (3.1) also holds and $g(X)$ also join-generates T , then ϕ is an isomorphism.

Definition 3.2. An *ideal* of a $(\vee, 0)$ -semilattice S is a lower subset I of S such that $0 \in I$ and $u \vee v \in I$ for all $u, v \in I$. We denote by $\text{Id } S$ the lattice of ideals of S .

Let $\phi: S \rightarrow T$ be a $(\vee, 0)$ -homomorphism. The *0-kernel* of ϕ is $\ker_0 \phi = \{a \in S \mid \phi(a) = 0\}$; it is an ideal of S . We say that ϕ *separates zero* if $\ker_0 \phi = \{0\}$.

Let P be a poset, let $\vec{S} = (S_p, \phi_{p,q} \mid p \leq q \text{ in } P)$ be a diagram of $(\vee, 0)$ -semilattices. An *ideal* of \vec{S} is a family $(I_p)_{p \in P}$ such that I_p is an ideal of S_p and $\phi_{p,q}(I_p) \subseteq I_q$ for all $p \leq q$ in P .

Let $\vec{\phi} = (\phi_p)_{p \in P}: \vec{S} \rightarrow \vec{T}$ be a natural transformation of P -indexed diagrams of $(\vee, 0)$ -semilattices. The *0-kernel* of $\vec{\phi}$ is $\ker_0 \vec{\phi} = (\ker_0 \phi_p)_{p \in P}$, it is an ideal of \vec{S} .

Lemma 3.3. *Let S be a $(\vee, 0)$ -semilattice, let $I \in \text{Id } S$. Put:*

$$\theta_I = \{(x, y) \in S^2 \mid (\exists u \in I)(x \vee u = y \vee u)\}.$$

The relation θ_I is a congruence of S .

Notation 3.4. We denote S/I the $(\vee, 0)$ -semilattice S/θ_I , where θ_I is the congruence defined in Lemma 3.3. Given $a \in S$, we denote by a/I the equivalent class of a for θ_I . The $(\vee, 0)$ -homomorphism $\phi: S \rightarrow S/I$, $a \mapsto a/I$ is *the canonical projection*. Notice that $\ker_0 \phi = I$.

If $I = \{0\}$, we identify S/I and S .

Lemma 3.5. *Let $\phi: S \rightarrow T$ be a $(\vee, 0)$ -homomorphism, and let $I \in \text{Id } S$ and $J \in \text{Id } T$ such that $\phi(I) \subseteq J$. There exists a unique map $\psi: S/I \rightarrow T/J$ such that $\psi(a/I) = \phi(a)/J$ for each $a \in S$. Moreover, ψ is a $(\vee, 0)$ -homomorphism.*

Notation 3.6. We say that ϕ *induces* the $(\vee, 0)$ -homomorphism $\psi: S/I \rightarrow T/J$ in Lemma 3.5.

Lemma 3.7. *Let P be a poset, let $\vec{S} = (S_p, \phi_{p,q} \mid p \leq q \text{ in } P)$ be a diagram of $(\vee, 0)$ -semilattices, and let \vec{I} be an ideal of \vec{S} . Denote by $\psi_{p,q}: S_p/I_p \rightarrow S_q/I_q$ the $(\vee, 0)$ -homomorphism induced by $\phi_{p,q}$, then $(S_p/I_p, \psi_{p,q} \mid p \leq q \text{ in } P)$ is a diagram of $(\vee, 0)$ -semilattices.*

Notation 3.8. We denote \vec{S}/\vec{I} the diagram $(S_p/I_p, \psi_{p,q} \mid p \leq q \text{ in } P)$ introduced in Lemma 3.7.

Lemma 3.9. *Let P be a poset, let $\vec{\phi}: \vec{S} \rightarrow \vec{T}$ be a natural transformation of P -indexed diagrams of $(\vee, 0)$ -semilattices, let $\vec{I} \in \text{Id } \vec{S}$ and $\vec{J} \in \text{Id } \vec{T}$ such that $\phi_p(I_p) \subseteq J_p$ for all $p \in P$. Denote by $\psi_p: S_p/I_p \rightarrow T_p/J_p$ the $(\vee, 0)$ -homomorphism induced by ϕ_p . Then $\vec{\psi}$ is a natural transformation from \vec{S}/\vec{I} to \vec{T}/\vec{J} .*

Notation 3.10. We say that $\vec{\phi}$ *induces* $\vec{\psi}: \vec{S}/\vec{I} \rightarrow \vec{T}/\vec{J}$, the natural transformation defined in Lemma 3.9.

Definition 3.11. A $(\vee, 0)$ -homomorphism $\phi: S \rightarrow T$ is *ideal-induced* if ϕ is surjective and for all $x, y \in S$ with $\phi(x) = \phi(y)$ there exists $z \in S$ such that $x \vee z = y \vee z$ and $\phi(z) = 0$.

Let P be a poset, let $\vec{S} = (S_p, \phi_{p,q} \mid p \leq q \text{ in } P)$ and $\vec{T} = (T_p, \psi_{p,q} \mid p \leq q \text{ in } P)$ be P -indexed diagrams of $(\vee, 0)$ -semilattices. A natural transformation $\vec{\pi} = (\pi_p)_{p \in P}: \vec{S} \rightarrow \vec{T}$ is *ideal-induced* if π_p is ideal-induced for each $p \in P$.

Remark 3.12. Let I be an ideal of a $(\vee, 0)$ -semilattice A , denote by $\pi: A \rightarrow A/I$ the canonical projection, then π is ideal-induced.

The next lemmas give a characterization of ideal-induced $(\vee, 0)$ -homomorphisms.

Lemma 3.13. *Let $\phi: S \rightarrow T$ be a $(\vee, 0)$ -homomorphism. The following statements are equivalent*

- (1) ϕ is ideal-induced.
- (2) The $(\vee, 0)$ -homomorphism $\psi: S/\ker_0 \phi \rightarrow T$ induced by ϕ is an isomorphism.

The following lemma expresses that, given a diagram \vec{S} of $(\vee, 0)$ -semilattices, the colimits of quotients of \vec{S} are the quotients of the colimits of \vec{S} .

Lemma 3.14. *Let P be a directed poset, let $\vec{S} = (S_p, \phi_{p,q} \mid p \leq q \text{ in } P)$ be a P -indexed diagram in $\mathbf{Sem}_{\vee,0}$, and let $(S, \phi_p \mid p \in P) = \varinjlim \vec{S}$ be a directed colimit cocone in $\mathbf{Sem}_{\vee,0}$. The following statements hold:*

- (1) Let \vec{I} be an ideal of \vec{S} . Then $I = \bigcup_{p \in P} \phi_p(I_p)$ is an ideal of S . Moreover, denote by $\psi_p: S_p/I_p \rightarrow S/I$ the $(\vee, 0)$ -homomorphism induced by ϕ_p , for each $p \in P$. The following is a directed colimit cocone:

$$(S/I, \psi_p \mid p \in P) = \varinjlim \vec{S}/\vec{I} \quad \text{in } \mathbf{Sem}_{\vee,0}.$$

- (2) Let $I \in \text{Id } S$. Put $I_p = \phi_p^{-1}(I)$ for each $p \in P$. Then $\vec{I} = (I_p)_{p \in P}$ is an ideal of \vec{S} , moreover $I = \bigcup_{p \in P} \phi_p(I_p)$.

Lemma 3.15. *Let $\pi: A \twoheadrightarrow B$ be a surjective morphism of algebras. The $(\vee, 0)$ -homomorphism $\text{Con}_c \pi$ is ideal-induced. Moreover, $\ker_0(\text{Con}_c \pi) = (\text{Con}_c A) \downarrow \ker \pi$.*

Proposition 3.16. *Let S and T be $(\vee, 0)$ -semilattices with T finite, let $\phi: S \rightarrow T$ be an ideal-induced $(\vee, 0)$ -homomorphism, and let $X \subseteq S$ finite. There exists a finite $(\vee, 0)$ -subsemilattice S' of S such that $X \subseteq S'$ and $\phi \upharpoonright S': S' \rightarrow T$ is ideal-induced.*

Proof. As ϕ is surjective and X is finite, there exists a finite $(\vee, 0)$ -subsemilattice Y of S such that $X \subseteq Y$ and $\phi(Y) = T$. Given $x, y \in Y$ with $\phi(x) = \phi(y)$ we fix $u_{x,y} \in S$ such that $\phi(u_{x,y}) = 0$ and $x \vee u_{x,y} = y \vee u_{x,y}$. Let U be the $(\vee, 0)$ -subsemilattice of S generated by $\{u_{x,y} \mid x, y \in Y \text{ and } \phi(x) = \phi(y)\}$. As $\phi(u) = 0$ for all generators, $\phi(u) = 0$ for each $u \in U$.

Let S' be the $(\vee, 0)$ -subsemilattice of S generated by $Y \cup U$. As S' is finitely generated, it is finite. As $Y \subseteq S'$, $\phi(S') = T$. Let $a, b \in S'$ such that $\phi(a) = \phi(b)$. There exist $x, y \in Y$ and $u, v \in U$ such that $a = x \vee u$ and $b = y \vee v$, thus $\phi(a) = \phi(x \vee u) = \phi(x) \vee \phi(u) = \phi(x)$. Similarly, $\phi(b) = \phi(y)$, hence $\phi(x) = \phi(y)$, moreover $x, y \in Y$, so $u_{x,y} \in U$. The element $w = u \vee v \vee u_{x,y}$ belongs to U , hence $\phi(w) = 0$, moreover $w \in S'$. From $x \vee u_{x,y} = y \vee u_{x,y}$ it follows that $a \vee w = b \vee w$. Therefore, $\phi \upharpoonright S'$ is ideal-induced. \square

4. PARTIAL ALGEBRAS

In this section we introduce a few basic properties of partial algebras. We fix a similarity type \mathcal{L} . Given $\ell \in \mathcal{L}$ we denote by $\text{ar}(\ell)$ the arity of ℓ .

Definition 4.1. A *partial algebra* A is a set (the *universe* of the partial algebra), given with a set $D_\ell = \text{Def}_\ell(A) \subseteq A^{\text{ar}(\ell)}$ and a map $\ell^A: D_\ell \rightarrow A$ called a *partial operation*, for each $\ell \in \mathcal{L}$.

Let $\ell \in \mathcal{L}$ be an n -ary operation. If $\vec{x} \in \text{Def}_\ell(A)$ we say that $\ell^A(\vec{x})$ is *defined in* A . We generalize this notion to terms in the usual way. For example, given binary operations ℓ_1 and ℓ_2 of a partial algebra A and $x, y, z \in A$,

$\ell_1^A(\ell_2^A(x, y), \ell_1^A(y, z))$ is defined in A if and only if $(x, y) \in \text{Def}_{\ell_2}(A)$, $(y, z) \in \text{Def}_{\ell_1}(A)$, and $(\ell_2^A(x, y), \ell_1^A(y, z)) \in \text{Def}_{\ell_1}(A)$.

Given a term t , we denote by $\text{Def}_t(A)$ the set of all tuples \vec{x} of A such that $t(\vec{x})$ is defined in A .

We denote $\ell(\vec{x})$ instead of $\ell^A(\vec{x})$ when there is no ambiguity. Any algebra A has a natural structure of partial algebra with $\text{Def}_\ell(A) = A^{\text{ar}(\ell)}$ for each $\ell \in \mathcal{L}$.

Definition 4.2. Let A, B be partial algebras. A *morphism of partial algebras* is a map $f: A \rightarrow B$ such that $f(\vec{x}) \in \text{Def}_\ell(B)$ and $\ell(f(\vec{x})) = f(\ell(\vec{x}))$, for all $\ell \in \mathcal{L}$ and all $\vec{x} \in \text{Def}_\ell(A)$.

The *category of partial algebras*, denoted by $\mathbf{PAI}g_{\mathcal{L}}$, is the category in which the objects are the partial algebras and the arrows are the above-mentioned morphisms of partial algebras.

A morphism $f: A \rightarrow B$ of partial algebras is *strong* if $(f(A))^{\text{ar}(\ell)} \subseteq \text{Def}_\ell(B)$ for each $\ell \in \mathcal{L}$.

A partial algebra A is *finite* if its universe is finite.

Remark 4.3. A morphism $f: A \rightarrow B$ of partial algebras is an isomorphism if and only if the following conditions are both satisfied

- (1) The map f is bijective.
- (2) The element $\ell(f(\vec{x}))$ is defined in B implies that $\ell(\vec{x})$ is defined in A , for each operation $\ell \in \mathcal{L}$ and each tuple \vec{x} of A .

We remind that the converse of (2) is always true.

Definition 4.4. Given a partial algebra A , a *partial subalgebra* B of A is a subset B of A endowed with a structure of partial algebra such that $\text{Def}_\ell(B) \subseteq \text{Def}_\ell(A)$ and $\ell^A(\vec{x}) = \ell^B(\vec{x})$ for all $\ell \in \mathcal{L}$ and all $\vec{x} \in \text{Def}_\ell(B)$. The inclusion map from A into B is a morphism of partial algebras called *the inclusion morphism*.

A partial subalgebra B of A is *full* if whenever $\ell \in \mathcal{L}$ and $\vec{x} \in B^{\text{ar}(\ell)}$ are such that $\ell^A(\vec{x})$ is defined and belongs to B , then $\ell(\vec{x})$ is defined in B . It is equivalent to the following equality:

$$\text{Def}_\ell(B) = \{\vec{x} \in \text{Def}_\ell(A) \cap B^{\text{ar}(\ell)} \mid \ell(\vec{x}) \in B\}, \quad \text{for each } \ell \in \mathcal{L}.$$

A partial subalgebra B of A is *strong* if the inclusion map is a strong morphism, that is, $B^{\text{ar}(\ell)} \subseteq \text{Def}_\ell(A)$ for each $\ell \in \mathcal{L}$.

An *embedding* of partial algebras is a one-to-one morphism of partial algebras.

Notation 4.5. Let $f: A \rightarrow B$ be a morphism of partial algebras, let X be a partial subalgebra of A . The set $f(X)$ can be endowed with a natural structure of partial algebra, by setting $\text{Def}_\ell(f(X)) = f(\text{Def}_\ell(X)) = \{f(\vec{x}) \mid \vec{x} \in \text{Def}_\ell(X)\}$, for each $\ell \in \mathcal{L}$. Similarly, let Y be a partial subalgebra of B . The set $f^{-1}(Y)$ can be endowed with a natural structure of partial algebra, by setting $\text{Def}_\ell(f^{-1}(Y)) = f^{-1}(\text{Def}_\ell(Y)) = \{\vec{x} \in A \mid f(\vec{x}) \in \text{Def}_\ell(Y)\}$, for each $\ell \in \mathcal{L}$.

Remark. Let $f: A \rightarrow B$ and $g: B \rightarrow C$ be morphisms of partial algebras, let X a sub-partial algebra of A , then $(g \circ f)(X) = g(f(X))$ as partial algebras. Let Z be a partial subalgebra of X , then $(g \circ f)^{-1}(Z) = f^{-1}(g^{-1}(Z))$ as partial algebras.

Let $f: A \rightarrow B$ be a morphism of partial algebras, let X be a partial subalgebra of A . Then X is a partial subalgebra of $f^{-1}(f(X))$. In particular $f^{-1}(f(A)) = A$ as partial algebras. Let Y be a partial subalgebra of B , then $f(f^{-1}(Y))$ is a partial subalgebra of Y .

If \mathcal{L} is infinite, then there are a finite partial algebra A (even with one element) and an infinite chain of partial subalgebras of A with union A . In particular, A is not finitely presented in the category $\mathbf{PAlg}_{\mathcal{L}}$.

A morphism $f: A \rightarrow B$ of partial algebras is strong if and only if $f(A)$ is a strong partial subalgebra of B .

Lemma 4.6. *An embedding $f: A \rightarrow B$ of partial algebras is an isomorphism if and only if $f(A) = B$ as partial algebras.*

Proof. Assume that f is an isomorphism, let g be its inverse. Notice that $f(A)$ is a partial subalgebra of B and $B = f(g(B))$ is a partial subalgebra of $f(A)$, therefore $B = f(A)$ as partial algebras.

Conversely, assume that $B = f(A)$ as partial algebras. Then f is surjective, moreover f is an embedding, so f is a bijection. Let $g = f^{-1}$ in \mathbf{Set} . Let $\vec{y} \in \text{Def}_{\ell}(B)$. As $B = f(A)$, there exists $\vec{x} \in \text{Def}_{\ell}(A)$ such that $f(\vec{x}) = \vec{y}$, thus $g(\vec{y}) = \vec{x} \in \text{Def}_{\ell}(A)$. Moreover $g(\ell(\vec{y})) = g(\ell(f(\vec{x}))) = g(f(\ell(\vec{x}))) = \ell(\vec{x}) = \ell(g(\vec{y}))$. \square

Notation 4.7. Let A be a partial algebra, let X be a subset of A . We define inductively, for each $n < \omega$,

$$\begin{aligned} \langle X \rangle_A^0 &= X \cup \{c \mid c \text{ is a constant of } \mathcal{L}\} \\ \langle X \rangle_A^{n+1} &= \langle X \rangle_A^n \cup \{\ell(\vec{x}) \mid \ell \in \mathcal{L}, \vec{x} \in \text{Def}_{\ell}(A), \vec{x} \text{ is an ar}(\ell)\text{-tuple of } \langle X \rangle_A^n\} \end{aligned}$$

We endow $\langle X \rangle_A^n$ with the induced structure of full partial subalgebra of A . If \mathcal{L} and X are both finite, then $\langle X \rangle_A^n$ is finite for each $n < \omega$. If A is understood, we shall simply denote this partial algebra by $\langle X \rangle^n$.

Definition 4.8. A partial algebra A satisfies an identity $t_1 = t_2$ if $t_1(\vec{x}) = t_2(\vec{x})$ for each tuple \vec{x} of A such that both $t_1(\vec{x})$ and $t_2(\vec{x})$ are defined in A . Otherwise we say that A fails $t_1 = t_2$.

Let \mathcal{V} be a variety of algebras, a partial algebra A is a *partial algebra of \mathcal{V}* if A satisfies all identities of \mathcal{V} .

Remark. Let A be a partial algebra, let $\ell \in \mathcal{L}$. If $\text{Def}_{\ell}(A) = \emptyset$ then A satisfies $\ell(\vec{x}) = y$, vacuously.

If A fails $t_1 = t_2$, then there exists a tuple \vec{x} of A such that $t_1(\vec{x})$ and $t_2(\vec{x})$ are both defined and $t_1(\vec{x}) \neq t_2(\vec{x})$.

Lemma 4.9. *The category $\mathbf{PAlg}_{\mathcal{L}}$ has all directed colimits. Moreover, given a directed poset P , a P -indexed diagram $\vec{A} = (A_p, f_{p,q} \mid p \leq q \text{ in } P)$ in $\mathbf{PAlg}_{\mathcal{L}}$, and a directed colimit cocone:*

$$(A, f_p \mid p \in P) = \varinjlim (A_p, f_{p,q} \mid p \leq q \text{ in } P), \quad \text{in } \mathbf{Set}, \quad (4.1)$$

the set A can be uniquely endowed with a structure of partial algebra such that:

- $\text{Def}_{\ell}(A) = \{f_p(\vec{x}) \mid p \in P \text{ and } \vec{x} \in \text{Def}_{\ell}(A_p)\}$, for each $\ell \in \mathcal{L}$;
- $\ell(f_p(\vec{x})) = f_p(\ell(\vec{x}))$ for each $p \in P$, all $\ell \in \mathcal{L}$, and all $\vec{x} \in \text{Def}_{\ell}(A_p)$.

Moreover, if A is endowed with this structure of partial algebra, the following statements hold:

- (1) $(A, f_p \mid p \in P) = \varinjlim (A_p, f_{p,q} \mid p \leq q \text{ in } P)$ in $\mathbf{PAlg}_{\mathcal{L}}$.
- (2) Assume that for each $\ell \in \mathcal{L}$, each $p \in P$, and each $\text{ar}(\ell)$ -tuple \vec{x} of A_p there exists $q \geq p$ such that $f_{p,q}(\vec{x}) \in \text{Def}_{\ell}(A_q)$. Then A is an algebra, that is, $\text{Def}_{\ell}(A) = A^{\text{ar}(\ell)}$ for each $\ell \in \mathcal{L}$.

- (3) If P has no maximal element and $f_{p,q}$ is strong for all $p < q$ in P , then A is an algebra.
- (4) $\text{Def}_t(A) = \{f_p(\vec{x}) \mid p \in P \text{ and } \vec{x} \in \text{Def}_t(A_p)\}$ for each term t of \mathcal{L} .
- (5) Let $t_1 = t_2$ be an identity. If A_p satisfies $t_1 = t_2$ for all $p \in P$, then A satisfies $t_1 = t_2$.

Proof. Put $\text{Def}_\ell(A) = \{f_p(\vec{x}) \mid p \in P \text{ and } \vec{x} \in \text{Def}_\ell(A_p)\}$, for each $\ell \in \mathcal{L}$.

Let $\ell \in \mathcal{L}$, let $\vec{x} \in \text{Def}_\ell(A)$. There exist $p \in P$ and $\vec{y} \in \text{Def}_\ell(A_p)$ such that $\vec{x} = f_p(\vec{y})$. We first show that $f_p(\ell(\vec{y}))$ does not depend on the choice of p and \vec{y} . Let $q \in P$ and $\vec{z} \in \text{Def}_\ell(A_q)$ such that $\vec{x} = f_q(\vec{z})$. As $f_p(\vec{y}) = \vec{x} = f_q(\vec{z})$, it follows from (4.1) that there exists $r \geq p, q$ such that $f_{p,r}(\vec{y}) = f_{q,r}(\vec{z})$. Therefore the following equalities hold:

$$f_p(\ell(\vec{y})) = f_r(f_{p,r}(\ell(\vec{y}))) = f_r(\ell(f_{p,r}(\vec{y}))) = f_r(\ell(f_{q,r}(\vec{z}))) = f_r(f_{q,r}(\ell(\vec{z}))) = f_q(\ell(\vec{z})).$$

Hence $\ell(f_p(\vec{y})) = f_p(\ell(\vec{y}))$ for all $p \in P$ and all $\vec{y} \in \text{Def}_\ell(A_p)$ uniquely define a partial operation $\ell: \text{Def}_\ell(A) \rightarrow A$. Moreover f_p is a morphism of partial algebras for each $p \in P$.

Let $(B, g_p \mid p \in P)$ be a cocone over $(A_p, f_{p,q} \mid p \leq q \text{ in } P)$ in $\mathbf{PAlg}_{\mathcal{L}}$. In particular, it is a cocone in \mathbf{Set} , so there exists a unique map $h: A \rightarrow B$ such that $h \circ f_p = g_p$ for each $p \in P$. Let $\ell \in \mathcal{L}$, let $\vec{x} \in \text{Def}_\ell(A)$. There exist $p \in P$ and $\vec{y} \in \text{Def}_\ell(A_p)$ such that $\vec{x} = f_p(\vec{y})$, thus $h(\vec{x}) = h(f_p(\vec{y})) = g_p(\vec{y})$. As g_p is a morphism of partial algebras and $\vec{y} \in \text{Def}_\ell(A_p)$, we obtain that $h(\vec{x}) \in \text{Def}_\ell(B)$. Moreover the following equalities hold:

$$\ell(h(\vec{x})) = \ell(g_p(\vec{y})) = g_p(\ell(\vec{y})) = h(f_p(\ell(\vec{y}))) = h(\ell(f_p(\vec{y}))) = h(\ell(\vec{x})).$$

Hence h is a morphism of partial algebras. Therefore:

$$(A, f_p \mid p \in P) = \varinjlim (A_p, f_{p,q} \mid p \leq q \text{ in } P) \quad \text{in } \mathbf{PAlg}_{\mathcal{L}}.$$

Assume that for each $\ell \in \mathcal{L}$, for all $p \in P$, and for all $\text{ar}(\ell)$ -tuples \vec{x} of A_p , there exists $q \geq p$ such that $f_{p,q}(\vec{x}) \in \text{Def}_\ell(A_q)$.

Let $\ell \in \mathcal{L}$, let \vec{x} be an $\text{ar}(\ell)$ -tuple of A . There exist $p \in P$ and a tuple \vec{y} of A_p such that $\vec{x} = f_p(\vec{y})$. Let $q \geq p$ such that $f_{p,q}(\vec{y}) \in \text{Def}_\ell(A_q)$. It follows that $\vec{x} = f_p(\vec{y}) = f_q(f_{p,q}(\vec{y}))$ belongs to $\text{Def}_\ell(A)$. Therefore A is an algebra.

The statement (3) follows directly from (2). The statement (4) is proved by a straightforward induction on terms, and (5) is an easy consequence of (4). \square

5. PREGAMPS

A *pregamp* is a partial algebra endowed with a semilattice-valued “distance” (cf. (1)-(3)) compatible with all operations of A (cf. (4)). It is a generalization of the notion of *semilattice-metric space* defined in [6, Section 5-1].

Definition 5.1. Let A be a partial algebra, let S be a $(\vee, 0)$ -semilattice. A *S-valued partial algebra distance* on A is a map $\delta: A^2 \rightarrow S$ such that:

- (1) $\delta(x, y) = 0$ if and only if $x = y$, for all $x, y \in A$.
- (2) $\delta(x, y) = \delta(y, x)$, for all $x, y \in A$.
- (3) $\delta(x, y) \leq \delta(x, z) \vee \delta(z, y)$, for all $x, y, z \in A$.
- (4) $\delta(\ell(\vec{x}), \ell(\vec{y})) \leq \bigvee_{k < \text{ar}(\ell)} \delta(x_k, y_k)$, for all $\ell \in \mathcal{L}$ and all $\vec{x}, \vec{y} \in \text{Def}_\ell(A)$.

Then we say that $\mathbf{A} = (A, \delta, S)$ is a *pregamp*. We shall generally write $\delta_{\mathbf{A}} = \delta$ and $\tilde{A} = S$.

The *pregamp* is *distance-generated* if it satisfies the following additional property:

- (5) S is join-generated by $\delta_{\mathbf{A}}(A^2)$. That is, for all $\alpha \in S$ there are $n \geq 0$ and n -tuples \vec{x}, \vec{y} of A such that $\alpha = \bigvee_{k < n} \delta_{\mathbf{A}}(x_k, y_k)$.

Example 5.2. Let A be an algebra. We remind the reader that $\Theta_A(x, y)$ denotes the smallest congruence that identifies x and y , for all $x, y \in A$. This defines a distance $\Theta_A: A^2 \rightarrow \text{Con}_c A$. Moreover, $(A, \Theta_A, \text{Con}_c A)$ is a distance-generated pregamp.

A straightforward induction argument on the length of the term t yields the following lemma.

Lemma 5.3. *Let \mathbf{A} be a pregamp, let t be an n -ary term, and let $\vec{x}, \vec{y} \in \text{Def}_t(A)$. The following inequality holds:*

$$\delta_{\mathbf{A}}(t(\vec{x}), t(\vec{y})) \leq \bigvee_{k < n} \delta_{\mathbf{A}}(x_k, y_k).$$

We say that $\delta_{\mathbf{A}}$ and t are compatible.

Definition 5.4. Let \mathbf{A} and \mathbf{B} be pregamps. A *morphism* from \mathbf{A} to \mathbf{B} is an ordered pair $\mathbf{f} = (f, \tilde{f})$ such that $f: A \rightarrow B$ is a morphism of partial algebras, $\tilde{f}: \tilde{A} \rightarrow \tilde{B}$ is a $(\vee, 0)$ -homomorphism, and $\delta_{\mathbf{B}}(f(x), f(y)) = \tilde{f}(\delta_{\mathbf{A}}(x, y))$ for all $x, y \in A$.

Given morphisms $\mathbf{f}: \mathbf{A} \rightarrow \mathbf{B}$ and $\mathbf{g}: \mathbf{B} \rightarrow \mathbf{C}$ of pregamps, the pair $\mathbf{g} \circ \mathbf{f} = (g \circ f, \tilde{g} \circ \tilde{f})$ is a morphism from \mathbf{A} to \mathbf{C} .

We denote by $\mathbf{PGamp}_{\mathcal{L}}$ the category of pregamps with the morphisms defined above.

We denote by \mathbf{P}_{ga} the functor from the category of \mathcal{L} -algebras to $\mathbf{PGamp}_{\mathcal{L}}$ that maps an algebra A to $(A, \Theta_A, \text{Con}_c A)$, and a morphism of algebras f to $(f, \text{Con}_c f)$. We denote by \mathbf{C}_{pg} the functor from $\mathbf{PGamp}_{\mathcal{L}}$ to $\mathbf{Sem}_{\vee, 0}$ that maps a pregamp \mathbf{A} to \tilde{A} , and maps a morphism of pregamps $\mathbf{f}: \mathbf{A} \rightarrow \mathbf{B}$ to the $(\vee, 0)$ -homomorphism \tilde{f} .

Remark 5.5. A morphism $\mathbf{f}: \mathbf{A} \rightarrow \mathbf{B}$ of pregamps is an isomorphism if and only if f is an isomorphism of partial algebras and \tilde{f} is an isomorphism of $(\vee, 0)$ -semilattices.

Notice that $\mathbf{C}_{\text{pg}} \circ \mathbf{P}_{\text{ga}} = \text{Con}_c$.

We leave to the reader the straightforward proof of the following lemma.

Lemma 5.6. *The category $\mathbf{PGamp}_{\mathcal{L}}$ has all directed colimits. Moreover, given a directed poset P , a P -indexed diagram $\vec{\mathbf{A}} = (\mathbf{A}_p, \mathbf{f}_{p,q} \mid p \leq q \text{ in } P)$ in $\mathbf{PGamp}_{\mathcal{L}}$, a directed colimit cocone $(A, f_p \mid p \in P) = \varinjlim (A_p, f_{p,q} \mid p \leq q \text{ in } P)$ in $\mathbf{PAlg}_{\mathcal{L}}$, and a directed colimit cocone $(\tilde{A}, \tilde{f}_p \mid p \in P) = \varinjlim (\tilde{A}_p, \tilde{f}_{p,q} \mid p \leq q \text{ in } P)$ in $\mathbf{Sem}_{\vee, 0}$, there exists a unique \tilde{A} -valued partial algebra distance $\delta_{\mathbf{A}}$ on A such that $\delta_{\mathbf{A}}(f_p(x), f_p(y)) = \tilde{f}_p(\delta_{\mathbf{A}_p}(x, y))$ for all $p \in P$ and all $x, y \in A_p$.*

Furthermore $\mathbf{A} = (A, \delta_{\mathbf{A}}, \tilde{A})$ is a pregamp, $\mathbf{f}_p: \mathbf{A}_p \rightarrow \mathbf{A}$ is a morphism of pregamps for each $p \in P$, and the following is a directed colimit cocone:

$$(\mathbf{A}, \mathbf{f}_p \mid p \in P) = \varinjlim (\mathbf{A}_p, \mathbf{f}_{p,q} \mid p \leq q \text{ in } P), \quad \text{in } \mathbf{PGamp}_{\mathcal{L}}.$$

Moreover if \mathbf{A}_p is distance-generated for each $p \in P$, then \mathbf{A} is distance-generated.

Remark 5.7. As an immediate application of Lemma 5.6, and the fact that Con_c preserves directed colimits, we obtain that both \mathbf{C}_{pg} and \mathbf{P}_{ga} preserve directed colimits.

Definition 5.8. An embedding $f: B \rightarrow A$ of pregamps is a morphism of pregamps such that f and \tilde{f} are both one-to-one.

A sub-pregamp of a pregamp A is a pregamp B such that B is a partial subalgebra of A , \tilde{B} is a $(\vee, 0)$ -subsemilattice of \tilde{A} , and $\delta_B = \delta_A \upharpoonright B^2$.

If $f: B \rightarrow A$ and $\tilde{f}: \tilde{B} \rightarrow \tilde{A}$ denote the inclusion maps, the morphism of pregamps $\mathbf{f} = (f, \tilde{f})$ is called *the canonical embedding*.

Notation 5.9. Let $\mathbf{f}: B \rightarrow A$ be a morphism of pregamps. Given a sub-pregamp C of B , the triple $\mathbf{f}(C) = (f(C), \delta_A \upharpoonright (f(C))^2, \tilde{f}(\tilde{C}))$ (see Notation 4.5) is a sub-pregamp of A .

Given C a sub-pregamp of A , we denote $\mathbf{f}^{-1}(C) = (f^{-1}(C), \delta_B \upharpoonright (f^{-1}(C))^2, \tilde{f}^{-1}(\tilde{C}))$, it is a sub-pregamp of B .

We leave to the reader the straightforward proof of the following description of sub-pregamps and embeddings.

Proposition 5.10. *The following statements hold.*

- (1) Let A be a pregamp, let B be a partial subalgebra of A , let \tilde{B} be a $(\vee, 0)$ -subsemilattice of \tilde{A} that contains $\delta_A(B^2)$. Put $\delta_B = \delta_A \upharpoonright B^2$. Then (B, δ_B, \tilde{B}) is a sub-pregamp of A . Moreover, all sub-pregamps of A are of this form.
- (2) Let $\mathbf{f}: B \rightarrow A$ be a morphism of pregamps. Then f is an embedding of partial algebras if and only if \tilde{f} separates 0. Moreover \mathbf{f} is an embedding if and only if \tilde{f} is an embedding.
- (3) Let $\mathbf{f}: B \rightarrow A$ be an embedding of pregamps. The restriction $\mathbf{f}: B \rightarrow \mathbf{f}(B)$ is an isomorphism of pregamps.

The following result appears in [9, Theorem 10.4]. It gives a description of finitely generated congruences of general algebra.

Lemma 5.11. *Let B be an algebra, let m be a positive integer, let $x, y \in B$, and let \vec{x}, \vec{y} be m -tuples of B . Then $\Theta_B(x, y) \leq \bigvee_{i < m} \Theta_B(x_i, y_i)$ if and only if there are a positive integer n , a list \vec{z} of parameters from B , and terms t_0, \dots, t_n such that*

$$\begin{aligned} x &= t_0(\vec{x}, \vec{y}, \vec{z}), \\ y &= t_n(\vec{x}, \vec{y}, \vec{z}), \\ t_j(\vec{y}, \vec{x}, \vec{z}) &= t_{j+1}(\vec{x}, \vec{y}, \vec{z}) \quad (\text{for all } j < n). \end{aligned}$$

The following lemma shows that the obvious direction of Lemma 5.11 holds for pregamps.

Lemma 5.12. *Let B be a pregamp, let m be a positive integer, let $x, y \in B$, and let \vec{x}, \vec{y} be m -tuples of B . Assume that there are a positive integer n , a list \vec{z} of parameters from B , and terms t_0, \dots, t_n such that the following equalities hold and all evaluations are defined*

$$\begin{aligned} x &= t_0(\vec{x}, \vec{y}, \vec{z}), \\ y &= t_n(\vec{x}, \vec{y}, \vec{z}), \\ t_j(\vec{y}, \vec{x}, \vec{z}) &= t_{j+1}(\vec{x}, \vec{y}, \vec{z}), \quad (\text{for all } j < n). \end{aligned}$$

Then $\delta_B(x, y) \leq \bigvee_{i < m} \delta_B(x_i, y_i)$.

Proof. As $\delta_{\mathbf{B}}$ is compatible with terms (cf. Lemma 5.3), and $\delta_{\mathbf{B}}(u, u) = 0$ for each $u \in B$, the following inequality holds:

$$\delta_{\mathbf{B}}(t_j(\vec{x}, \vec{y}, \vec{z}), t_j(\vec{y}, \vec{x}, \vec{z})) \leq \bigvee_{k < n} \delta_{\mathbf{B}}(x_k, y_k), \quad \text{for all } j < n.$$

Hence:

$$\delta_{\mathbf{B}}(x, y) \leq \bigvee_{j < n} \delta_{\mathbf{B}}(t_j(\vec{x}, \vec{y}, \vec{z}), t_j(\vec{y}, \vec{x}, \vec{z})) \leq \bigvee_{k < n} \delta_{\mathbf{B}}(x_k, y_k). \quad \square$$

The following definition expresses that whenever two elements of A are identified by a ‘‘congruence’’ of A , then there is a ‘‘good reason’’ for this in B (cf. Lemma 5.11).

Definition 5.13. A morphism $f: \mathbf{A} \rightarrow \mathbf{B}$ of pregamps is *congruence-tractable* if for all $m < \omega$ and for all $x, y, x_0, y_0, \dots, x_{m-1}, y_{m-1}$ in A such that:

$$\delta_{\mathbf{B}}(x, y) \leq \bigvee_{k < m} \delta_{\mathbf{B}}(x_k, y_k),$$

there are a positive integer n , a list \vec{z} of parameters from B , and terms t_0, \dots, t_n such that the following equations are satisfied in B (in particular, all the corresponding terms are defined).

$$\begin{aligned} f(x) &= t_0(f(\vec{x}), f(\vec{y}), \vec{z}), \\ f(y) &= t_n(f(\vec{x}), f(\vec{y}), \vec{z}), \\ t_j(f(\vec{y}), f(\vec{x}), \vec{z}) &= t_{j+1}(f(\vec{x}), f(\vec{y}), \vec{z}) \quad (\text{for all } j < n). \end{aligned}$$

Lemma 5.14. Let P be directed poset and let $\vec{\mathbf{A}} = (\mathbf{A}_p, \mathbf{f}_{p,q} \mid p \leq q \text{ in } P)$ be a direct system of pregamps. Assume that for each $p \in P$ there exists $q \geq p$ in P such that $\mathbf{f}_{p,q}$ is congruence-tractable. Let:

$$(\mathbf{A}, \mathbf{f}_p \mid p \in P) = \varinjlim (\mathbf{A}_p, \mathbf{f}_{p,q} \mid p \leq q \text{ in } P), \quad \text{in } \mathbf{PGamp}_{\mathcal{L}}.$$

If A is an algebra then the following statements hold:

- (1) Let $x, y \in A$, let $m < \omega$, let $x_0, y_0, \dots, x_{m-1}, y_{m-1}$ in A . The following two inequalities are equivalent:

$$\delta_{\mathbf{A}}(x, y) \leq \bigvee_{k < m} \delta_{\mathbf{A}}(x_k, y_k), \quad (5.1)$$

$$\Theta_{\mathbf{A}}(x, y) \leq \bigvee_{k < m} \Theta_{\mathbf{A}}(x_k, y_k). \quad (5.2)$$

- (2) There exists a unique $(\vee, 0)$ -homomorphism $\phi: \text{Con}_{\mathbf{c}} A \rightarrow \tilde{A}$ such that:

$$\phi(\Theta_{\mathbf{A}}(x, y)) = \delta_{\mathbf{A}}(x, y), \quad \text{for all } x, y \in A.$$

Moreover ϕ is an embedding.

- (3) If \mathbf{A} is distance-generated, then the $(\vee, 0)$ -homomorphism ϕ above is an isomorphism.

Proof. Lemma 5.6 and Lemma 4.9 imply that the following are directed colimits cocones

$$(A, f_p \mid p \in P) = \varinjlim (A_p, f_{p,q} \mid p \leq q \text{ in } P), \quad \text{in } \mathbf{PAlg}_{\mathcal{L}}. \quad (5.3)$$

$$(A, f_p \mid p \in P) = \varinjlim (A_p, f_{p,q} \mid p \leq q \text{ in } P), \quad \text{in } \mathbf{Set}. \quad (5.4)$$

$$(\tilde{A}, \tilde{f}_p \mid p \in P) = \varinjlim (\tilde{A}_p, \tilde{f}_{p,q} \mid p \leq q \text{ in } P), \quad \text{in } \mathbf{Sem}_{\vee, 0}. \quad (5.5)$$

(1) Let $x, y \in A$, let $m < \omega$, and let \vec{x}, \vec{y} be m -tuples of A .

Assume that (5.1) holds. It follows from (5.4) that there are $p \in P$, $x', y' \in A$, and m -tuples \vec{x}', \vec{y}' of A_p , such that $x = f_p(x')$, $y = f_p(y')$, $\vec{x} = f_p(\vec{x}')$, and $\vec{y} = f_p(\vec{y}')$. The inequality (5.1) can be written

$$\delta_{\mathbf{A}}(f_p(x'), f_p(y')) \leq \bigvee_{k < m} \delta_{\mathbf{A}}(f_p(x'_k), f_p(y'_k)).$$

This implies:

$$\tilde{f}_p(\delta_{\mathbf{A}_p}(x', y')) \leq \tilde{f}_p\left(\bigvee_{k < m} \delta_{\mathbf{A}_p}(x'_k, y'_k)\right).$$

Hence, it follows from (5.5) that there exists $q \geq p$ with:

$$\tilde{f}_{p,q}(\delta_{\mathbf{A}_p}(x', y')) \leq \tilde{f}_{p,q}\left(\bigvee_{k < m} \delta_{\mathbf{A}_p}(x'_k, y'_k)\right),$$

so, changing p to q , x' to $f_{p,q}(x')$, y' to $f_{p,q}(y')$, \vec{x}' to $f_{p,q}(\vec{x}')$, and \vec{y}' to $f_{p,q}(\vec{y}')$, we can assume that:

$$\delta_{\mathbf{A}_p}(x', y') \leq \bigvee_{k < m} \delta_{\mathbf{A}_p}(x'_k, y'_k).$$

Let $q \geq p$ in P such that $f_{p,q}$ is congruence-tractable. There are a positive integer n , a list \vec{z} of parameters from A_q , and terms t_0, \dots, t_n such that the following equations are satisfied in A_q :

$$\begin{aligned} f_{p,q}(x') &= t_0(f_{p,q}(\vec{x}'), f_{p,q}(\vec{y}'), \vec{z}), \\ f_{p,q}(y') &= t_n(f_{p,q}(\vec{x}'), f_{p,q}(\vec{y}'), \vec{z}), \\ t_k(f_{p,q}(\vec{y}'), f_{p,q}(\vec{x}'), \vec{z}) &= t_{k+1}(f_{p,q}(\vec{x}'), f_{p,q}(\vec{y}'), \vec{z}), \quad (\text{for all } k < n). \end{aligned}$$

Hence, applying f_q , we obtain

$$\begin{aligned} x &= t_0(\vec{x}, \vec{y}, f_q(\vec{z})), \\ y &= t_n(\vec{x}, \vec{y}, f_q(\vec{z})), \\ t_k(\vec{y}, \vec{x}, f_q(\vec{z})) &= t_{k+1}(\vec{x}, \vec{y}, f_q(\vec{z})), \quad (\text{for all } k < n). \end{aligned}$$

Therefore, it follows from Lemma 5.11 that (5.2) holds.

Conversely, assume that (5.2) holds. It follows from Lemma 5.11 that there are a positive integer n , a list \vec{z} of parameters from A , and terms t_0, \dots, t_n such that

$$\begin{aligned} x &= t_0(\vec{x}, \vec{y}, \vec{z}), \\ y &= t_n(\vec{x}, \vec{y}, \vec{z}), \\ t_j(\vec{y}, \vec{x}, \vec{z}) &= t_{j+1}(\vec{x}, \vec{y}, \vec{z}), \quad (\text{for all } j < n). \end{aligned}$$

We conclude, using Lemma 5.12, that (5.1) holds.

As $\text{Con}_c A$ is generated by $\{\Theta_A(x, y) \mid x, y \in A\}$, the statement (2) follows from Proposition 3.1. Moreover if we assume that \mathbf{A} is distance-generated, that is \tilde{A} is join-generated by $\{\delta_{\mathbf{A}}(x, y) \mid x, y \in A\}$, then ϕ is an isomorphism. \square

As an immediate application, we obtain that a ‘‘true’’ directed colimit of ‘‘good’’ preamps is an algebra together with its congruences.

Corollary 5.15. *Let P be directed poset with no maximal element and let $\vec{A} = (\mathbf{A}_p, \mathbf{f}_{p,q} \mid p \leq q \text{ in } P)$ be a P -indexed diagram of distance-generated pregamps. If $\mathbf{f}_{p,q}$ is congruence-tractable and $f_{p,q}$ is strong for all $p < q$ in P , then there exists a unique $(\vee, 0)$ -homomorphism $\phi: \text{Con}_c A \rightarrow \vec{A}$ such that:*

$$\phi(\Theta_A(x, y)) = \delta_{\mathbf{A}}(x, y) \quad \text{for all } x, y \in A.$$

Moreover, ϕ is an isomorphism.

Definition 5.16. An ideal of a pregamp \mathbf{A} is an ideal of \vec{A} . Denote by $\text{Id } \mathbf{A} = \text{Id } \vec{A}$ the set of all ideals of \mathbf{A} .

Let P be a poset, let $\vec{A} = (\mathbf{A}_p, \mathbf{f}_{p,q} \mid p \leq q \text{ in } P)$ be a P -indexed diagram in $\mathbf{PGamp}_{\mathcal{L}}$. An ideal of \vec{A} is an ideal of $(\vec{A}_p, \vec{f}_{p,q} \mid p \leq q \text{ in } P)$ (cf. Definition 3.2).

Definition 5.17. Let $\pi: \mathbf{A} \rightarrow \mathbf{B}$ be a morphism of pregamps. The 0-kernel of π , denoted by $\ker_0 \pi$, is the 0-kernel of $\vec{\pi}$ (cf. Definition 3.2).

Let P be a poset and let $\vec{\pi} = (\pi_p)_{p \in P}: \vec{A} \rightarrow \vec{B}$ be a natural transformation of P -indexed diagrams of pregamps. The 0-kernel of $\vec{\pi}$ is $\vec{I} = (\ker_0 \pi_p)_{p \in P}$.

Remark 5.18. The 0-kernel of π is an ideal of \mathbf{A} . Similarly the 0-kernel of $\vec{\pi}$ is an ideal of \vec{A} .

If $\pi: A \rightarrow B$ is a morphism of algebras, then $\ker_0 \mathbf{P}_{\text{ga}}(\pi)$ is the set of all compact congruences of A below $\ker \pi$, that is, $\ker_0 \mathbf{P}_{\text{ga}}(\pi) = (\text{Con}_c A) \downarrow \ker \pi$.

Definition 5.19. A morphism $\mathbf{f}: \mathbf{A} \rightarrow \mathbf{B}$ of pregamps, is *ideal-induced* if $f(A) = B$ as partial algebras and \vec{f} is ideal-induced. In that case we say that \mathbf{B} is an *ideal-induced image* of \mathbf{A} .

Let P be a poset, let \vec{A} and \vec{B} be P -indexed diagrams of pregamps. A natural transformation $\vec{f} = (\mathbf{f}_p)_{p \in P}: \vec{A} \rightarrow \vec{B}$ is *ideal-induced* if \mathbf{f}_p is ideal-induced for each $p \in P$.

Remark 5.20. A morphism $\mathbf{f}: \mathbf{A} \rightarrow \mathbf{B}$ of pregamps is ideal-induced if \vec{f} is ideal-induced, f is surjective, and for each $\ell \in \mathcal{L}$ and each tuple \vec{b} of B , $\ell(\vec{b})$ is defined in B if and only if there exists a tuple \vec{a} in A such that $\vec{b} = f(\vec{a})$ and $\ell(\vec{a})$ is defined in A .

If $f: A \rightarrow B$ is a surjective morphism of algebras, then $\mathbf{P}_{\text{ga}}(f)$ is ideal-induced.

If $\mathbf{f}: \mathbf{A} \rightarrow \mathbf{B}$ and $\mathbf{g}: \mathbf{B} \rightarrow \mathbf{C}$ are ideal-induced morphisms of pregamps, then $\mathbf{g} \circ \mathbf{f}$ is ideal-induced.

The following proposition gives a description of *quotients* of pregamps.

Proposition 5.21. *Let \mathbf{A} be a pregamp and let $I \in \text{Id } \vec{A}$. The binary relation $\theta_I = \{(x, y) \in A^2 \mid \delta_{\mathbf{A}}(x, y) \in I\}$ is an equivalence relation on A . Given $a \in A$ denote by a/I the θ_I -equivalence class containing a , and set $A/I = A/\theta_I$. We can define a structure of partial algebra on A/I in the following way. Given $\ell \in \mathcal{L}$, we put:*

$$\begin{aligned} \text{Def}_{\ell}(A/I) &= \{\vec{x}/I \mid \vec{x} \in \text{Def}_{\ell}(A)\}, \\ \ell^{A/I}(\vec{x}/I) &= \ell^A(\vec{x})/I, \quad \text{for all } \vec{x} \in \text{Def}_{\ell}(A). \end{aligned}$$

Moreover $\delta_{\mathbf{A}/I}: (A/I)^2 \rightarrow \vec{A}/I$, $(x/I, y/I) \mapsto \delta_{\mathbf{A}}(x, y)/I$ defines an \vec{A}/I -valued partial algebra distance, and the following statements hold:

- (1) $\mathbf{A}/I = (A/I, \delta_{\mathbf{A}/I}, \vec{A}/I)$ is a pregamp.

- (2) Put $\pi: A \rightarrow A/I$, $x \mapsto x/I$, and denote by $\tilde{\pi}: \tilde{A} \rightarrow \tilde{A}/I$, $d \mapsto d/I$ the canonical projection. Then $\boldsymbol{\pi} = (\pi, \tilde{\pi})$ is an ideal-induced morphism of pregamps from \mathbf{A} to \mathbf{A}/I .
- (3) The 0-kernel of $\boldsymbol{\pi}$ is I .
- (4) If \mathbf{A} is distance-generated, then \mathbf{A}/I is distance-generated.

Proof. The relation θ_I is reflexive (it follows from Definition 5.1(1)), symmetric (see Definition 5.1(2)) and transitive (see Definition 5.1(3)), thus it is an equivalence relation.

Let $\ell \in \mathcal{L}$, let $\vec{x}, \vec{y} \in \text{Def}_\ell(A)$ such that $x_k/I = y_k/I$ for each $k < \text{ar}(\ell)$. It follows from Definition 5.1(4) that $\delta_{\mathbf{A}}(\ell^A(\vec{x}), \ell^A(\vec{y})) \leq \bigvee_{k < \text{ar}(\ell)} \delta_{\mathbf{A}}(x_k, y_k) \in I$, so $\ell^A(\vec{x})/I = \ell^A(\vec{y})/I$. Therefore the partial operation $\ell^{A/I}: \text{Def}_\ell(A/I) \rightarrow A/I$ is well defined.

Let $x, x', y, y' \in A$, assume that $x/I = x'/I$ and $y/I = y'/I$. The following inequality holds:

$$\delta_{\mathbf{A}}(x, y) \leq \delta_{\mathbf{A}}(x, x') \vee \delta_{\mathbf{A}}(x', y') \vee \delta_{\mathbf{A}}(y', y).$$

However, $\delta_{\mathbf{A}}(x, x')$ and $\delta_{\mathbf{A}}(y, y')$ both belong to I , hence $\delta_{\mathbf{A}}(x, y)/I \leq \delta_{\mathbf{A}}(x', y')/I$. Similarly $\delta_{\mathbf{A}}(x', y')/I \leq \delta_{\mathbf{A}}(x, y)/I$. So the map $\delta_{\mathbf{A}/I}: (A/I)^2 \rightarrow \tilde{A}/I$ is well defined.

Let $x, y \in A$, the following equivalences hold:

$$\delta_{\mathbf{A}/I}(x/I, y/I) = 0/I \iff \delta_{\mathbf{A}}(x, y) \in I \iff x/I = y/I.$$

That is, Definition 5.1(1) holds. Each of the conditions of Definition 5.1(2)-(5) for $\delta_{\mathbf{A}}$ implies its analogue for $\delta_{\mathbf{A}/I}$.

It is easy to check that $\boldsymbol{\pi}$ is well defined and is a morphism of pregamps. \square

Notation 5.22. The notations \mathbf{A}/I , A/I , and $\delta_{\mathbf{A}/I}$ used in Proposition 5.21 will be used throughout the paper. The map $\boldsymbol{\pi}$ is the canonical projection.

If $I = \{0\}$, we identify \mathbf{A}/I and \mathbf{A} .

If X is a partial subalgebra of A , then we denote $X/I = \{x/I \mid x \in X\}$, with its natural structure of partial subalgebra of A , inherited from X , with $\text{Def}_\ell(X/I) = \{\vec{x}/I \mid \vec{x} \in \text{Def}_\ell(X)\}$ for each $\ell \in \mathcal{L}$. That is $X/I = \pi(X)$ as partial algebras.

Let \mathbf{B} be a sub-pregamp of \mathbf{A} and let I be a common ideal of \mathbf{A} and \mathbf{B} . Then we identify the quotient \mathbf{B}/I with the corresponding sub-pregamp of \mathbf{A}/I .

Remark. It is easy to construct a pregamp \mathbf{A} , a term t , a tuple \vec{x} of A , and an ideal I of \tilde{A} , such that $t(\vec{x})$ is not defined in A , but $t(\vec{x}/I)$ is defined in A/I .

The following proposition gives a description on how morphisms of pregamps factorize through quotients. It is related to Lemma 3.5.

Proposition 5.23. *Let $f: \mathbf{A} \rightarrow \mathbf{B}$ be a morphism of pregamps, let $I \in \text{Id } \mathbf{A}$, and let $J \in \text{Id } \mathbf{B}$. If $\tilde{f}(I) \subseteq J$, then the following maps are well defined:*

$$\begin{aligned} g: A/I &\rightarrow B/J \\ x/I &\mapsto f(x)/J, \\ \tilde{g}: \tilde{A}/I &\rightarrow \tilde{B}/J \\ \alpha/I &\mapsto \tilde{f}(\alpha)/J. \end{aligned}$$

Moreover, $\mathbf{g} = (g, \tilde{g})$ is a morphism of pregamps from \mathbf{A}/I to \mathbf{B}/J . If $\pi_I: \mathbf{A} \rightarrow \mathbf{A}/I$ and $\pi_J: \mathbf{A} \rightarrow \mathbf{A}/I$ denote the canonical projections, then the following diagram commutes:

$$\begin{array}{ccc} \mathbf{A} & \xrightarrow{f} & \mathbf{B} \\ \pi_I \downarrow & & \downarrow \pi_J \\ \mathbf{A}/I & \xrightarrow{g} & \mathbf{B}/J \end{array}$$

Proof. Observe that $\tilde{g}: \tilde{\mathbf{A}}/I \rightarrow \tilde{\mathbf{B}}/J$ is the $(\vee, 0)$ -homomorphism induced by \tilde{f} . Let $x, y \in A$ such that $x/I = y/I$, that is, $\delta_{\mathbf{A}}(x, y) \in I$. It follows that $\delta_{\mathbf{B}}(f(x), f(y)) = \tilde{f}(\delta_{\mathbf{A}}(x, y)) \in J$, so $f(x)/J = f(y)/J$. Therefore the map g is well defined.

Let $\ell \in \mathcal{L}$, let $\vec{a} \in \text{Def}_{\ell}(\mathbf{A}/I)$, and let $\vec{x} \in \text{Def}_{\ell}(A)$ such that $\vec{a} = \vec{x}/I$. The following equalities hold:

$$g(\ell(\vec{x}/I)) = g(\ell(\vec{x})/I) = f(\ell(\vec{x}))/J = \ell(f(\vec{x}))/J = \ell(f(\vec{x})/J) = \ell(g(\vec{x}/I)).$$

Thus $g(\ell(\vec{a})) = \ell(g(\vec{a}))$. Therefore g is a morphism of partial algebras.

Let $x, y \in A$. It is easy to check $\tilde{g}(\delta_{\mathbf{A}/I}(x/I, y/I)) = \delta_{\mathbf{B}/J}(g(x/J), g(y/J))$. Therefore $\mathbf{g}: \mathbf{A}/I \rightarrow \mathbf{B}/J$ is a morphism of pregamps. Moreover $\pi_J \circ \mathbf{f} = \mathbf{g} \circ \pi_I$ is obvious. \square

Notation 5.24. We say that \mathbf{f} induces $\mathbf{g}: \mathbf{A}/I \rightarrow \mathbf{B}/J$, the morphism of Proposition 5.23.

Let P be a poset, let $\vec{\mathbf{A}} = (\mathbf{A}_p, \mathbf{f}_{p,q} \mid p \leq q \text{ in } P)$ be a P -indexed diagram in $\mathbf{PGamp}_{\mathcal{L}}$, let $\vec{I} = (I_p)_{p \in P}$ be an ideal of $\vec{\mathbf{A}}$, and let $\mathbf{g}_{p,q}: \mathbf{A}_p/I_p \rightarrow \mathbf{A}_q/I_q$ the morphism induced by $\mathbf{f}_{p,q}$, for all $p \leq q$ in P . We denote by $\vec{\mathbf{A}}/\vec{I} = (\mathbf{A}_p/I_p, \mathbf{g}_{p,q} \mid p \leq q \text{ in } P)$.

The diagram $\vec{\mathbf{A}}/\vec{I}$ is a *quotient* of $\vec{\mathbf{A}}$.

Remark 5.25. It is easy to check that $\vec{\mathbf{A}}/\vec{I}$ is indeed a diagram. Given $p \leq q \leq r$ in P and $x \in A_p$ the following equalities hold:

$$g_{q,r}(g_{p,q}(x/I_p)) = g_{q,r}(f_{p,q}(x)/I_q) = f_{q,r}(f_{p,q}(x))/I_r = f_{p,r}(x)/I_r = g_{p,r}(x/I_p).$$

Proposition 5.23 can be easily extended to diagrams in the following way. It is also related to Lemma 3.7.

Proposition 5.26. *Let P be a poset, let $\vec{\mathbf{A}} = (\mathbf{A}_p, \mathbf{f}_{p,q} \mid p \leq q \text{ in } P)$ and $\vec{\mathbf{B}} = (\mathbf{B}_p, \mathbf{g}_{p,q} \mid p \leq q \text{ in } P)$ be P -indexed diagrams in $\mathbf{PGamp}_{\mathcal{L}}$. Let \vec{I} be an ideal of $\vec{\mathbf{A}}$, let \vec{J} be an ideal of $\vec{\mathbf{B}}$. Let $\vec{\xi} = (\xi_p)_{p \in P}: \vec{\mathbf{A}} \rightarrow \vec{\mathbf{B}}$ be a natural transformation such that $\xi_p(I_p) \subseteq J_p$ for each $p \in P$. Denote by $\chi_p: \mathbf{A}_p/I_p \rightarrow \mathbf{B}_p/J_p$ the morphism induced by ξ_p , for each $p \in P$. Then $\vec{\chi} = (\chi_p)_{p \in P}$ is a natural transformation from $\vec{\mathbf{A}}/\vec{I}$ to $\vec{\mathbf{B}}/\vec{J}$.*

Notation 5.27. With the notation of Proposition 5.26. We say that $\vec{\chi}: \vec{\mathbf{A}}/\vec{I} \rightarrow \vec{\mathbf{B}}/\vec{J}$ is induced by $\vec{\xi}$.

The following lemma expresses that ideal-induced images of pregamps correspond, up to isomorphism, to quotients of pregamps. It is related to Lemma 3.13.

Lemma 5.28. *Let $\mathbf{f}: \mathbf{A} \rightarrow \mathbf{B}$ be a morphism of pregamps. The following statements are equivalent:*

- (1) \mathbf{f} is ideal-induced.
- (2) \mathbf{f} induces an isomorphism $\mathbf{g}: \mathbf{A}/\ker_0 \mathbf{f} \rightarrow \mathbf{B}$.

Proof. Denote $\pi: \mathbf{A} \rightarrow \mathbf{A}/\ker_0 \mathbf{f}$ the canonical projection, so $\mathbf{g} \circ \pi = \mathbf{f}$.

Assume that \mathbf{f} is ideal-induced. As $\tilde{f}: \tilde{A} \rightarrow \tilde{B}$ is ideal-induced, Lemma 3.13 implies that \tilde{f} induces an isomorphism $\tilde{g}: \tilde{A}/\ker_0 \tilde{f} \rightarrow \tilde{B}$. It follows that \tilde{g} separates 0, thus (cf. Proposition 5.10(2)) g is an embedding. Moreover $g(\mathbf{A}/\ker_0 \mathbf{f}) = g(\pi(\mathbf{A})) = \mathbf{f}(\mathbf{A}) = \mathbf{B}$ as partial algebras. Therefore it follows from Lemma 4.6 that g is an isomorphism of partial algebras, thus \mathbf{g} is an isomorphism of pregamps (cf. Remark 5.5).

Assume that \mathbf{g} is an isomorphism. It follows that \tilde{g} is an isomorphism, so Lemma 3.13 implies that \tilde{f} is ideal-induced. Moreover g is an isomorphism, thus $\mathbf{f}(\mathbf{A}) = g(\pi(\mathbf{A})) = g(\mathbf{A}/\ker_0 \mathbf{f}) = \mathbf{B}$ as partial algebras. Therefore \mathbf{f} is ideal-induced. \square

The following proposition expresses that a quotient of a quotient is a quotient. It follows from Lemma 5.28, together with the fact that a composition of ideal-induced morphisms of pregamps is ideal-induced.

Proposition 5.29. *Let \mathbf{A} be a pregamp, let I be an ideal of \mathbf{A} , let J be an ideal of \mathbf{A}/I . Then $(\mathbf{A}/I)/J$ is isomorphic to a quotient of \mathbf{A} .*

The following results expresses that, up to isomorphism, quotients of sub-pregamps are sub-pregamps of quotients.

Proposition 5.30. *Let \mathbf{A} be a pregamp, let \mathbf{B} be a sub-pregamp of \mathbf{A} , and let $I \in \text{Id } \mathbf{B}$. Then there exist $J \in \text{Id } \mathbf{A}$, a sub-pregamp \mathbf{C} of \mathbf{A}/J , and an isomorphism $\mathbf{f}: \mathbf{B}/I \rightarrow \mathbf{C}$.*

Let \mathbf{A} be a pregamp, let $I \in \text{Id } \mathbf{A}$, and let \mathbf{B} be a sub-pregamp of \mathbf{A}/I . There exists a sub-pregamp \mathbf{C} of \mathbf{A} such that \mathbf{B} is isomorphic to some quotient of \mathbf{C} .

Proof. Let \mathbf{A} be a pregamp, let \mathbf{B} be a sub-pregamp of \mathbf{A} , let $I \in \text{Id } \mathbf{B}$. Put $J = \tilde{A} \downarrow I$. As I is an ideal of \tilde{B} , it is directed, therefore J is an ideal of \tilde{A} .

Let $\mathbf{f}: \mathbf{B} \rightarrow \mathbf{A}$ be the canonical embedding. Notice that $\tilde{f}(I) \subseteq J$; denote by $\mathbf{g}: \mathbf{B}/I \rightarrow \mathbf{A}/J$ the morphism induced by \mathbf{f} (cf. Proposition 5.23).

Let $d, d' \in \tilde{B}$ such that $\tilde{g}(d/I) = \tilde{g}(d'/I)$, that is, $d/J = d'/J$, so there exists $u \in J$ such that $d \vee u = d' \vee u$. As $J = \tilde{A} \downarrow I$, there exists $v \in I$ such that $u \leq v$, hence $d \vee v = d' \vee v$, that is, $d/I = d'/I$. Therefore \tilde{g} is an embedding. It follows from Proposition 5.10 that $\mathbf{g} = (g, \tilde{g})$ is an embedding and induces an isomorphism $\mathbf{B}/I \rightarrow \mathbf{g}(\mathbf{B}/I)$; the latter is a sub-pregamp of \mathbf{A}/J .

Now let $I \in \text{Id } \mathbf{A}$ and let \mathbf{B} be a sub-pregamp of \mathbf{A}/I . Denote by $\pi: \mathbf{A} \rightarrow \mathbf{A}/I$ the canonical projection, put $\mathbf{C} = \pi^{-1}(\mathbf{B})$ (cf. Notation 5.9). As π is ideal-induced, it is easy to check that $\pi(\mathbf{C}) = \mathbf{B}$, and the restriction $\pi \upharpoonright \mathbf{C} \rightarrow \mathbf{B}$ is ideal-induced. \square

The following lemma, in conjunction with Lemma 3.14, proves that, given a diagram of pregamps $\vec{\mathbf{A}}$, the quotients of colimits of $\vec{\mathbf{A}}$ are colimits of quotients of $\vec{\mathbf{A}}$.

Lemma 5.31. *Let P be directed poset and let $\vec{\mathbf{A}} = (\mathbf{A}_p, \mathbf{f}_{p,q} \mid p \leq q \text{ in } P)$ be a P -indexed diagram in $\mathbf{PGamp}_{\mathcal{L}}$. Let $(\mathbf{A}, \mathbf{f}_p \mid p \in P) = \varinjlim (\mathbf{A}_p, \mathbf{f}_{p,q} \mid p \leq q \text{ in } P)$*

be a directed colimit cocone in $\mathbf{PGamp}_{\mathcal{L}}$. Let $\vec{I} = (I_p)_{p \in P}$ be an ideal of \vec{A} . Then $I = \bigcup_{p \in P} \tilde{f}_p(I_p)$ is an ideal of \mathbf{A} .

Let $g_p: \mathbf{A}_p/I_p \rightarrow \mathbf{A}/I$ be the morphism induced by f_p , let $g_{p,q}: \mathbf{A}_p/I_p \rightarrow \mathbf{A}_q/I_q$ be the morphism induced by $f_{p,q}$, for all $p \leq q$ in P . The following is a directed colimit cocone:

$$(\mathbf{A}/I, g_p \mid p \in P) = \varinjlim (\mathbf{A}_p/I_p, f_{p,q} \mid p \leq q \text{ in } P) \text{ in } \mathbf{PGamp}_{\mathcal{L}}.$$

Proof. Lemma 4.9 and Lemma 5.6 imply that the following are colimits cocones:

$$(A, f_p \mid p \in P) = \varinjlim (A_p, f_{p,q} \mid p \leq q \text{ in } P) \text{ in } \mathbf{Set}, \quad (5.6)$$

$$(\vec{A}, \tilde{f}_p \mid p \in P) = \varinjlim (\vec{A}_p, \tilde{f}_{p,q} \mid p \leq q \text{ in } P) \text{ in } \mathbf{Sem}_{\vee,0} \quad (5.7)$$

$$(A, f_p \mid p \in P) = \varinjlim (A_p, f_{p,q} \mid p \leq q \text{ in } P) \text{ in } \mathbf{PAlg}_{\mathcal{L}} \quad (5.8)$$

Moreover, Lemma 3.14 implies that I is an ideal of \vec{A} and that the following is a directed colimit cocone:

$$(\vec{A}/I, \tilde{g}_p \mid p \in P) = \varinjlim (\vec{A}_p/I_p, \tilde{g}_{p,q} \mid p \leq q \text{ in } P) \text{ in } \mathbf{Sem}_{\vee,0} \quad (5.9)$$

Let $p \in P$, let $x, y \in A_p$ such that $g_p(x/I_p) = g_p(y/I_p)$. It follows that $f_p(x)/I = f_p(y)/I$, that is, $\delta_{\mathbf{A}}(f_p(x), f_p(y)) \in I$. So there exist $q \in P$ and $\alpha \in \vec{A}_q$ such that $\tilde{f}_p(\delta_{\mathbf{A}_p}(x, y)) = \delta_{\mathbf{A}}(f_p(x), f_p(y)) = \tilde{f}_q(\alpha)$. It follows from (5.7) that there exists $r \geq p, q$ such that $\delta_{\mathbf{A}_r}(f_{p,r}(x), f_{p,r}(y)) = \tilde{f}_{p,r}(\delta_{\mathbf{A}_p}(x, y)) = \tilde{f}_{q,r}(\alpha)$. However, $\tilde{f}_{q,r}(\alpha) \in \tilde{f}_{q,r}(I_q) \subseteq I_r$, so $f_{p,r}(x)/I_r = f_{p,r}(y)/I_r$, and so $g_{p,r}(x/I_p) = g_{p,r}(y/I_p)$. Moreover $A/I = \bigcup_{p \in P} f_p(A_p)/I = \bigcup_{p \in P} g_p(A_p/I_p)$. Hence the following is a directed colimit cocone:

$$(A/I, g_p \mid p \in P) = \varinjlim (A_p/I_p, g_{p,q} \mid p \leq q \text{ in } P) \text{ in } \mathbf{Set}.$$

Let $\ell \in \mathcal{L}$. The following equalities hold:

$$\text{Def}_{\ell}(A/I) = \text{Def}_{\ell}(A)/I = \bigcup_{p \in P} f_p(\text{Def}_{\ell}(A_p))/I = \bigcup_{p \in P} g_p(\text{Def}_{\ell}(A_p/I_p)).$$

Let $p \in P$, let $\vec{a} \in \text{Def}_{\ell}(A_p/I)$. Then, as g_p is a morphism of partial algebras, $g_p(\ell(\vec{a})) = \ell(g_p(\vec{a}))$. So, by Lemma 4.9, the following is a directed colimit cocone:

$$(A/I, g_p \mid p \in P) = \varinjlim (A_p/I_p, g_{p,q} \mid p \leq q \text{ in } P) \text{ in } \mathbf{PAlg}_{\mathcal{L}}.$$

As f_p is a morphism of pregamps, $\delta_{A/I}(g_p(x), g_p(y)) = \tilde{g}_p(\delta_{\mathbf{A}_p/I_p}(x, y))$ for all $p \in P$ and all $x, y \in A_p/I_p$, thus Lemma 5.6 implies that the following is a directed colimit cocone:

$$(A/I, g_p \mid p \in P) = \varinjlim (\mathbf{A}_p/I_p, g_{p,q} \mid p \leq q \text{ in } P) \text{ in } \mathbf{PGamp}_{\mathcal{L}}. \quad \square$$

Definition 5.32. A pregamp \mathbf{A} satisfies an identity $t_1 = t_2$ if A/I satisfies $t_1 = t_2$ for each $I \in \text{Id } \mathbf{A}$.

Let \mathcal{V} be a variety of algebras. A pregamp \mathbf{A} is a *pregamp of \mathcal{V}* if it satisfies all identities of \mathcal{V} .

Remark. It is not hard to construct a pregamp \mathbf{A} , an identity $t_1 = t_2$, and an ideal I of \vec{A} such that A satisfies $t_1 = t_2$, but A/I fails $t_1 = t_2$.

A pregamp \mathbf{A} satisfies an identity $t_1 = t_2$ if and only if for each ideal-induced morphism $f: \mathbf{A} \rightarrow \mathbf{B}$ of pregamps, the partial algebra B satisfies $t_1 = t_2$.

Definition 5.33. Let \mathcal{V} be a variety of algebras. The category of pregamps of \mathcal{V} , denoted by $\mathbf{PGamp}(\mathcal{V})$, is the full subcategory of $\mathbf{PGamp}_{\mathcal{L}}$ in which the objects are all the pregamps of \mathcal{V} .

As an immediate application of Lemma 5.31 and Lemma 3.14, we obtain that the class of all pregamps that satisfy a given identity is closed under directed colimits.

Corollary 5.34. Let \mathcal{V} be a variety of algebras and let P be a directed poset. Let $\tilde{\mathbf{A}} = (\mathbf{A}_p, \mathbf{f}_{p,q} \mid p \leq q \text{ in } P)$ be a P -indexed diagram in $\mathbf{PGamp}(\mathcal{V})$. Let $(\mathbf{A}, \mathbf{f}_p \mid p \in P) = \varinjlim (\mathbf{A}_p, \mathbf{f}_{p,q} \mid p \leq q \text{ in } P)$ be a directed colimit cocone in $\mathbf{PGamp}_{\mathcal{L}}$. Then \mathbf{A} is a pregamp of \mathcal{V} .

Similarly, it follows from Proposition 5.29 and Proposition 5.30 that the class of all pregamps that satisfy a given identity is closed under ideal-induced images and sub-pregamps.

Corollary 5.35. Let \mathbf{A} be a pregamp of a variety \mathcal{V} , let \mathbf{B} be a pregamp, let $\mathbf{f}: \mathbf{A} \rightarrow \mathbf{B}$ be an ideal-induced morphism of pregamps, then \mathbf{B} is a pregamp of \mathcal{V} . Furthermore, every sub-pregamp of \mathbf{A} is a pregamp of \mathcal{V} .

6. GAMPS

A *gamp* of a variety \mathcal{V} is a pregamp that “belongs” to \mathcal{V} (cf. (1)), together with a partial subalgebra (cf. (2)). The main interest of this new notion is to express later some additional properties that reflect properties of algebras (cf. Definition 6.3). It is a generalization of the notion of a *semilattice-metric cover as defined in [6, Section 5-1]*.

Definition 6.1. Let \mathcal{V} be a variety of \mathcal{L} -algebras. A *gamp* (resp., a *gamp of \mathcal{V}*) is a quadruple $\mathbf{A} = (A^*, A, \delta_{\mathbf{A}}, \tilde{A})$ such that

- (1) $(A, \delta_{\mathbf{A}}, \tilde{A})$ is a pregamp (resp., a pregamp of \mathcal{V}) (cf. Definitions 5.1 and 5.32).
- (2) A^* is a partial subalgebra of A .

A *realization* of \mathbf{A} is an ordered pair (A', χ) such that $A' \in \mathcal{V}$, A is a partial subalgebra of A' , $\chi: \tilde{A} \rightarrow \text{Conc } A'$ is a $(\vee, 0)$ -embedding, and $\chi(\delta_{\mathbf{A}}(x, y)) = \Theta_{A'}(x, y)$ for all $x, y \in A$. A realization is *isomorphic* if χ is an isomorphism.

A gamp \mathbf{A} is *finite* if both A and \tilde{A} are finite.

Let \mathbf{A} and \mathbf{B} be gamps. A morphism $\mathbf{f}: (\mathbf{A}, \delta_{\mathbf{A}}, \tilde{A}) \rightarrow (\mathbf{B}, \delta_{\mathbf{B}}, \tilde{B})$ of pregamps is a *morphism of gamps* from \mathbf{A} to \mathbf{B} if $f(A^*)$ is a partial subalgebra of B^* .

The *category of gamps of \mathcal{V}* , denoted by $\mathbf{Gamp}(\mathcal{V})$, is the category in which the objects are the gamps of \mathcal{V} and the arrows are the morphisms of gamps.

A *subgamp* of a gamp \mathbf{A} is a gamp $\mathbf{B} = (B^*, B, \delta_{\mathbf{B}}, \tilde{B})$ such that B^* is a partial subalgebra of A^* , B is a partial subalgebra of A , $\delta_{\mathbf{B}} = \delta_{\mathbf{A}} \upharpoonright B^2$, and \tilde{B} is a $(\vee, 0)$ -subsemilattice of \tilde{A} . Let $f: B \rightarrow A$ and $\tilde{f}: \tilde{B} \rightarrow \tilde{A}$ be the inclusion maps. The ordered pair (f, \tilde{f}) is a morphism of gamps from \mathbf{B} to \mathbf{A} , called *the canonical embedding*.

Remark. A gamp might have no realization. A realization of a finite gamp does not need to be finite.

Let $\mathbf{f}: \mathbf{A} \rightarrow \mathbf{B}$ be a morphism of gamps, let (A', χ) be a realization of \mathbf{A} , and let (B', ξ) be a realization of \mathbf{B} . There might not exist any morphism $g: A' \rightarrow B'$.

Definition 6.2. A *gamp of lattices* is a gamp of the variety of all lattices.

Let \mathbf{B} be a gamp of lattices. A *chain* of \mathbf{B} is a sequence x_0, x_1, \dots, x_{n-1} of B^* such that $x_i \wedge x_j = x_i$ in B for all $i \leq j < n$. We sometime denote such a chain as $x_0 \leq x_1 \leq \dots \leq x_{n-1}$. If $x_i \neq x_j$ for all $i < j < n$, we denote the chain as $x_0 < x_1 < \dots < x_{n-1}$.

Let $u < v$ be a chain of \mathbf{B} , we say that v is a *cover* of u , and then we write $u \prec v$, if there is no chain $u < x < v$ in \mathbf{B} .

The following properties for a gamp come from algebra. It follows from Definition 6.3(1) that there are many operations defined in A . With (2) or (3) all “congruences” have a set of “generators”. Condition (4) expresses that whenever two elements are identified by a “congruence” of A^* , then there is a “good reason” for this in A (cf. Lemma 5.11). Conditions (6) and (7) are related to the transitive closure of relations. Condition (8) is related to congruence n -permutability (cf. Proposition 2.1).

Definition 6.3. A gamp \mathbf{A} is *strong* if the following holds:

- (1) A^* is a strong partial subalgebra of A (cf. Definition 4.4).

A gamp \mathbf{A} is *distance-generated* if it satisfies the following condition:

- (2) Every element of \tilde{A} is a finite join of elements of the form $\delta_{\mathbf{A}}(x, y)$ where $x, y \in A^*$.

A gamp \mathbf{A} of lattices is *distance-generated with chains* if

- (3) For all $\alpha \in \tilde{A}$ there are a positive integer n , and chains $x_0 < y_0, x_1 < y_1, \dots, x_{n-1} < y_{n-1}$ of \mathbf{A} such that $\alpha = \bigvee_{k < n} \delta_{\mathbf{A}}(x_k, y_k)$.

A gamp \mathbf{A} is *congruence-tractable* (cf. Lemma 5.11) if

- (4) For all $x, y, x_0, y_0, \dots, x_{m-1}, y_{m-1}$ in A^* , if $\delta_{\mathbf{A}}(x, y) \leq \bigvee_{k < m} \delta_{\mathbf{A}}(x_k, y_k)$ then there are a positive integer n , a list \vec{z} of parameters from A , and terms t_0, \dots, t_n such that the following equations are satisfied in A .

$$\begin{aligned} x &= t_0(\vec{x}, \vec{y}, \vec{z}), \\ y &= t_n(\vec{x}, \vec{y}, \vec{z}), \\ t_k(\vec{y}, \vec{x}, \vec{z}) &= t_{k+1}(\vec{x}, \vec{y}, \vec{z}) \quad (\text{for all } k < n). \end{aligned}$$

A morphism $\mathbf{f}: \mathbf{A} \rightarrow \mathbf{B}$ of gamps is *strong* if

- (5) $f(A)$ is a strong partial subalgebra of B^* (cf. Definition 4.4).

A morphism $\mathbf{f}: \mathbf{A} \rightarrow \mathbf{B}$ of gamps is *congruence-cuttable* if

- (6) $f(A)$ is a partial sublattice of B^* and given a finite subset X of \tilde{B} and $x, y \in A$ with $\delta_{\mathbf{A}}(f(x), f(y)) \leq \bigvee X$, there are $n < \omega$ and $f(x) = x_0, \dots, x_n = f(y)$ in B^* such that $\delta_{\mathbf{A}}(x_k, x_{k+1}) \in \downarrow X$ for all $k < n$.

A morphism $\mathbf{f}: \mathbf{A} \rightarrow \mathbf{B}$ of gamps of the variety of all lattices is *congruence-cuttable with chains* if

- (7) $f(A)$ is a partial sublattice of B^* and given a finite subset X of \tilde{B} and $x, y \in A$ with $\delta_{\mathbf{A}}(f(x), f(y)) \leq \bigvee X$, there is a chain $x_0 < \dots < x_n$ of \mathbf{B} such that $x_0 = f(x) \wedge f(y)$, $x_n = f(x) \vee f(y)$, and $\delta_{\mathbf{A}}(x_k, x_{k+1}) \in \downarrow X$ for all $k < n$.

Let $n \geq 1$ be an integer. A gamp \mathbf{A} is *congruence n -permutable* if the following statement holds:

- (8) For all $x_0, x_1, \dots, x_n \in A^*$, there are $x_0 = y_0, y_1, \dots, y_n = x_n$ in A such that:

$$\delta_{\mathbf{A}}(y_k, y_{k+1}) \leq \bigvee (\delta_{\mathbf{A}}(x_i, x_{i+1}) \mid i < n \text{ even}), \quad \text{for all } k < n \text{ odd,}$$

$$\delta_{\mathbf{A}}(y_k, y_{k+1}) \leq \bigvee (\delta_{\mathbf{A}}(x_i, x_{i+1}) \mid i < n \text{ odd}), \quad \text{for all } k < n \text{ even.}$$

The following lemma shows that chains in strong gamps of lattices behave the same way as chains in lattices.

Lemma 6.4. *Let $x_0 < \dots < x_n$ a chain of a strong gamp of lattices \mathbf{B} . The equalities $x_i \wedge x_j = x_j \wedge x_i = x_i$ and $x_i \vee x_j = x_j \vee x_i = x_j$ hold in B for all $i \leq j \leq n$.*

Moreover the following statements hold:

$$\delta_{\mathbf{B}}(x_k, x_{k'}) \leq \delta_{\mathbf{B}}(x_i, x_j), \quad \text{for all } i \leq k \leq k' \leq j \leq n. \quad (6.1)$$

$$\delta_{\mathbf{B}}(x_i, x_j) = \bigvee_{i \leq k < j} \delta_{\mathbf{B}}(x_k, x_{k+1}), \quad \text{for all } i \leq j \leq n. \quad (6.2)$$

Proof. Let $i \leq j \leq n$. As $x_i, x_j \in B^*$, all the elements $x_i \wedge x_j, x_j \wedge x_i, x_i \vee x_j$, and $x_j \vee x_i$ are defined in B .

As $u \wedge v = v \wedge u$ is an identity of lattices, it follows that $x_j \wedge x_i = x_i \wedge x_j = x_i$.

As $x_j \wedge x_i = x_i, (x_i \wedge x_j) \vee x_j = x_i \vee x_j$ in B , and $(u \wedge v) \vee v = v$ is an identity of lattices, $x_i \vee x_j = (x_i \wedge x_j) \vee x_j = x_j$. Similarly $x_j \vee x_i = x_j$.

Let $i \leq k \leq k' \leq j \leq n$. As $x_k \wedge x_{k'} = x_k$ and $x_j \wedge x_{k'} = x_{k'}$, we obtain from Definition 5.1(4) the inequality:

$$\delta_{\mathbf{B}}(x_k, x_{k'}) = \delta_{\mathbf{B}}(x_k \wedge x_{k'}, x_j \wedge x_{k'}) \leq \delta_{\mathbf{B}}(x_k, x_j).$$

Similarly, as $x_k = x_k \vee x_i$ and $x_j = x_j \vee x_i$, the inequality $\delta_{\mathbf{B}}(x_k, x_j) \leq \delta_{\mathbf{B}}(x_i, x_j)$ holds. Therefore (6.1) holds.

Let $i < j \leq n$. Definition 5.1(3) implies the inequality:

$$\delta_{\mathbf{B}}(x_i, x_j) \leq \bigvee_{i \leq k < j} \delta_{\mathbf{B}}(x_k, x_{k+1}).$$

Moreover (6.1) implies $\delta_{\mathbf{B}}(x_k, x_{k+1}) \leq \delta_{\mathbf{B}}(x_i, x_j)$ for all $i \leq k < j$, it follows that (6.2) is true. \square

The following proposition gives a description of *quotient* of gamps.

Proposition 6.5. *Let \mathbf{A} be a gamp of \mathcal{V} and let I be an ideal of \tilde{A} . Let $(A, \delta_{\mathbf{A}}, \tilde{A})/I = (A/I, \delta_{\mathbf{A}/I}, \tilde{A}/I)$ be the quotient pregamp and set $A^*/I = \{a/I \mid a \in A^*\}$ (cf. Notation 5.22). The following statements hold:*

- (1) $\mathbf{A}/I = (A^*/I, A/I, \delta_{\mathbf{A}/I}, \tilde{A}/I)$ is a gamp of \mathcal{V} .
- (2) The canonical projection $\pi: (A, \delta_{\mathbf{A}}, \tilde{A}) \rightarrow (A/I, \delta_{\mathbf{A}/I}, \tilde{A}/I)$ of pregamps is a morphism of gamps from \mathbf{A} to \mathbf{A}/I .
- (3) If (A', χ) is a realization of \mathbf{A} in \mathcal{V} , then $(A'/\bigvee \chi(I), \chi')$ is a realization of \mathbf{A}/I in \mathcal{V} , where

$$\chi': \tilde{A}/I \rightarrow \text{Conc}(A'/\bigvee \chi(I))$$

$$d/I \mapsto \chi(d)/\bigvee \chi(I).$$

Moreover, if (A', χ) is an isomorphic realization of \mathbf{A} , then $(A'/\bigvee \chi(I), \chi')$ is an isomorphic realization of \mathbf{A}/I .

- (4) If \mathbf{A} is strong, then \mathbf{A}/I is strong.
- (5) If \mathbf{A} is distance-generated (resp., distance-generated with chains) then \mathbf{A}/I is distance-generated (resp., distance-generated with chains).
- (6) If \mathbf{A} is distance-generated and congruence-tractable, then \mathbf{A}/I is congruence-tractable.
- (7) Let $n \geq 2$ an integer. If \mathbf{A} is congruence n -permutable then \mathbf{A}/I is congruence n -permutable.
- (8) If \mathbf{A} is a gamp of lattices and $x_0 \leq x_1 \leq \dots \leq x_n$ is a chain of \mathbf{A} , then $x_0/I \leq x_1/I \leq \dots \leq x_n/I$ is a chain of \mathbf{A}/I .

Proof. The statement (1) follows from Corollary 5.35. Denote by $\pi: (A, \delta_{\mathbf{A}}, \tilde{A}) \rightarrow (A/I, \delta_{\mathbf{A}/I}, \tilde{A}/I)$ the canonical projection of pregamps. The fact that $\pi(A^*) = A^*/I$ as partial algebras follows from the definition of A^*/I . Thus (3) holds.

Let (A', χ) be a realization of \mathbf{A} in \mathcal{V} , let $d, d' \in \tilde{A}$ such that $d/I = d'/I$. Hence there exists $u \in I$ such that $d \vee u = d' \vee u$, it follows that $\chi(d)/\chi(u) = \chi(d')/\chi(u)$, hence $\chi(d)/\bigvee \chi(I) = \chi(d')/\bigvee \chi(I)$. Therefore the map χ' is well defined. It is easy to check that χ' is a $(\vee, 0)$ -homomorphism. Assume that $\chi'(d/I) \leq \chi'(d'/I)$ for some $d, d' \in \tilde{A}$. Hence $\chi(d)/\bigvee \chi(I) \leq \chi(d')/\bigvee \chi(I)$, so $\chi(d) \leq \chi(d') \vee \bigvee \chi(I)$. However, $\chi(d)$ is a compact congruence of A' , so there exist $u \in I$ such that $\chi(d) \leq \chi(d') \vee \chi(u) = \chi(d' \vee u)$, as χ is an embedding, it follows that $d \leq d' \vee u$, so $d/I \leq d'/I$. Therefore χ' is an embedding.

Let $x, y \in A$. The following equivalences hold:

$$\begin{aligned}
x/I = y/I &\iff \delta_{\mathbf{A}}(x, y) \in I \\
&\iff \chi(\delta_{\mathbf{A}}(x, y)) \leq \bigvee \chi(I) \\
&\iff \Theta_{A'}(x, y) \leq \bigvee \chi(I) \\
&\iff x/\bigvee \chi(I) = y/\bigvee \chi(I).
\end{aligned}$$

So we can identify A/I with the corresponding subset of $A'/\bigvee \chi(I)$. Moreover, given $\vec{a} \in \text{Def}_{\ell}(A/I)$, there exists $\vec{x} \in \text{Def}_{\ell}(A)$ such that $\vec{a} = \vec{x}/I$, hence $\ell(\vec{a}) = \ell(\vec{x})/I$ is identified with $\ell(\vec{x})/\bigvee \chi(I) = \ell(\vec{x}/\bigvee \chi(I))$. So this identification preserves the operations.

Now assume that the realization is isomorphic. Then χ is surjective, thus χ' is surjective, and thus bijective, hence the realization $(A'/\bigvee \chi(I), \chi')$ is isomorphic. Therefore (3) holds.

The proofs of the statements (4), (5), (7), and (8) are straightforward.

Assume that \mathbf{A} is distance-generated and congruence-tractable. Let $x, y \in A^*$, let $m < \omega$, let \vec{x}, \vec{y} be m -tuples of A^* . Assume that:

$$\delta_{\mathbf{A}/I}(x/I, y/I) \leq \bigvee_{k < m} \delta_{\mathbf{A}/I}(x_k/I, y_k/I).$$

It follows that there exists $u \in I$ with:

$$\delta_{\mathbf{A}}(x, y) \leq u \vee \bigvee_{k < m} \delta_{\mathbf{A}}(x_k, y_k).$$

However, as \mathbf{A} is distance-generated, there exist $x'_0, \dots, x'_{p-1}, y'_0, \dots, y'_{p-1}$ in A^* such that $u = \bigvee_{k < p} \delta_{\mathbf{A}}(x'_k, y'_k)$. As $\delta_{\mathbf{A}}(x'_k, y'_k) \leq u \in I$, $x'_k/I = y'_k/I$ for all $k < p$.

Moreover the following inequality holds:

$$\delta_{\mathbf{A}}(x, y) \leq \bigvee_{k < m} \delta_{\mathbf{A}}(x_k, y_k) \vee \bigvee_{k < p} \delta_{\mathbf{A}}(x'_k, y'_k).$$

As \mathbf{A} is congruence-tractable, there are a positive integer n , a list \vec{z} of parameters from A , and terms t_0, \dots, t_n such that, the following equations are satisfied in A :

$$\begin{aligned} x &= t_0(\vec{x}, \vec{x}', \vec{y}, \vec{y}', \vec{z}), \\ y &= t_n(\vec{x}, \vec{x}', \vec{y}, \vec{y}', \vec{z}), \\ t_k(\vec{y}, \vec{y}', \vec{x}, \vec{x}', \vec{z}) &= t_{k+1}(\vec{x}, \vec{x}', \vec{y}, \vec{y}', \vec{z}) \quad (\text{for all } k < n). \end{aligned}$$

Put $t'_k(\vec{a}, \vec{b}, \vec{c}, \vec{d}) = t_k(\vec{a}, \vec{d}, \vec{b}, \vec{c}, \vec{z})$, for all tuples $\vec{a}, \vec{b}, \vec{c}, \vec{d}$ and all $k \leq n$. As $\vec{x}'/I = \vec{y}'/I$ the following equations are satisfied in A/I :

$$\begin{aligned} x/I &= t'_0(\vec{x}/I, \vec{y}/I, \vec{z}/I, \vec{x}'/I), \\ y/I &= t'_n(\vec{x}/I, \vec{y}/I, \vec{z}/I, \vec{x}'/I), \\ t'_k(\vec{y}/I, \vec{x}/I, \vec{z}/I, \vec{x}'/I) &= t'_{k+1}(\vec{x}/I, \vec{y}/I, \vec{z}/I, \vec{x}'/I) \quad (\text{for all } k < n). \end{aligned}$$

Therefore \mathbf{A}/I is congruence-tractable. \square

Definition 6.6. The gamp \mathbf{A}/I described in Proposition 6.5 is a *quotient of \mathbf{A}* , the morphism π is *the canonical projection*.

The following proposition describes how morphisms factorize through quotients of gamps.

Proposition 6.7. *Let $\mathbf{f}: \mathbf{A} \rightarrow \mathbf{B}$ be a morphism of gamps, let I be an ideal of \tilde{A} , and let J be an ideal of \tilde{B} . Assume that $\tilde{f}(I) \subseteq J$ and denote by*

$$\mathbf{g}: (A/I, \delta_{\mathbf{A}/I}, \tilde{A}/I) \rightarrow (B/J, \delta_{\mathbf{B}/J}, \tilde{B}/J)$$

the morphism of pregamps induced by \mathbf{f} . The following statements hold.

- (1) $\mathbf{g}: \mathbf{A}/I \rightarrow \mathbf{B}/J$ is a morphism of gamps.
- (2) If \mathbf{f} is strong, then \mathbf{g} is strong.
- (3) If \mathbf{f} is congruence-cutttable then \mathbf{g} is congruence-cutttable.
- (4) If \mathbf{A} and \mathbf{B} are gamps of lattices and \mathbf{f} is congruence-cutttable with chains, then \mathbf{g} is congruence-cutttable with chains.

Proof. The equality $g(A^*/I) = f(A^*)/J$ of partial algebras holds. Moreover, as $f(A^*)$ is a partial subalgebra of B , $g(A^*/I)$ is a partial subalgebra of B/J . Therefore (1) holds.

The statement (2) follows from the definitions of a quotient gamp (cf. Proposition 6.5) and of a strong morphism (Definition 6.3).

Assume that \mathbf{f} is congruence-cutttable. As $f(A)$ is a partial subalgebra of B^* it follows that $g(A/I)$ is a partial subalgebra of B^*/J . Let X be a finite subset of \tilde{B} , let $x, y \in A$ such that $\delta_{\mathbf{B}/J}(g(x/I), g(y/I)) \leq \bigvee X/I$. If $X = \emptyset$, then $x/I = y/I$, hence the case is immediate.

If $X \neq \emptyset$, let $u \in J$ such that $\delta_{\mathbf{B}}(g(x), g(y)) \leq u \vee \bigvee X$. Put $X' = X \cup \{u\}$. There are $n < \omega$ and $f(x) = x_0, \dots, x_n = f(y)$ in B^* such that $\delta_{\mathbf{B}}(x_k, x_{k+1}) \in \downarrow X'$ for each $k < n$. If $\delta_{\mathbf{B}}(x_k, y_k) \leq u$, then $\delta_{\mathbf{B}/J}(x_k/J, y_k/J) = \delta_{\mathbf{B}}(x_k, y_k)/J = 0/J \in \downarrow X'/J$. Otherwise $\delta_{\mathbf{B}}(x_k, y_k) \in \downarrow X$, thus $\delta_{\mathbf{B}/J}(x_k/J, y_k/J) = \delta_{\mathbf{B}}(x_k, y_k)/J \in \downarrow X/J$. Therefore (3) holds.

The proof of (4) is similar to the proof of (3). \square

We introduce in the following definitions a functor $\mathbf{G}: \mathcal{V} \rightarrow \mathbf{Gamp}(\mathcal{V})$, a functor $\mathbf{C}: \mathbf{Gamp}(\mathcal{V}) \rightarrow \mathbf{Sem}_{\mathcal{V},0}$ and functors $\mathbf{P}_{\text{gl}}, \mathbf{P}_{\text{gr}}: \mathbf{Gamp}(\mathcal{V}) \rightarrow \mathbf{PGamp}(\mathcal{V})$.

Definition 6.8. Let A be a member of a variety of algebras \mathcal{V} . Then the quadruple $\mathbf{G}(A) = (A, A, \Theta_A, \text{Con}_c A)$ is a gamp of \mathcal{V} (we recall that $\Theta_A(x, y)$ denotes the smallest congruence that identifies x and y). If $f: A \rightarrow B$ is a morphism of algebras, then $\mathbf{G}(f) = (f, \text{Con}_c f)$ is a morphism of gamps from $\mathbf{G}(A)$ to $\mathbf{G}(B)$. It defines a functor from the category \mathcal{V} to the category $\mathbf{Gamp}(\mathcal{V})$.

A gamp \mathbf{A} is an algebra of \mathcal{V} if \mathbf{A} is isomorphic to $\mathbf{G}(B)$ for some $B \in \mathcal{V}$.

Let \mathbf{A} be a gamp of \mathcal{V} , we set $\mathbf{C}(\mathbf{A}) = \tilde{\mathbf{A}}$. Let $\mathbf{f}: \mathbf{A} \rightarrow \mathbf{B}$ be a morphism of gamps of \mathcal{V} , we set $\mathbf{C}(\mathbf{f}) = \tilde{\mathbf{f}}$. This defines a functor $\mathbf{C}: \mathbf{Gamp}(\mathcal{V}) \rightarrow \mathbf{Sem}_{\mathcal{V},0}$.

Let \mathbf{A} be a gamp of \mathcal{V} , we set $\mathbf{P}_{\text{gr}}(\mathbf{A}) = (A, \delta_{\mathbf{A}}, \tilde{\mathbf{A}})$. Let $\mathbf{f}: \mathbf{A} \rightarrow \mathbf{B}$ be a morphism of gamps of \mathcal{V} , we put $\mathbf{P}_{\text{gr}}(\mathbf{f}) = \mathbf{f}$ as a morphism of pregamps. This defines a functor $\mathbf{P}_{\text{gr}}: \mathbf{Gamp}(\mathcal{V}) \rightarrow \mathbf{PGamp}(\mathcal{V})$.

Let \mathbf{A} be a gamp of \mathcal{V} , we set $\mathbf{P}_{\text{gl}}(\mathbf{A}) = (A^*, \delta_{\mathbf{A}} \upharpoonright (A^*)^2, \tilde{\mathbf{A}})$. Let $\mathbf{f}: \mathbf{A} \rightarrow \mathbf{B}$ be a morphism of gamps of \mathcal{V} , we denote by $\mathbf{P}_{\text{gl}}(\mathbf{f})$ the restriction $(f, \tilde{\mathbf{f}}): \mathbf{P}_{\text{gl}}(\mathbf{A}) \rightarrow \mathbf{P}_{\text{gl}}(\mathbf{B})$. This defines a functor $\mathbf{P}_{\text{gl}}: \mathbf{Gamp}(\mathcal{V}) \rightarrow \mathbf{PGamp}(\mathcal{V})$.

Remark 6.9. The following equations, between the functors introduced in Definition 5.4 and Definition 6.8, are satisfied:

$$\begin{aligned} \mathbf{C} \circ \mathbf{G} &= \text{Con}_c = \mathbf{C}_{\text{pg}} \circ \mathbf{P}_{\text{ga}}, \\ \mathbf{P}_{\text{gr}} \circ \mathbf{G} &= \mathbf{P}_{\text{gl}} \circ \mathbf{G} = \mathbf{P}_{\text{ga}}, \\ \mathbf{C}_{\text{pg}} \circ \mathbf{P}_{\text{gr}} &= \mathbf{C}_{\text{pg}} \circ \mathbf{P}_{\text{gl}} = \mathbf{C}. \end{aligned}$$

If A is a subalgebra of B , then, in general, $\mathbf{G}(A)$ is not a subgamp of $\mathbf{G}(B)$. The different ‘‘congruences’’ of a subgamp can be extended in a natural way to different ‘‘congruences’’ of the gamp.

If a gamp \mathbf{A} is an algebra, then there is a unique, up to isomorphism, algebra $B \in \mathcal{V}$ such that $\mathbf{A} \cong \mathbf{G}(B)$. Indeed A is an algebra and $\mathbf{A} \cong \mathbf{G}(A)$.

Let $B \in \mathcal{V}$ and let I be an ideal of $\text{Con}_c B$. There is an isomorphism $\mathbf{f}: \mathbf{G}(B)/I \cong \mathbf{G}(B/\bigvee I)$ satisfying $f(x/I) = x/\bigvee I$ and $\tilde{\mathbf{f}}(\alpha/I) = \alpha/\bigvee I$ for each $x \in B$ and each $\alpha \in \text{Con}_c B$.

Let B be an algebra. Then B is congruence n -permutable if and only if $\mathbf{G}(B)$ is congruence n -permutable (this follows immediately from Proposition 2.1).

Lemma 6.10. *Let \mathcal{V} be a variety of algebras. The category $\mathbf{Gamp}(\mathcal{V})$ has all directed colimits. Suppose that we are given a directed poset P , a P -indexed diagram $\vec{\mathbf{A}} = (\mathbf{A}_p, \mathbf{f}_{p,q} \mid p \leq q \text{ in } P)$ in $\mathbf{Gamp}(\mathcal{V})$, and a directed colimit cocone:*

$$((A, \delta_{\mathbf{A}}, \tilde{\mathbf{A}}), \mathbf{f}_p \mid p \in P) = \varinjlim ((A_p, \delta_{\mathbf{A}_p}, \tilde{\mathbf{A}}_p), \mathbf{f}_{p,q} \mid p \leq q \text{ in } P) \quad \text{in } \mathbf{PGamp}_{\mathcal{L}}.$$

Put $A^* = \bigcup_{p \in P} f(A_p^*)$ with its natural structure of partial algebra (cf. Lemma 4.9), then $\mathbf{A} = (A^*, A, \delta_{\mathbf{A}}, \tilde{\mathbf{A}})$ is a gamp of \mathcal{V} , $\mathbf{f}_p: \mathbf{A}_p \rightarrow \mathbf{A}$ is a morphism of gamps, and the following is a directed colimit cocone:

$$(\mathbf{A}, \mathbf{f}_p \mid p \in P) = \varinjlim (\mathbf{A}_p, \mathbf{f}_{p,q} \mid p \leq q \text{ in } P), \quad \text{in } \mathbf{Gamp}(\mathcal{V}).$$

Moreover the following statements hold:

- (1) If \mathbf{A}_p is distance-generated for each $p \in P$, then \mathbf{A} is distance-generated.
- (2) If \mathcal{V} is a variety of lattices and \mathbf{A}_p is distance-generated with chains for each $p \in P$, then \mathbf{A} is distance-generated with chains.

- (3) Let n be a positive integer. If \mathbf{A}_p is congruence n -permutable for each $p \in P$, then \mathbf{A} is congruence n -permutable.

Proof. It follows from Corollary 5.34 that $(A, \delta_{\mathbf{A}}, \tilde{A})$ is a pregamp of \mathcal{V} . Moreover A^* is a partial subalgebra of A . Hence $\mathbf{A} = (A^*, A, \delta_{\mathbf{A}}, \tilde{A})$ is a gamp of \mathcal{V} . As $f(A_p^*)$ is a partial subalgebra of A^* , $\mathbf{f}_p: \mathbf{A}_p \rightarrow \mathbf{A}$ is a morphism of gamps. It is easy to check that the following is a directed colimit cocone:

$$(\mathbf{A}, \mathbf{f}_p \mid p \in P) = \varinjlim (\mathbf{A}_p, \mathbf{f}_{p,q} \mid p \leq q \text{ in } P), \quad \text{in } \mathbf{Gamp}(\mathcal{V}).$$

Assume that \mathbf{A}_p is distance-generated for each $p \in P$. Let $\alpha \in \tilde{A}$, then there are $p \in P$ and $\beta \in \tilde{A}_p$ such that $\alpha = \tilde{f}_p(\beta)$. As \mathbf{A}_p is distance-generated, there are an integer $n \geq 0$ and n -tuple \vec{x}, \vec{y} of A_p^* such that $\beta = \bigvee_{k < n} \delta_{\mathbf{A}_p}(x_k, y_k)$. Therefore the following equalities hold:

$$\alpha = \tilde{f}_p(\beta) = \tilde{f}_p \left(\bigvee_{k < n} \delta_{\mathbf{A}_p}(x_k, y_k) \right) = \bigvee_{k < n} \tilde{f}_p(\delta_{\mathbf{A}_p}(x_k, y_k)) = \bigvee_{k < n} \delta_{\mathbf{A}}(f_p(x_k), f_p(y_k)).$$

Thus \mathbf{A} is distance-generated.

The proofs of (2) and (3) are similar. \square

As an immediate application we obtain the following corollary.

Corollary 6.11. *The functors \mathbf{G} , \mathbf{C} , \mathbf{P}_{gl} , and \mathbf{P}_{gr} preserves directed colimits.*

Proof. It follows from the description of directed colimits of gamps (cf. Lemma 6.10) and pregamps (cf. Lemma 5.6) that \mathbf{C} , \mathbf{P}_{gl} , and \mathbf{P}_{gr} preserve directed colimits. As $\text{Con}_{\mathbf{c}}$ preserves directed colimits, \mathbf{G} also preserves directed colimits. \square

Definition 6.12. Let \mathcal{V} be a variety of algebras, let P be a poset, and let $\vec{\mathbf{A}} = (\mathbf{A}_p, \mathbf{f}_{p,q} \mid p \leq q \text{ in } P)$ in $\mathbf{Gamp}(\mathcal{V})$. An *ideal* of $\vec{\mathbf{A}}$ is an ideal of $\mathbf{C} \circ \vec{\mathbf{A}}$. It consists of a family $\vec{I} = (I_p)_{p \in P}$ such that I_p is an ideal of \tilde{A}_p and $\tilde{f}_{p,q}(I_p) \subseteq I_q$ for all $p \leq q$ in P .

We denote by $\vec{\mathbf{A}}/\vec{I} = (\mathbf{A}_p/I_p, \mathbf{g}_{p,q} \mid p \leq q \text{ in } P)$, where $\mathbf{g}_{p,q}: \mathbf{A}_p/I_p \rightarrow \mathbf{A}_q/I_q$ is induced by $\mathbf{f}_{p,q}$, for all $p \leq q$ in P .

The diagram $\vec{\mathbf{A}}/\vec{I}$ is a *quotient* of $\vec{\mathbf{A}}$.

Definition 6.13. Let \mathcal{V} be a variety of algebras, let P be a poset. A *partial lifting* in \mathcal{V} is a diagram $\vec{\mathbf{A}} = (\mathbf{A}_p, \mathbf{f}_{p,q} \mid p \leq q \text{ in } P)$ in $\mathbf{Gamp}(\mathcal{V})$ such that the following statements hold:

- (1) The gamp \mathbf{A}_p is strong, congruence-tractable, distance-generated, and has an isomorphic realization (cf. Definitions 6.3 and 6.1), for each $p \in P$.
- (2) The morphisms $\mathbf{f}_{p,q}$ is strong and congruence-cutttable (cf. Definition 6.3), for all $p < q$ in P .

The partial lifting is a *lattice partial lifting* if \mathcal{V} is a variety of lattices, \mathbf{B}_p is distance-generated with chains for each $p \in P$, and $\mathbf{f}_{p,q}$ is congruence-cutttable with chains for all $p < q$ in P .

A partial lifting $\vec{\mathbf{A}}$ is *congruence n -permutable* if \mathbf{A}_p is congruence n -permutable for each $p \in P$.

Let $\vec{S} = (S_p, \sigma_{p,q})$ be a diagram in $\mathbf{Sem}_{\mathcal{V},0}$. A *partial lifting of \vec{S}* is a partial lifting of $\vec{\mathbf{A}}$ such that $\mathbf{C} \circ \vec{\mathbf{A}} \cong \vec{S}$.

Remark 6.14. If $\vec{\mathbf{A}}$ is a partial lifting of \vec{S} , then there exists a diagram $\vec{\mathbf{A}}' \cong \vec{\mathbf{A}}$ such that $\mathbf{C} \circ \vec{\mathbf{A}}' = \vec{S}$. Hence we can assume that $\mathbf{C} \circ \vec{\mathbf{A}} = \vec{S}$.

The following result expresses the fact that a subdiagram or a quotient of a partial lifting is a partial lifting.

Lemma 6.15. *Let \mathcal{V} be a variety of algebras, let P be a poset, and let $\vec{\mathbf{A}}$ be a partial lifting in \mathcal{V} of a diagram \vec{S} . The following statements hold:*

- (1) *Let \vec{I} be an ideal of $\vec{\mathbf{A}}$; then $\vec{\mathbf{A}}/\vec{I}$ is a partial lifting of \vec{S}/I .*
- (2) *Let $Q \subseteq P$; then $\vec{\mathbf{A}} \upharpoonright Q$ is a partial lifting of $\vec{S} \upharpoonright Q$.*

Proof. The statement (1) follows from Proposition 6.5 and Proposition 6.7. The statement (2) is immediate. \square

Lemma 6.16. *Let \mathcal{V} be a variety of algebras, let P be a directed poset with no maximal element, and let $\vec{\mathbf{A}} = (\mathbf{A}_p, \mathbf{f}_{p,q} \mid p \leq q \text{ in } P)$ be a partial lifting in \mathcal{V} . Consider a colimit cocone:*

$$(\mathbf{A}, \mathbf{f}_p \mid p \in P) = \varinjlim \vec{\mathbf{A}} \quad \text{in } \mathbf{Gamp}(\mathcal{V}).$$

Then \mathbf{A} is an algebra in \mathcal{V} . Moreover, for any positive integer n , if all \mathbf{A}_p are congruence n -permutable, then the algebra corresponding to \mathbf{A} is congruence n -permutable.

Proof. The morphism $\mathbf{f}_{p,q}: (A_p, \delta_{\mathbf{A}_p}, \tilde{A}_p) \rightarrow (A_q, \delta_{\mathbf{A}_q}, \tilde{A}_q)$ of pregamps is both strong and congruence-tractable for all $p < q$ in P . It follows from the description of colimits in $\mathbf{Gamp}(\mathcal{V})$ (cf. Lemma 6.10 and Corollary 5.15) that \mathbf{A} is an algebra and there is an isomorphism $\phi: \text{Con}_c \mathbf{A} \rightarrow \tilde{\mathbf{A}}$ satisfying:

$$\phi(\Theta_{\mathbf{A}}(x, y)) = \delta_{\mathbf{A}}(x, y), \quad \text{for all } x, y \in \mathbf{A}.$$

As $\mathbf{A}^* = \bigcup_{p \in P} \mathbf{f}_p(\mathbf{A}_p^*)$ and $\mathbf{f}_{p,q}(\mathbf{A}_p^*) \subseteq \mathbf{A}_q^*$ for all $p < q$ in P , it follows that $\mathbf{A}^* = \mathbf{A}$. Therefore $(\text{id}_{\mathbf{A}}, \phi): \mathbf{G}(\mathbf{A}) \rightarrow \mathbf{A}$ is an isomorphism of gamps.

Now assume that all \mathbf{A}_p are congruence n -permutable. It follows from Lemma 6.10 that \mathbf{A} is congruence n -permutable. As \mathbf{A} is an algebra, the conclusion follows from Proposition 2.1. \square

7. LOCALLY FINITE PROPERTIES

The aim of this section is to prove Lemma 7.8, which is a special version of Buttress Lemma [6] adapted to gamps.

We use the following generalizations of (2), (3), (4), (6), and (7) of Definition 6.3.

Definition 7.1. Fix a gamp \mathbf{A} in a variety \mathcal{V} , a $(\vee, 0)$ -semilattice S , and a $(\vee, 0)$ -homomorphism $\phi: \tilde{\mathbf{A}} \rightarrow S$. The gamp \mathbf{A} is *distance-generated through ϕ* if the following statement holds:

- (2') For each $s \in S$ there are $n < \omega$ and n -tuple \vec{x}, \vec{y} of \mathbf{A}^* such that:

$$s = \bigvee_{k < n} \phi(\delta_{\mathbf{A}}(x_k, y_k))$$

If \mathbf{A} is a gamp of lattices, we say \mathbf{A} is *distance-generated with chains through ϕ* if

(3') For each $s \in S$ there are $n < \omega$ and chains $x_0 < y_0, \dots, x_{n-1} < y_{n-1}$ of \mathbf{A} such that:

$$s = \bigvee_{k < n} \phi(\delta_{\mathbf{A}}(x_k, y_k))$$

The gamp \mathbf{A} is *congruence-tractable through ϕ* if the following statement holds:

(4') Let $x, y, x_0, y_0, \dots, x_{m-1}, y_{m-1}$ in A^* , if $\phi(\delta_{\mathbf{A}}(x, y)) \leq \bigvee_{k < m} \phi(\delta_{\mathbf{A}}(x_k, y_k))$ then there are a positive integer n , a list \vec{z} of parameters from A , and terms t_0, \dots, t_n such that, the following equations are satisfied in A :

$$\begin{aligned} \phi(\delta_{\mathbf{A}}(x, t_0(\vec{x}, \vec{y}, \vec{z}))) &= 0, \\ \phi(\delta_{\mathbf{A}}(y, t_n(\vec{x}, \vec{y}, \vec{z}))) &= 0, \\ \phi(\delta_{\mathbf{A}}(t_j(\vec{y}, \vec{x}, \vec{z}), t_{j+1}(\vec{x}, \vec{y}, \vec{z}))) &= 0 \quad (\text{for all } j < n). \end{aligned}$$

A morphism $\mathbf{f}: \mathbf{U} \rightarrow \mathbf{A}$ of gamps is *congruence-cutttable through ϕ* if the following statement holds:

(6') Given $X \subseteq S$, given x, y in U , if $\phi(\tilde{f}(\delta_{\mathbf{A}}(x, y))) \leq \bigvee X$ then there exist $n < \omega$ and $f(x) = x_0, x_1, \dots, x_n = f(y)$ in A^* such that $\delta_{\mathbf{A}}(x_k, x_{k+1}) \in S \downarrow X$ for all $k < n$.

A morphism $\mathbf{f}: \mathbf{U} \rightarrow \mathbf{A}$ of gamps of lattices is *congruence-cutttable with chains through ϕ* if the following statement holds:

(7') Given $X \subseteq S$, given $x, y \in U$, if $\phi(\tilde{f}(\delta_{\mathbf{A}}(x, y))) \leq \bigvee X$ then there are $n < \omega$ and a chain $x_0 < x_1 < \dots < x_n$ of \mathbf{A} such that $x_0 = f(x) \wedge f(y)$, $x_n = f(x) \vee f(y)$, and $\phi(\delta_{\mathbf{A}}(x_k, x_{k+1})) \in S \downarrow X$ for all $k < n$.

Remark 7.2. A gamp \mathbf{A} is distance-generated through $\text{id}_{\bar{A}}$ if and only if \mathbf{A} is distance-generated.

A gamp \mathbf{A} is congruence-tractable through $\text{id}_{\bar{A}}$ if and only if \mathbf{A} is congruence-tractable.

A morphism $\mathbf{f}: \mathbf{U} \rightarrow \mathbf{A}$ of gamps is congruence-cutttable through $\text{id}_{\bar{A}}$ if and only if \mathbf{f} is congruence-cutttable.

A morphism $\mathbf{f}: \mathbf{U} \rightarrow \mathbf{A}$ of gamps of lattice is congruence-cutttable with chains through $\text{id}_{\bar{A}}$ if and only if \mathbf{f} is congruence-cutttable with chains.

Lemma 7.3. *Let \mathbf{A} be a gamp in a variety of algebras \mathcal{V} , let $\phi: \tilde{A} \rightarrow S$ an ideal-induced $(\vee, 0)$ -homomorphism, and put $I = \ker_0 \phi$.*

- (1) *If \mathbf{A} is distance-generated through ϕ , then \mathbf{A}/I is distance-generated.*
- (2) *If \mathbf{A} is distance-generated with chains through ϕ , then \mathbf{A}/I is distance-generated with chains.*
- (3) *If \mathbf{A} is congruence-tractable through ϕ then \mathbf{A}/I is congruence-tractable.*

Proof. As ϕ is ideal-induced, it induces an isomorphism $\xi: \tilde{A}/I \rightarrow S$.

Assume that \mathbf{A} is distance-generated through ϕ . Let $d \in \tilde{A}/I$, put $s = \xi(d)$. There are $n < \omega$ and n -tuples \vec{x}, \vec{y} in A^* such that $s = \bigvee_{k < n} \phi(\delta_{\mathbf{A}}(x_k, y_k))$. It follows that $s = \bigvee_{k < n} \xi(\delta_{\mathbf{A}/I}(x_k/I, y_k/I))$, so $d = \bigvee_{k < n} \delta_{\mathbf{A}/I}(x_k/I, y_k/I)$. Therefore \mathbf{A}/I is distance-generated.

The case where \mathbf{A} is distance-generated with chains through ϕ is similar.

Assume that \mathbf{A} is congruence-tractable through ϕ . Let $x, y, x_0, y_0, \dots, x_{m-1}, y_{m-1}$ in A^* such that:

$$\delta_{\mathbf{A}/I}(x/I, y/I) \leq \bigvee_{k < m} \delta_{\mathbf{A}/I}(x_k/I, y_k/I).$$

Thus $\phi(\delta_{\mathbf{A}/I}(x, y)) \leq \bigvee_{k < m} \phi(\delta_{\mathbf{A}}(x_k, y_k))$. Hence there are a positive integer n , a list \vec{z} of parameters from A , and terms t_0, \dots, t_n such that the following equations are satisfied in A :

$$\begin{aligned} \phi(\delta_{\mathbf{A}}(x, t_0(\vec{x}, \vec{y}, \vec{z}))) &= 0, \\ \phi(\delta_{\mathbf{A}}(y, t_n(\vec{x}, \vec{y}, \vec{z}))) &= 0, \\ \phi(\delta_{\mathbf{A}}(t_k(\vec{y}, \vec{x}, \vec{z}), t_{k+1}(\vec{x}, \vec{y}, \vec{z}))) &= 0 \quad (\text{for all } k < n). \end{aligned}$$

Those equations imply that the following equations are satisfied in A/I :

$$\begin{aligned} x/I &= t_0(\vec{x}/I, \vec{y}/I, \vec{z}/I), \\ y/I &= t_n(\vec{x}/I, \vec{y}/I, \vec{z}/I), \\ t_k(\vec{y}/I, \vec{x}/I, \vec{z}/I) &= t_{k+1}(\vec{x}/I, \vec{y}/I, \vec{z}/I) \quad (\text{for all } k < n). \end{aligned}$$

Therefore \mathbf{A}/I is congruence-tractable. \square

The proof of the following lemma is similar.

Lemma 7.4. *Let $\mathbf{f}: \mathbf{U} \rightarrow \mathbf{A}$ be a morphism of gamps, let I be an ideal of \mathbf{U} , and let $\phi: \tilde{A} \rightarrow S$ be an ideal-induced $(\vee, 0)$ -homomorphism. Put $J = \ker_0 \phi$. Denote by $\mathbf{g}: \mathbf{U}/I \rightarrow \mathbf{A}/J$ the morphism of gamps induced by \mathbf{f} . The following statements hold:*

- (1) *If \mathbf{f} is congruence-cutttable through ϕ , then \mathbf{g} is congruence-cutttable.*
- (2) *Assume that \mathbf{f} is a morphism of gamps of lattices. If \mathbf{f} is congruence-cutttable with chains through ϕ , then \mathbf{g} is congruence-cutttable with chains.*

We shall now define *locally finite properties* for an algebra B , as properties that are satisfied by “many” finite subgamps of $\mathbf{G}(B)$.

Definition 7.5. Let B be an algebra. A *locally finite property* for B is a property (P) in a subgamp \mathbf{A} of $\mathbf{G}(B)$ such that there exists a finite $X \subseteq B$ satisfying that for every finite full partial subalgebra A^* of B that contains X , there exists a finite $Y \subseteq B$ such that for every finite full partial subalgebra A of B that contains $A^* \cup Y$ and every finite $(\vee, 0)$ -subsemilattice \tilde{A} of $\text{Con}_c B$ that contains $\text{Con}_c^A(B)$, the subgamp $(A^*, A, \Theta_B, \tilde{A})$ of $\mathbf{G}(B)$ satisfies (P) .

Proposition 7.6. *Any finite conjunction of locally finite properties is locally finite.*

Lemma 7.7. *Let B be an algebra and denote by \mathcal{L} the similarity type of B . The properties (1)-(8) in \mathbf{A} are locally finite for B :*

Assume that \mathcal{L} is finite.

- (1) *\mathbf{A} is strong.*

Fix an ideal-induced $(\vee, 0)$ -homomorphism $\phi: \text{Con}_c B \rightarrow S$, with S finite.

- (2) *\mathbf{A} is distance-generated through $\phi \upharpoonright \tilde{A}$.*

Fix an ideal-induced $(\vee, 0)$ -homomorphism $\phi: \text{Con}_c B \rightarrow S$ with S finite, and assume that B is a lattice.

- (3) *\mathbf{A} is distance-generated with chains through $\phi \upharpoonright \tilde{A}$.*

Fix an ideal-induced $(\vee, 0)$ -homomorphism $\phi: \text{Con}_c B \rightarrow S$, with S finite.

- (4) *\mathbf{A} is congruence-tractable through $\phi \upharpoonright \tilde{A}$.*

Assume that \mathcal{L} is finite, fix a finite gamp \mathbf{U} , fix a morphism $\mathbf{f}: \mathbf{U} \rightarrow \mathbf{G}(B)$ of gamps.

(5) The restriction $\mathbf{f}: \mathbf{U} \rightarrow \mathbf{A}$ is strong.

Fix an ideal-induced $(\vee, 0)$ -homomorphism $\phi: \text{Con}_c B \rightarrow S$ where S is finite, a finite gamp \mathbf{U} , and a morphism $\mathbf{f}: \mathbf{U} \rightarrow \mathbf{G}(B)$ of gamps.

(6) The restriction of $\mathbf{f}: \mathbf{U} \rightarrow \mathbf{A}$ is congruence-cutttable through $\phi \upharpoonright \tilde{A}$.

Assume that B is a lattice. Fix an ideal-induced $(\vee, 0)$ -homomorphism $\phi: \text{Con}_c B \rightarrow S$, where S is finite. Fix a finite gamp \mathbf{U} , fix a morphism $\mathbf{f}: \mathbf{U} \rightarrow \mathbf{G}(B)$ of gamps.

(7) The restriction of $\mathbf{f}: \mathbf{U} \rightarrow \mathbf{A}$ is congruence-cutttable with chains through $\phi \upharpoonright \tilde{A}$.

Let $n \geq 2$ be an integer. Assume that B is congruence n -permutable.

(8) \mathbf{A} is congruence n -permutable.

Proof. (1) Put $X = \emptyset$ and let A^* be a finite full partial subalgebra of B . Put $Y = \{\ell^B(\vec{x}) \mid \ell \in \mathcal{L} \text{ and } \vec{x} \text{ is an } \text{ar}(\ell)\text{-tuple of } A^*\}$. As A^* and \mathcal{L} are both finite, Y is also finite. Let A be a finite full partial subalgebra of B that contains $A^* \cup Y$, let \tilde{A} be a finite $(\vee, 0)$ -subsemilattice of $\text{Con}_c B$ containing $\text{Con}_c^A(B)$. As $\ell(\vec{x}) \in Y \subseteq A$, it is defined in A , for each $\ell \in \mathcal{L}$ and each $\text{ar}(\ell)$ -tuple \vec{x} of A .

(2) Let $s \in S$. As ϕ is surjective, there exists $\theta \in \text{Con}_c B$ such that $s = \phi(\theta)$. So there exist $n < \omega$ and n -tuple \vec{x}, \vec{y} of B such that $s = \phi(\bigvee_{k < n} \Theta_B(x_k, y_k))$. Put $X_s = \{x_0, \dots, x_{n-1}, y_0, \dots, y_{n-1}\}$.

Put $X = \bigcup_{s \in S} X_s$. As S is finite and X_s is finite for all $s \in S$, X is finite. Let A^* be a finite full partial subalgebra of B that contains X . Put $Y = \emptyset$. Let A be a finite full partial subalgebra of B that contains $A^* \cup Y$. Let \tilde{A} be a finite $(\vee, 0)$ -subsemilattice of $\text{Con}_c B$ containing $\text{Con}_c^A(B)$. By construction $(A^*, A, \Theta_B, \tilde{A})$ satisfies (2).

The proof that (3) is a locally finite property is similar.

(4) Put $X = \emptyset$, let A^* be a finite full partial subalgebra of B . Denote by E the set of all quadruples (x, y, \vec{x}, \vec{y}) such that the following statements are satisfied:

- $x, y \in A^*$.
- \vec{x} and \vec{y} are m -tuples of A^* , for some $m < \omega$.
- $\phi(\Theta_B(x, y)) \leq \bigvee_{k < m} \phi(\Theta_B(x_k, y_k))$.
- $(x_i, y_i) \neq (x_j, y_j)$ for all $i < j < m$.

Put $I = \ker_0 \phi$ and let $(x, y, \vec{x}, \vec{y}) \in E$. As ϕ is ideal-induced, there exists $\alpha \in I$ such that $\Theta_B(x, y) \leq \bigvee_{k < m} \Theta_B(x_k, y_k) \vee \alpha$. Let \vec{x}', \vec{y}' be p -tuples of B such that $\alpha = \bigvee_{k < p} \Theta(x'_k, y'_k)$. Hence:

$$\Theta_B(x, y) \leq \bigvee_{k < m} \Theta_B(x_k, y_k) \vee \bigvee_{k < p} \Theta_B(x'_k, y'_k).$$

It follows from Lemma 5.11 that there are a positive integer n , a list \vec{z} of parameters from B , and terms t_0, \dots, t_n such that the following equalities hold in B :

$$x = t_0(\vec{x}, \vec{x}', \vec{y}, \vec{y}', \vec{z}), \quad (7.1)$$

$$y = t_n(\vec{x}, \vec{x}', \vec{y}, \vec{y}', \vec{z}), \quad (7.2)$$

$$t_j(\vec{y}, \vec{y}', \vec{x}, \vec{x}', \vec{z}) = t_{j+1}(\vec{x}, \vec{x}', \vec{y}, \vec{y}', \vec{z}), \quad \text{for all } j < n. \quad (7.3)$$

Put $t'_j(\vec{a}, \vec{b}, \vec{c}, \vec{d}) = t_j(\vec{a}, \vec{d}, \vec{b}, \vec{d}, \vec{c})$ for all tuples $\vec{a}, \vec{b}, \vec{c}, \vec{d}$ of B of appropriate length. The following inequalities follow from the compatibility of Θ_B with terms:

$$\Theta_B(t'_j(\vec{x}, \vec{y}, \vec{z}, \vec{x}'), t_j(\vec{x}, \vec{x}', \vec{y}, \vec{y}', \vec{z})) \leq \bigvee_{k < p} \Theta_B(x'_k, y'_k) = \alpha, \quad (7.4)$$

$$\Theta_B(t'_j(\vec{y}, \vec{x}, \vec{z}, \vec{x}'), t_j(\vec{y}, \vec{y}', \vec{x}, \vec{x}', \vec{z})) \leq \bigvee_{k < p} \Theta_B(x'_k, y'_k) = \alpha. \quad (7.5)$$

As $\phi(\alpha) = 0$, it follows from (7.4) and (7.1) that:

$$\phi(\Theta_B(x, t'_0(\vec{x}, \vec{y}, \vec{z}, \vec{x}'))) = 0$$

Set $\vec{z}' = (\vec{z}, \vec{x}')$. As $\phi(\alpha) = 0$, it follows from (7.1)-(7.5) that

$$\phi(\Theta_B(x, t'_0(\vec{x}, \vec{y}, \vec{z}'))) = 0,$$

$$\phi(\Theta_B(y, t'_n(\vec{x}, \vec{y}, \vec{z}'))) = 0,$$

$$\phi(\Theta_B(t'_j(\vec{y}, \vec{x}, \vec{z}'), t'_{j+1}(\vec{x}, \vec{y}, \vec{z}'))) = 0 \quad (\text{for all } j < n).$$

Let $Y_{(x,y,\vec{x},\vec{y})}$ be a finite partial subalgebra of B such that both $t'_j(\vec{x}, \vec{y}, \vec{z}, \vec{x}')$ and $t'_j(\vec{y}, \vec{x}, \vec{z}, \vec{x}')$ are defined in $Y_{(x,y,\vec{x},\vec{y})}$, for each $j \leq n$.

Put $Y = \bigcup \{Y_e \mid e \in E\}$. As Y_e is finite for each $e \in E$ and E is finite, Y is finite. Let A be a finite full partial subalgebra of B that contains $Y \cup A^*$, and let \tilde{A} be a finite $(\vee, 0)$ -subsemilattice of $\text{Con}_c B$ containing $\text{Con}_c^A(B)$. It is not hard to verify that $(A^*, A, \Theta_B, \tilde{A})$ satisfies (4).

(5) As U is finite, the set $X_1 = \langle f(U) \rangle_B^1$ (cf. Notation 4.7) is also finite. As \tilde{U} is finite and each element of $\tilde{f}(\tilde{U})$ is a compact congruence of B , we can choose a finite subset X_2 of B such that $\tilde{f}(\tilde{U}) \subseteq \text{Con}_c^{X_2}(B)$. The set $X = X_1 \cup X_2$ is finite. Let A^* be a finite full partial subalgebra of B containing X . Put $Y = \emptyset$, let A be a full partial subalgebra of B containing $Y \cup A^*$, and let \tilde{A} be a finite $(\vee, 0)$ -subsemilattice of $\text{Con}_c B$ containing $\text{Con}_c^A(B)$. The following containments hold:

$$\tilde{f}(\tilde{U}) \subseteq \text{Con}_c^{X_2}(B) \subseteq \text{Con}_c^A(B) \subseteq \tilde{A}.$$

Moreover $\langle f(U) \rangle_B^1 \subseteq A^*$.

The proofs that (6), (7), and (8) are locally finite properties are similar. \square

The following lemma is an analogue for gamps of the *Buttress Lemma* (cf. [6]).

Lemma 7.8. *Let \mathcal{V} be a variety of algebras in a finite similarity type, let P be a lower finite poset, let $(S_p)_{p \in P}$ be a family of finite $(\vee, 0)$ -semilattices, let $B \in \mathcal{V}$, and let $(\phi_p)_{p \in P}$ be a family of $(\vee, 0)$ -homomorphisms where $\phi_p: \text{Con}_c B \rightarrow S_p$ is ideal-induced for each $p \in P$. There exists a diagram $\vec{A} = (\mathbf{A}_p, \mathbf{f}_{p,q} \mid p \leq q \text{ in } P)$ of finite subgamps of $\mathbf{G}(B)$ such that the following statements hold:*

- (1) $\phi_p \upharpoonright \tilde{A}_p$ is ideal-induced for each $p \in P$.
- (2) \mathbf{A}_p is strong, distance-generated through $\phi_p \upharpoonright \tilde{A}_p$, and congruence-tractable through $\phi_p \upharpoonright \tilde{A}_p$, for each $p \in P$.
- (3) \mathbf{A}_p is a subgamp of \mathbf{A}_q and $\mathbf{f}_{p,q}$ is the canonical embedding, for all $p \leq q$ in P .
- (4) $\mathbf{f}_{p,q}$ is strong and congruence-cuttable through $\phi_q \upharpoonright \tilde{A}_q$, for all $p < q$ in P .

If B is a lattice, then we can construct $\tilde{\mathbf{A}}$ such that $\mathbf{f}_{p,q}$ is congruence-cutttable with chains through $\phi_q \upharpoonright \tilde{A}_q$ for all $p < q$ in P , and \mathbf{B}_p is distance-generated with chains through ϕ_p for each $p \in P$.

If B is congruence n -permutable (where n is a positive integer), then we can construct $\tilde{\mathbf{A}}$ such that \mathbf{A}_p is congruence n -permutable for each $p \in P$.

Proof. Let $r \in P$, suppose having constructed a diagram $(\mathbf{A}_p, \mathbf{f}_{p,q} \mid p \leq q < r)$ of finite subgamps of $\mathbf{G}(B)$ such that the following statements hold:

- (1) $\phi_p \upharpoonright \tilde{A}_p$ is ideal-induced for each $p < r$.
- (2) \mathbf{A}_p is strong, distance-generated through $\phi_p \upharpoonright \tilde{A}_p$, and congruence-tractable through $\phi_p \upharpoonright \tilde{A}_p$, for each $p < r$.
- (3) \mathbf{A}_p is a subgamp of \mathbf{A}_q and $\mathbf{f}_{p,q}$ is the canonical embedding, for all $p \leq q < r$.
- (4) $\mathbf{f}_{p,q}$ is strong and congruence-cutttable through $\phi_q \upharpoonright \tilde{A}_q$, for all $p < q < r$.

The following property in \mathbf{A} a subgamp of $\mathbf{G}(B)$ is locally finite (see Proposition 7.6 and Lemma 7.7)

- (F) \mathbf{A} is strong, distance-generated through $\phi_r \upharpoonright \tilde{A}$, congruence-tractable through $\phi_r \upharpoonright \tilde{A}$, and the canonical embedding $\mathbf{f}: \mathbf{A}_p \rightarrow \mathbf{A}$ is strong and congruence-cutttable through $\phi_r \upharpoonright \tilde{A}$, for all $p < r$.

Thus there exist finite partial subalgebras A_r^* and A_r of B such that for each \tilde{A} containing $\text{Con}_c^{A_r}(B)$, the gamp $(A_r^*, A_r, \Theta_B, \tilde{A})$ satisfies (F). Moreover it follows from Proposition 3.16 that there exists a finite $(\vee, 0)$ -subsemilattice \tilde{A}_r of $\text{Con}_c B$, such that $\text{Con}_c^{A_r}(B) \subseteq \tilde{A}_r$ and $\phi_r \upharpoonright \tilde{A}_r$ is ideal-induced.

Set $\mathbf{A}_r = (A_r^*, A_r, \Theta_B, \tilde{A}_r)$, and $\mathbf{f}_{p,r}$ the canonical embedding for each $p \leq r$. By construction, the diagram $(\mathbf{A}_p, \mathbf{f}_{p,q} \mid p \leq q \leq r)$ satisfies the required conditions. The construction of $(\mathbf{A}_p, \mathbf{f}_{p,q} \mid p \leq q \text{ in } P)$ follows by induction.

If B is a lattice, then we can add to the property (F) the condition $\mathbf{f}: \mathbf{A}_p \rightarrow \mathbf{A}$ is congruence-cutttable with chains through $\phi_r \upharpoonright \tilde{A}$ for each $p < r$, and \mathbf{B}_r is distance-generated with chains through ϕ_r .

If B is congruence n -permutable, then we can add to the property (F) the condition \mathbf{A}_r is congruence n -permutable. \square

8. NORM-COVERINGS AND LIFTERS

We introduced the following definition in [3].

Definition 8.1. A finite subset V of a poset U is a *kernel* if for every $u \in U$, there exists a largest element $v \in V$ such that $v \leq u$.

We say that U is *supported* if every finite subset of U is contained in a kernel of U .

It is not hard to verify that this definition of a supported poset is equivalent to the one used in [6].

The following definition introduced in [3] also appears in [6] in a weaker form. Nevertheless, in the context of \aleph_0 -lifters (cf. Definition 8.3), all these definitions are equivalent.

Definition 8.2. A *norm-covering* of a poset P is a pair (U, ∂) , where U is a supported poset and $\partial: U \rightarrow P$, $u \mapsto \partial u$ is an isotone map.

We say that an ideal \mathbf{u} of U is *sharp* if the set $\{\partial u \mid u \in \mathbf{u}\}$ has a largest element, which we shall then denote by $\partial \mathbf{u}$. We shall denote by $\text{Id}_s U$ the set of all sharp ideals of U , partially ordered by inclusion.

We remind the reader the following definition introduced in [6].

Definition 8.3. Let P be a poset. An \aleph_0 -*lifter* of P is a pair (U, \mathbf{U}) , where U is a norm-covering of P and \mathbf{U} is a subset of $\text{Id}_s U$ satisfying the following properties:

- (1) The set $\mathbf{U}^- = \{\mathbf{u} \in \mathbf{U} \mid \partial \mathbf{u} \text{ is not maximal in } P\}$ is lower finite, that is, the set $\mathbf{U} \downarrow \mathbf{u}$ is finite for each $\mathbf{u} \in \mathbf{U}^-$.
- (2) For every map $S: \mathbf{U}^- \rightarrow [U]^{<\omega}$, there exists an isotone map $\sigma: P \rightarrow \mathbf{U}$ such that
 - (a) the map σ is a *section* of ∂ , that is, $\partial \sigma(p) = p$ holds for each $p \in P$;
 - (b) the containment $S(\sigma(p)) \cap \sigma(q) \subseteq \sigma(p)$ holds for all $p < q$ in P . (*Observe that $\sigma(p)$ belongs to \mathbf{U}^- .*)

We say that P is \aleph_0 -*liftable* if it has an \aleph_0 -lifter.

The existence of lifters is related to the following infinite combinatorial statement introduced in [5].

Definition 8.4. For cardinals κ, λ and a poset P , let $(\kappa, <\lambda) \rightsquigarrow P$ hold if for every mapping $F: \mathfrak{P}(\kappa) \rightarrow [\kappa]^{<\lambda}$, there exists a one-to-one map $f: P \hookrightarrow \kappa$ such that

$$F(f(P \downarrow p)) \cap f(P \downarrow q) \subseteq f(P \downarrow p), \quad \text{for all } p \leq q \text{ in } P.$$

Notice that in case P is lower finite, it is sufficient to verify the conclusion above for all $F: [\kappa]^{<\omega} \rightarrow [\kappa]^{<\lambda}$ isotone and all $p < q$ in P .

Given a poset P , we introduce a new poset which looks like a lexicographical product of P with a tree.

Definition 8.5. Let P be a poset with a smallest element, let $X \subseteq P$, let $\vec{R} = (R_x)_{x \in X}$ be a family of sets, let $\alpha \leq \omega$. Consider the following poset:

$$T = \{(n, \vec{x}, \vec{r}) \mid n < \alpha, \vec{x} \in X^n, \text{ and } \vec{r} \in R_{x_0} \times \cdots \times R_{x_{n-1}}\},$$

ordered by $(m, \vec{y}, \vec{s}) \leq (n, \vec{x}, \vec{r})$ if and only if $m \leq n$, $\vec{x} = \vec{y} \upharpoonright m$, and $\vec{r} = \vec{s} \upharpoonright m$. Recall that $\vec{y} \upharpoonright m = (y_0, \dots, y_{m-1})$. Given $t = (n, \vec{x}, \vec{r}) \in T$ and $m \leq n$, we set $t \upharpoonright m = (m, \vec{x} \upharpoonright m, \vec{r} \upharpoonright m)$.

Put:

$$A = P \boxtimes_{\vec{R}} \alpha = T \times P = \bigcup_{n < \alpha} \bigcup_{\vec{x} \in X^n} (\{n\} \times \{\vec{x}\} \times (R_{x_0} \times \cdots \times R_{x_{n-1}}) \times P).$$

Any element of A can be written (n, \vec{x}, \vec{r}, p) with $n < \alpha$, $\vec{x} \in X^n$ and $\vec{r} \in R_{x_0} \times \cdots \times R_{x_{n-1}}$.

We define an order on A by $(m, \vec{y}, \vec{s}, q) \leq (n, \vec{x}, \vec{r}, p)$ if and only if the following conditions hold:

- (1) $(m, \vec{y}, \vec{s}) \leq (n, \vec{x}, \vec{r})$.
- (2) If $m = n$ then $p \leq q$.
- (3) If $m < n$ then $p \leq y_m$.

Remark 8.6. In the context of Definition 8.5, notice that T is a lower finite tree. Indeed, if $(n, \vec{x}, \vec{r}) \in T$, then $T \downarrow (n, \vec{x}, \vec{r}) = \{(m, \vec{x} \upharpoonright m, \vec{r} \upharpoonright m) \mid m \leq n\}$ is a chain of length n . The tree T is called *the tree associated to $P \boxtimes_{\vec{R}} \alpha$* .

The following statements hold:

- (1) If P is lower finite, then A is lower finite.
- (2) The inequalities $\text{card } A \leq \aleph_0 + \text{card } P + \sum_{x \in X} \text{card } R_x$ holds.
- (3) The inequalities $\text{card } T \leq \aleph_0 + \sum_{x \in X} \text{card } R_x$ holds.

Remark 8.7. In the context of Definition 8.5, if $a < b$ in A , then there are $t = (n, \vec{x}, \vec{r}) \in T$, $p, q \in P$, and $m \leq n$ such that $a = (t \upharpoonright m, p)$ and $b = (t, q)$. Moreover, if $m < n$, then $(t \upharpoonright m, p) < (t \upharpoonright (m+1), 0) \leq (t, q)$. It follows easily that $a \prec b$ if and only if exactly one of the following statements holds:

- (1) $m = n$ and $p \prec q$.
- (2) $n = m + 1$, $p = x_m$, and $q = 0$.

As a consequence, we obtain immediately that *If P , X , and all R_x are finite, then each $a \in A$ has only finitely many covers.*

Lemma 8.8. *Let $\kappa \geq \lambda$ be infinite cardinals, let P be a lower finite κ -small poset with a smallest element, let $X \subseteq P$, let $\vec{R} = (R_x)_{x \in X}$ be a family of κ -small sets, and let $\alpha \leq \omega$. If $(\kappa, < \lambda) \rightsquigarrow P$, then $(\kappa, < \lambda) \rightsquigarrow P \boxtimes_{\vec{R}} \alpha$.*

Proof. Denote by T the tree associated to $A = P \boxtimes_{\vec{R}} \alpha$. It follows from Remark 8.6 together with the assumptions on cardinalities that the following inequalities hold:

$$\text{card } T \leq \aleph_0 + \sum_{x \in X} \text{card } R_x \leq \aleph_0 + \sum_{x \in X} \kappa \leq \kappa.$$

Thus there exists a partition $(K_t)_{t \in T}$ of κ such that $\text{card } K_t = \kappa$ for each $t \in T$.

Notice that A is lower finite. Let $F: [\kappa]^{< \omega} \rightarrow [\kappa]^{< \lambda}$ isotone, let $t \in T$. Assume having constructed, for each $s < t$, a one-to-one map $\sigma_s: P \hookrightarrow K_s$ such that setting

$$S_s = \{\sigma_{s \upharpoonright m}(p) \mid m < n \text{ and } p \leq x_m\}, \text{ for all } s = (n, \vec{x}, \vec{r}) \leq t,$$

the following containments hold:

$$\text{rng } \sigma_s \subseteq K_s - F(S_s), \text{ for each } s < t,$$

$$F(\sigma_s(P \downarrow p) \cup S_s) \cap \sigma_s(P \downarrow q) \subseteq \sigma_s(P \downarrow p), \text{ for all } p \leq q \text{ in } P \text{ and all } s < t.$$

Put $F_t(U) = F(U \cup S_t) - F(S_t)$ for each $U \in [K_t - F(S_t)]^{< \omega}$. As S_t is finite, this defines a map $F_t: [K_t - F(S_t)]^{< \omega} \rightarrow [K_t - F(S_t)]^{< \lambda}$. As $F(S_t)$ is λ -small, $\text{card}(K_t - F(S_t)) = \kappa$, moreover $(\kappa, < \lambda) \rightsquigarrow P$, so there exists a one-to-one map $\sigma_t: P \hookrightarrow K_t - F(S_t)$ such that:

$$F(\sigma_t(P \downarrow p) \cup S_t) \cap \sigma_t(P \downarrow q) \subseteq \sigma_t(P \downarrow p), \text{ for all } p \leq q \text{ in } P.$$

Therefore we construct, by induction on t , a one-to-one map $\sigma_t: P \hookrightarrow K_t$ for each $t \in T$, such that setting

$$S_t = \{\sigma_{t \upharpoonright m}(p) \mid m < n \text{ and } p \leq x_m\}, \text{ for all } t = (n, \vec{x}, \vec{r}) \in T, \quad (8.1)$$

the following containments hold:

$$\text{rng } \sigma_t \subseteq K_t - F(S_t), \text{ for each } t \in T, \quad (8.2)$$

$$F(\sigma_t(P \downarrow p) \cup S_t) \cap \sigma_t(P \downarrow q) \subseteq \sigma_t(P \downarrow p), \text{ for all } p \leq q \text{ in } P \text{ and all } t \in T. \quad (8.3)$$

For $(t, p) \in A$, set $\sigma(t, p) = \sigma_t(p)$. This defines a map $\sigma: A \rightarrow \kappa$. Let $a = (s, p)$ and $b = (t, q)$ in A such that $\sigma(a) = \sigma(b)$. It follows from (8.2) that $\sigma(a) \in K_s$ and $\sigma(b) \in K_t$. As $(K_u)_{u \in T}$ is a partition of κ , we obtain $s = t$. Moreover $\sigma_t(p) = \sigma(a) = \sigma(b) = \sigma_t(q)$, so, as σ_t is one-to-one, $p = q$, and so $a = b$. Therefore σ is one-to-one.

Let $a = (t, p) \in A$, with $t = (n, \vec{x}, \vec{r}) \in T$. It follows from the definition of A that:

$$A \downarrow a = \{(t \upharpoonright m, q) \in A \mid m < n \text{ and } q \in P \downarrow x_m\} \cup \{(t, q) \in A \mid q \in P \downarrow p\}.$$

Thus, from (8.1), we see that

$$\sigma(A \downarrow a) = S_t \cup \sigma_t(P \downarrow p), \quad \text{for each } a = (t, p) \in A. \quad (8.4)$$

As $(K_t)_{t \in T}$ is a partition, it follows from (8.1) and (8.2) that $K_t \cap S_t = \emptyset$, thus, by (8.4),

$$\sigma(A \downarrow a) \cap K_t = \sigma_t(P \downarrow p), \quad \text{for each } a = (t, p) \in A. \quad (8.5)$$

Let $a \prec b$ in A . There are two cases to consider (cf. Remark 8.7). First assume that $a = (t, p)$ and $b = (t, q)$ with $p \prec q$ and $t = (n, \vec{x}, \vec{r}) \in T$. Let $c \leq b$ with $\sigma(c) \in F(\sigma(A \downarrow a))$. We can write $c = (t \upharpoonright m, p')$, with $m \leq n$. Suppose first that $m < n$. As $c \leq b$, $p' \leq x_m$, thus $c < a$. So $\sigma(c) \in \sigma(A \downarrow a)$. Now suppose that $m = n$. It follows from (8.2) that

$$\sigma(c) = \sigma_t(p') \in F(\sigma(A \downarrow a)) \cap \sigma(A \downarrow b) \cap (K_t - F(S_t)).$$

So, from (8.4) and (8.5) we obtain

$$\sigma(c) \in F(\sigma_t(P \downarrow p) \cup S_t) \cap \sigma_t(P \downarrow q).$$

Thus (8.3) implies that $\sigma(c) \in \sigma_t(P \downarrow p)$, from (8.4) we obtain $\sigma(c) \in \sigma(A \downarrow a)$. Therefore the containment $F(\sigma(A \downarrow a)) \cap \sigma(A \downarrow b) \subseteq \sigma(A \downarrow a)$ holds.

Now assume that $t = (n+1, \vec{x}, \vec{r}) \in T$, $b = (t, 0)$ and $a = (t \upharpoonright n, x_n)$. Let $c \leq b$ such that $\sigma(c) \in F(\sigma(A \downarrow a))$. As $c \leq b$ there are $m \leq n+1$ and $p \in P$ such that $c = (t \upharpoonright m, p)$. If $m \leq n$, then $p \leq x_m$, thus $c \leq a$, so $\sigma(c) \in \sigma(A \downarrow a)$. If $m = n+1$, then $c = b$. From (8.1) and (8.4) we obtain $\sigma(A \downarrow a) = S_{t \upharpoonright n} \cup \sigma_{t \upharpoonright n}(P \downarrow x_n) = S_t$. Therefore $\sigma_t(0) = \sigma(c) \in F(S_t)$, in contradiction with (8.2). So the containment $F(\sigma(A \downarrow a)) \cap \sigma(A \downarrow b) \subseteq \sigma(A \downarrow a)$ holds. \square

Corollary 8.9. *For an integer $m > 1$, put*

$$B_m(\leq 2) = \{X \in \mathfrak{P}(m) \mid \text{either } \text{card } X \leq 2 \text{ or } X = m\}.$$

Let P be a $(\vee, 0)$ -semilattice embeddable, as a poset, into $B_m(\leq 2)$. Let $X \subseteq P$, let $\vec{R} = (R_x)_{x \in X}$ be a family of finite sets, and let $\alpha \leq \omega$. There exists an \aleph_0 -lifter (U, \mathcal{U}) of $A = P \boxtimes_{\vec{R}} \alpha$ such that U has cardinality \aleph_2 .

Proof. It follows from [8], see also [2, Theorem 46.2], that $(\aleph_2, 2, \aleph_0) \rightarrow m$, so [5, Proposition 5.2] implies $(\aleph_2, < \aleph_0) \rightsquigarrow B_m(\leq 2)$, and so, from [5, Lemma 3.2] we obtain $(\aleph_2, < \aleph_0) \rightsquigarrow P$. It follows from Lemma 8.8 that $(\aleph_2, < \aleph_0) \rightsquigarrow A$. The conclusion follows from Remark 8.7 together with [6, Lemma 3-5.5]. \square

9. THE CONDENSATE LIFTING LEMMA FOR GAMPS

In this section we apply the Armature Lemma from [6] together with Lemma 7.8 to prove a special case of the Condensate Lifting Lemma for gamps.

Definition 9.1. Let \mathcal{A} and \mathcal{S} be categories, let $\Phi: \mathcal{A} \rightarrow \mathcal{S}$ be a functor. We introduce the following statements:

- (CLOS) \mathcal{A} has all small directed colimits.
- (PROD) Any two objects of \mathcal{A} have a product in \mathcal{A} .
- (CONT) The functor Φ preserves all small directed colimits.

Remark 9.2. Given a norm-covering X of a poset P and a category \mathcal{A} that satisfies both (CLOS) and (PROD), we can construct an object $\mathbf{F}(X) \otimes \vec{A}$ which is a directed colimit of finite products of objects in \vec{A} , together with morphisms

$$\pi_{\mathbf{x}}^X \otimes \vec{A}: \mathbf{F}(X) \otimes \vec{A} \rightarrow A_{\partial \mathbf{x}}$$

for each $\mathbf{x} \in \text{Id}_s X$.

Moreover if \mathcal{A} is a class of algebras closed under finite products and directed colimits, then $\pi_{\mathbf{x}}^X \otimes \vec{A}$ is surjective, and $\text{card}(\mathbf{F}(X) \otimes \vec{A}) \leq \text{card} X + \sum_{p \in P} A_p$. For more details about this construction, we refer the reader to [6, Chapter 2].

In the following theorem, we refer the reader to Definition 6.13 for the definition of a partial lifting.

Theorem 9.3. *Let \mathcal{V} and \mathcal{W} be varieties of algebras such that \mathcal{W} has finite similarity type, let (X, \mathbf{X}) be an \aleph_0 -lifter of a poset P , let $\vec{A} = (A_p, f_{p,q})$ be a diagram in \mathcal{V} such that $\text{Conc}_c A_p$ is finite for each $p \in P^\neq$, let $B \in \mathcal{W}$ such that $\text{Conc}_c B \cong \text{Conc}_c(\mathbf{F}(X) \otimes \vec{A})$. Then there exists a partial lifting of $\text{Conc}_c \circ \vec{A}$ in \mathcal{W} such that \mathbf{B}_p is finite for each $p \in P^\neq$ and \mathbf{B}_p is a quotient of $\mathbf{G}(B)$ for each $p \in \text{Max } P$.*

Proof. Denote by \mathcal{S} the category of all $(\vee, 0)$ -semilattices with $(\vee, 0)$ -homomorphisms. The category \mathcal{V} satisfies (CLOS) and (PROD), so $\mathbf{F}(X) \otimes \vec{A}$ is well defined (cf. [6, Section 3-1]). The functor Conc_c satisfies (CONT). Let $\chi: \text{Conc}_c B \rightarrow \text{Conc}_c(\mathbf{F}(X) \otimes \vec{A})$ be an isomorphism and put $\rho_{\mathbf{x}} = \text{Conc}_c(\pi_{\mathbf{x}}^X \otimes \vec{A}) \circ \chi$ for each $\mathbf{x} \in \mathbf{X}$. As $\pi_{\mathbf{x}}^X \otimes \vec{A}$ is surjective and χ is an isomorphism, it follows that $\rho_{\mathbf{x}}$ is ideal-induced.

Lemma 7.8 implies that there exists a diagram $\vec{B} = (\mathbf{B}_{\mathbf{x}}, \mathbf{g}_{\mathbf{x}, \mathbf{y}} \mid \mathbf{x} \leq \mathbf{y} \text{ in } \mathbf{X}^\neq)$ of finite subgroups of $\mathbf{G}(B)$ in \mathcal{W} such that the following statements hold:

- (1) $\rho_{\mathbf{x}} \upharpoonright \vec{B}_{\mathbf{x}}$ is ideal-induced for each $\mathbf{x} \in \mathbf{X}^\neq$.
- (2) $\mathbf{B}_{\mathbf{x}}$ is strong, distance-generated through $\rho_{\mathbf{x}} \upharpoonright \vec{B}_{\mathbf{x}}$, and congruence-tractable through $\rho_{\mathbf{x}} \upharpoonright \vec{B}_{\mathbf{x}}$, for each $\mathbf{x} \in \mathbf{X}^\neq$.
- (3) $\mathbf{g}_{\mathbf{x}, \mathbf{y}}$ is the canonical embedding, for all $\mathbf{x} \leq \mathbf{y}$ in \mathbf{X}^\neq .
- (4) $\mathbf{g}_{\mathbf{x}, \mathbf{y}}$ is strong and congruence-cutttable through $\rho_{\mathbf{y}} \upharpoonright \vec{B}_{\mathbf{y}}$, for all $\mathbf{x} < \mathbf{y}$ in \mathbf{X}^\neq .

We extend this diagram to an \mathbf{X} -indexed diagram, with $\mathbf{B}_{\mathbf{y}} = \mathbf{G}(B)$ and defining $\mathbf{g}_{\mathbf{x}, \mathbf{y}}$ as the canonical embedding for each $\mathbf{y} \in \mathbf{X} - \mathbf{X}^\neq$ and each $\mathbf{x} \leq \mathbf{y}$. Thus $\mathbf{C} \circ \mathbf{B}$ is an \mathbf{X} -indexed diagram in the comma category $\mathcal{S} \downarrow \text{Conc}_c B$, moreover $\mathbf{C}(\mathbf{B}_{\mathbf{x}}) = \vec{B}_{\mathbf{x}}$ is finite for each $\mathbf{x} \in \mathbf{X}^\neq$. Therefore, it follows from the Armature Lemma [6] that there exists an isotone section $\sigma: P \hookrightarrow \mathbf{X}$ such that the family $(\rho_{\sigma(p)} \upharpoonright \vec{B}_{\sigma(p)})_{p \in P}$ is a natural transformation from $(\mathbf{C}(\mathbf{B}_{\sigma(p)}), \mathbf{C}(\mathbf{g}_{\sigma(p), \sigma(q)}) \mid p \leq q \text{ in } P)$ to $\text{Conc}_c \circ \vec{A}$.

Put $\vec{B}' = (\mathbf{B}_{\sigma(p)}, \mathbf{g}_{\sigma(p), \sigma(q)} \mid p \leq q \text{ in } P)$ and $I_p = \ker_0 \rho_{\sigma(p)}$, for each $p \in P$. This defines an ideal $\vec{I} = (I_p)_{p \in P}$ of \vec{B}' . Moreover, as $\rho_{\sigma(p)} \upharpoonright \vec{B}_{\sigma(p)}$ is ideal-induced for each $p \in P$, these morphisms induce a natural equivalence $(\mathbf{C} \circ \vec{B}') / \vec{I} \cong \text{Conc}_c \circ \vec{A}$ (cf. Lemma 3.13).

Denote by $\mathbf{h}_{p,q}: \mathbf{B}_{\sigma(p)} / I_p \rightarrow \mathbf{B}_{\sigma(q)} / I_q$ the morphism induced by $\mathbf{g}_{\sigma(p), \sigma(q)}$. It follows from Proposition 6.5 that $\mathbf{B}_{\sigma(p)} / I_p$ is strong for each $p \in P$, and it follows from Proposition 6.7 that $\mathbf{h}_{p,q}$ is strong for all $p < q$ in P . Lemma 7.3 implies that

$\mathbf{B}_{\sigma(p)}/I_p$ is distance-generated and congruence-tractable for each $p \in P$. From Lemma 7.4 we obtain that $\mathbf{h}_{p,q}$ is congruence-cutttable for all $p < q$ in P .

Let $p \in P$, let $\chi: \tilde{\mathbf{B}}_{\sigma(p)} \rightarrow \text{Con}_c B$ be the inclusion map. As $\mathbf{B}_{\sigma(p)}$ is a subgamp of $\mathbf{G}(B)$, (B, χ) is a realization of $\mathbf{B}_{\sigma(p)}$, thus it induces a realization $(B/\bigvee I_p, \chi')$ where $\chi': \tilde{\mathbf{B}}_p/I_p \rightarrow B/\bigvee I$ satisfies $\chi'(d/I_p) = d/\bigvee I_p$ (cf. Proposition 6.5). As $\rho_{\sigma(p)} \upharpoonright \tilde{\mathbf{B}}_{\sigma(p)}$ is ideal-induced it is easy to check that χ' is surjective, hence it defines an isomorphic realization.

Let p a maximal element of P . From $\mathbf{B}_{\sigma(p)} = \mathbf{G}(B)$ it follows that $\mathbf{B}_{\sigma(p)}/I_p$ is a quotient of $\mathbf{G}(B)$.

Therefore $\vec{\mathbf{B}}/\vec{I}$ is a partial lifting of $\text{Con}_c \circ \vec{A}$ in \mathcal{W} . Moreover, if $p \in P^=$ then $\mathbf{B}_{\sigma(p)}$ is finite, thus $\mathbf{B}_{\sigma(p)}/I_p$ is finite. \square

Remark 9.4. Use the notation of Theorem 9.3. A small modification of the proof above shows that if \mathcal{W} is a variety of lattices, then we can construct a lattice partial lifting $\vec{\mathbf{B}}$ of $\text{Con}_c \circ \vec{A}$ in \mathcal{W} . Moreover, for any integer $n \geq 2$, if B is congruence n -permutable then all \mathbf{B}_p can be chosen congruence n -permutable.

Corollary 9.5. *Let \mathcal{V} be a locally finite variety of algebras, let \mathcal{W} be a variety of algebras with finite similarity type. The following statements are equivalent:*

- (1) $\text{crit}(\mathcal{V}; \mathcal{W}) > \aleph_0$.
- (2) *Let T be a countable lower finite tree and let \vec{A} be a T -indexed diagram of finite algebras in \mathcal{V} . Then $\text{Con}_c \circ \vec{A}$ has a partial lifting in \mathcal{W} .*
- (3) *Let \vec{A} be a ω -indexed diagram of finite algebras in \mathcal{V} . Then $\text{Con}_c \circ \vec{A}$ has a partial lifting in \mathcal{W} .*

Proof. Assume that (1) holds. Let T be a countable lower finite tree and let \vec{A} be a T -indexed diagram of finite algebras in \mathcal{V} . It follows from [3, Corollary 4.7] that there exists an \aleph_0 -lifter (X, \mathbf{X}) of T such that $\text{card } X = \aleph_0$. Hence the following inequalities hold:

$$\text{card}(\mathbf{F}(X) \otimes \vec{A}) \leq \text{card } X + \sum_{p \in T} A_p \leq \aleph_0 \sum_{p \in T} \aleph_0 = \aleph_0.$$

Thus, as $\text{crit}(\mathcal{V}; \mathcal{W}) > \aleph_0$, there exists $B \in \mathcal{W}$ such that $\text{Con}_c B \cong \text{Con}_c(\mathbf{F}(X) \otimes \vec{A})$. It follows from Theorem 9.3 that there exists a partial lifting of $\text{Con}_c \circ \vec{A}$ in \mathcal{W} .

The implication (2) \implies (3) is immediate.

Assume that (3) holds and let $A \in \mathcal{V}$ such that $\text{card } \text{Con}_c A \leq \aleph_0$. By replacing A with one of its subalgebras we can assume that $\text{card } A \leq \aleph_0$ (see [3, Lemma 3.6]). As \mathcal{V} is locally finite, there exists an increasing sequence $(A_k)_{k < \omega}$ of finite subalgebras of A with union A . Denote by $f_{i,j}: A_i \rightarrow A_j$ the inclusion map, for all $i \leq j < \omega$. Put $\vec{A} = (A_i, f_{i,j} \mid i \leq j < \omega)$. Let $\vec{\mathbf{B}}$ be a partial lifting of $\text{Con}_c \circ \vec{A}$ in \mathcal{W} , let \mathbf{B} be the directed colimit of $\vec{\mathbf{B}}$ in $\mathbf{Gamp}(\mathcal{V})$. As \mathbf{C} and Con_c both preserve directed colimits, the following isomorphisms hold:

$$\mathbf{C}(\mathbf{B}) \cong \mathbf{C}(\varinjlim \vec{\mathbf{B}}) \cong \varinjlim (\mathbf{C} \circ \vec{\mathbf{B}}) \cong \varinjlim (\text{Con}_c \circ \vec{A}) \cong \text{Con}_c \varinjlim \vec{A} \cong \text{Con}_c A.$$

Moreover, it follows from Lemma 6.16 that \mathbf{B} is an algebra in \mathcal{W} , that is, there exists $B \in \mathcal{W}$ such that $\mathbf{B} \cong \mathbf{G}(B)$, so $\text{Con}_c B = \mathbf{C}(\mathbf{G}(B)) \cong \mathbf{C}(\mathbf{B}) \cong \text{Con}_c A$. Therefore $\text{crit}(\mathcal{V}; \mathcal{W}) > \aleph_0$. \square

10. THE CHAIN DIAGRAM OF A LATTICE

From now on this paper a *partial lifting* will always be a lattice partial lifting (cf. Definition 6.13). The aim of this section is, given a partial sublattice K of a lattice L , to construct a diagram of lattices \vec{A} , such that whenever $\text{Con}_c \circ \vec{A}$ has a partial lifting in a variety \mathcal{V} , then K is a partial sublattice of some $L' \in \mathcal{V} \cup \mathcal{V}^d$.

We shall construct in Definition 10.8 a diagram \vec{A} , called the chain diagram of K in L . In Theorem 10.12 we shall prove that if $\text{Con}_c \circ \vec{A}$ has a partial lifting with enough direct congruence chains (cf. Definition 10.5) then K is a partial sublattice of a partial lattice in the partial lifting. In Lemma 10.7 we shall give diagrams that “force” congruence chains to be direct, then in Lemma 10.13, using Definition 10.2 and Lemma 10.4, we shall glue all these diagrams together to obtain a diagram \vec{A}' , such that whenever \vec{B} is a partial lifting of $\text{Con}_c \circ \vec{A}'$ with enough congruence chains, then K is a partial sublattice of a quotient of either \mathbf{B}_1 or its dual.

Remark 10.1. The dual of a partial lattice B is the partial lattice B^d , with the same set of elements B , $\text{Def}_\vee(B^d) = \text{Def}_\wedge(B)$, $\text{Def}_\wedge(B^d) = \text{Def}_\vee(B)$, $x \vee_{B^d} y = x \wedge_B y$ for all $(x, y) \in \text{Def}_\vee(B^d)$, and $x \wedge_{B^d} y = x \vee_B y$ for all $(x, y) \in \text{Def}_\wedge(B^d)$.

The dual of a gamp \mathbf{B} of lattices is $\mathbf{B}^d = (B^{*d}, B^d, \delta_{\mathbf{B}}, \tilde{B})$.

Let \mathcal{C} be a set of nonempty finite chains. We put:

$$\begin{aligned} \mathcal{J}_1(\mathcal{C}) &= \{\emptyset\} \cup \{\{C\} \mid C \in \mathcal{C}\}, \\ \mathcal{J}_2(\mathcal{C}) &= \mathcal{J}_1(\mathcal{C}) \cup \{\{C, D\} \subseteq \mathcal{C} \mid \text{either } \text{lh}(C) = \text{lh}(D) = 2 \text{ or } C \subseteq D\}, \\ \mathcal{J}(\mathcal{C}) &= \mathcal{J}_2(\mathcal{C}) \cup \{\top\}, \end{aligned}$$

ordered by inclusion on $\mathcal{J}_2(\mathcal{C})$ and with $P \leq \top$ for each $P \in \mathcal{J}(\mathcal{C})$. Put

$$E_P = \begin{cases} C & \text{if } P = \{C\} \\ \{0, 1\} & \text{if } P = \emptyset \end{cases}, \quad \text{for each } P \in \mathcal{J}_1(\mathcal{C}).$$

Let $e_{\emptyset, \{C\}}: \{0, 1\} \rightarrow C$ be the bounds-preserving morphism, and $e_{P, P} = \text{id}_{E_P}$ for each $P \in \mathcal{J}_1(\mathcal{C})$. Thus $\vec{E}(\mathcal{C}) = (E_P, e_{P, Q} \mid P \leq Q \text{ in } \mathcal{J}_1(\mathcal{C}))$ is a diagram of lattices, with 0, 1-homomorphisms of lattices.

The following definition describes the way that our diagrams will be glued together.

Definition 10.2. Let T be a finite nonempty set, let I be a finite poset, let J be a lower subset of I . Let $\vec{A}^t = (A_i^t, f_{i,j}^t \mid i \leq j \text{ in } I)$ be a diagram of algebras for all $t \in T$, such that $\vec{A}^s \upharpoonright J = \vec{A}^t \upharpoonright J$ for all $s, t \in T$, that is

- (1) The equality $A_j^s = A_j^t$ holds for all $s, t \in T$ and all $j \in J$.
- (2) The equality $f_{i,j}^s = f_{i,j}^t$ holds for all $s, t \in T$ and all $i \leq j$ in J .

Set

$$A_i = \begin{cases} A_i^t & \text{for any } t \in T, \text{ if } i \in J \\ \prod_{t \in T} A_i^t & \text{if } i \in I - J \end{cases}, \quad \text{for each } i \in I$$

If $i \in J$, denote $\pi_i^t = \text{id}_{A_i} = \text{id}_{A_i^t}$. If $i \in I - J$, denote $\pi_i^t: A_i \rightarrow A_i^t$ the canonical projection. Let

$$f_{i,j}: A_i \rightarrow A_j$$

$$x \mapsto \begin{cases} f_{i,j}^t(x) & \text{for any } t \in T, \text{ if } i, j \in J \\ (f_{i,j}^t(x))_{t \in T} & \text{if } i \in J \text{ and } j \in I - J \\ (f_{i,j}^t(x_t))_{t \in T} & \text{if } i, j \in I - J \text{ and } x = (x_t)_{t \in T} \end{cases}, \quad \text{for all } i \leq j \text{ in } I$$

Then $\vec{A} = (A_i, f_{i,j} \mid i \leq j \text{ in } I)$ is the product of $(\vec{A}^t)_{t \in T}$ over J , and $\vec{\pi}^t = (\pi_i^t)_{i \in I}: \vec{A} \rightarrow \vec{A}^t$ is the canonical projection.

Remark 10.3. Let \mathcal{K} be a class of bounded lattices closed under finite products. Consider the objects of Definition 10.2, assume that \vec{A}^t is a diagram in \mathcal{K} for each $t \in T$. Then \vec{A} the product of $(\vec{A}^t)_{t \in T}$ over J is a diagram in \mathcal{K} .

Let $i \in I$ such that A_i^t is finite for each $t \in T$, then A_i is finite.

Using the construction of the product in Definition 10.2 we see that $\vec{A} \upharpoonright J = \vec{A}^t \upharpoonright J$ for each $t \in T$.

The following lemma gives a way to extend diagrams indexed by $\mathcal{J}(\mathcal{C})$ to $\mathcal{J}(\mathcal{C}')$ for $\mathcal{C} \subseteq \mathcal{C}'$.

Lemma 10.4. *Let \mathcal{K} be a class of bounded lattices, let $\mathcal{C} \subseteq \mathcal{C}'$ be sets of finite chains in \mathcal{K} . Then $\mathcal{J}(\mathcal{C}) \subseteq \mathcal{J}(\mathcal{C}')$ and $\mathcal{J}_1(\mathcal{C}) \subseteq \mathcal{J}_1(\mathcal{C}')$. Furthermore, consider a diagram $\vec{B} = (B_P, g_{P,Q} \mid P \leq Q \text{ in } \mathcal{J}(\mathcal{C}))$ in \mathcal{K} (with 0, 1-homomorphisms of lattices). If $\vec{B} \upharpoonright \mathcal{J}_1(\mathcal{C}) = \vec{E}(\mathcal{C})$, then there exists a diagram \vec{B}' of \mathcal{K} , indexed by $\mathcal{J}(\mathcal{C}')$, such that $\vec{B}' \upharpoonright \mathcal{J}(\mathcal{C}) = \vec{B}$ and $\vec{B}' \upharpoonright \mathcal{J}_1(\mathcal{C}') = \vec{E}(\mathcal{C}')$.*

Proof. It is sufficient to establish the result in case $\mathcal{C}' = \mathcal{C} \cup \{C'\}$, for some chain C' . For $P \in \mathcal{J}(\mathcal{C}')$, let:

$$B'_P = \begin{cases} B_P & \text{if } P \in \mathcal{J}(\mathcal{C}) \\ B_{\{C\}} = C & \text{if } P = \{C, C'\}, \text{ with } C \in \mathcal{C}, \\ C' & \text{if } P = \{C'\} \end{cases}$$

For a finite chain C , put

$$f_C: C \rightarrow \mathbf{2}$$

$$x \mapsto \begin{cases} 0 & \text{if } x < 1_C, \\ 1 & \text{if } x = 1_C. \end{cases}$$

Let $P \leq Q$ in $\mathcal{J}(\mathcal{C}')$. Set $g'_{P,P} = \text{id}_{B'_P}$, and if $P < Q$,

$$g'_{P,Q} = \begin{cases} g_{P,Q} & \text{if } P, Q \in \mathcal{J}(\mathcal{C}), \\ g_{\{C\}, \top} & \text{if } P = \{C, C'\} \text{ and } Q = \top, \text{ with } C \in \mathcal{C}, \\ e_{\emptyset, \{C'\}} & \text{if } P = \emptyset \text{ and } Q = \{C'\}, \\ e_{\emptyset, \{C\}} & \text{if } P = \emptyset \text{ and } Q = \{C, C'\}, \text{ with } C \in \mathcal{C}, \\ e_{\emptyset, \{C\}} \circ f_{C'} & \text{if } P = \{C'\} \text{ and } Q = \{C, C'\}, \text{ with } C \in \mathcal{C}, \\ \text{id}_C & \text{if } P = \{C\} \text{ and } Q = \{C, C'\}, \text{ with } C \in \mathcal{C}, \\ g_{\emptyset, \top} \circ f_{C'} & \text{if } P = \{C'\} \text{ and } Q = \top. \end{cases}$$

The elements of $\mathcal{J}(\mathcal{C}') - \mathcal{J}(\mathcal{C})$ are $\{C'\}$ and $\{C, C'\}$ where either $\text{lh}(C) = \text{lh}(C') = 2$ or $C' \subseteq C$ or $C \subseteq C'$, with $C \in \mathcal{C}$. Let $P < Q < R$ in $\mathcal{J}(\mathcal{C}')$. In particular P belongs to $\mathcal{J}_1(\mathcal{C}')$. If $P, Q, R \in \mathcal{J}(\mathcal{C})$, then:

$$g'_{Q,R} \circ g'_{P,Q} = g_{Q,R} \circ g_{P,Q} = g_{P,R} = g'_{P,R}.$$

Now we assume that $\{P, Q, R\} \not\subseteq \mathcal{J}(\mathcal{C})$. Assume that $P = \emptyset$ and $Q = \{C'\}$. Then either $R = \top$ or $R = \{C, C'\}$ for some $C \in \mathcal{C}$. If $R = \top$ then:

$$g'_{Q,R} \circ g'_{P,Q} = g_{\emptyset, \top} \circ f_{C'} \circ e_{\emptyset, \{C'\}} = g_{\emptyset, \top} = g'_{P,R}.$$

If $R = \{C, C'\}$ for some $C \in \mathcal{C}$, then:

$$g'_{Q,R} \circ g'_{P,Q} = e_{\emptyset, \{C\}} \circ f_{C'} \circ e_{\emptyset, \{C'\}} = g_{\emptyset, \{C\}} = g'_{P,R}.$$

Assume that $P = \emptyset$ and $Q = \{C\}$, where $C \in \mathcal{C}$. Then $R \notin \mathcal{J}(\mathcal{C})$, so $R = \{C, C'\}$, thus:

$$g'_{Q,R} \circ g'_{P,Q} = \text{id}_C \circ g_{\emptyset, \{C\}} = g_{\emptyset, \{C\}} = e_{\emptyset, \{C\}} = g'_{P,R}.$$

If $P = \emptyset$ and $Q = \{C_1, C_2\} \subseteq \mathcal{C}$, with $C_1 \neq C_2$, then $R = \top$, so $P, Q, R \in \mathcal{J}(\mathcal{C})$, a contradiction. If $P = \emptyset$ and $Q = \{C, C'\}$, for some $C \in \mathcal{C}$, then $R = \top$, thus:

$$g'_{Q,R} \circ g'_{P,Q} = g_{\{C\}, \top} \circ e_{\emptyset, \{C\}} = g_{\emptyset, \top} = g'_{P,R}.$$

If $P = \{C\}$, for some $C \in \mathcal{C}$, then $R = \top$, and as P, Q, R are not all in $\mathcal{J}(\mathcal{C})$, $Q = \{C, C'\}$, thus:

$$g'_{Q,R} \circ g'_{P,Q} = g_{\{C\}, \top} \circ \text{id}_{B_{\{C\}}} = g_{\{C\}, \top} = g'_{P,R}$$

If $P = \{C'\}$, then $R = \top$, and $Q = \{C, C'\}$ for some $C \in \mathcal{C}$, thus:

$$g'_{Q,R} \circ g'_{P,Q} = g_{\{C\}, \top} \circ e_{\emptyset, \{C\}} \circ f_{C'} = g_{\emptyset, \top} \circ f_{C'} = g'_{P,R}.$$

It follows that $\vec{B}' = (B'_P, g'_{P,Q} \mid P \leq Q \text{ in } \mathcal{J}(\mathcal{C}'))$ is a diagram in \mathcal{K} . Moreover, by construction $\vec{B}' \upharpoonright \mathcal{J}(\mathcal{C}) = \vec{B}$ and $\vec{B}' \upharpoonright \mathcal{J}_1(\mathcal{C}') = \vec{E}(\mathcal{C}')$. \square

The following definition involves the *chains* (in gamps of lattices) introduced in Definition 6.2.

Definition 10.5. Let \mathbf{B} be a gamp of lattices such that \tilde{B} is a finite Boolean lattice. A chain $x_0 < x_1 < \dots < x_n$ of \mathbf{B} is a *congruence chain* of \mathbf{B} if there exists a bijection $\sigma: n \rightarrow \text{At } \tilde{B}$ such that $\delta_{\mathbf{B}}(x_k, x_{k+1}) = \sigma(k)$ for each $k < n$.

Let $C = \{c_0, \dots, c_n\}$ be a chain with $c_0 < c_1 < \dots < c_n$ and let $\xi: \tilde{B} \rightarrow \text{Con}_c C$ be an isomorphism. A congruence chain $x_0 < \dots < x_n$ of \mathbf{B} is *direct for* (ξ, C) if $\xi(\delta_{\mathbf{B}}(x_k, x_{k+1})) = \Theta_C(c_k, c_{k+1})$ for all $k < n$. We simply say that $x_0 < x_1 < \dots < x_n$ is a *direct congruence chain of* \mathbf{B} in case ξ and C are both understood.

In both cases x_0 and x_n are the *extremities* of the congruence chain.

Given a lattice B with a finite Boolean congruence lattice, a *congruence chain of* B is a congruence chain of $\mathbf{G}(B)$.

In particular, if $x_0 < x_1 < \dots < x_n$ of \mathbf{B} is a congruence chain of \mathbf{B} , then $x_i \in B^*$ for each $i \leq n$ (cf. Definition 6.2).

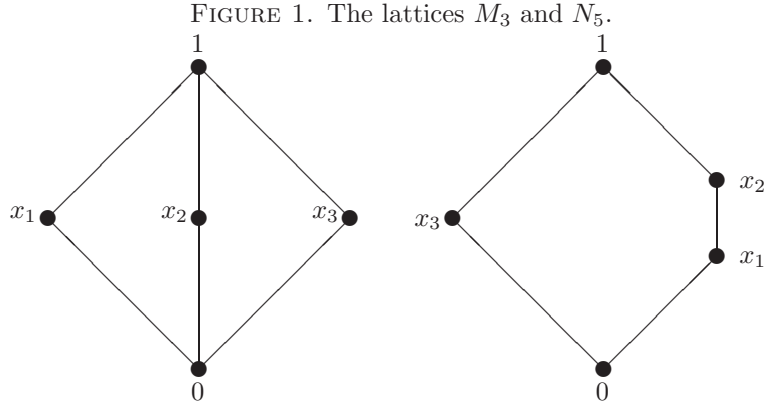
Remark 10.6. Let $x_0 < x_1 < \dots < x_n$ be a congruence chain of a strong gamp \mathbf{B} of lattices. It follows from Lemma 6.4 that $x_i \vee x_j = x_j$ in B for all $0 \leq i \leq j \leq n$.

Let C be a chain of length 2, let \mathbf{B} be a strong gamp of lattices, and let $\xi: \tilde{B} \rightarrow \text{Con}_c C$ be an isomorphism. A congruence chain $x_0 < x_1 < x_2$ of \mathbf{B} is either direct

or dually direct, that is, either $x_0 < x_1 < x_2$ is a direct congruence chain of \mathbf{B} or $x_2 <_{\mathbf{B}^d} x_1 <_{\mathbf{B}^d} x_0$ is a direct congruence chain of \mathbf{B}^d .

Given a lattice B with a finite Boolean congruence lattice, a chain $x_0 < x_1 < \dots < x_n$ is a congruence chain of B if and only if there exists a bijection $\sigma: n \rightarrow \text{At } \tilde{B}$ such that $\Theta_B(x_k, x_{k+1}) = \sigma(k)$ for each $k < n$.

A finite chain C is a congruence chain of a lattice B if and only if B is a congruence-preserving extension of C . In particular if B is a finite distributive lattice, then every maximal chain of B is a congruence chain of B .



In our next lemma we shall construct a diagram of lattices that “forces” congruence chains to be direct.

Lemma 10.7. *Let $K = N_5$ or $K = M_3$, with vertices labeled as in the Figure 1, let \mathcal{K} be a class of bounded lattices closed under finite products such that every bounded sublattice of K belongs to \mathcal{K} . Let $C_1, C_2, C_3 \in \mathcal{K}$ be distinct finite chains with extremities 0 and 1 such that both C_1 and C_2 have length 2 while either C_3 has length 2 or $C_1, C_2 \subseteq C_3$. Put $\mathcal{C} = \{C_1, C_2, C_3\}$. Then there exists a diagram \vec{A} of finite lattices of \mathcal{K} with 0, 1-lattice homomorphisms, indexed by $\mathcal{J}(\mathcal{C})$, such that:*

- (1) *The equality $\vec{A} \upharpoonright \mathcal{J}_1(\mathcal{C}) = \vec{E}(\mathcal{C})$ holds.*
- (2) *Let \vec{B} be a partial lifting of $\text{Con}_c \circ \vec{A}$ and let $u \neq v$ in B_\emptyset^* . If $\mathbf{B}_{\{C_k\}}$ contains a direct congruence chain with extremities $g_{\emptyset, \{C_k\}}(u)$ and $g_{\emptyset, \{C_k\}}(v)$ for each $k \in \{1, 2\}$, then every congruence chain of $\mathbf{B}_{\{C_3\}}$ with extremities $g_{\emptyset, \{C_3\}}(u)$ and $g_{\emptyset, \{C_3\}}(v)$ is direct.*

Proof. We can assume that $C_1 = \{0, x_1, 1\}$ and $C_2 = \{0, x_2, 1\}$. Let $0 = y_0 < y_1 < y_2 < \dots < y_n = 1$ be the elements of C_3 . Put $D_3 = \{0, x_3, 1\}$ and set

$$T = \{t \mid t: C_3 \rightarrow D_3 \text{ is isotone and surjective}\}.$$

Let $t \in T$. Put:

$$\begin{aligned} A_\emptyset^t &= A_\emptyset = \{0, 1\}, \\ A_{\{C_i\}}^t &= A_{\{C_i\}} = C_i && \text{for } 1 \leq i \leq 3 \\ A_{\{C_i, C_j\}}^t &= \{0, x_i, x_j, 1\} && \text{for } 1 \leq i < j \leq 3 \\ A_\top^t &= K. \end{aligned}$$

For $P \leq Q$ in $\mathcal{J}(\mathcal{C})$, let $f_{P,Q}^t: A_P^t \rightarrow A_Q^t$ be the inclusion map if $P \neq \{C_3\}$ and $Q \neq \{C_3\}$, otherwise let:

$$f_{P,Q}^t = \begin{cases} \text{id}_{A_P} & \text{if } P = Q = \{C_3\}, \\ t & \text{if } P = \{C_3\} \text{ and } Q > P, \\ e_{\emptyset, C_3} & \text{if } P = \emptyset \text{ and } Q = \{C_3\}. \end{cases}$$

Let $P < Q < R$ in $\mathcal{J}(\mathcal{C})$. As all the maps involved are 0, 1-homomorphisms and $A_\emptyset^t = \{0, 1\}$, if $P = \emptyset$ then $f_{P,R}^t = f_{Q,R}^t \circ f_{P,Q}^t$. Now assume that $P > \emptyset$. If $P \neq \{C_3\}$, then $f_{P,Q}^t$, $f_{Q,R}^t$, and $f_{P,R}^t$ are all inclusion maps, thus $f_{P,R}^t = f_{Q,R}^t \circ f_{P,Q}^t$. If $P = \{C_3\}$, then $f_{P,Q}^t = t$, the morphism $f_{Q,R}^t$ is the inclusion map, and $f_{P,R}^t = t$, so $f_{P,R}^t = f_{Q,R}^t \circ f_{P,Q}^t$. Thus $\vec{A}^t = (A_P^t, f_{P,Q}^t \mid P \leq Q \text{ in } \mathcal{J}(\mathcal{C}))$ is a diagram of finite lattices of \mathcal{K} . Moreover by construction $\vec{A}^t \upharpoonright \mathcal{J}_1(\mathcal{C}) = \vec{E}(\mathcal{C})$.

Let $\vec{A} = (A_P, f_{P,Q} \mid P \leq Q \text{ in } \mathcal{J}(\mathcal{C}))$ be the product of $(\vec{A}^t)_{t \in T}$ over $\mathcal{J}_1(\mathcal{C})$ (cf. Definition 10.2). Hence $\vec{A} \upharpoonright \mathcal{J}_1(\mathcal{C}) = \vec{E}(\mathcal{C})$ and it follows from Remark 10.3 that \vec{A} is a diagram of finite lattices in \mathcal{K} .

Let $\vec{B} = (\mathbf{B}_P, \mathbf{g}_{P,Q} \mid P \leq Q \text{ in } \mathcal{J}(\mathcal{C}))$ be a partial lifting of $\text{Con}_c \circ \vec{A}$. We can assume that $\mathbf{C} \circ \vec{B} = \text{Con}_c \circ \vec{A}$. Set $\delta_P = \delta_{\mathbf{B}_P}$, for each $P \in \mathcal{J}(\mathcal{C})$. Let $u \neq v$ in B_\emptyset^* with $u \wedge v = u$. Assume that $\mathbf{B}_{\{C_k\}}$ contains a direct congruence chain $g_{\emptyset, \{C_k\}}(u) < x'_k < g_{\emptyset, \{C_k\}}(v)$ for $k = 1, 2$. Let $g_{\emptyset, \{C_3\}}(u) = y'_0 < y'_1 < \dots < y'_n = g_{\emptyset, \{C_3\}}(v)$ be a congruence chain of $\mathbf{B}_{\{C_3\}}$.

Let $\sigma: \{0, 1, \dots, n-1\} \rightarrow \{0, 1, \dots, n-1\}$ be a bijection such that:

$$\Theta_{\{C_3\}}(y_i, y_{i+1}) = \delta_{\{C_3\}}(y'_{\sigma(i)}, y'_{\sigma(i+1)}), \quad \text{for all } i < n \quad (10.1)$$

Assume that σ is not the identity map. Let i be minimal such that $\sigma(i) \neq i$. The minimality of i implies that $\sigma(k) = k$ for all $k < i$, thus:

$$\Theta_{C_3}(y_0, y_i) = \bigvee_{k < i} \Theta_{C_3}(y_k, y_{k+1}) = \bigvee_{k < i} \delta_{\{C_3\}}(y'_k, y'_{k+1})$$

It follows from Lemma 6.4 that the following equality holds:

$$\Theta_{C_3}(y_0, y_i) = \delta_{\{C_3\}}(y'_0, y'_i). \quad (10.2)$$

Set $j = \sigma^{-1}(i)$. Then

$$\Theta_{C_3}(y_j, y_{j+1}) = \delta_{\{C_3\}}(y'_i, y'_{i+1}). \quad (10.3)$$

It follows from Lemma 6.4, (10.2), and (10.3) that the following equality holds:

$$\delta_{\{C_3\}}(y'_0, y'_{i+1}) = \Theta_{C_3}(y_0, y_i) \vee \Theta_{C_3}(y_j, y_{j+1}). \quad (10.4)$$

Using the minimality of i , we obtain that $i < j$. Hence σ defines by restriction a bijection from $\{i, i+1, \dots, n-1\} - \{j\}$ onto $\{i+1, i+2, \dots, n-1\}$, and so

$$\begin{aligned} \Theta_{C_3}(y_i, y_j) \vee \Theta_{C_3}(y_{j+1}, y_n) &= \bigvee \{ \Theta_{C_3}(y_k, y_{k+1}) \mid k \in \{i, i+1, \dots, n-1\} - \{j\} \} \\ &= \bigvee \{ \delta_{\{C_3\}}(y'_{\sigma(k)}, y'_{\sigma(k+1)}) \mid k \in \{i, i+1, \dots, n-1\} - \{j\} \} \\ &= \bigvee \{ \delta_{\{C_3\}}(y'_s, y'_{s+1}) \mid s \in \{i+1, i+2, \dots, n-1\} \}. \end{aligned}$$

It follows from Lemma 6.4 that the following equation is satisfied:

$$\delta_{\{C_3\}}(y'_{i+1}, y'_n) = \Theta_{C_3}(y_i, y_j) \vee \Theta_{C_3}(y_{j+1}, y_n). \quad (10.5)$$

Let:

$$t: C_3 \rightarrow D_3$$

$$y_k \mapsto \begin{cases} 0 & \text{if } k \leq i \\ x_3 & \text{if } i < k \leq j, \quad \text{for all } k < n. \\ 1 & \text{if } j < k \end{cases}$$

Let $\vec{\pi} = (\pi_P)_{P \in \mathcal{J}(\mathcal{C})}: \vec{A} \rightarrow \vec{A}^t$ be the canonical projection. The vector $\vec{\chi} = \text{Conc} \circ \vec{\pi}$ is an ideal-induced natural transformation from $\mathbf{C} \circ \vec{\mathbf{B}}$ to $\text{Conc} \circ \vec{A}^t$. Denote $\delta'_P = \chi_P \circ \delta_P$ for each $P \in \mathcal{J}(\mathcal{C})$. Notice that $A_P^t = A_P$, $\pi_P = \text{id}_{A_P}$, $\chi_P = \text{id}_{\text{Conc } A_P}$, and $\delta'_P = \delta_P$ for each $P \in \mathcal{J}_1(\mathcal{C})$. Let $P \leq Q$ in $\mathcal{J}(\mathcal{C})$ with $P \in \mathcal{J}_1(\mathcal{C})$. Let $a, b \in B_P$. The following equations are satisfied:

$$\begin{aligned} \delta'_Q(g_{P,Q}(a), g_{P,Q}(b)) &= \chi_Q((\text{Conc } f_{P,Q})(\delta_P(a, b))) \\ &= (\text{Conc } f_{P,Q}^t)(\chi_P(\delta_P(a, b))) \\ &= (\text{Conc } f_{P,Q}^t)(\delta_P(a, b)). \end{aligned} \quad (10.6)$$

As $g_{\emptyset, \{C_k\}}(u) < x'_k < g_{\emptyset, \{C_k\}}(v)$ is a direct congruence chain of $\mathbf{B}_{\{C_k\}}$, we obtain

$$\delta_{\{C_k\}}(g_{\emptyset, \{C_k\}}(u), x'_k) = \Theta_{C_k}(0, x_k). \quad (10.7)$$

Let $k \in \{1, 2\}$, let $Q \geq \{C_k\}$ in $\mathcal{J}(\mathcal{C})$. The following equalities hold:

$$\begin{aligned} \delta'_Q(g_{\emptyset, Q}(u), g_{\{C_k\}, Q}(x'_k)) &= (\text{Conc } f_{\{C_k\}, Q}^t)(\delta_{\{C_k\}}(g_{\emptyset, \{C_k\}}(u), x'_k)) \quad \text{by (10.6).} \\ &= (\text{Conc } f_{\{C_k\}, Q}^t)(\Theta_{C_k}(0, x_k)) \quad \text{by (10.7).} \\ &= \Theta_{A_Q^t}(0, x_k), \end{aligned}$$

hence

$$\delta'_Q(g_{\emptyset, Q}(u), g_{\{C_k\}, Q}(x'_k)) = \Theta_{A_Q^t}(0, x_k), \quad \text{for } k = 1, 2, \text{ and } Q \geq \{C_k\}. \quad (10.8)$$

Let $k \in \{1, 2\}$ and set $Q = \{C_k, C_3\}$. Put $u' = g_{\{C_3\}, Q}(y'_0) = g_{\emptyset, Q}(u)$, $v' = g_{\{C_3\}, Q}(y'_n) = g_{\emptyset, Q}(v)$, and $a = g_{\{C_3\}, Q}(y'_{i+1})$. The following equalities hold:

$$\begin{aligned} \delta'_Q(u', a) &= \delta'_Q(g_{\{C_3\}, Q}(y'_0), g_{\{C_3\}, Q}(y'_{i+1})) \\ &= (\text{Conc } t)(\delta_{\{C_3\}}(y'_0, y'_{i+1})) \quad \text{by (10.6), as } f_{\{C_3\}, Q}^t = t. \\ &= (\text{Conc } t)(\Theta_{C_3}(y_0, y_i) \vee \Theta_{C_3}(y_j, y_{j+1})) \quad \text{by (10.4).} \\ &= \Theta_{A_Q^t}(t(y_0), t(y_i)) \vee \Theta_{A_Q^t}(t(y_j), t(y_{j+1})) \\ &= \Theta_{A_Q^t}(x_3, 1), \end{aligned}$$

hence

$$\delta'_Q(u', a) = \Theta_{A_Q^t}(x_3, 1). \quad (10.9)$$

Similarly, it follows from (10.5) that

$$\delta'_Q(a, v') = \Theta_{A_Q^t}(0, x_3). \quad (10.10)$$

Put $b = g_{\{C_k\}, Q}(x'_k)$. It follows from (10.8) that

$$\delta'_Q(u', b) = \Theta_{A_Q^t}(0, x_k) = \Theta_{A_Q^t}(x_3, 1). \quad (10.11)$$

With a similar argument we obtain:

$$\delta'_Q(b, v') = \Theta_{A_Q^t}(0, x_3) \quad (10.12)$$

The equations (10.9) and (10.11) imply:

$$\delta'_Q(a, b) \subseteq \delta'_Q(a, u') \vee \delta'_Q(u', b) = \Theta_{A_Q^t}(x_3, 1).$$

Similarly, from (10.10) and (10.12), we obtain:

$$\delta'_Q(a, b) \subseteq \delta'_Q(a, v') \vee \delta'_Q(v', b) = \Theta_{A_Q^t}(0, x_3).$$

Therefore, the following containments hold:

$$\delta'_Q(a, b) \subseteq \Theta_{A_Q^t}(x_3, 1) \cap \Theta_{A_Q^t}(0, x_3) = \mathbf{0}_{A_Q^t},$$

that is, $\delta'_Q(g_{\{C_3\}, Q}(y'_{i+1}), g_{\{C_k\}, Q}(x'_k)) = \mathbf{0}_{A_Q^t}$. Hence

$$\delta'_\top(g_{\{C_3\}, \top}(y'_{i+1}), g_{\{C_k\}, \top}(x'_k)) = \mathbf{0}_{A_\top^t}, \quad \text{for all } k \in \{1, 2\}.$$

So $\delta'_\top(g_{\{C_1\}, \top}(x'_1), g_{\{C_2\}, \top}(x'_2)) = \mathbf{0}_{A_\top^t}$. Put $R = \{C_1, C_2\}$, set $x''_1 = g_{\{C_1\}, R}(x'_1)$ and $x''_2 = g_{\{C_2\}, R}(x'_2)$. The following equalities hold:

$$\begin{aligned} \mathbf{0}_{A_\top^t} &= \delta'_\top(g_{\{C_1\}, \top}(x'_1), g_{\{C_2\}, \top}(x'_2)) \\ &= (\text{Conc } f_{R, \top}^t)(\delta'_R(g_{\{C_1\}, R}(x'_1), g_{\{C_2\}, R}(x'_2))) \\ &= (\text{Conc } f_{R, \top}^t)(\delta'_R(x''_1, x''_2)). \end{aligned}$$

Moreover, as $f_{R, \top}^t$ is an embedding, the map $\text{Conc } f_{R, \top}^t$ separates 0, and so

$$\delta'_R(x''_1, x''_2) = \mathbf{0}_{A_R^t}. \quad (10.13)$$

The following equalities hold:

$$\begin{aligned} \Theta_{A_R^t}(0, x_1) &= \delta'_R(g_{\emptyset, R}(u), x''_1) && \text{by (10.8)} \\ &= \delta'_R(g_{\emptyset, R}(u), x''_1) \vee \delta'_R(x''_1, x''_2) && \text{by (10.13)} \\ &= \delta'_R(g_{\emptyset, R}(u), x''_2) \vee \delta'_R(x''_1, x''_2) \\ &= \delta'_R(g_{\emptyset, R}(u), x''_2) && \text{by (10.13)} \\ &= \Theta_{A_R^t}(0, x_2) && \text{by (10.8)}. \end{aligned}$$

For $K = N_5$, the lattice A_R^t is the three-element chain $0 < x_1 < x_2 < 1$. For $K = M_3$, the lattice A_R^t is the square $0 < x_1, x_2 < 1$. In both cases, $\Theta_{A_R^t}(0, x_1) \neq \Theta_{A_R^t}(0, x_2)$, a contradiction; thus σ is the identity map, so the congruence chain $g_{\emptyset, C_3}(u) = y'_0 < y'_1 < \dots < y'_n = g_{\emptyset, C_3}(v)$ is direct. \square

Definition 10.8. Let L be a nontrivial bounded lattice, let \mathcal{C} be a set of spanning finite chains of L . Set $A_\emptyset = \{0, 1\}$, set $A_\top = L$, and let A_P be the sublattice of L generated by $\bigcup_{C \in P} C$ for each $P \in \mathcal{J}(\mathcal{C}) - \{\emptyset, \top\}$. Let $f_{P, Q}: A_P \rightarrow A_Q$ be the inclusion map, for all $P \leq Q$ in $\mathcal{J}(\mathcal{C})$. Then $\vec{A} = (A_P, f_{P, Q} \mid P \leq Q \text{ in } \mathcal{J}(\mathcal{C}))$ is the \mathcal{C} -chain diagram of L .

Let K be a spanning partial sublattice of L . We denote by \mathcal{C}_K be the set of all spanning chains of K of length either 2 or 3. The chain diagram of K in L is the \mathcal{C}_K -chain diagram of L .

Let $C \in \mathcal{C}$, hence $A_{\{C\}} = C$. Thus $\vec{A} \upharpoonright \mathcal{J}_1(\mathcal{C}) = \vec{E}(\mathcal{C})$.

Let $P \in \mathcal{J}(\mathcal{C}) - \{\top\}$. If $P = \emptyset$, then $A_P = \{0, 1\}$ is finite. If $P = \{C\}$, then $A_P = C$ is finite. If $P = \{C, D\}$, with $\text{lh}(C) = 2 = \text{lh}(D)$ or $C \subseteq D$, the lattice A_P generated by $C \cup D$ is finite and distributive. Hence A_P is finite and distributive for each $P < \top$ in $\mathcal{J}(\mathcal{C})$.

Remark 10.9. Use the objects and notations of Definition 10.8. Let \mathcal{K} be a class of bounded lattices, assume that L and all bounded lattices generated by two elements (see Figure 2) belong to \mathcal{K} . Then \vec{A} is a diagram in \mathcal{K} .

Remark 10.10. Let $\vec{A} = (A_i, f_{i,j} \mid i \leq j \text{ in } I)$ be a diagram indexed by a poset I , let $\vec{B} = (\mathbf{B}_i, g_{i,j} \mid i \leq j \text{ in } I)$ be a partial lifting of $\text{Con}_c \circ \vec{A}$. We can assume that $\mathbf{C} \circ \vec{B} = \text{Con}_c \circ \vec{A}$ (cf. Remark 6.14). Let $i < j$ in I , let x, y in A_i , let x', y' in B_i , assume that $\delta_{\mathbf{B}_i}(x', y') = \Theta_{A_i}(x, y)$. Then:

$$\begin{aligned} \delta_{\mathbf{B}_j}(g_{i,j}(x'), g_{i,j}(y')) &= (\text{Con}_c f_{i,j})(\delta_{\mathbf{B}_i}(x', y')) \\ &= (\text{Con}_c f_{i,j})(\Theta_{A_i}(x, y)) \\ &= \Theta_{A_j}(f_{i,j}(x), f_{i,j}(y)). \end{aligned}$$

In particular, if both $f_{i,j}$ and $g_{i,j}$ are inclusion maps, then $\delta_{\mathbf{B}_j}(x', y') = \Theta_{A_j}(x, y)$.

The following lemma is a first step to prove Theorem 10.12; it handles a few particular cases of meets and joins. It also shows that different congruence chains of a partial lifting are identical provided they correspond to the same element of the chain diagram.

We refer the reader to Definition 6.13 for partial liftings.

Lemma 10.11. *Let L be a bounded lattice, let \mathcal{C} be a set of spanning chains of L , let \vec{A} be the \mathcal{C} -chain diagram of L , and let $\vec{B} = (\mathbf{B}_P, g_{P,Q} \mid P \leq Q \text{ in } \mathcal{J}(\mathcal{C}))$ be a partial lifting such that $g_{P,Q}$ is the inclusion map for all $P \leq Q$ in $\mathcal{J}(\mathcal{C})$. Let $\xi: \mathbf{C} \circ \vec{B} \rightarrow \text{Con}_c \circ \vec{A}$ be a natural equivalence, let u, v in B_\emptyset , let $x_1 \neq x_2$ in $L - \{0, 1\}$. Set $C_1 = \{0, x_1, 1\}$ and $C_2 = \{0, x_2, 1\}$. Assume $\mathcal{C} \supseteq \{C_1, C_2\}$, let $u < y_k < v$ be a direct congruence chain of $\mathbf{B}_{\{C_k\}}$ for each $k \in \{1, 2\}$. Then the following statements hold:*

- (1) $\xi_{\{C_1, C_2\}}(\delta_{\mathbf{B}_{\{C_1, C_2\}}}(y_1, y_2)) = \Theta_A(x_1, x_2)$.
- (2) The elements u, v, y_1, y_2 are pairwise distinct.
- (3) If $x_1 \wedge x_2 = 0$, then $y_1 \wedge y_2 = u$ in $B_{\{C_1, C_2\}}$.
- (4) If $x_1 \vee x_2 = 1$, then $y_1 \vee y_2 = v$ in $B_{\{C_1, C_2\}}$.
- (5) If $x_1 < x_2$ then $y_1 \wedge y_2 = y_1$ in $B_{\{C_1, C_2\}}$; that is, $y_1 < y_2$ is a chain of $\mathbf{B}_{\{C_1, C_2\}}$.
- (6) Assume that $x_1 < x_2$ and $D = \{0, x_1, x_2, 1\} \in \mathcal{C}$. Let $u < y'_1 < y'_2 < v$ be a direct congruence chain of $\mathbf{B}_{\{D\}}$. Then $y_1 = y'_1$ and $y_2 = y'_2$.

Proof. We can assume that $\mathbf{C} \circ \vec{B} = \text{Con}_c \circ \vec{A}$ and thus that ξ is the identity. Put $\delta_R = \delta_{\mathbf{B}_R}$ for each $R \in \mathcal{J}(\mathcal{C})$. Put $P = \{C_1, C_2\}$.

As $u < y_k < v$ is a direct congruence chain of $\mathbf{B}_{\{C_k\}}$ and $\{C_k\} \subseteq P$, the following equalities hold:

$$\delta_P(u, y_k) = \Theta_{A_P}(0, x_k), \quad \text{for all } k \in \{1, 2\}. \quad (10.14)$$

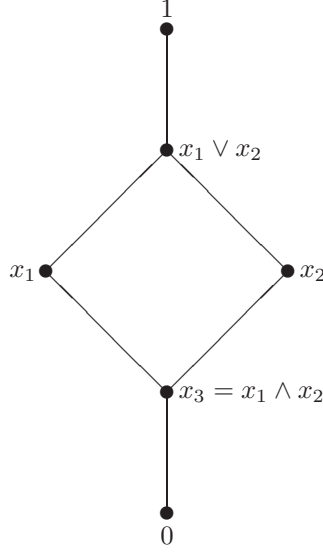
$$\delta_P(y_k, v) = \Theta_{A_P}(x_k, 1), \quad \text{for all } k \in \{1, 2\}. \quad (10.15)$$

Notice that $\Theta_{A_P}(0, x_1) \cap \Theta_{A_P}(x_1, 1) = \mathbf{0}_{A_P}$, thus:

$$\begin{aligned} \delta_P(y_1, y_2) &= \delta_P(y_1, y_2) \vee (\Theta_{A_P}(0, x_1) \cap \Theta_{A_P}(x_1, 1)) \\ &= \delta_P(y_1, y_2) \vee (\delta_P(u, y_1) \cap \delta_P(y_1, v)) \quad \text{by (10.14) and (10.15)}. \end{aligned}$$

Therefore, as $\text{Con}_c A_P$ is distributive, we obtain

$$\delta_P(y_1, y_2) = (\delta_P(y_1, y_2) \vee \delta_P(u, y_1)) \cap (\delta_P(y_1, y_2) \vee \delta_P(y_2, v)) \quad (10.16)$$

FIGURE 2. The free bounded lattice generated by x_1 and x_2 .

Moreover, the following equalities hold:

$$\begin{aligned}
 \delta_P(y_1, y_2) \vee \delta_P(u, y_1) &= \delta_P(u, y_1) \vee \delta_P(u, y_2) \\
 &= \Theta_{A_P}(0, x_1) \vee \Theta_{A_P}(0, x_2) && \text{by (10.14).} \\
 &= \Theta_{A_P}(0, x_1 \vee x_2) && \text{see Figure 2.} \quad (10.17)
 \end{aligned}$$

With a similar argument we obtain

$$\delta_P(y_1, y_2) \vee \delta_P(y_2, v) = \Theta_{A_P}(x_1 \wedge x_2, 1) \quad (10.18)$$

It follows from (10.17) and (10.18) that:

$$\delta_P(y_1, y_2) = \Theta_{A_P}(0, x_1 \vee x_2) \cap \Theta_{A_P}(x_1 \wedge x_2, 1) = \Theta_{A_P}(x_1, x_2).$$

As $u < y_k < v$ is a congruence chain, it follows that $u \neq v$, $u \neq y_k$, and $y_k \neq v$ for all $k \in \{1, 2\}$. Moreover, as $\delta_P(y_1, y_2) = \Theta_{A_P}(x_1, x_2) \neq \mathbf{0}_{A_P}$, we get $y_1 \neq y_2$.

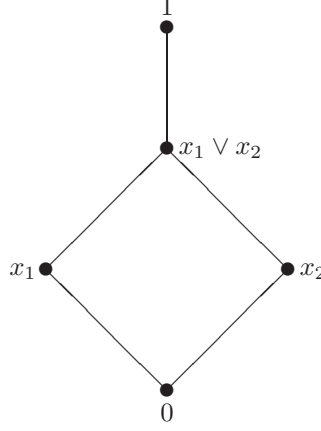
Assume that $x_1 \wedge x_2 = 0$, so the lattice A_P is (a quotient of) the lattice of Figure 3.

As $u < y_k < v$ is a congruence chain of $\mathbf{B}_{\{C_k\}}$, we get that $u \wedge y_k = u$ in B_P for each $k \in \{1, 2\}$. As $y_1, y_2 \in B_P^*$, $y_1 \wedge y_2$ is defined in B_P . Thus the following containments hold:

$$\begin{aligned}
 \delta_P(u, y_1 \wedge y_2) &\subseteq \delta_P(u, y_1) \cap \delta_P(u, y_2) \\
 &= \Theta_{A_P}(0, x_1) \cap \Theta_{A_P}(0, x_2) && \text{by (10.14)} \\
 &= \mathbf{0}_{A_P}
 \end{aligned}$$

and so $y_1 \wedge y_2 = u$ in B_P . Similarly, if $x_1 \vee x_2 = 1$, using Lemma 6.4 and (10.15), we obtain $y_1 \vee y_2 = v$ in B_P .

Now assume that $x_1 < x_2$. So the lattice A_P is a chain. Moreover, as u, y_1, y_2, v belong to B_P^* and \mathbf{B}_P is a strong gamp, the elements $y_1 \wedge y_2$, $u \wedge y_1 = u$, $u \wedge y_2 = u$, and $v \wedge y_1 = y_1$ are defined in B_P . Thus $\delta_P(y_1 \wedge y_2, y_1) \subseteq \delta_P(u, y_1 \wedge y_2) \vee \delta_P(u, y_1) =$

FIGURE 3. The lattice A_P , if $x_1 \wedge x_2 = 0$.

$\delta_P(u, y_1)$. Hence:

$$\begin{aligned} \delta_P(y_1 \wedge y_2, y_1) &\subseteq \delta_P(u, y_1) \cap \delta_P(y_2, v) \\ &= \Theta_{A_P}(0, x_1) \cap \Theta_{A_P}(x_2, 1) \quad \text{by (10.14) and (10.15)} \\ &= \mathbf{0}_{A_P} \end{aligned}$$

therefore $y_1 \wedge y_2 = y_1$ in B_P .

Put $D = \{0, x_1, x_2, 1\}$, assume that $D \in \mathcal{C}$, let $u < y'_1 < y'_2 < v$ be a direct congruence chain of $\mathbf{B}'_{\{D\}}$. Let $k \in \{1, 2\}$. The following containment holds:

$$\delta_{\{C_k, D\}}(y_k \wedge y'_k, y_k) \subseteq \delta_{\{C_k, D\}}(y_k \wedge y'_k, u) \vee \delta_{\{C_k, D\}}(u, y_k).$$

Moreover $\delta_{\{C_k, D\}}(y_k \wedge y'_k, u) = \delta_{\{C_k, D\}}(y_k \wedge y'_k, u \wedge y'_k) \subseteq \delta_{\{C_k, D\}}(u, y_k)$. Therefore:

$$\delta_{\{C_k, D\}}(y_k \wedge y'_k, y_k) \subseteq \delta_{\{C_k, D\}}(u, y_k). \quad (10.19)$$

The following containment also holds:

$$\delta_{\{C_k, D\}}(y_k \wedge y'_k, y_k) = \delta_{\{C_k, D\}}(y_k \wedge y'_k, y_k \wedge v) \subseteq \delta_{\{C_k, D\}}(y'_k, v). \quad (10.20)$$

The containments (10.19) and (10.20) imply:

$$\begin{aligned} \delta_{\{C_k, D\}}(y_k \wedge y'_k, y_k) &\subseteq \delta_{\{C_k, D\}}(u, y_k) \cap \delta_{\{C_k, D\}}(y'_k, v) \\ &= \Theta_{A_{\{C_k, D\}}}(0, x_k) \cap \Theta_{A_{\{C_k, D\}}}(x_k, 1) \\ &= \mathbf{0}_{A_{\{C_k, D\}}}. \end{aligned}$$

Thus $y_k = y_k \wedge y'_k$. Similarly $y'_k = y_k \wedge y'_k$, thus $y_k = y'_k$. \square

Given a partial lifting $\vec{\mathbf{B}}$ of $\text{Con}_c \circ \vec{\mathbf{A}}$ with enough direct congruence chains, where $\vec{\mathbf{A}}$ is the chain diagram of some partial sublattice K of L , we can now construct a partial lattice in $\vec{\mathbf{B}}$ isomorphic to K .

Theorem 10.12. *Let L be a bounded lattice and let K be a spanning partial sublattice of L . Let $\vec{\mathbf{A}}$ be the chain diagram of K in L (that is, the \mathcal{C}_K -chain diagram of L , see Definition 10.8), let $\vec{\mathbf{B}} = (\mathbf{B}_P, \mathbf{g}_{P,Q} \mid P \leq Q \text{ in } \mathcal{I}(\mathcal{C}_K))$ be a partial lifting and let $\vec{\xi}: \mathcal{C} \circ \vec{\mathbf{B}} \rightarrow \text{Con}_c \circ \vec{\mathbf{A}}$ be a natural equivalence. Let u, v in B_\emptyset . Assume that for each $C \in \mathcal{C}_K$ there exists a direct congruence chain of $\mathbf{B}_{\{C\}}$ with extremities $g_{\emptyset, \{C\}}(u)$ and $g_{\emptyset, \{C\}}(v)$.*

Given $x \in K - \{0, 1\}$, set $C_x = \{0, x, 1\}$ and let $g_{\emptyset, \{C_x\}}(u) < t_x < g_{\emptyset, \{C_x\}}(v)$ be a direct congruence chain of \mathbf{B}_{C_x} . Put:

$$\begin{aligned} h: K &\rightarrow B_{\top} \\ x &\mapsto g_{\{C_x\}, \top}(t_x) \\ 0 &\mapsto g_{\emptyset, \top}(u) \\ 1 &\mapsto g_{\emptyset, \top}(v) \end{aligned}$$

The map h is an embedding of partial lattices and $(h, \xi_{\top}): (K, K, \Theta_L, \text{Con}_c L) \rightarrow \mathbf{B}_{\top}$ is an embedding of gamps.

Proof. In this proof we set $\mathcal{C} = \mathcal{C}_K$. As $f_{P,Q}$ is an embedding of lattices and $\mathcal{J}(\mathcal{C}_K)$ has a largest element, we can assume that $g_{P,Q}$ is the inclusion map for all $P \leq Q$ in $\mathcal{J}(\mathcal{C})$. We can also assume that $\mathbf{C} \circ \vec{\mathbf{B}} = \text{Con}_c \circ \vec{\mathbf{A}}$. Denote $\delta_P = \delta_{\mathbf{B}_P}$ for each $P \in \mathcal{J}(\mathcal{C})$. With those assumptions the map h can be described in the following way

$$\begin{aligned} h: K &\rightarrow B_{\top} \\ x &\mapsto t_x \\ 0 &\mapsto u \\ 1 &\mapsto v \end{aligned}$$

It follows from Lemma 10.11(2) that the map h is one-to-one.

Let $x_1, x_2 \in K$. If either $x_1 = 0$, $x_1 = 1$, $x_2 = 0$, $x_2 = 1$, or $x_1 = x_2$, then $h(x_1 \wedge x_2) = h(x_1) \wedge h(x_2)$ and $h(x_1 \vee x_2) = h(x_1) \vee h(x_2)$. Now assume that $x_1 \neq x_2$, $x_1 \notin \{0, 1\}$, and $x_2 \notin \{0, 1\}$. Set $C_k = \{0, x_k, 1\}$, set $y_k = h(x_k)$. Hence $u < y_k < v$ is a direct congruence chain of $\mathbf{B}_{\{C_k\}}$ for all $k \in \{1, 2\}$. If $x_1 \wedge x_2 = 0$, it follows from Lemma 10.11(3) that $y_1 \wedge y_2 = u$ in $\mathbf{B}_{\{C_1, C_2\}}$, so it also holds in B_{\top} , hence $h(x_1 \wedge x_2) = h(0) = u = y_1 \wedge y_2 = h(x_1) \wedge h(x_2)$. Similarly, if $x_1 \vee x_2 = 1$, then $h(x_1 \vee x_2) = h(x_1) \vee h(x_2)$, and, if $x_1 < x_2$ then $h(x_1) \wedge h(x_2) = h(x_1)$, so $h(x_1 \wedge x_2) = h(x_1) = h(x_1) \wedge h(x_2)$ and $h(x_1 \vee x_2) = h(x_2) = h(x_1) \vee h(x_2)$.

Assume that x_1 and x_2 are incomparable and that $x_1 \wedge x_2 \in K - \{0\}$. Set $x_3 = x_1 \wedge x_2$. Put $C_k = \{0, x_k, 1\}$, set $y_k = h(x_k)$, hence $u < y_k < v$ is a direct congruence chain of $\mathbf{B}'_{\{C_k\}}$, for all $k \in \{1, 2, 3\}$. Put $D_k = \{0, x_3, x_k, 1\}$ for all $k \in \{1, 2\}$. Let $u < y'_3 < y'_2 < v$ be a direct congruence chain of $\mathbf{B}_{\{D_2\}}$ and let $u < y''_3 < y''_1 < v$ be a direct congruence chain of $\mathbf{B}_{\{D_1\}}$. It follows from Lemma 10.11(6) that $y_3 = y'_3 = y''_3$, $y_2 = y'_2$, and $y_1 = y''_1$. Hence $u < y_3 < y_k < v$ is a direct congruence chain of $\mathbf{B}_{\{D_k\}}$ for all $k \in \{1, 2\}$. Thus the following equalities hold:

$$\delta_{\{C_1, D_2\}}(y_3, y_2) = \Theta_{A_{\{C_1, D_2\}}}(x_3, x_2), \quad (10.21)$$

$$\delta_{\{C_1, D_2\}}(u, y_1) = \Theta_{A_{\{C_1, D_2\}}}(0, x_1). \quad (10.22)$$

As $u < y_3 < y_1 < v$ and $u < y_3 < y_2 < v$ are chains of $\mathbf{B}_{\{C_1, D_2\}}$, it follows that

$$\begin{aligned} \delta_{\{C_1, D_2\}}(y_3, y_1 \wedge y_2) &= \delta_{\{C_1, D_2\}}(y_3 \wedge y_2, y_1 \wedge y_2) \\ &\subseteq \delta_{\{C_1, D_2\}}(y_3, y_1) \\ &= \delta_{\{C_1, D_2\}}(u \vee y_3, y_1 \vee y_3) \\ &\subseteq \delta_{\{C_1, D_2\}}(u, y_1) \end{aligned}$$

and $\delta_{\{C_1, D_2\}}(y_3, y_1 \wedge y_2) = \delta_{\{C_1, D_2\}}(y_1 \wedge y_3, y_1 \wedge y_2) \subseteq \delta_{\{C_1, D_2\}}(y_3, y_2)$. Therefore the following containments hold:

$$\begin{aligned} \delta_{\{C_1, D_2\}}(y_3, y_1 \wedge y_2) &\subseteq \delta_{\{C_1, D_2\}}(u, y_1) \cap \delta_{\{C_1, D_2\}}(y_3, y_2) \\ &= \Theta_{A_{\{C_1, D_2\}}}(0, x_1) \cap \Theta_{A_{\{C_1, D_2\}}}(x_3, x_2) \quad \text{by (10.21) and (10.22)} \\ &= \mathbf{0}_{A_{\{C_1, D_2\}}}, \quad \text{see Figure 2.} \end{aligned}$$

Thus $y_3 = y_1 \wedge y_2$ in B_\top , so $h(x_1 \wedge x_2) = h(x_3) = y_3 = y_1 \wedge y_2 = h(x_1) \wedge h(x_2)$. Similarly if x_1 and x_2 are incomparable and $x_1 \vee x_2 \in K - \{1\}$ then $h(x_1 \vee x_2) = h(x_1) \vee h(x_2)$. Hence h is a morphism of partial lattices from K to B_\top .

Let $x, y \in K - \{0, 1\}$. Lemma 10.11(1) implies that:

$$\delta_{\{C_x, C_y\}}(h(x), h(y)) = \delta_{\{C_x, C_y\}}(t_x, t_y) = \Theta_{A_{\{C_x, C_y\}}}(x, y).$$

Therefore $\delta_\top(h(x), h(y)) = \Theta_L(x, y)$.

Let $x \in K$. From $\delta_{\{C_x\}}(u, t_x) = \Theta_{C_x}(0, x)$ it follows that $\delta_\top(h(0), h(x)) = \Theta_L(0, x)$. A similar argument gives $\delta_\top(h(x), h(1)) = \Theta_L(x, 1)$. Moreover, $\delta_\emptyset(u, v) = \Theta_{A_\emptyset}(0, 1)$, which implies in turn $\delta_\top(f(0), f(1)) = \Theta_L(0, 1)$.

Therefore $(h, \xi_\top): (K, K, \Theta_L, \text{Con}_c L) \rightarrow \mathbf{B}_\top$ is an embedding of gamps. \square

Gluing the chain diagram of a partial lattice K of L and the “directing” diagrams constructed in Lemma 10.7, we obtain a result similar to Theorem 10.12. We still need to assume the existence of enough congruence chains but these no longer need to be direct. As our directing diagrams “force” all congruence chains to be either direct or dually direct, our result is stated up to dualization.

Lemma 10.13. *Let \mathcal{K} be a class of bounded lattices closed under finite products and containing all bounded lattices generated by two elements. Assume that either M_3 or N_5 belongs to \mathcal{K} . Let $L \in \mathcal{K}$, let K be a spanning finite partial sublattice of L , such that $0, 1 \in K$ and K has at least five elements. There exists a direct system $\vec{A} = (A_P, f_{P,Q} \mid P \leq Q \text{ in } \mathcal{J}(\mathcal{C}_K))$ of \mathcal{K} (with 0,1-lattice homomorphisms) such that:*

- (1) *The lattice A_P is finite and distributive, for each $P < \top$ in $\mathcal{J}(\mathcal{C})$.*
- (2) *If L is finite then A_\top is finite, otherwise $\text{card } A_\top = \text{card } L$.*
- (3) *The equality $\vec{A} \upharpoonright \mathcal{J}_1(\mathcal{C}) = \vec{E}(\mathcal{C})$ holds.*
- (4) *Let $\vec{B} = (\mathbf{B}_P, \mathbf{g}_{P,Q} \mid P \leq Q \text{ in } \mathcal{J}(\mathcal{C}))$ be a partial lifting of $\text{Con}_c \circ \vec{A}$, let u, v in B_\emptyset . If $B'_{\{C\}}$ contains a congruence chain with extremities $g_{\emptyset, \{C\}}(u)$ and $g_{\emptyset, \{C\}}(v)$ for each $C \in \mathcal{C}$, then there exists a subgamp of a quotient of either \mathbf{B}_\top or its dual isomorphic to $(K, K, \Theta_L, \text{Con}_c L)$.*

Proof. Put $\mathcal{C} = \mathcal{C}_K$. The \mathcal{C} -chain diagram \vec{A}^0 of L is a diagram in \mathcal{K} .

Let $C_1, C_2, D \in \mathcal{C}$ be pairwise distinct chains such that C_1 and C_2 both have length 2, and either D has length 2 or $C_1, C_2 \subseteq D$.

As $M_3 \in \mathcal{K}$ or $N_5 \in \mathcal{K}$, it follows from Lemma 10.7 that there exists a diagram $\vec{H}^{C_1, C_2, D}$ of finite lattices of \mathcal{K} indexed by $\mathcal{J}(\{C_1, C_2, D\})$ such that:

- (1) *The equality $\vec{H}^{C_1, C_2, D} \upharpoonright \mathcal{J}_1(\{C_1, C_2, D\}) = \vec{E}(\{C_1, C_2, D\})$ holds.*
- (2) *Let $(\mathbf{B}_P, \mathbf{g}_{P,Q} \mid P \leq Q \text{ in } \mathcal{J}(\{C_1, C_2, D\}))$ be a partial lifting of $\text{Con}_c \circ \vec{H}^{C_1, C_2, D}$, let u, v in B_\emptyset . Assume $\mathbf{B}_{\{C_k\}}$ contains a direct congruence chain with extremities $g_{\emptyset, \{C_k\}}(u)$ and $g_{\emptyset, \{C_k\}}(v)$ for $k \in \{1, 2\}$. If $\mathbf{B}_{\{D\}}$ contains a congruence chain with extremities $g_{\emptyset, \{D\}}(u)$ and $g_{\emptyset, \{D\}}(v)$ then it is also direct.*

It follows from Lemma 10.4 that there exists a direct system $\vec{A}^{C_1, C_2, D}$ of finite lattices in \mathcal{K} indexed by $\mathcal{J}(\mathcal{C})$, such that:

$$\begin{aligned}\vec{A}^{C_1, C_2, D} \upharpoonright \mathcal{J}(\{C_1, C_2, D\}) &= \vec{H}^{C_1, C_2, D} \\ \vec{A}^{C_1, C_2, D} \upharpoonright \mathcal{J}_1(\mathcal{C}) &= \vec{E}(\mathcal{C}).\end{aligned}$$

Let T be the set with elements 0 and (C_1, C_2, D) where $C_1, C_2, D \in \mathcal{C}$ are pairwise distinct and either D has length 2 or $C_1, C_2 \subseteq D$. Let $\vec{A} = (A_P, f_{P, Q} \mid P \leq Q \text{ in } \mathcal{J}(\mathcal{C}))$ be the product of $(\vec{A}^t)_{t \in T}$ over $\mathcal{J}_1(\mathcal{C})$ (cf. Definition 10.2) and let $\vec{\pi}^t = (\pi_P^t)_{P \in \mathcal{J}(\mathcal{C})}: \vec{A} \rightarrow \vec{A}^t$ be the canonical projection for each $t \in T$ (see Definition 10.2).

It follows from Remark 10.3 that \vec{A} is a diagram in \mathcal{K} and the lattice A_P is finite for each $P < \top$ in $\mathcal{J}(\mathcal{C})$. The lattice A_\top is a finite product of finite lattices and L , so (2) holds. From the definition of the product over $\mathcal{J}_1(\mathcal{C})$ (cf. Definition 10.2), it follows that $\vec{A} \upharpoonright \mathcal{J}_1(\mathcal{C}) = \vec{A}^0 \upharpoonright \mathcal{J}_1(\mathcal{C}) = \vec{E}(\mathcal{C})$.

Let $\vec{B} = (\mathbf{B}_P, \mathbf{g}_{P, Q} \mid P \leq Q \text{ in } \mathcal{J}(\mathcal{C}))$ be a partial lifting of $\text{Con}_{\mathcal{C}} \circ \vec{A}$. We can assume that $\mathbf{C} \circ \vec{B} = \text{Con}_{\mathcal{C}} \circ \vec{A}$. Let $u, v \in B_\emptyset$. Suppose that $\mathbf{B}_{\{C\}}$ contains a congruence chain with extremities $g_{\emptyset, \{C\}}(u)$ and $g_{\emptyset, \{C\}}(v)$ for each $C \in \mathcal{C}$. In the rest of our proof, all congruence chains of $\mathbf{B}_{\{C\}}$ will have extremities $g_{\emptyset, \{C\}}(u)$ and $g_{\emptyset, \{C\}}(v)$.

Given $t \in T$, we put $\vec{I}^t = \ker_0 \vec{\pi}^t$, that is

$$I_P^t = \ker_0 \mathbf{G}(\pi_P^t) = \ker_0(\text{Con}_{\mathcal{C}} \pi_P^t) = (\text{Con}_{\mathcal{C}} A_P) \downarrow \ker \pi_P^t, \quad \text{for each } P \in \mathcal{J}(\mathcal{C}).$$

Let $\vec{\chi}^t: (\text{Con}_{\mathcal{C}} \circ \vec{A}) / \vec{I}^t \rightarrow \text{Con}_{\mathcal{C}} \circ \vec{A}^t$ induced by $\text{Con}_{\mathcal{C}} \circ \vec{\pi}^t$. It follows from Lemma 3.13 that $\vec{\chi}^t$ is a natural equivalence. Moreover $I_P^t = \{0\}$, $A_P = A_P^t$, and $\chi_P^t = \text{id}_{\text{Con}_{\mathcal{C}} A_P}$ for each $P \in \mathcal{J}_1(\mathcal{C})$ and each $t \in T$.

Claim. *Let $C_1, C_2 \in \mathcal{C}$ be distinct chains of length 2, let $D \in \mathcal{C} - \{C_1, C_2\}$ such that either D has length 2 or $C_1, C_2 \subseteq D$. Assume that $\mathbf{B}_{\{C_k\}}$ contains a direct congruence chain for $k = 1, 2$. Then $\mathbf{B}_{\{D\}}$ contain a direct congruence chain.*

Proof of Claim. Put $t = (C_1, C_2, D)$. The restriction $(\vec{B} / \vec{I}^t) \upharpoonright \mathcal{J}(\{C_1, C_2, D\})$ is a partial lifting of $\vec{A}^t \upharpoonright \mathcal{J}(\{C_1, C_2, D\}) = \vec{H}^t$ induced by the restriction of $\vec{\chi}^t$. As $\chi_P^t = \text{id}_{\text{Con}_{\mathcal{C}} A_P}$ and $I_P^t = \{0\}$, it follows that the direct congruence chains of \mathbf{B}_P / I_P^t are the same for both structures for each $P \in \{\{C_1\}, \{C_2\}, \{C_3\}\}$. It follows from (2) that every congruence chain of $\mathbf{B}_{\{D\}}$ is direct. \square Claim.

Let C be a chain of length 2. A congruence chain of $\mathbf{B}_{\{C\}}$ is either direct or dually direct. As K has at least five elements, there are at least three chains of length 2 in \mathcal{C} . Therefore, up to changing \vec{B} to its dual, we can take $C_1, C_2 \in \mathcal{C}$ distinct of length 2, such that $\mathbf{B}_{\{C_k\}}$ has a direct congruence chain. Let $D \in \mathcal{J}(\mathcal{C}) - \{C_1, C_2\}$ be a chain of length 2. It follows from the Claim that all congruence chains of $\mathbf{B}_{\{D\}}$ are direct. Therefore $\mathbf{B}_{\{D\}}$ has a direct congruence chain for every $D \in \mathcal{J}(\mathcal{C})$ of length 2.

Let $D = \{0, x_1, x_2, 1\} \in \mathcal{J}(\mathcal{C})$ be a chain of length 3, put $C_k = \{0, x_k, 1\}$ for $k = 1, 2$. As $\mathbf{B}_{\{C_k\}}$ has a direct congruence chain for $k = 1, 2$, it follows from the Claim that $\mathbf{B}_{\{D\}}$ has a direct congruence chain.

Thus, applying Theorem 10.12, to \vec{B} / \vec{I}^0 which is a partial lifting of $\text{Con}_{\mathcal{C}} \circ \vec{A}^0$, we obtain a subgamp of a $\mathbf{B}_\top / I_\top^0$ isomorphic to $(K, K, \Theta_L, \text{Con}_{\mathcal{C}} L)$. \square

The following lemma gives a way to find a partial sublattice of some lattice in a variety but not in another.

Lemma 10.14. *Let \mathcal{V} be a variety of bounded lattices, let \mathcal{W} be variety of lattices. If $\mathcal{V} \not\subseteq \mathcal{W}$ and $\mathcal{V} \not\subseteq \mathcal{W}^d$ then there are a countable bounded lattice $L \in \mathcal{V}$ and a finite spanning partial sublattice K of L such that K is not a partial sublattice of any lattice of $\mathcal{W} \cup \mathcal{W}^d$.*

Proof. Assume that $\mathcal{V} \not\subseteq \mathcal{W}$ and $\mathcal{V} \not\subseteq \mathcal{W}^d$. Let $t_1 = t_2$ be an identity satisfied in \mathcal{W} but not satisfied in \mathcal{V} , let $t'_1 = t'_2$ be an identity satisfied in \mathcal{W}^d but not in \mathcal{V} . Let $L \in \mathcal{V}$ be a countable bounded lattice that fails both $t_1 = t_2$ and $t'_1 = t'_2$, let K be a finite spanning partial sublattice of L which fails both $t_1 = t_2$ and $t'_1 = t'_2$. As K does not satisfy $t_1 = t_2$, it is not a partial sublattice of any lattice of \mathcal{W} . Similarly K it is not a partial sublattice of any lattice of \mathcal{W}^d . \square

Lemma 10.14 and Lemma 10.13 are the mains tools of this paper to construct a diagram liftable in a variety and not in another one.

11. A LARGER DIAGRAM

The aim of this section is, given a diagram \vec{A} indexed by a poset I , to construct a new diagram \vec{A}' (cf. Lemma 11.1) indexed by a larger poset J (cf. Definition 8.5), such that the existence of a partial lifting of $\text{Con}_c \circ \vec{A}'$ in some “good” variety implies the existence of a partial lifting of $\text{Con}_c \circ \vec{A}$ with many congruence chains (cf. Lemma 11.2).

Lemma 11.1. *Let \mathcal{K} be a class of bounded lattices, let I be a poset with a smallest element 0 , let $\vec{A} = (A_i, f_{i,j})_{i \leq j \text{ in } I}$ be a direct system of \mathcal{K} such that $A_0 = \mathbf{2}$, let $X \subseteq I - \{0\}$ such that A_x is a finite chain of length at least 2, for each $x \in X$, and let $\alpha \leq \omega$ be an ordinal. Put*

$$P_x = \{p: A_x \rightarrow \mathbf{2} \mid p \text{ is isotone and preserves bounds}\}, \quad \text{for each } x \in X.$$

Set $\vec{P} = (P_x)_{x \in X}$ and $J = I \boxtimes_{\vec{P}} \alpha$, and denote by T its associated tree. Put:

$$A'_{(t,i)} = A_i, \quad \text{for each } (t,i) \in J.$$

Let $t = (n, \vec{x}, \vec{p}) \in T$, let $\mathbf{a} = (t \upharpoonright m, i) \leq \mathbf{b} = (t, j)$ in J . If $m = n$ we put $f'_{\mathbf{a},\mathbf{b}} = f_{i,j}$. If $m < n$ we put $f'_{\mathbf{a},\mathbf{b}} = f_{0,j} \circ p_m \circ f_{i,x_m}$. Then $\vec{A}' = (A'_{\mathbf{a}}, f'_{\mathbf{a},\mathbf{b}} \mid \mathbf{a} \leq \mathbf{b} \text{ in } J)$ is a J -indexed diagram in \mathcal{K} .

Proof. Let $\mathbf{a} \leq \mathbf{b} \leq \mathbf{c}$ in J . There are $t = (n, \vec{x}, \vec{p}) \in T$, $\ell \leq m \leq n$, and $i, j, k \in I$ such that $\mathbf{a} = (t \upharpoonright \ell, i)$, $\mathbf{b} = (t \upharpoonright m, j)$, and $\mathbf{c} = (t, k)$. If $\ell = m = n$, then the following equalities hold:

$$f'_{\mathbf{b},\mathbf{c}} \circ f'_{\mathbf{a},\mathbf{b}} = f_{j,k} \circ f_{i,j} = f_{i,k} = f'_{\mathbf{a},\mathbf{c}}.$$

If $\ell < m = n$, then the following equalities hold:

$$f'_{\mathbf{b},\mathbf{c}} \circ f'_{\mathbf{a},\mathbf{b}} = f_{j,k} \circ f_{0,j} \circ p_\ell \circ f_{i,x_\ell} = f_{0,k} \circ p_\ell \circ f_{i,x_\ell} = f'_{\mathbf{a},\mathbf{c}}.$$

If $\ell = m < n$, then the following equalities hold:

$$f'_{\mathbf{b},\mathbf{c}} \circ f'_{\mathbf{a},\mathbf{b}} = f_{0,k} \circ p_m \circ f_{j,x_m} \circ f_{i,j} = f_{0,k} \circ p_m \circ f_{i,x_m} = f'_{\mathbf{a},\mathbf{c}}.$$

Assume $\ell < m < n$. Notice that $p_m \circ f_{j,x_m} \circ f_{0,j}: \mathbf{2} \rightarrow \mathbf{2}$ is the identity map (as it preserves bounds). Therefore the following equalities hold:

$$f'_{\mathbf{b},\mathbf{c}} \circ f'_{\mathbf{a},\mathbf{b}} = f_{0,k} \circ p_m \circ f_{j,x_m} \circ f_{0,j} \circ p_\ell \circ f_{i,x_\ell} = f_{0,k} \circ p_\ell \circ f_{i,x_\ell} = f'_{\mathbf{a},\mathbf{c}}.$$

Thus $\vec{A}' = (A'_{\mathbf{a}}, f'_{\mathbf{a},\mathbf{b}} \mid \mathbf{a} \leq \mathbf{b} \text{ in } J)$ is a J -indexed diagram in \mathcal{K} . \square

For the statement of the following lemma, we remind the reader that partial liftings are introduced in Definition 6.13.

Lemma 11.2. *We use the notation of Lemma 11.1, with $\alpha = \omega$. Let \mathcal{V} be a variety of lattices such that every simple lattice in \mathcal{V} contains a prime interval. Let $(\mathbf{B}_{\mathbf{a}}, g_{\mathbf{a},\mathbf{b}} \mid \mathbf{a} \leq \mathbf{b} \text{ in } J)$ be a partial lifting of $\text{Con}_c \circ \vec{A}'$ such that $\mathbf{B}_{\mathbf{a}}$ is finite for each $\mathbf{a} \in J^\neq$. Then there are $t \in T$ and u, v in $B_{(t,0)}$ such that for each $x \in X$, the gamp $\mathbf{B}_{(t,x)}$ has a congruence chain with extremities $g_{(t,0),(t,x)}(u)$ and $g_{(t,0),(t,x)}(v)$.*

Proof. Denote by T the tree associated to $I \boxtimes_{\vec{P}} \alpha$. Let $t = (n, \vec{x}, \vec{p}) \in T$; then $J_t = \{(t, i) \mid i \in I\}$ is a subposet of J . Moreover, the assignment $I \rightarrow J_t, i \mapsto (t, i)$ is an isomorphism, and it induces an isomorphism of the diagrams $\vec{A}' \upharpoonright J_t$ and \vec{A} .

Assume that for all $t \in T$ and for every chain $u < v$ of $\mathbf{B}_{(t,0)}$, there exists $x \in X$ such that $\mathbf{B}_{(t,x)}$ has no congruence chain with extremities $g_{(t,0),(t,x)}(u)$ and $g_{(t,0),(t,x)}(v)$.

Our aim is to construct a sequence $\vec{x} = (x_k)_{k < \omega}$ of X , $\vec{p} \in \prod_{k < \omega} P_{x_k}$, and $\mathbf{a}_n = (n, \vec{x} \upharpoonright n, \vec{p} \upharpoonright n, 0)$ for each $n < \omega$, such that for each $m < \omega$ and for each chain $u < v$ in $\mathbf{B}_{\mathbf{a}_m}$, there are $n > m$ and $z \in B_{\mathbf{a}_n}^*$ such that $g_{\mathbf{a}_m, \mathbf{a}_n}(u) < z < g_{\mathbf{a}_m, \mathbf{a}_n}(v)$ is a chain of $\mathbf{B}_{\mathbf{a}_n}$.

We can assume that $\mathbf{C} \circ \vec{B} = \text{Con}_c \circ \vec{A}'$ (cf. Remark 6.14). We set $\delta_{\mathbf{a}} = \delta_{\mathbf{B}_{\mathbf{a}}}$, for each $\mathbf{a} \in J$.

Assume having already constructed $\vec{x} \in X^n$ and $\vec{p} \in P_{x_0} \times \cdots \times P_{x_{n-1}}$ for some $n < \omega$. Set $\mathbf{a}_m = (m, \vec{x} \upharpoonright m, \vec{p} \upharpoonright m, 0)$ for each $m \leq n$. As $A'_{\mathbf{a}_n} = A_0 = \mathbf{2}$ there exists a non-zero congruence of $A'_{\mathbf{a}_n}$. Moreover, $\mathbf{B}_{\mathbf{a}_n}$ is distance-generated with chains, thus there exist a chain $u < v$ of $\mathbf{B}_{\mathbf{a}_n}$. Using the finiteness of $\mathbf{B}_{\mathbf{a}_n}$ we can construct a covering $u' \prec v'$ in $\mathbf{B}_{\mathbf{a}_n}$.

Let $m \leq n$ minimal such that there exist $u \prec v$ in $\mathbf{B}_{\mathbf{a}_m}$ with $g_{\mathbf{a}_m, \mathbf{a}_n}(u) \prec g_{\mathbf{a}_m, \mathbf{a}_n}(v)$. Let u, v be such elements, thus $u \neq v$, so $\delta_{\mathbf{a}_n}(u, v) \neq \mathbf{0}_{A_{\mathbf{a}_n}}$, however $A_{\mathbf{a}_n} = \mathbf{2}$, therefore $\delta_{\mathbf{a}_n}(u, v) = \mathbf{1}_{A_{\mathbf{a}_n}}$. Put $u' = g_{\mathbf{a}_m, \mathbf{a}_n}(u)$, and $v' = g_{\mathbf{a}_m, \mathbf{a}_n}(v)$. The following equalities hold:

$$\delta_{\mathbf{a}_n}(u', v') = (\text{Con}_c f_{\mathbf{a}_m, \mathbf{a}_n})(\delta_{\mathbf{a}_m}(u, v)) = (\text{Con}_c f_{\mathbf{a}_m, \mathbf{a}_n})(\mathbf{1}_{A_{\mathbf{a}_m}}) = \mathbf{1}_{A_{\mathbf{a}_n}}$$

Put $t = (n, \vec{x}, \vec{p})$. Let $y \in X$ such that $\mathbf{B}_{(t,y)}$ has no congruence chain with extremities $g_{\mathbf{a}_n, (t,y)}(u')$ and $g_{\mathbf{a}_n, (t,y)}(v')$. Set $\mathbf{y} = (t, y)$. Put $S = \text{At Con}_c A_{\mathbf{y}}$, we recall that $A_{\mathbf{y}} = A_y$ is a chain. So the following equalities hold:

$$\delta_{\mathbf{a}_n}(g_{\mathbf{a}_n, \mathbf{y}}(u'), g_{\mathbf{a}_n, \mathbf{y}}(v')) = (\text{Con}_c f_{0, \mathbf{y}})(\delta_{\mathbf{a}_n}(u', v')) = (\text{Con}_c f_{0, \mathbf{y}})(\mathbf{1}_{A_0}) = \mathbf{1}_{A_{\mathbf{y}}} = \bigvee S$$

As $g_{\mathbf{a}_n, \mathbf{y}}$ is congruence-cutttable with chains, there exists a chain $g_{\mathbf{a}_n, \mathbf{y}}(u') = u_0 < u_1 < \cdots < u_\ell = g_{\mathbf{a}_n, \mathbf{y}}(v')$ of $\mathbf{B}_{\mathbf{y}}$ such that for each $k < \ell$ there exists $\alpha \in S$ such that $\delta_{\mathbf{y}}(u_k, u_{k+1}) \leq \alpha$; moreover, as α is an atom, $\delta_{\mathbf{y}}(u_k, u_{k+1}) = \alpha$.

Let $\alpha \in S$. If there does not exist $k < \ell$ such that $\alpha = \delta_{\mathbf{y}}(u_k, u_{k+1})$, then:

$$\bigvee (S - \{\alpha\}) \geq \bigvee_{k < \ell} \delta_{\mathbf{y}}(u_k, u_{k+1}) \geq \delta_{\mathbf{y}}(u_0, u_\ell) = \mathbf{1}_{A_{\mathbf{y}}} \geq \alpha,$$

a contradiction as S is the set of atoms of the Boolean lattice $\text{Con}_c A_{\mathbf{y}}$. Hence for each $\alpha \in S$ there exists $k < \ell$ such that $\alpha = \delta_{\mathbf{y}}(u_k, u_{k+1})$. However, $g_{\mathbf{a}_n, \mathbf{y}}(u') = u_0 < u_1 < \cdots < u_\ell = g_{\mathbf{a}_n, \mathbf{y}}(v')$ is not a congruence chain of $\mathbf{B}_{\mathbf{y}}$, so there are $k <$

$k' < \ell$ and $\alpha \in S$ such that $\alpha = \delta_{\mathbf{y}}(u_k, u_{k+1}) = \delta_{\mathbf{y}}(u_{k'}, u_{k'+1})$. Let $p_n: A_{\mathbf{y}} \twoheadrightarrow A_0 = \mathbf{2}$ be isotone such that $(\text{Conc } p_n)(\alpha) = \mathbf{1}_{A_0}$.

Put $x_n = \mathbf{y}$, $\mathbf{a}_{n+1} = (n+1, 0, (\vec{x}, x_n), (\vec{p}, p_n))$, $u'' = g_{\mathbf{a}_m, \mathbf{a}_{n+1}}(u)$, $v'' = g_{\mathbf{a}_m, \mathbf{a}_{n+1}}(v)$, and $z = g_{\mathbf{y}, \mathbf{a}_{n+1}}(u_{k+1})$. Notice that $u'' = g_{\mathbf{a}_m, \mathbf{a}_{n+1}}(u) = g_{\mathbf{y}, \mathbf{a}_{n+1}}(u_0)$, $f'_{\mathbf{y}, \mathbf{a}_{n+1}} = p_{n+1}$, and $\delta_{\mathbf{a}_{n+1}}(u_0, u_{k+1}) \geq \alpha$, thus

$$\begin{aligned} \delta_{\mathbf{a}_{n+1}}(u'', z) &= \delta_{\mathbf{a}_{n+1}}(g_{\mathbf{y}, \mathbf{a}_{n+1}}(u_0), g_{\mathbf{y}, \mathbf{a}_{n+1}}(u_{k+1})) \\ &= (\text{Conc } f'_{\mathbf{y}, \mathbf{a}_{n+1}})(\delta_{\mathbf{a}_{n+1}}(u_0, u_{k+1})) \\ &\geq (\text{Conc } p_{n+1})(\alpha) \\ &= \mathbf{1}_{A_0}. \end{aligned}$$

Hence $z \neq u''$. Similarly $z \neq v''$, so $u'' = g_{\mathbf{a}_m, \mathbf{a}_{n+1}}(u) < z < g_{\mathbf{a}_m, \mathbf{a}_{n+1}}(v) = v''$ is a chain of $\mathbf{B}_{\mathbf{a}_{n+1}}$.

Arguing by induction, we thus construct a subdiagram

$$(\mathbf{B}_{\mathbf{a}_m}, \mathbf{g}_{\mathbf{a}_m, \mathbf{a}_n} \mid m \leq n < \omega)$$

of $\vec{\mathbf{B}}$. It follows from the construction that for all $m < \omega$ and all $u \prec v$ in $\mathbf{B}_{\mathbf{a}_m}$ there are $n > m$ and a chain $g_{\mathbf{a}_m, \mathbf{a}_n}(u) < z < g_{\mathbf{a}_m, \mathbf{a}_n}(v)$ of $\mathbf{B}_{\mathbf{a}_n}$.

Let $\vec{\mathbf{D}} = (\mathbf{2}, \text{id}_{\mathbf{2}})_{m \leq n < \omega}$. It is easy to check that $(\mathbf{B}_{\mathbf{a}_m}, \mathbf{g}_{\mathbf{a}_m, \mathbf{a}_n} \mid m \leq n < \omega)$ is a partial lifting of $\vec{\mathbf{D}}$. Hence it follows from Lemma 6.16 that

$$\mathbf{B} = \varinjlim (\mathbf{B}_{\mathbf{a}_m}, \mathbf{g}_{\mathbf{a}_m, \mathbf{a}_n} \mid m \leq n < \omega)$$

is a lattice in \mathcal{V} and $\text{Conc } \mathbf{B} \cong \mathbf{2}$, that is, \mathbf{B} is simple. By construction, there does not exist $u \prec v$ in \mathbf{B} ; a contradiction.

So there are $t \in T$ and $u < v$ in B_t such that for each $x \in X$, the partial lattice $B'_{(t,x)}$ has a congruence chain with extremities $g_{(t,0), (t,x)}(u)$ and $g_{(t,0), (t,x)}(v)$. \square

Theorem 11.3. *Let \mathcal{K} be a class of bounded lattices closed under finite products and directed colimits such that either $M_3 \in \mathcal{K}$ or $N_5 \in \mathcal{K}$ and every bounded lattice generated by two elements belongs to \mathcal{K} . Let $L \in \mathcal{K}$ and let K be a finite partial sublattice of L . There exists a lattice $A \in \mathcal{K}$ such that the following statements hold:*

- $\text{card } A = \aleph_2 + \text{card } L$.
- *Let B be a lattice such that every simple lattice in $\mathbf{Var } B$ contains a prime interval. If $\text{Conc } B \cong \text{Conc } A$, then K is a partial sublattice of a quotient of either B or its dual.*

Proof. Denote $\mathcal{C} = \mathcal{C}_K$. It follows from Lemma 10.13 that there exists a direct system $\vec{A} = (A_P, f_{P,Q})_{P \leq Q \text{ in } \mathcal{J}(\mathcal{C})}$ of \mathcal{K} (with 0, 1-lattice homomorphisms) such that:

- (1) The lattice A_P is finite, for each $P < \top$ in $\mathcal{J}(\mathcal{C})$.
- (2) $\text{card } A_{\top} \leq \text{card } L + \aleph_0$.
- (3) The equality $\vec{A} \upharpoonright \mathcal{J}_1(\mathcal{C}) = \vec{E}(\mathcal{C})$ holds.
- (4) Let $(\mathbf{B}_P, \mathbf{g}_{P,Q} \mid P \leq Q \text{ in } \mathcal{J}(\mathcal{C}))$ be a partial lifting of $\text{Conc } \circ \vec{A}$, let u, v in B_{\emptyset} . If $\mathbf{B}_{\{C\}}$ contains a congruence chain with extremities $g_{\emptyset, \{C\}}(u)$ and $g_{\emptyset, \{C\}}(v)$ for each $C \in \mathcal{C}$, then there exists a one-to-one morphism of partial lattices from K to a quotient of either B_{\top} or its dual.

Put $I = \mathcal{J}(\mathcal{C})$, set $X = \{\{C\} \mid C \in \mathcal{C}\}$, put $P_{\{C\}} = \{p: C \rightarrow \mathbf{2} \mid p \text{ is isotone}\}$, for each $C \in \mathcal{C}$, as in Lemma 11.1. Put $J = I \boxtimes_{\overline{P}} \omega$. Let \vec{A}' as in Lemma 11.1. Then A'_a is finite for each $a \in J^=$ and $\text{card } A'_a \leq \text{card } L + \aleph_0$ for each $a \in \text{Max } J$.

It follows from Corollary 8.9 that there exists an \aleph_0 -lifter (U, \mathbf{U}) of J such that $\text{card } U = \aleph_2$. Put $A = \mathbf{F}(U) \otimes \vec{A}'$ (cf. Remark 9.2). The following inequality holds:

$$\text{card } A \leq \text{card } U + \sum_{j \in J} \text{card } A'_j = \aleph_2 + \text{card } L.$$

Let B be a lattice such that $\text{Conc } B \cong \text{Conc } A = \text{Conc } (\mathbf{F}(U) \otimes \vec{A}')$ and every simple lattice in $\mathcal{W} = \mathbf{Var } B$ contains a prime interval. Thus it follows from Theorem 9.3 and Remark 9.4 that there exists $\vec{B} = (\mathbf{B}_a, g_{a,b} \mid a \leq b \text{ in } J)$ a (lattice) partial lifting of $\text{Conc } \circ \vec{A}'$, such that \mathbf{B}_a is finite for each $a \in J^=$, and \mathbf{B}_a is a quotient of $\mathbf{G}(B)$ for each $a \in \text{Max } J$.

Now Lemma 11.2 implies that there exists $(b_i)_{i \in I}$ in J such that the diagram $(\mathbf{B}_{b_i}, g_{b_i, b_j} \mid i \leq j \text{ in } I)$ is a partial lifting of $\text{Conc } \circ \vec{A}$, and there exists a chain $u < v$ in \mathbf{B}_{b_0} such that \mathbf{B}_{b_x} has a congruence chain with extremities $g_{b_0, b_x}(u)$ and $g_{b_0, b_x}(v)$ for each $x \in X$. So it follows from (4) that K is a partial sublattice of a quotient of either \mathbf{B}_{b_\top} or its dual. Notice that b_\top is maximal in J , so \mathbf{B}_{b_\top} is a quotient of $\mathbf{G}(B)$. \square

Remark 11.4. The lattice A constructed in Theorem 11.3 is a directed colimit of finite products of lattices each of which is either L , M_3 , N_5 , or a 0,1-lattice generated by two elements (cf. Figures 1 and 2).

Corollary 11.5. *Let \mathcal{V} be a variety of bounded lattices and let \mathcal{W} be a variety of lattices such that every simple lattice in \mathcal{W} contains a prime interval. Let $n \geq 4$, let L be a congruence n -permutable lattice in \mathcal{V} . If $L \notin \mathcal{W} \cup \mathcal{W}^d$, then there exists a congruence n -permutable lattice $A \in \mathcal{V}$ such that $\text{Conc } A \not\cong \text{Conc } B$ for any $B \in \mathcal{W}$.*

Proof. With a proof similar to the one of Lemma 10.14, we can find a finite spanning partial sublattice K of L such that K does not embed into any lattice of $\mathcal{W} \cup \mathcal{W}^d$. Changing L to one of its congruence n -permutable sublattices that contains K , we can assume that L is countable.

Denote by \mathcal{K} the class of all congruence n -permutable lattices in \mathcal{V} . Notice that if \mathcal{V} contains neither M_3 nor N_5 , then \mathcal{V} is distributive, so the only possibility for \mathcal{W} is the trivial variety and the result holds in that case. Thus we can assume that either $M_3 \in \mathcal{K}$ or $N_5 \in \mathcal{K}$. Moreover, as $n \geq 4$, every bounded lattice generated by two elements is congruence n -permutable and so it belongs to \mathcal{K} . Theorem 11.3 implies that there exists $A \in \mathcal{V}$ such that $\text{card } A = \aleph_2$ and $\text{Conc } A$ has no lifting in \mathcal{W} . \square

Theorem 11.6. *Let \mathcal{V} be a variety of bounded lattices and let \mathcal{W} be a variety of lattices. If every simple lattice in \mathcal{W} contains a prime interval, then one of the following statements holds:*

- (1) $\text{crit}(\mathcal{V}; \mathcal{W}) \leq \aleph_2$.
- (2) $\mathcal{V} \subseteq \mathcal{W}$.
- (3) $\mathcal{V} \subseteq \mathcal{W}^d$.

Proof. We can assume that $M_3 \in \mathcal{V}$ or $N_5 \in \mathcal{V}$ (the result is trivial otherwise). Assume that $\mathcal{V} \not\subseteq \mathcal{W}$ and $\mathcal{V} \not\subseteq \mathcal{W}^d$. It follows from Lemma 10.14 that there are a

countable lattice $L \in \mathcal{V}$ and a finite spanning partial sublattice K of L such that K is not a partial sublattice of any lattice of $\mathcal{W} \cup \mathcal{W}^d$. Let A be the lattice constructed in Theorem 11.3, so $\text{card } A = \aleph_2$ and $\text{Con}_c A$ cannot have a lifting in \mathcal{W} , hence $\text{crit}(\mathcal{V}; \mathcal{W}) \leq \aleph_2$. \square

Remark 11.7. In the proof of Theorem 11.6, if $\mathcal{V} \not\subseteq \mathcal{W}$ and $\mathcal{V} \not\subseteq \mathcal{W}^d$, we construct a diagram \vec{A}' in \mathcal{V} indexed by a poset J , such that $\text{Con}_c \circ \vec{A}'$ has no partial lifting in \mathcal{W} , in particular it has no lifting in \mathcal{W} . Moreover the poset I is lower finite and A'_i is finite for each $i \in I^=$ and countable otherwise.

If we assume that simple lattices of \mathcal{W} have length bounded by some integer α , then we can “cut” the diagram \vec{A}' , taking $J = I \boxtimes_{\vec{P}} \alpha$. Thus we obtain a finite diagram \vec{A}' such that $\text{Con}_c \circ \vec{A}'$ has no lifting in \mathcal{W} . If we also assume that \mathcal{V} is finitely generated, then \vec{A}' is a finite diagram of finite lattices, since we can choose the partial lattice K as a sublattice of a finite lattice L in Lemma 10.14.

This partially answers [14, Problem 5], in particular for finitely generated varieties of lattices.

Corollary 11.8. *Let \mathcal{V} and \mathcal{W} be varieties of lattices such that every simple lattice in \mathcal{W} contains a prime interval (this holds, in particular, in case \mathcal{W} is finitely generated). Then either $\text{crit}(\mathcal{V}; \mathcal{W}) \leq \aleph_2$ or $\text{Con}_c \mathcal{V} \subseteq \text{Con}_c \mathcal{W}$.*

This also solves [14, Problem 6]. Notice that this question is misformulated: indeed, denoting by K the chain of length 2 and by L the chain of length 3, then $\text{Con}_c(\mathbf{Var } K) = \text{Con}_c(\mathbf{Var } L)$, but $K \not\cong L$ and $K \not\cong L^d$. However, the following statement holds.

Corollary 11.9. *Let K and L be finite subdirectly irreducible lattices. If $\text{Con}_c(\mathbf{Var } K) = \text{Con}_c(\mathbf{Var } L)$, then either $K \cong L$ or $K \cong L^d$.*

Proof. As $\text{Con}_c(\mathbf{Var } K) \subseteq \text{Con}_c(\mathbf{Var } L)$, it follows from Theorem 11.6 that either $\mathbf{Var } K \subseteq \mathbf{Var } L$ or $\mathbf{Var } K \subseteq \mathbf{Var } L^d$. Similarly, as $\text{Con}_c(\mathbf{Var } K) \supseteq \text{Con}_c(\mathbf{Var } L)$, it follows from Theorem 11.6 that either $\mathbf{Var } L \subseteq \mathbf{Var } K$ or $\mathbf{Var } L \subseteq \mathbf{Var } K^d$.

Up to changing L to its dual, we can assume that $\mathbf{Var } K \subseteq \mathbf{Var } L$. If $\mathbf{Var } L \subseteq \mathbf{Var } K^d$, then the following containments hold:

$$\mathbf{Var } L \subseteq \mathbf{Var } K^d \subseteq \mathbf{Var } L^d \subseteq \mathbf{Var } (K^d)^d = \mathbf{Var } K.$$

In both cases, $\mathbf{Var } L \subseteq \mathbf{Var } K$, thus $\mathbf{Var } L = \mathbf{Var } K$.

As K and L are both finite and subdirectly irreducible, it follows from Jónsson’s Lemma that K is a quotient of a sublattice of L and that L is a quotient of a sublattice of K . Therefore we can conclude $K \cong L$. \square

Corollary 11.10. *Let \mathcal{V} and \mathcal{W} be varieties of lattices such that every simple lattice in \mathcal{W} contains a prime interval. If $\text{Con}_c \mathcal{V} \subseteq \text{Con}_c \mathcal{W}$ then either $\mathcal{V} \subseteq \mathcal{W}$ or $\mathcal{V} \subseteq \mathcal{W}^d$.*

Baker proves in [1] that there exist continuously many varieties of locally finite modular lattices. By using products of projective planes of different characteristics, we obtain the corollary below.

Corollary 11.11. *There are continuously many congruence classes of locally finite varieties of modular lattices.*

By using [4, Theorem 3.11], which gives a lower bound for some critical points, we obtain the following result.

Corollary 11.12. *Let \mathcal{V} be a locally finite variety of modular lattices in which every simple lattice has length at most n and let F be a field. If $\mathcal{V} \not\subseteq \mathbf{Var}(\text{Sub } F^n)$, then $\text{crit}(\mathcal{V}; \mathbf{Var}(\text{Sub } F^n)) = \aleph_2$.*

There is an algorithm that, given finite lattices K, L decides whether $\mathbf{Var } K \subseteq \mathbf{Var } L$. Therefore we can partially solve [6, Problem 4] for lattices and critical point \aleph_2 .

Corollary 11.13. *There is an algorithm that, given finite lattices K, L , decides whether $\text{crit}(\mathbf{Var } K; \mathbf{Var } L) \leq \aleph_2$ or $\text{crit}(\mathbf{Var } K; \mathbf{Var } L) = \infty$.*

12. FUNCTORIAL RESULTS

In this section we study the existence of a functor $\Psi: \mathcal{V} \rightarrow \mathcal{W}$ between varieties of lattices such that $\text{Con}_c \circ \Psi \cong \text{Con}_c$. We prove that such a functor exists if and only if either $\mathcal{V} \subseteq \mathcal{W}$ or $\mathcal{V} \subseteq \mathcal{W}^d$. However, the functor itself does not need to be equivalent to either the inclusion or dualization. The paper [7] provides many such examples, using the *lattice tensor product* \boxtimes . Let S be a simple bounded lattice, denote by \mathcal{L} the variety of all lattices. Then:

$$\begin{aligned} \Psi: \mathcal{L} &\rightarrow \mathcal{L} \\ A &\mapsto S \boxtimes A, && \text{for each lattice } A \in \mathcal{L}, \\ f &\mapsto S \boxtimes f, && \text{for each morphism of lattices } f, \end{aligned}$$

is a functor and $\text{Con}_c \circ \Psi$ is naturally equivalent to Con_c .

The following lemma grants the existence of congruence chains (cf. Definition 10.5 and Remark 10.6).

Lemma 12.1. *Let A and B be lattices, let $f: A \rightarrow B$ be a morphism of lattices, and let $\pi_0, \pi_1: B \rightarrow A$ be morphisms of lattices. Assume that $\pi_0 \circ f = \pi_1 \circ f = \text{id}_A$ and $\text{Con}_c B \cong \mathbf{2}^2$ with coatoms $\ker \pi_0$ and $\ker \pi_1$. There are $u < v$ in A and a congruence chain of B with extremities $f(u)$ and $f(v)$.*

Proof. Set $\alpha_k = \ker \pi_k$, denote by β_k the complement of α_k in the Boolean lattice $\text{Con}_c B$, that is $\beta_k = \alpha_{1-k}$ for each $k < 2$. Let $u < v$ in A . The following equalities hold:

$$(\text{Con}_c \pi_k)(\Theta_B(f(u), f(v))) = \Theta_A(\pi_k(f(u)), \pi_k(f(v))) = \Theta_A(u, v) = \mathbf{1}_A.$$

This implies the containment $\Theta_B(f(u), f(v)) \supseteq \beta_k$ for each $k < 2$. Therefore:

$$\Theta_B(f(u), f(v)) = \mathbf{1}_B = \beta_0 \vee \beta_1.$$

There are an integer $n > 0$ and a chain $f(u) = x_0 < x_1 < \dots < x_n = f(v)$ of B such that $\Theta_B(x_i, x_{i+1}) \in \{\beta_0, \beta_1\}$ for all $i < n$. Up to permuting π_0 and π_1 , we can assume that $\Theta_B(x_0, x_1) = \beta_0$, that is, $\pi_0(x_1) > \pi_0(x_0) = u$ and $\pi_1(x_1) = \pi_1(x_0) = u$.

Put $v' = \pi_0(x_1)$, put $t_1 = x_1 \wedge f(v')$. Hence $\pi_0(t_1) = \pi_0(x_1) \wedge \pi_0(f(v')) = v' \wedge v' = v'$ and $\pi_1(t_1) = \pi_1(x_1) \wedge \pi_1(f(v')) = u \wedge v' = u$, thus the following equalities hold:

$$\begin{aligned} u &= \pi_0(f(u)) = \pi_1(f(u)) = \pi_1(t_1), \\ v' &= \pi_0(f(v')) = \pi_1(f(v')) = \pi_0(t_1). \end{aligned}$$

Therefore $\Theta(f(u), t_1) = \beta_0$ and $\Theta(t_1, f(v')) = \beta_1$, that is $f(u) < t_1 < f(v')$ is a congruence chain of B . \square

Theorem 12.2. *Let \mathcal{K} be a class of bounded lattices containing every chain of length either 1, 2, or 3, let \mathcal{W} be a variety of lattices, and let $\Psi: \mathcal{K} \rightarrow \mathcal{W}$ be a functor such that $\text{Con}_c \circ \Psi \cong \text{Con}_c$. Then, up to changing \mathcal{W} to its dual and composing Ψ with dualization, \mathcal{W} contains \mathcal{K} , and if $\Phi: \mathcal{K} \rightarrow \mathcal{W}$ denotes the inclusion functor, there exists a natural transformation $(\varepsilon_L)_{L \in \mathcal{K}}: \Phi \rightarrow \Psi$ such that ε_L is a congruence-preserving embedding from L into $\Psi(L)$, for each $L \in \mathcal{K}$.*

Proof. Fix a natural equivalence $\vec{\xi} = (\xi_L)_{L \in \mathcal{K}}: \text{Con}_c \circ \Psi \rightarrow \text{Con}_c$.

Let $\mathbf{2} = \{0, 1\}$ be a chain. In this proof, each time we define a morphism $f_{P,Q}$ or $f_{P,Q}^k$ of lattices, we also denote $g_{P,Q} = \Psi(f_{P,Q})$ or $g_{P,Q}^k = \Psi(f_{P,Q}^k)$. Given any finite chain C , we denote by $f_{\mathbf{2},C}: \mathbf{2} \rightarrow C$ the only 0, 1-homomorphism of lattices.

Let $C = \{0, x_1, 1\}$ be a chain with $0 < x_1 < 1$. Consider the following 0, 1-homomorphisms of lattices

$$\begin{array}{ccc} f_{C,\mathbf{2}}^0: C \rightarrow \mathbf{2} & & f_{C,\mathbf{2}}^1: C \rightarrow \mathbf{2} \\ x_1 \mapsto 0 & & x_1 \mapsto 1. \end{array}$$

Notice that $\text{Con}_c \Psi(C) \cong \text{Con}_c C \cong \mathbf{2}^2$, $\{\ker g_{C,\mathbf{2}}^0, \ker g_{C,\mathbf{2}}^1\} = \text{At } \text{Con}_c \Psi(C)$, and $f_{C,\mathbf{2}}^0 \circ f_{\mathbf{2},C} = f_{C,\mathbf{2}}^1 \circ f_{\mathbf{2},C} = \text{id}_{\mathbf{2}}$. It follows from Lemma 12.1 that there are $u < v$ in $\Psi(\mathbf{2})$ and a congruence chain $g_{\mathbf{2},C}(u) = z_0^C < z_1^C < z_2^C = g_{\mathbf{2},C}(v)$ of $\Psi(C)$. Up to changing \mathcal{W} to its dual and dualizing Ψ , we can assume that $z_0^C < z_1^C < z_2^C$ is a direct congruence chain for (ξ_C, C) , that is,

$$\xi_C(\Theta_{\Psi(C)}(z_0^C, z_1^C)) = \Theta_C(0, x_1) \text{ and } \xi_C(\Theta_{\Psi(C)}(z_1^C, z_2^C)) = \Theta_C(x_1, 1). \quad (12.1)$$

As $\ker f_{C,\mathbf{2}}^0 = \Theta_C(0, x_1)$, $g_{C,\mathbf{2}}^0(z_1^C) = g_{C,\mathbf{2}}^0(z_0^C)$, similarly $g_{C,\mathbf{2}}^1(z_1^C) = g_{C,\mathbf{2}}^1(z_2^C)$. Therefore the following equalities hold:

$$g_{C,\mathbf{2}}^0(z_1^C) = g_{C,\mathbf{2}}^0(z_0^C) = u = g_{C,\mathbf{2}}^1(z_0^C) \quad (12.2)$$

$$g_{C,\mathbf{2}}^1(z_1^C) = g_{C,\mathbf{2}}^1(z_2^C) = v = g_{C,\mathbf{2}}^0(z_2^C) \quad (12.3)$$

Given chains D and D' of the same length, we denote by $f_{D,D'}: D \rightarrow D'$ the only isomorphism.

Let D be a chain of length 2, we denote $z_k^D = g_{C,D}(z_k^C)$ for all $k < 3$. The following equalities hold:

$$z_0^D = g_{C,D}(z_0^C) = g_{C,D}(g_{\mathbf{2},C}(u)) = g_{C,D} \circ f_{\mathbf{2},C}(u) = g_{\mathbf{2},D}(u).$$

With a similar argument we obtain $z_3^D = g_{\mathbf{2},D}(v)$. Moreover $g_{\mathbf{2},D}(u) = z_0^D < z_1^D < z_2^D = g_{\mathbf{2},D}(v)$ is a direct congruence chain of $\Psi(D)$.

Let $D = \{0, y_1, y_2, 1\}$ be a chain with $0 < y_1 < y_2 < 1$. Consider the following 0, 1-homomorphisms of lattices

$$\begin{array}{ccc} f_{C,D}^1: C \rightarrow D & & f_{C,D}^2: C \rightarrow D \\ x_1 \mapsto y_1 & & x_1 \mapsto y_2 \\ f_{D,\mathbf{2}}^1: D \rightarrow \mathbf{2} & & f_{D,\mathbf{2}}^2: D \rightarrow \mathbf{2} & & f_{D,\mathbf{2}}^3: D \rightarrow \mathbf{2} \\ y_1 \mapsto 1 & & y_1 \mapsto 0 & & y_1 \mapsto 0 \\ y_2 \mapsto 1 & & y_2 \mapsto 1 & & y_2 \mapsto 0 \end{array}$$

With a simple computation we obtain

$$f_{C,\mathbf{2}}^0 = f_{D,\mathbf{2}}^2 \circ f_{C,D}^1 = f_{D,\mathbf{2}}^3 \circ f_{C,D}^1 = f_{D,\mathbf{2}}^3 \circ f_{C,D}^2 \quad (12.4)$$

$$f_{C,\mathbf{2}}^1 = f_{D,\mathbf{2}}^1 \circ f_{C,D}^1 = f_{D,\mathbf{2}}^1 \circ f_{C,D}^2 = f_{D,\mathbf{2}}^2 \circ f_{C,D}^2 \quad (12.5)$$

Put $z_0^D = g_{2,D}(u)$, $z_1^D = g_{C,D}^1(z_1^C)$, $z_2^D = g_{C,D}^2(z_1^C)$, and $z_3^D = g_{2,D}(v)$. As $f_{D,2}^k \circ f_{2,D} = \text{id}_2$, the following equalities hold:

$$g_{D,2}^k(z_0^D) = g_{D,2}^k(g_{2,D}(u)) = u, \quad \text{for all } k \in \{1, 2, 3\}. \quad (12.6)$$

Similarly we obtain

$$g_{D,2}^k(z_3^D) = g_{D,2}^k(g_{2,D}(v)) = v, \quad \text{for all } k \in \{1, 2, 3\}. \quad (12.7)$$

The following equalities hold:

$$\begin{aligned} g_{D,2}^1(z_1^D) &= g_{D,2}^1 \circ g_{C,D}^1(z_1^C) && \text{as } z_1^D = g_{C,D}^1(z_1^C). \\ &= g_{C,2}^1(z_1^C) && \text{by (12.5).} \\ &= u && \text{by (12.2).} \end{aligned}$$

With similar arguments (12.2), (12.3), (12.4), (12.5), (12.6), and (12.7) implies the following equalities

$$u = g_{D,2}^1(z_0^D) = g_{D,2}^2(z_0^D) = g_{D,2}^3(z_0^D) = g_{D,2}^2(z_1^D) = g_{D,2}^3(z_1^D) = g_{D,2}^3(z_2^D). \quad (12.8)$$

$$v = g_{D,2}^1(z_3^D) = g_{D,2}^2(z_3^D) = g_{D,2}^3(z_3^D) = g_{D,2}^1(z_1^D) = g_{D,2}^1(z_2^D) = g_{D,2}^2(z_2^D). \quad (12.9)$$

Notice that $g_{D,2}^k(z_0^D) \leq g_{D,2}^k(z_1^D) \leq g_{D,2}^k(z_2^D) \leq g_{D,2}^k(z_3^D)$ for all $k \in \{1, 2, 3\}$ and $z_i^D \neq z_j^D$ for $0 \leq i < j \leq 3$. As $\bigcap (\ker g_{D,2}^k \mid k \in \{1, 2, 3\}) = \mathbf{0}_{\Psi(D)}$, this implies $z_0^D < z_1^D < z_2^D < z_3^D$.

The following equalities hold:

$$\xi_D(\ker g_{D,2}^1) = \ker f_{D,2}^1 = \Theta_D(y_1, y_3), \quad (12.10)$$

$$\xi_D(\ker g_{D,2}^2) = \ker f_{D,2}^2 = \Theta_D(y_0, y_1) \vee \Theta_D(y_2, y_3) \quad (12.11)$$

$$\xi_D(\ker g_{D,2}^3) = \ker f_{D,2}^3 = \Theta_D(y_0, y_2). \quad (12.12)$$

Therefore,

$$\begin{aligned} \xi_D(\Theta_{\psi(D)}(z_0^D, z_1^D)) &\subseteq \xi_D(\ker g_{D,2}^2) \cap \xi_D(\ker g_{D,2}^3) && \text{by (12.8)} \\ &= (\Theta_D(y_0, y_1) \vee \Theta_D(y_2, y_3)) \cap \Theta_D(y_0, y_2) && \text{by (12.11), (12.12)} \\ &= \Theta_D(y_0, y_1) \end{aligned}$$

Similarly (12.8), (12.9), (12.10), (12.11), and (12.12) implies that:

$$\xi_D(\Theta_{\psi(D)}(z_i^D, z_{i+1}^D)) = \Theta_D(y_i, y_{i+1}), \quad \text{for all } i < 4.$$

Therefore, $g_{2,D}(u) = z_0^D < z_1^D < z_2^D < z_3^D = g_{2,D}(v)$ is a direct congruence chain of $\Psi(D)$.

We have constructed for each chain D of \mathcal{K} of length either 2 or 3 a direct congruence chain of $\Psi(D)$ with extremities $g_{2,D}(u)$ and $g_{2,D}(v)$. Let $L \in \mathcal{K}$, let \mathcal{C} be the set of all chains of L of length either 2 or 3. Let $\vec{A} = (A_P, h_{P,Q}^L \mid P \leq Q \in \mathcal{J}(\mathcal{C}))$ be the \mathcal{C} -chain diagram of L . Put $C_x = \{0, x, 1\}$ for each $x \in L - \{0, 1\}$. As $\vec{B} = \mathbf{G} \circ \Psi \circ \vec{A}$ is a partial lifting of $\text{Con}_c \circ \vec{A}$ it follows from Theorem 10.12 that the map

$$\begin{aligned} \varepsilon_L: L &\rightarrow B_{\top} = \Psi(L) \\ 0 &\mapsto \Psi(h_{0,\top}^L)(u) \\ 1 &\mapsto \Psi(h_{0,\top}^L)(v) \\ x &\mapsto \Psi(h_{C_x,\top}^L)(z_1^{C_x}) \end{aligned}$$

is an embedding of lattices. Moreover $(\varepsilon_L, \xi_L): (L, L, \Theta_L, \text{Con}_c L) \rightarrow \mathbf{B}_\top = \mathbf{G}(\Psi(L))$ is an embedding. As ξ_L is an isomorphism, it follows that $\text{Con}_c \varepsilon_L: \text{Con}_c L \rightarrow \text{Con}_c(\Psi(L))$ is an isomorphism.

Let $p: K \rightarrow L$ be a morphism of lattices in \mathcal{K} . Let $x \in K - \{0, 1\}$. Notice that the restriction of $p: C_x \rightarrow C_{p(x)}$ is the only isomorphism, thus $p \circ h_{C_x, \top}^K = h_{C_{p(x)}, \top}^L \circ f_{C_x, C_{p(x)}}$. Therefore the following equalities hold:

$$\begin{aligned} \Psi(p)(\varepsilon_K(x)) &= \Psi(p \circ h_{C_x, \top}^K)(z_1^{C_x}) \\ &= \Psi(h_{C_{p(x)}, \top}^L \circ f_{C_x, C_{p(x)}})(z_1^{C_x}) \\ &= \Psi(h_{C_{p(x)}, \top}^L)(g_{C_x, C_{p(x)}}(z_1^{C_x})) \\ &= \Psi(h_{C_{p(x)}, \top}^L)(z_1^{C_{p(x)}}) \\ &= \varepsilon_L(p(x)). \end{aligned}$$

Similarly $\Psi(p)(\varepsilon_K(0)) = \varepsilon_L(p(0))$ and $\Psi(p)(\varepsilon_K(1)) = \varepsilon_L(p(1))$. So $\vec{\varepsilon}$ is a natural transformation. \square

Remark 12.3. Let $C = \{0, x_1, 1\}$ be a chain of length 3, let $f: \mathbf{2} \rightarrow C$ be the only 0, 1-homomorphism of lattices. Given an element a in a bounded lattice L , denote by $f_{a,L}: C \rightarrow L$ the only 0, 1-homomorphism such that $f_{a,L}(x_1) = a$.

It follows from the proof of Theorem 12.2 that given $u < v$ in $\Psi(\mathbf{2})$ and a direct congruence chain $\Psi(f)(u) < z_1 < \Psi(f)(v)$ of $\Psi(C)$, we can construct $\vec{\varepsilon}: \Phi \rightarrow \Psi$ satisfying the conclusion of the theorem, and such that $\varepsilon_L(a) = \Psi(f_{a,L})(z_1)$, for each $L \in \mathcal{K}$ and each $a \in L$.

Corollary 12.4. *Let \mathcal{V} and \mathcal{W} be varieties of lattices, let $\Psi: \mathcal{V} \rightarrow \mathcal{W}$ be a functor such that $\text{Con}_c \circ \Psi \cong \text{Con}_c$. Then either $\mathcal{V} \subseteq \mathcal{W}$ or $\mathcal{V} \subseteq \mathcal{W}^d$.*

Corollary 12.5. *Let \mathcal{N}_5 be the variety of lattices generated by the five-element non-modular lattice. Denote by \mathcal{M} the variety of all modular lattices. Then there exists no functor $\Psi: \mathcal{N}_5 \rightarrow \mathcal{M}$ such that $\text{Con}_c \circ \Psi \cong \text{Con}_c$.*

Corollary 12.6. *Let \mathcal{W} be a variety of lattices, let K be bounded lattice that belongs neither to \mathcal{V} nor to its dual variety. Consider the diagram \vec{A} which has objects K and a chain C of length 3, and arrows all 0, 1-lattice homomorphisms from C to K together with all 0, 1-lattice homomorphisms from C to C . Then $\text{Con}_c \circ \vec{A}$ is not liftable in \mathcal{V} .*

13. AN OPEN PROBLEM

The most natural problem brought by the present work is whether the assumption about simple members of \mathcal{W} is necessary for getting the result of Theorem 11.6. It is not even known whether $\text{Con}_c \mathcal{V} \subseteq \text{Con}_c \mathcal{W}$ implies that \mathcal{V} is contained either in \mathcal{W} or its dual, for any varieties \mathcal{V} and \mathcal{W} of lattices. For example, it is not known whether $\text{Con}_c \mathcal{N}_5 \subseteq \text{Con}_c \mathcal{M}$, although Corollary 12.5 suggests that this would not occur for any ‘‘obvious’’ reason.

More generally, even in crossover contexts (i.e., \mathcal{V} and \mathcal{W} may not share the same similarity type), it is not completely unplausible that the chain diagram (cf. Definition 10.8) could be tailored to further varieties of algebras. If this could be done, then it might be possible to prove that for varieties of many other structures than just lattices, the containment $\text{Con}_c \mathcal{V} \subseteq \text{Con}_c \mathcal{W}$ could always be expressed *via*

a suitable amount of interpretability of members of \mathcal{V} in \mathcal{W} . Hence, at least up to a suitable notion of interpretability, $\text{Con}_c \mathcal{V}$ would always determine \mathcal{V} . This would be a most satisfactory answer to Question (Q2).

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