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Landau operator on the quaternionic field

Azzouz Zinoun ^{*}, Dominique Kazmierowski [†]
and Ahmed Intissar [‡]

Abstract

The Landau operator on the quaternionic field is a differential operator $\mathcal{H}_{\vec{B}}$ acting on $C^\infty(\mathbb{H}; \mathbb{C})$, defined as the Fourier transform of the sub-Laplacian associated to the quaternionic Heisenberg group $\mathfrak{Sm}\mathbb{H} \times_\omega \mathbb{H}$, where $\mathbb{H} \sim \mathbb{R}^4$ is viewed as the space of quaternions, ω is a canonical $\mathfrak{Sm}\mathbb{H}$ -valued symplectic form on \mathbb{H} , $\mathfrak{Sm}\mathbb{H} \sim \mathbb{R}^3$ and $\vec{B} \in \mathfrak{Sm}\mathbb{H}$ is fixed. $\mathcal{H}_{\vec{B}}$ is the Hamiltonian of a charged particle in \mathbb{H} , interacting with a uniform magnetic field \vec{B} . By a suitable orthogonal change of basis in \mathbb{H} , $\mathcal{H}_{\vec{B}}$ is transformed into an other Landau operator $\mathcal{H}_{\|\vec{B}\|}$ which is much simpler, $\|\vec{B}\|$ is the norm of the magnetic field \vec{B} . This new operator is the Hamiltonian of two superposed uncoupled complex harmonic oscillators.

1 Introduction

Landau operator arises in both physics and mathematics, and appears in the framework of the study of charged particles acted on by magnetic fields. The aim of this paper is to present from a mathematical point of view an operator analogue to the well known quantum mechanical Hamiltonian of a charged particle moving in a uniform magnetic field [1]; this operator will be denoted by $\mathcal{H}_{\vec{B}}$. $\mathcal{H}_{\vec{B}}$ acts on $C^\infty(\mathbb{H}, \mathbb{C})$, where \mathbb{H} is the quaternion space

^{*}Laboratoire Phlam, UMR-CNRS 8523, UFR de Physique Université Lille I — Sciences et Technologies, 59655 Villeneuve d'Ascq Cedex, France

E-mail: Azzouz.Zinoun@univ-lille1.fr

[†]Université Lille I — Sciences et Technologies, 59655 Villeneuve d'Ascq Cedex, France

E-mail: dominique.kazmierowski@gmail.com

[‡]Département de Mathématiques Faculté des Sciences de Rabat BP 1014, Morocco

E-mail: intissar@fsr.ac.ma

and \mathbb{C} is the complex field and \vec{B} is a uniform magnetic field. In Section 2, we give a brief review of the Schrödinger equation. In Section 3, we construct the Heisenberg group and its Heisenberg algebra from which we derive the Landau operator. In Section 4, we show that the Landau operator $\mathcal{H}_{\vec{B}}$ is defined as the Fourier transform of the sub-Laplacian associated to the quaternionic Heisenberg group of dimension 7; we give the expressions of the Landau operator both in real and complex form; we define also the angular momentum operator on $\mathbb{H} \sim \mathbb{R}^4$. In section 5, we state a theorem which allows us to give a canonical expression $\mathcal{H}_{\|\vec{B}\|}$ of $\mathcal{H}_{\vec{B}}$, which is more simpler to study. In Section 6, the symmetry group and the gauge invariance of the Landau operator are discussed. Section 7 relates this work with studies by other people on a quantum system in a uniform magnetic field.

2 Physical motivation and a brief review of Schrödinger equation

The standard procedure for extending the equations of motion of classical mechanics to the quantum Schrödinger equation (in position space) is through a generalization of the Hamiltonian formulation of classical mechanics. For the free particle in one dimension this consist of simply making the identification

$$H_{classical} = \frac{p^2}{2m} \implies \hat{H} = \frac{\hat{p}^2}{2m}. \quad (1)$$

The classical Hamiltonian function appropriate for a charged particle (of charge q and mass m) acted on by external electric and magnetic fields (in three dimensions) is given by

$$H_{classical} = \frac{1}{2m}(\vec{p} - q\vec{A})^2 + q\phi. \quad (2)$$

The corresponding quantum mechanical Hamiltonian is obtained by replacing the momentum variable by its operator counterpart giving the Schrödinger equation

$$\hat{H}\psi(\vec{r}, t) = E\psi(\vec{r}, t) = i\hbar\frac{\partial}{\partial t}\psi(\vec{r}, t) \quad (3)$$

where

$$\hat{H} = \frac{1}{2m}(\hat{p} - q\hat{A}(\vec{r}, t))^2 + q\hat{\phi}(\vec{r}, t) \quad (4)$$

and $\vec{r} \in \mathbb{R}^3$ is the position and $t \in \mathbb{R}^+$ is the time.

One must be careful of the ordering of any differential operators, so we find

$$\begin{aligned} [\hat{p} - q\hat{A}(\vec{r}, t)]^2\psi(\vec{r}, t) &= -\hbar^2\vec{\nabla}^2\psi(\vec{r}, t) + iq\hbar\vec{\nabla} \cdot [\hat{A}(\vec{r}, t)\psi(\vec{r}, t)] \\ &\quad + iq\hbar\hat{A}(\vec{r}, t) \cdot [\vec{\nabla}\psi(\vec{r}, t)] + q^2[\hat{A}(\vec{r}, t) \cdot \hat{A}(\vec{r}, t)]\psi(\vec{r}, t). \end{aligned} \quad (5)$$

When the magnetic field \vec{B} is uniform we can take the potential vector on the following form $\hat{A} = \frac{1}{2}(\vec{B} \wedge \hat{r})$, where \hat{r} is the position vector operator. For the sake of simplicity, we will forget the ‘hat’ symbol on the operators. Suppose the Euclidian oriented space \mathbb{R}^3 is endowed with the canonical basis (e'_1, e'_2, e'_3) and coordinates (x_1, x_2, x_3) , the components of the momentum vector operator are $(-i\frac{\partial}{\partial x_1}, -i\frac{\partial}{\partial x_2}, -i\frac{\partial}{\partial x_3})$. Let (B_1, B_2, B_3) denote the components of the magnetic field, then the potential vector has components (A_1, A_2, A_3) given by

$$\begin{aligned} A_1 &= \frac{1}{2}(B_2x_3 - B_3x_2) \\ A_2 &= \frac{1}{2}(B_3x_1 - B_1x_3) \\ A_3 &= \frac{1}{2}(B_1x_2 - B_2x_1). \end{aligned}$$

We define the differential 1-form $A = A_1 dx_1 + A_2 dx_2 + A_3 dx_3$. If we identify at each point (x_1, x_2, x_3) the basis $(dx_2 \wedge dx_3, dx_3 \wedge dx_1, dx_1 \wedge dx_2)$ of the 2-forms with the canonical basis (e'_1, e'_2, e'_3) of \mathbb{R}^3 , then we get $dA = \vec{B}$ since the magnetic field is uniform. The corresponding Hamiltonian operator H at $x = (x_1, x_2, x_3)$ is then equal to

$$\begin{aligned} H &= -\frac{\hbar^2}{2m} \left\{ \left(\frac{\partial}{\partial x_1} - i\frac{q}{\hbar}A_1 \right)^2 + \left(\frac{\partial}{\partial x_2} - i\frac{q}{\hbar}A_2 \right)^2 + \left(\frac{\partial}{\partial x_3} - i\frac{q}{\hbar}A_3 \right)^2 \right\} \\ &= -\frac{\hbar^2}{2m} \left\{ \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2} - i\frac{q}{\hbar}B_1 \left(x_2 \frac{\partial}{\partial x_3} - x_3 \frac{\partial}{\partial x_2} \right) \right. \\ &\quad \left. - i\frac{q}{\hbar}B_2 \left(x_3 \frac{\partial}{\partial x_1} - x_1 \frac{\partial}{\partial x_3} \right) - i\frac{q}{\hbar}B_3 \left(x_1 \frac{\partial}{\partial x_2} - x_2 \frac{\partial}{\partial x_1} \right) \right. \\ &\quad \left. - \frac{q^2}{4\hbar^2} \|\vec{B}\|^2 \|\vec{x}\|^2 + \frac{q^2}{4\hbar^2} (\vec{B} \cdot \vec{x})^2 \right\}. \end{aligned} \quad (6)$$

The orbital momentum is :

$$\vec{L} = \vec{r} \wedge \vec{p} = \begin{pmatrix} L_1 \\ L_2 \\ L_3 \end{pmatrix} = -i\hbar \begin{pmatrix} x_2 \frac{\partial}{\partial x_3} - x_3 \frac{\partial}{\partial x_2} \\ x_3 \frac{\partial}{\partial x_1} - x_1 \frac{\partial}{\partial x_3} \\ x_1 \frac{\partial}{\partial x_2} - x_2 \frac{\partial}{\partial x_1} \end{pmatrix}.$$

The Hamiltonian H can be written in the following form :

$$H = -\frac{\hbar^2}{2m} \left\{ \Delta + \frac{q}{\hbar^2} \vec{L} \cdot \vec{B} - \frac{q^2}{4\hbar^2} \|\vec{B}\|^2 \|\vec{x}\|^2 + \frac{q^2}{4\hbar^2} (\vec{B} \cdot \vec{x})^2 \right\}$$

If \vec{B} is directed along e'_3 then :

$$\begin{aligned} H &= -\frac{\hbar^2}{2m} \left\{ \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2} - i\frac{q}{\hbar} B_3 \left(x_1 \frac{\partial}{\partial x_2} - x_2 \frac{\partial}{\partial x_1} \right) \right. \\ &\quad \left. - \frac{q^2}{4\hbar^2} B_3^2 (x_1^2 + x_2^2) \right\} \\ &= H_{\perp} + H_{//}. \end{aligned}$$

In the (x_1, x_2) -plane, the particle is described by the operator

$$H_{\perp} = -\frac{\hbar^2}{2m} \left\{ \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} - i\frac{q}{\hbar} B_3 \left(x_1 \frac{\partial}{\partial x_2} - x_2 \frac{\partial}{\partial x_1} \right) - \frac{q^2}{4\hbar^2} B_3^2 (x_1^2 + x_2^2) \right\}.$$

H_{\perp} is called the *Landau operator*.

The operator $H_{//} = -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x_3^2}$ is the Hamiltonian along the x_3 -axis.

If we introduce the complex variable $z = x_1 + ix_2$, the Landau operator becomes

$$H_{\perp} = -\frac{\hbar^2}{2m} \left\{ 4 \frac{\partial}{\partial z} \frac{\partial}{\partial \bar{z}} + \frac{q}{\hbar} B_3 \left(z \frac{\partial}{\partial z} - \bar{z} \frac{\partial}{\partial \bar{z}} \right) - \frac{q^2}{4\hbar^2} B_3^2 |z|^2 \right\}.$$

All these expressions of H represent the well known Hamiltonian of a charged particle in a uniform magnetic field (see [1] and [2]).

3 Mathematical point of view and the statement of the main results.

The space we work in is the Euclidean space \mathbb{R}^4 identified with the quaternionic space \mathbb{H} . In the basis (e_0, e_1, e_2, e_3) , an element x of \mathbb{R}^4 is written in

the following form :

$$x = x_0 e_0 + x_1 e_1 + x_2 e_2 + x_3 e_3 \quad (7)$$

$$= (x_0, \vec{x}) \quad (8)$$

where $\vec{x} = (x_1, x_2, x_3) \in \mathbb{R}^3 \sim \Im m \mathbb{H}$.

Let $\{1, \mathbf{i}, \mathbf{j}, \mathbf{k}\}$ be the basis of \mathbb{H} , with

$$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = -1$$

$$\mathbf{ij} = -\mathbf{ji}, \quad \mathbf{ik} = -\mathbf{ki}, \quad \mathbf{jk} = -\mathbf{kj}.$$

An element x of \mathbb{H} is written as :

$$x = x_0 + x_1 \mathbf{i} + x_2 \mathbf{j} + x_3 \mathbf{k} \quad (9)$$

$$= (x_0, \Im m x). \quad (10)$$

A $U(1)$ potential will be given by functions $A_\mu(x) \in \mathbb{R}$. It will be convenient if we go further to work with the differential 1-form on \mathbb{R}^4 defined by

$$A(x) = \sum_{\alpha=0}^3 A_\alpha(x) dx_\alpha. \quad (11)$$

The 1-form A has a geometric significance: it is the connection form which is used to define covariant derivatives. We shall also write the curvature F as an exterior 2-form

$$F = \frac{1}{2} F_{\mu\nu} dx_\mu \wedge dx_\nu. \quad (12)$$

The 2-form F is computed from A by

$$\begin{aligned} F &= dA = \sum_{\mu=0}^3 dA_\mu \wedge dx_\mu \\ &= \frac{1}{2} \sum_{\mu,\nu} F_{\mu\nu} dx_\mu \wedge dx_\nu \\ &= \frac{1}{2} \left(\frac{\partial A_1}{\partial x_0} - \frac{\partial A_0}{\partial x_1} \right) dx_0 \wedge dx_1 + \frac{1}{2} \left(\frac{\partial A_2}{\partial x_0} - \frac{\partial A_0}{\partial x_2} \right) dx_0 \wedge dx_2 + \\ &\quad \frac{1}{2} \left(\frac{\partial A_3}{\partial x_0} - \frac{\partial A_0}{\partial x_3} \right) dx_0 \wedge dx_3 + \frac{1}{2} \left(\frac{\partial A_2}{\partial x_1} - \frac{\partial A_1}{\partial x_2} \right) dx_1 \wedge dx_2 + \\ &\quad \frac{1}{2} \left(\frac{\partial A_3}{\partial x_2} - \frac{\partial A_2}{\partial x_3} \right) dx_2 \wedge dx_3 + \frac{1}{2} \left(\frac{\partial A_1}{\partial x_3} - \frac{\partial A_3}{\partial x_1} \right) dx_3 \wedge dx_1. \quad (13) \end{aligned}$$

In this work we are concerned by a uniform magnetic field $\vec{B} \in \mathbb{R}^3$. Since \mathbb{R}^3 is identified with $\Im m\mathbb{H}$, we write

$$\begin{aligned} B_q &= B_1\mathbf{i} + B_2\mathbf{j} + B_3\mathbf{k} \\ &= \vec{B} \end{aligned}$$

(B_q represents the quaternion version of the magnetic field $\vec{B} \in \mathbb{R}^3$). A matrix representation of $\mathbf{i}, \mathbf{j}, \mathbf{k}$ on \mathbb{R}^4 is:

$$\mathbf{i} = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \quad (14)$$

$$\mathbf{j} = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \quad (15)$$

$$\mathbf{k} = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}. \quad (16)$$

The matrix associated to the magnetic field \vec{B} is :

$$\Omega_{\vec{B}} = \begin{pmatrix} 0 & -B_1 & -B_2 & -B_3 \\ B_1 & 0 & -B_3 & B_2 \\ B_2 & B_3 & 0 & -B_1 \\ B_3 & -B_2 & B_1 & 0 \end{pmatrix}. \quad (17)$$

Since we deal with uniform fields, the matrix $\Omega_{\vec{B}}$ can be obtained from (13) by considering that the coefficients of the 2-form F are constants and that F is self-dual 2-form. We have

$$\frac{1}{2} \left(\frac{\partial A_1}{\partial x_0} - \frac{\partial A_0}{\partial x_1} \right) = E_1, \quad \frac{1}{2} \left(\frac{\partial A_2}{\partial x_0} - \frac{\partial A_0}{\partial x_2} \right) = E_2, \quad \frac{1}{2} \left(\frac{\partial A_3}{\partial x_0} - \frac{\partial A_0}{\partial x_3} \right) = E_3 \quad (18)$$

$$\frac{1}{2} \left(\frac{\partial A_3}{\partial x_2} - \frac{\partial A_2}{\partial x_3} \right) = B_1, \quad \frac{1}{2} \left(\frac{\partial A_1}{\partial x_3} - \frac{\partial A_3}{\partial x_1} \right) = B_2, \quad \frac{1}{2} \left(\frac{\partial A_2}{\partial x_1} - \frac{\partial A_1}{\partial x_2} \right) = B_3 \quad (19)$$

The matrix associated to F is :

$$\Omega_{\vec{E}, \vec{B}} = \begin{pmatrix} 0 & -E_1 & -E_2 & -E_3 \\ E_1 & 0 & -B_3 & B_2 \\ E_2 & B_3 & 0 & -B_1 \\ E_3 & -B_2 & B_1 & 0 \end{pmatrix} \quad (20)$$

Note the analogy of the tensor $\Omega_{\vec{E},\vec{B}}$ and the electromagnetic tensor ($F^{\mu\nu}$), which one can derive from the Maxwell equations (we work in the Euclidean space \mathbb{R}^4 instead of the Minkowski space $\mathbb{R}^{3,1}$). This matrix may be interpreted as the matrix associated to an electric and a magnetic uniform fields

$$\vec{E} = \begin{pmatrix} E_1 \\ E_2 \\ E_3 \end{pmatrix}, \quad \vec{B} = \begin{pmatrix} B_1 \\ B_2 \\ B_3 \end{pmatrix}.$$

We define the connection A associated to these uniform fields by its potential vector

$$\vec{A} = \Omega_{\vec{E},\vec{B}}\vec{x}$$

where

$$\vec{x} = \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{pmatrix} \in \mathbb{R}^4,$$

$$\Omega_{\vec{E},\vec{B}}\vec{x} = \begin{pmatrix} A_0 \\ A_1 \\ A_2 \\ A_3 \end{pmatrix} = \begin{pmatrix} -E_1x_1 - E_2x_2 - E_3x_3 \\ B_1x_0 - B_3x_2 + B_2x_3 \\ B_2x_0 + B_3x_1 - B_1x_3 \\ B_3x_0 - B_2x_1 + B_1x_2 \end{pmatrix}$$

Let \star be the Hodge operator; the self-duality of the 2-form F is: $\star F = F$, this implies $\vec{E} = \vec{B}$, hence the matrix (20) becomes :

$$\Omega_{B_q} = \begin{pmatrix} 0 & -B_1 & -B_2 & -B_3 \\ B_1 & 0 & -B_3 & B_2 \\ B_2 & B_3 & 0 & -B_1 \\ B_3 & -B_2 & B_1 & 0 \end{pmatrix}$$

which is exactly the matrix (17). Then,

$$\Omega_{B_q}\vec{x} = \begin{pmatrix} A_0 \\ A_1 \\ A_2 \\ A_3 \end{pmatrix} = \begin{pmatrix} -B_1x_1 - B_2x_2 - B_3x_3 \\ B_1x_0 - B_3x_2 + B_2x_3 \\ B_2x_0 + B_3x_1 - B_1x_3 \\ B_3x_0 - B_2x_1 + B_1x_2 \end{pmatrix}. \quad (21)$$

The Landau operator H_{B_q} is obtained from (4) and (5) (Quantum Mechanics

result), (for simplicity we take $\hbar = 1$, $q = 1$, and $2m = 1$):

$$\begin{aligned}\mathcal{H}_{B_q} &= - \left[\left(\frac{\partial}{\partial x_0} + iA_0 \right)^2 + \left(\frac{\partial}{\partial x_1} + iA_1 \right)^2 \right. \\ &\quad \left. + \left(\frac{\partial}{\partial x_2} + iA_2 \right)^2 + \left(\frac{\partial}{\partial x_3} + iA_3 \right)^2 \right] \\ &= -\Delta - 2i \langle \Omega_{B_q} \vec{x}, \nabla \rangle + \|\Omega_{B_q} \vec{x}\|^2\end{aligned}\quad (22)$$

which is the Hamiltonian of a charged particle in $\mathbb{H} \sim \mathbb{R}^4$ acted on by a uniform magnetic field $B_q \in \mathfrak{Sm}\mathbb{H} \sim \mathbb{R}^3$.

3.1 Heisenberg group

We will derive the Landau operator (22) from a Lie group and its associated Lie algebra approach. This Landau operator on the quaternionic field may be viewed as the Fourier transform of the sub-Laplacian associated to the quaternionic Heisenberg group $\mathfrak{Sm}\mathbb{H} \times_{\omega} \mathbb{H}$ of dimension 7, where $\mathbb{H} \sim \mathbb{R}^4$ is viewed as the space of quaternions, ω is a canonical $\mathfrak{Sm}\mathbb{H}$ -valued symplectic form on \mathbb{H} , $\mathfrak{Sm}\mathbb{H} \sim \mathbb{R}^3$ and $\nu \in \mathfrak{Sm}\mathbb{H}$ is fixed. This Heisenberg group is associated to the exact sequence

$$0 \longrightarrow \mathfrak{Sm}\mathbb{H} \longrightarrow \mathfrak{Sm}\mathbb{H} \times_{\omega} \mathbb{H} \longrightarrow \mathbb{H} \longrightarrow 0$$

that is,

$$0 \longrightarrow \mathbb{R}^3 \longrightarrow \mathbb{R}^3 \times_{\omega} \mathbb{R}^4 \longrightarrow \mathbb{R}^4 \longrightarrow 0.$$

3.1.1 Canonical form on \mathbb{H}

Define a 2-form ω on \mathbb{H} by

$$\omega(x, y) = \frac{1}{2} (y\bar{x} - x\bar{y})$$

for every $x, y \in \mathbb{H}$, where $\bar{x} = x_0 1 - x_1 \mathbf{i} - x_2 \mathbf{j} - x_3 \mathbf{k}$ is the quaternionic conjugate of x . It is easily checked that $\omega(x, y) \in \mathfrak{Sm}(\mathbb{H})$ can be written

$$\omega(x, y) = \omega_1(x, y)\mathbf{i} + \omega_2(x, y)\mathbf{j} + \omega_3(x, y)\mathbf{k}$$

where $\omega_1, \omega_2, \omega_3$ are real valued 2-forms on \mathbb{H} .

$$\begin{aligned}\omega_1(x, y) &= x_0 y_1 - x_1 y_0 + x_2 y_3 - x_3 y_2 \\ \omega_2(x, y) &= x_0 y_2 - x_2 y_0 + x_3 y_1 - x_1 y_3 \\ \omega_3(x, y) &= x_0 y_3 - x_3 y_0 + x_1 y_2 - x_2 y_1.\end{aligned}$$

$\omega_1, \omega_2, \omega_3$ are symplectic forms on \mathbb{R}^4 , so that ω can be viewed as a \mathbb{R}^3 -valued symplectic form on \mathbb{R}^4 given by

$$\omega(x, y) = \omega_1(x, y)e'_1 + \omega_2(x, y)e'_2 + \omega_3(x, y)e'_3$$

where (e'_1, e'_2, e'_3) is the canonical basis of \mathbb{R}^3 corresponding to the basis $(\mathbf{i}, \mathbf{j}, \mathbf{k})$ of $\mathfrak{Im}(\mathbb{H})$. In the following we consider the quaternionic Heisenberg group $N_\omega = \mathbb{R}^3 \times_\omega \mathbb{R}^4$ associated to the symplectic form ω defined above. Note that N_ω is topologically $\mathbb{R}^3 \times \mathbb{R}^4 \cong \mathbb{R}^7$.

3.1.2 Heisenberg group associated to ω

Let (t, x) be an element of $(\mathbb{R}^3 \times \mathbb{R}^4)$, where:

$$t \in \mathbb{R}^3, t = t_1e'_1 + t_2e'_2 + t_3e'_3$$

$$x \in \mathbb{R}^4, x = x_0e_0 + x_1e_1 + x_2e_2 + x_3e_3.$$

We define the multiplication law \cdot_ω in $\mathbb{R}^3 \times \mathbb{R}^4$ by

$$(t, x) \cdot_\omega (t', x') = (t + t' + \omega(x, x'), x + x').$$

It is easily verified that $(\mathbb{R}^3 \times \mathbb{R}^4, \cdot_\omega)$ is the Heisenberg group associated to the form ω . A matrix representation of the Heisenberg group element (t, x) is :

$$\zeta(t, x) = \begin{pmatrix} 1 & 0 & 0 & -x_1 & x_0 & -x_3 & x_2 & t_1 \\ 0 & 1 & 0 & -x_2 & x_3 & x_0 & -x_1 & t_2 \\ 0 & 0 & 1 & -x_3 & -x_2 & x_1 & x_0 & t_3 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & x_0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & x_1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & x_2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & x_3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

It is easily checked that

$$\zeta(t, x)\zeta(t', y) = \zeta(t + t' + \omega(x, y), x + y)$$

and $\zeta(t, x)$ is an invertible matrix with inverse $\zeta(-t, -x)$. Hence $N_\omega = \mathbb{R}^3 \times_\omega \mathbb{R}^4$ may be viewed as a Lie subgroup of the affine group of \mathbb{R}^7 :

$$N_\omega \subset \text{Aff}(\mathbb{R}^7) \subset \text{GL}(\mathbb{R}^8).$$

3.1.3 Lie algebra of the Heisenberg group

One can determine in the usual manner the infinitesimal generators which characterize the Lie algebra of the Heisenberg group. We obtain a Heisenberg algebra \mathfrak{h} of dimension 7 generated by the following vector fields:

$$F_0 = \frac{\partial}{\partial x_0} - x_1 \frac{\partial}{\partial t_1} - x_2 \frac{\partial}{\partial t_2} - x_3 \frac{\partial}{\partial t_3} \quad (23)$$

$$F_1 = \frac{\partial}{\partial x_1} + x_0 \frac{\partial}{\partial t_1} + x_3 \frac{\partial}{\partial t_2} - x_2 \frac{\partial}{\partial t_3} \quad (24)$$

$$F_2 = \frac{\partial}{\partial x_2} - x_3 \frac{\partial}{\partial t_1} + x_0 \frac{\partial}{\partial t_2} + x_1 \frac{\partial}{\partial t_3} \quad (25)$$

$$F_3 = \frac{\partial}{\partial x_3} + x_2 \frac{\partial}{\partial t_1} - x_1 \frac{\partial}{\partial t_2} + x_0 \frac{\partial}{\partial t_3} \quad (26)$$

and

$$T_1 = \frac{\partial}{\partial t_1}, \quad T_2 = \frac{\partial}{\partial t_2}, \quad T_3 = \frac{\partial}{\partial t_3}.$$

The commutation relations between the generators are:

$$\begin{aligned} [T_\lambda, T_\mu] &= 0 \quad (\lambda, \mu = 1, 2, 3), \\ [F_\alpha, T_\lambda] &= 0 \quad (\alpha = 0, \dots, 3; \quad \lambda = 1, 2, 3) \end{aligned}$$

and

$$[F_\alpha, F_\beta] = 2T_\gamma$$

where $(\alpha\beta\gamma)$ is a circular permutation of (123) . The center of the algebra \mathfrak{h} is the abelian ideal generated by T_1 , T_2 and T_3 . This algebra is a 2-step nilpotent Lie algebra.

4 The Landau operator $\mathcal{H}_{\vec{v}}$

In the universal enveloping algebra $\mathcal{U}(\mathfrak{h})$ of \mathfrak{h} , we define the following quadratic operator, which is the Laplace element in $\mathcal{U}(\mathfrak{h})$:

$$\mathcal{Q} = F_0^2 + F_1^2 + F_2^2 + F_3^2 + T_1^2 + T_2^2 + T_3^2.$$

Explicitly, \mathcal{Q} takes the following form:

$$\begin{aligned} \mathcal{Q} &= \left(\frac{\partial}{\partial x_0} - x_1 \frac{\partial}{\partial t_1} - x_2 \frac{\partial}{\partial t_2} - x_3 \frac{\partial}{\partial t_3} \right)^2 + \left(\frac{\partial}{\partial x_1} + x_0 \frac{\partial}{\partial t_1} + x_3 \frac{\partial}{\partial t_2} - x_2 \frac{\partial}{\partial t_3} \right)^2 \\ &+ \left(\frac{\partial}{\partial x_2} - x_3 \frac{\partial}{\partial t_1} + x_0 \frac{\partial}{\partial t_2} + x_1 \frac{\partial}{\partial t_3} \right)^2 + \left(\frac{\partial}{\partial x_3} + x_2 \frac{\partial}{\partial t_1} - x_1 \frac{\partial}{\partial t_2} + x_0 \frac{\partial}{\partial t_3} \right)^2 \\ &+ \frac{\partial^2}{\partial t_1^2} + \frac{\partial^2}{\partial t_2^2} + \frac{\partial^2}{\partial t_3^2} \end{aligned} \quad (27)$$

and may be written

$$\mathcal{Q} = \Delta_{sub}^{N_\omega} + \Delta^{\mathbb{R}^3}$$

where

$$\Delta^{\mathbb{R}^3} = \frac{\partial^2}{\partial t_1^2} + \frac{\partial^2}{\partial t_2^2} + \frac{\partial^2}{\partial t_3^2}$$

is the standard Laplacian on \mathbb{R}^3 and $\Delta_{sub}^{N_\omega}$ is the sub-Laplacian associated to the Heisenberg group $N_\omega = \mathbb{R}^3 \times_\omega \mathbb{R}^4$. The partial Fourier transform $\mathcal{F}(\mathcal{Q})$ of \mathcal{Q} on the t -variable is obtained by replacing $\frac{\partial}{\partial t_\lambda}$ by $i\nu_\lambda$, $\lambda = 1, 2, 3$, (where $\vec{\nu} = (\nu_1, \nu_2, \nu_3)$ is the dual variable of $\vec{t} = (t_1, t_2, t_3)$); this defines an operator denoted by $\mathcal{F}(\mathcal{Q})$ which has the form:

$$\begin{aligned} \mathcal{F}(\mathcal{Q}) &= \mathcal{F}(\Delta_{sub}^{N_\omega}) + \mathcal{F}(\Delta^{\mathbb{R}^3}) \\ &= \nu_1^2 + \nu_2^2 + \nu_3^2 \\ &+ \left(\frac{\partial}{\partial x_0} + i(-\nu_1 x_1 - \nu_2 x_2 - \nu_3 x_3) \right)^2 + \left(\frac{\partial}{\partial x_1} + i(\nu_1 x_0 - \nu_3 x_2 + \nu_2 x_3) \right)^2 \\ &+ \left(\frac{\partial}{\partial x_2} + i(\nu_2 x_0 + \nu_3 x_1 - \nu_1 x_3) \right)^2 + \left(\frac{\partial}{\partial x_3} + i(\nu_3 x_0 - \nu_2 x_1 + \nu_1 x_2) \right)^2 \end{aligned} \quad (28)$$

with

$$\mathcal{F}(\Delta^{\mathbb{R}^3}) = \nu_1^2 + \nu_2^2 + \nu_3^2 \quad (29)$$

and

$$\begin{aligned} \mathcal{F}(\Delta_{sub}^{N_\omega}) &= \\ &\left(\frac{\partial}{\partial x_0} + i(-\nu_1 x_1 - \nu_2 x_2 - \nu_3 x_3) \right)^2 + \left(\frac{\partial}{\partial x_1} + i(\nu_1 x_0 - \nu_3 x_2 + \nu_2 x_3) \right)^2 \\ &+ \left(\frac{\partial}{\partial x_2} + i(\nu_2 x_0 + \nu_3 x_1 - \nu_1 x_3) \right)^2 + \left(\frac{\partial}{\partial x_3} + i(\nu_3 x_0 - \nu_2 x_1 + \nu_1 x_2) \right)^2. \end{aligned} \quad (30)$$

By identifying ν_λ with B_λ , $\lambda = 1, 2, 3$ we have

$$\mathcal{H}_{\vec{\nu}} = -\mathcal{F}(\Delta_{sub}^{N_\omega}) \quad (31)$$

which is exactly the expression (22); then we can say that the variables (t_1, t_2, t_3) are the dual variables of the components of the magnetic field $\vec{B} = (B_1, B_2, B_3) = (\nu_1, \nu_2, \nu_3)$; and that the Landau operator is equal to minus the partial Fourier transform of the sub-laplacian associated to the Heisenberg group $\mathbb{R}^3 \times_\omega \mathbb{R}^4$.

$\mathcal{F}(\Delta^{\mathbb{R}^3})$ is the intensity of the magnetic field.

4.1 Angular momentum in \mathbb{R}^4

The momentum operator of a particle in \mathbb{R}^3 with coordinates (x_1, x_2, x_3) is defined by $\hat{p} = -i\hbar\widehat{\nabla}$ and its components are:

$$\hat{p}_x = -i\hbar\frac{\partial}{\partial x_1}, \quad \hat{p}_y = -i\hbar\frac{\partial}{\partial x_2}, \quad \hat{p}_z = -i\hbar\frac{\partial}{\partial x_3}.$$

The angular momentum operator \widehat{l} is defined by $\widehat{l} = \widehat{x} \wedge \widehat{p}$, where \widehat{x} is the position operator of the particle defined by its components $(\hat{x}_1, \hat{x}_2, \hat{x}_3)$. Thus $\widehat{x} \wedge \widehat{p}$ has components

$$\begin{aligned} \hat{l}_x &= -i\hbar \left(x_2 \frac{\partial}{\partial x_3} - x_3 \frac{\partial}{\partial x_2} \right) \\ \hat{l}_y &= -i\hbar \left(x_3 \frac{\partial}{\partial x_1} - x_1 \frac{\partial}{\partial x_3} \right) \\ \hat{l}_z &= -i\hbar \left(x_1 \frac{\partial}{\partial x_2} - x_2 \frac{\partial}{\partial x_1} \right) \end{aligned}$$

and these operators verify the commutation relations:

$$[\hat{l}_x, \hat{l}_y] = i\hat{l}_z, \quad [\hat{l}_y, \hat{l}_z] = i\hat{l}_x, \quad [\hat{l}_z, \hat{l}_x] = i\hat{l}_y.$$

In \mathbb{R}^4 the position operator is :

$$\widehat{x} = x_0 e_0 + x_1 e_1 + x_2 e_2 + x_3 e_3$$

and the nabla operator

$$\widehat{\partial}_x = \frac{\partial}{\partial x_0} e_0 + \frac{\partial}{\partial x_1} e_1 + \frac{\partial}{\partial x_2} e_2 + \frac{\partial}{\partial x_3} e_3.$$

Here \widehat{x} and $\widehat{\partial}_x$ are meant to be elements of the free module

$$\mathcal{A} \langle e_0, e_1, e_2, e_3 \rangle \cong \mathcal{A} \otimes_{\mathbb{R}} \mathbb{R} \langle e_0, e_1, e_2, e_3 \rangle$$

with basis (e_0, e_1, e_2, e_3) and coefficients in the Weyl algebra \mathcal{A} of linear operators on $C^\infty(\mathbb{R}^4; \mathbb{C})$ with generators: the identity map on $C^\infty(\mathbb{R}^4; \mathbb{C})$, multiplication by x_α , partial derivatives $\frac{\partial}{\partial x_\alpha}$ and relations: $[\frac{\partial}{\partial x_\alpha}, x_\beta] = \delta_{\alpha\beta}$, for $\alpha, \beta = 0, \dots, 3$, the other commutators being zero. We now consider the exterior product $\widehat{x} \wedge \widehat{\partial}_x$, belonging to the module $\Lambda^2(\mathcal{A} \langle e_0, e_1, e_2, e_3 \rangle)$ which is

contained in $\mathcal{A} \otimes_{\mathbb{R}} \Lambda^2(\mathbb{R}^4)$:

$$\begin{aligned}\widehat{x} \wedge \widehat{\partial}_x &= \left(x_0 \frac{\partial}{\partial x_1} - x_1 \frac{\partial}{\partial x_0} \right) e_0 \wedge e_1 + \left(x_2 \frac{\partial}{\partial x_3} - x_3 \frac{\partial}{\partial x_2} \right) e_2 \wedge e_3 + \\ &+ \left(x_0 \frac{\partial}{\partial x_2} - x_2 \frac{\partial}{\partial x_0} \right) e_0 \wedge e_2 + \left(x_3 \frac{\partial}{\partial x_1} - x_1 \frac{\partial}{\partial x_3} \right) e_3 \wedge e_1 + \\ &+ \left(x_0 \frac{\partial}{\partial x_3} - x_3 \frac{\partial}{\partial x_0} \right) e_0 \wedge e_3 + \left(x_1 \frac{\partial}{\partial x_2} - x_2 \frac{\partial}{\partial x_1} \right) e_1 \wedge e_2.\end{aligned}$$

In the orthonormal direct basis (e_0, e_1, e_2, e_3) of \mathbb{R}^4 , the Hodge \star -operator is a linear involution of $\Lambda^2(\mathbb{R}^4)$ (i.e. $\star^2 = id_{\mathbb{R}^4}$) given by

$$\begin{aligned}\star e_0 \wedge e_1 &= e_2 \wedge e_3 \\ \star e_0 \wedge e_2 &= e_3 \wedge e_1 \\ \star e_0 \wedge e_3 &= e_1 \wedge e_2.\end{aligned}$$

We have:

$$\begin{aligned}\widehat{x} \wedge \widehat{\partial}_x + \star \left(\widehat{x} \wedge \widehat{\partial}_x \right) &= \\ &= \left(x_0 \frac{\partial}{\partial x_1} - x_1 \frac{\partial}{\partial x_0} + x_2 \frac{\partial}{\partial x_3} - x_3 \frac{\partial}{\partial x_2} \right) (e_0 \wedge e_1 + e_2 \wedge e_3) + \\ &+ \left(x_0 \frac{\partial}{\partial x_2} - x_2 \frac{\partial}{\partial x_0} + x_3 \frac{\partial}{\partial x_1} - x_1 \frac{\partial}{\partial x_3} \right) (e_0 \wedge e_2 + e_3 \wedge e_1) + \\ &+ \left(x_0 \frac{\partial}{\partial x_3} - x_3 \frac{\partial}{\partial x_0} + x_1 \frac{\partial}{\partial x_2} - x_2 \frac{\partial}{\partial x_1} \right) (e_0 \wedge e_3 + e_1 \wedge e_2). \quad (32)\end{aligned}$$

The space of self-adjoint elements of $\Lambda^2(\mathbb{R}^4)$ is a 3-dimensional vector space with basis

$$(e_0 \wedge e_1 + e_2 \wedge e_3, e_0 \wedge e_2 + e_3 \wedge e_1, e_0 \wedge e_3 + e_1 \wedge e_2),$$

which we identify with the basis (e'_1, e'_2, e'_3) of \mathbb{R}^3 , so that the self-adjoint elements of $\Lambda^2(\mathcal{A}\langle e_0, e_1, e_2, e_3 \rangle)$ can be viewed as elements of $\mathcal{A}\langle e'_1, e'_2, e'_3 \rangle \cong \mathcal{A} \otimes_{\mathbb{R}} \mathbb{R}\langle e'_1, e'_2, e'_3 \rangle$.

We define the angular momentum operator of a particle in \mathbb{R}^4 as the element \widehat{l} of $\mathcal{A} \otimes_{\mathbb{R}} \mathbb{C}\langle e'_1, e'_2, e'_3 \rangle$ given by

$$\widehat{l} = -i\hbar \left(\widehat{x} \wedge \widehat{\partial}_x + \star \left(\widehat{x} \wedge \widehat{\partial}_x \right) \right).$$

The components of \widehat{l} are :

$$\begin{aligned} l_1 &= -i\hbar \left(x_0 \frac{\partial}{\partial x_1} - x_1 \frac{\partial}{\partial x_0} + x_2 \frac{\partial}{\partial x_3} - x_3 \frac{\partial}{\partial x_2} \right) \\ l_2 &= -i\hbar \left(x_0 \frac{\partial}{\partial x_2} - x_2 \frac{\partial}{\partial x_0} + x_3 \frac{\partial}{\partial x_1} - x_1 \frac{\partial}{\partial x_3} \right) \\ l_3 &= -i\hbar \left(x_0 \frac{\partial}{\partial x_3} - x_3 \frac{\partial}{\partial x_0} + x_1 \frac{\partial}{\partial x_2} - x_2 \frac{\partial}{\partial x_1} \right) \end{aligned}$$

The components of \widehat{l} in terms of the complex variables $z'_1 = x_0 + ix_1$ and $z'_2 = x_2 + ix_3$ are :

$$\begin{aligned} l_1 &= \hbar \left(z'_1 \frac{\partial}{\partial z'_1} - \bar{z}'_1 \frac{\partial}{\partial \bar{z}'_1} + z'_2 \frac{\partial}{\partial z'_2} - \bar{z}'_2 \frac{\partial}{\partial \bar{z}'_2} \right) \\ l_2 &= -i\hbar \left(\bar{z}'_1 \frac{\partial}{\partial z'_2} - z'_2 \frac{\partial}{\partial \bar{z}'_1} - z'_2 \frac{\partial}{\partial \bar{z}'_1} + z'_1 \frac{\partial}{\partial \bar{z}'_2} \right) \\ l_3 &= \hbar \left(\bar{z}'_1 \frac{\partial}{\partial z'_2} - \bar{z}'_2 \frac{\partial}{\partial z'_1} - z'_1 \frac{\partial}{\partial \bar{z}'_2} + z'_2 \frac{\partial}{\partial \bar{z}'_1} \right). \end{aligned}$$

The expressions (22) and (31) can be written in terms of the angular momentum as follows

$$\mathcal{H}_{\vec{\nu}} = -\Delta + 2 \langle \vec{\nu}, \vec{l} \rangle + \|\vec{\nu}\|^2 \|x\|_{\mathbb{R}^4} \quad (33)$$

4.2 Complex form of $\mathcal{H}_{\vec{\nu}} = \mathcal{H}_{\vec{B}}$

Since \mathbb{R}^4 is isomorphic to \mathbb{C}^2 , we define the complex coordinates on \mathbb{R}^4 by $z'_1 = x_0 + ix_1$ and $z'_2 = x_2 + ix_3$; we have:

$$\begin{aligned} \frac{\partial}{\partial z'_1} &= \frac{1}{2} \left(\frac{\partial}{\partial x_0} - i \frac{\partial}{\partial x_1} \right), & \frac{\partial}{\partial \bar{z}'_1} &= \frac{1}{2} \left(\frac{\partial}{\partial x_0} + i \frac{\partial}{\partial x_1} \right) \\ \frac{\partial}{\partial z'_2} &= \frac{1}{2} \left(\frac{\partial}{\partial x_2} - i \frac{\partial}{\partial x_3} \right), & \frac{\partial}{\partial \bar{z}'_2} &= \frac{1}{2} \left(\frac{\partial}{\partial x_2} + i \frac{\partial}{\partial x_3} \right) \end{aligned}$$

then

$$\begin{aligned} \mathcal{H}_{\vec{\nu}} &= -4 \frac{\partial^2}{\partial z'_1 \partial \bar{z}'_1} - 4 \frac{\partial^2}{\partial z'_2 \partial \bar{z}'_2} + 2\nu_1 \left(z'_1 \frac{\partial}{\partial z'_1} - \bar{z}'_1 \frac{\partial}{\partial \bar{z}'_1} + z'_2 \frac{\partial}{\partial z'_2} - \bar{z}'_2 \frac{\partial}{\partial \bar{z}'_2} \right) \\ &\quad - 2i\nu_2 \left(\bar{z}'_1 \frac{\partial}{\partial z'_2} - z'_2 \frac{\partial}{\partial \bar{z}'_1} - z'_2 \frac{\partial}{\partial \bar{z}'_1} + z'_1 \frac{\partial}{\partial \bar{z}'_2} \right) \\ &\quad + 2\nu_3 \left(\bar{z}'_1 \frac{\partial}{\partial z'_2} - \bar{z}'_2 \frac{\partial}{\partial z'_1} + z'_2 \frac{\partial}{\partial \bar{z}'_1} - z'_1 \frac{\partial}{\partial \bar{z}'_2} \right) + |\vec{\nu}|^2 (|z'_1|^2 + |z'_2|^2). \end{aligned}$$

By introducing the angular momentum \vec{l} , the precedent formula may be condensed into

$$\mathcal{H}_{\vec{\nu}} = -4 \frac{\partial^2}{\partial z'_1 \partial \bar{z}'_1} - 4 \frac{\partial^2}{\partial z'_2 \partial \bar{z}'_2} + 2 \langle \vec{\nu}, \vec{l} \rangle + |\vec{\nu}|^2 (|z'_1|^2 + |z'_2|^2).$$

5 Canonical Landau operator $\mathcal{H}_{\|\vec{\nu}\|}$

5.1 Canonical form of the matrix $\Omega_{\vec{\nu}}$

As we have seen above, for each uniform magnetic field $\vec{B} = \vec{\nu} = (\nu_1, \nu_2, \nu_3)$, we associate the linear operator $\Omega_{\vec{\nu}}$ acting on \mathbb{R}^4 , defined by its matrix in the canonical basis (e_0, e_1, e_2, e_3)

$$\Omega_{\vec{\nu}} = \begin{pmatrix} 0 & -\nu_1 & -\nu_2 & -\nu_3 \\ \nu_1 & 0 & -\nu_3 & \nu_2 \\ \nu_2 & \nu_3 & 0 & -\nu_1 \\ \nu_3 & -\nu_2 & \nu_1 & 0 \end{pmatrix}$$

We remark that:

$$\Omega_{\vec{\nu}}^2 = -\|\vec{\nu}\|^2 \mathbb{I}_4 \quad (34)$$

\mathbb{I}_4 is the identity matrix on \mathbb{R}^4 .

Since $\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k} = -1$, as particular solutions of the equation (34) we have:

$$\Omega_{\vec{\nu}} = \pm \|\vec{\nu}\| \mathbf{i} \quad (35)$$

$$\Omega_{\vec{\nu}} = \pm \|\vec{\nu}\| \mathbf{j} \quad (36)$$

$$\Omega_{\vec{\nu}} = \pm \|\vec{\nu}\| \mathbf{k} \quad (37)$$

We have three particular orthogonal complex structures ; these solutions have a physical interpretation, since $\mathfrak{Sm}\mathbb{H} \sim \mathbb{R}^3$, the equation (35) means that the magnetic field is taken along the x_1 axis, the equation (36) means that the magnetic field is taken along the x_2 axis and the equation (37) means that the magnetic field is taken along the x_3 axis. We will prove that, for any $\vec{\nu} \in \mathbb{R}^3$ there exist an orthogonal transformation which passes from $\Omega_{\vec{\nu}}$ to $\Omega_{\|\vec{\nu}\|} = \|\vec{\nu}\| \mathbf{i}$ or to $\Omega_{\|\vec{\nu}\|} = \|\vec{\nu}\| \mathbf{j}$ or to $\Omega_{\|\vec{\nu}\|} = \|\vec{\nu}\| \mathbf{k}$; $\mathbf{i}, \mathbf{j}, \mathbf{k}$ are represented by the matrices (14), (15) and (16).

Theorem 5.1. *For every $\vec{\nu} = (\nu_1, \nu_2, \nu_3) \in \mathbb{R}^3$, there exists $\mathcal{R} \in \text{SO}(4)$ such that*

$$\mathcal{R} \Omega_{\vec{\nu}} \mathcal{R}^{-1} = \Omega_{\|\vec{\nu}\|} \quad (38)$$

Proof. We give the proof for the case $\Omega_{\|\vec{v}\|} = \|\vec{v}\|\mathbf{i}$; the proof for the other cases is similar to this one. We denote as above (e_0, e_1, e_2, e_3) the canonical basis of oriented Euclidian space \mathbb{R}^4 and we identify a linear operator in \mathbb{R}^4 with its matrix in this basis.

1. If $\nu_2 = \nu_3 = 0$ then

$$\Omega_{\vec{v}} = \begin{pmatrix} 0 & -\nu_1 & 0 & 0 \\ \nu_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\nu_1 \\ 0 & 0 & \nu_1 & 0 \end{pmatrix}$$

and if $\nu_1 \geq 0$, we have $\Omega_{\|\vec{v}\|} = \Omega_{\vec{v}}$ and we can choose for instance $\mathcal{R} = \mathbb{I}_4$. If $\nu_1 < 0$, we have $\Omega_{\|\vec{v}\|} = -\Omega_{\vec{v}}$ and we can choose

$$\mathcal{R} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

This matrix evidently belongs to $SO(4)$.

2. Suppose $\nu_2^2 + \nu_3^2 \neq 0$ and set $\lambda = \sqrt{\nu_2^2 + \nu_3^2}$. Define

$$\begin{aligned} \varepsilon_0 &:= e_0 \\ \varepsilon_1 &:= -\frac{1}{\|\vec{v}\|} \Omega_{\vec{v}} e_0 \end{aligned}$$

hence

$$\begin{aligned} \Omega_{\vec{v}} \varepsilon_0 &= -\|\vec{v}\| \varepsilon_1 \\ \Omega_{\vec{v}} \varepsilon_1 &= -\frac{1}{\|\vec{v}\|} \Omega_{\vec{v}}^2 e_0 = \|\vec{v}\| \varepsilon_0. \end{aligned}$$

We seek now ε_2 as a linear combination of e_2 and e_3 , $\varepsilon_2 = ae_2 + be_3$, with the orthonormality conditions:

$$\|\varepsilon_2\|^2 = 1, \quad \langle \varepsilon_1, \varepsilon_2 \rangle_{\mathbb{R}^4} = 0.$$

We get thus $a = \pm \frac{\nu_3}{\lambda}$ and $b = \mp \frac{\nu_2}{\lambda}$, we write ε_2 in the form: $\varepsilon_2 = \frac{\nu_3}{\lambda} e_2 - \frac{\nu_2}{\lambda} e_3$. The vector ε_3 is defined as

$$\varepsilon_3 = -\frac{1}{\|\vec{v}\|} \Omega_{\vec{v}} \varepsilon_2,$$

that is $\varepsilon_3 = -\frac{\lambda}{\|\vec{v}\|}e_1 + \frac{\nu_1\nu_2}{\lambda\|\vec{v}\|}e_2 + \frac{\nu_1\nu_3}{\lambda\|\vec{v}\|}e_3$. We can easily verify that $\Omega_{\vec{v}} \varepsilon_2 = -\|\vec{v}\|\varepsilon_3$, $\Omega_{\vec{v}} \varepsilon_3 = \|\vec{v}\|\varepsilon_2$ and that the matrix

$$\mathcal{R} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{\nu_1}{\|\vec{v}\|} & 0 & -\frac{\lambda}{\|\vec{v}\|} \\ 0 & \frac{\nu_2}{\|\vec{v}\|} & \frac{\nu_3}{\lambda} & \frac{\nu_1\nu_2}{\lambda\|\vec{v}\|} \\ 0 & \frac{\nu_3}{\|\vec{v}\|} & -\frac{\nu_2}{\lambda} & \frac{\nu_1\nu_3}{\lambda\|\vec{v}\|} \end{pmatrix}$$

is orthogonal with determinant 1. \square

Remark 5.2. One can easily show that the centralizer of $\Omega_{\|\vec{v}\|}$ in $SO(4)$ is $U(2)$. For a fixed $\vec{v} \in \mathbb{R}^3 \setminus \{0\}$, the set of orthonormal changes of basis $\mathcal{B} = \{\mathcal{R} \in SO(4) \mid \mathcal{R} \Omega_{\|\vec{v}\|} \mathcal{R}^{-1} = \Omega_{\vec{v}}\}$ can be written $\mathcal{B} = \mathcal{R}_0 U(2)$ for some $\mathcal{R}_0 \in \mathcal{B}$. The quotient space $SO(4)/U(2)$ is the so called space of orthogonal complex structures on \mathbb{R}^4 and it is diffeomorphic to the projective line $C\mathbb{P}^1$ (i.e. the sphere S^2).

Remark 5.3. The construction of \mathcal{R} can be done in the following way. Let (e'_1, e'_2, e'_3) be the canonical basis of the Euclidian oriented space \mathbb{R}^3 endowed with the vector product \wedge . For $\vec{v} = (\nu_1, \nu_2, \nu_3) \in \mathbb{R}^3$ such that $\nu_2^2 + \nu_3^2 \neq 0$ (i.e. $\frac{1}{\|\vec{v}\|}\vec{v} \neq e'_1$) we set $\lambda = \sqrt{\nu_2^2 + \nu_3^2}$ and we define

$$\varepsilon'_1 = \frac{1}{\|\vec{v}\|} \vec{v}, \quad (39)$$

$$\varepsilon'_2 = \frac{1}{\|\varepsilon'_1 \wedge e_1\|} \varepsilon'_1 \wedge e'_1, \quad (40)$$

$$\varepsilon'_3 = \varepsilon'_1 \wedge \varepsilon'_2. \quad (41)$$

The relating matrix between the basis $(\varepsilon'_1, \varepsilon'_2, \varepsilon'_3)$ and (e'_1, e'_2, e'_3) is

$$\begin{pmatrix} \frac{\nu_1}{\|\vec{v}\|} & 0 & -\frac{\lambda}{\|\vec{v}\|} \\ \frac{\nu_2}{\|\vec{v}\|} & \frac{\nu_3}{\lambda} & \frac{\nu_1\nu_2}{\lambda\|\vec{v}\|} \\ \frac{\nu_3}{\|\vec{v}\|} & -\frac{\nu_2}{\lambda} & \frac{\nu_1\nu_3}{\lambda\|\vec{v}\|} \end{pmatrix}$$

so that $(\varepsilon'_1, \varepsilon'_2, \varepsilon'_3)$ is easily seen to be a direct orthonormal basis of \mathbb{R}^3 . We consider the embedding $\Phi : \mathbb{R}^3 \hookrightarrow \mathbb{R}^4$ given by $\Phi(x_1e'_1 + x_2e'_2 + x_3e'_3) = x_0e_0 + x_1e_1 + x_2e_2 + x_3e_3$. Thus $\mathcal{R}_{\vec{v}} := (e_0, \Phi(\varepsilon'_1), \Phi(\varepsilon'_2), \Phi(\varepsilon'_3))$ is a direct orthonormal basis of \mathbb{R}^4 which represents an element of the quotient space $SO(4)/U(2)$. Denote σ the coset of $U(2)$ in $SO(4)/U(2)$. When restricted to the punctured sphere $S^2 \setminus \{\varepsilon'_1\}$ of \mathbb{R}^3 i.e. to the set $\{\vec{v} \in \mathbb{R}^3 \mid \|\vec{v}\| = 1, \vec{v} \neq e'_1\}$, the assignment

$$\mathcal{R} : \begin{array}{ccc} S^2 \setminus \{\varepsilon'_1\} & \longrightarrow & SO(4)/U(2) \setminus \{\sigma\} \\ \vec{v} & \longmapsto & \mathcal{R}_{\vec{v}} \end{array}$$

is one-to-one and continuous, as we can see by the formulas (39), (40) and (41). Since $\mathcal{R}_{\varepsilon'_1}$ can be identified with σ , \mathcal{R} extends to an homeomorphism $\mathcal{R} : S^2 \rightarrow SO(4)/U(2)$. In other words: each direction $\vec{\nu} \in S^2$ defines an orthogonal complex structure on \mathbb{R}^4 to which will correspond a quaternionic Landau operator $\mathcal{H}_{\vec{\nu}}$.

5.2 canonical form of the Landau operator

Recall that by a suitable orthogonal change of basis $\mathcal{R} : (e_\alpha)_{\alpha=0}^3 \rightarrow (e'_\alpha)_{\alpha=0}^3$ of \mathbb{R}^4 , we have

$$\mathcal{R}\Omega_{\vec{\nu}}\mathcal{R}^{-1} = \begin{pmatrix} 0 & -\|\vec{\nu}\| & 0 & 0 \\ \|\vec{\nu}\| & 0 & 0 & 0 \\ 0 & 0 & 0 & -\|\vec{\nu}\| \\ 0 & 0 & \|\vec{\nu}\| & 0 \end{pmatrix} = \Omega_{\|\vec{\nu}\|} = \|\vec{\nu}\|\mathbf{i}$$

where \mathbf{i} is the element of the base of $\Im m\mathbb{H}$ represented by (14) and

$$\|\vec{\nu}\| = \nu_1^2 + \nu_2^2 + \nu_3^2.$$

Let (y_0, y_1, y_2, y_3) denote the new coordinates in the basis $(e'_\alpha)_{\alpha=0}^3$. We consider the connection defined by

$$A' = \begin{pmatrix} 0 & -\|\vec{\nu}\| & 0 & 0 \\ \|\vec{\nu}\| & 0 & 0 & 0 \\ 0 & 0 & 0 & -\|\vec{\nu}\| \\ 0 & 0 & \|\vec{\nu}\| & 0 \end{pmatrix} \begin{pmatrix} y_0 \\ y_1 \\ y_2 \\ y_3 \end{pmatrix}$$

that is

$$A' = \begin{pmatrix} -\|\vec{\nu}\|y_1 \\ \|\vec{\nu}\|y_0 \\ -\|\vec{\nu}\|y_2 \\ \|\vec{\nu}\|y_3 \end{pmatrix} = \begin{pmatrix} A'_0 \\ A'_1 \\ A'_2 \\ A'_3 \end{pmatrix}$$

and the covariant derivative is :

$$\nabla_\alpha^{A'} = \frac{\partial}{\partial y_\alpha} + iA'_\alpha(y).$$

The corresponding Hamilton operator is

$$\begin{aligned} \mathcal{H}_{\|\vec{\nu}\|} &= -\left(\frac{\partial}{\partial y_0} - i\|\vec{\nu}\|y_1\right)^2 - \left(\frac{\partial}{\partial y_1} + i\|\vec{\nu}\|y_0\right)^2 - \left(\frac{\partial}{\partial y_2} - i\|\vec{\nu}\|y_3\right)^2 - \left(\frac{\partial}{\partial y_3} + i\|\vec{\nu}\|y_2\right)^2 \\ &= -\Delta_{(y_0, y_1, y_2, y_3)}^{\mathbb{R}^4} - 2i\|\vec{\nu}\| \left(y_0 \frac{\partial}{\partial y_1} - y_1 \frac{\partial}{\partial y_0} + y_2 \frac{\partial}{\partial y_3} - y_3 \frac{\partial}{\partial y_2} \right) \\ &\quad + \|\vec{\nu}\|^2 \left(y_0^2 + y_1^2 + y_2^2 + y_3^2 \right). \end{aligned}$$

One can show from Section 4 that the Landau operator $\mathcal{H}_{\|\vec{v}\|}$ may be defined as the partial Fourier Transform of the sub-Laplacian associated to the Heisenberg group $\mathbb{R} \times \mathbb{R}^4$; the Lie algebra \mathfrak{h}' of this group is generated by the following vector fields:

$$\begin{aligned} F'_0 &= \frac{\partial}{\partial y_0} - y_1 \frac{\partial}{\partial t'} , & F'_1 &= \frac{\partial}{\partial y_1} + y_0 \frac{\partial}{\partial t'} , \\ F'_2 &= \frac{\partial}{\partial y_2} - y_3 \frac{\partial}{\partial t'} , & F'_3 &= \frac{\partial}{\partial y_3} + y_2 \frac{\partial}{\partial t'} \end{aligned}$$

and

$$T' = \frac{\partial}{\partial t'} .$$

The quotient of \mathfrak{h} by the ideal generated by T_2, T_3 is none other than the Heisenberg algebra \mathfrak{h}' of dimension 5.

The partial Fourier transform with respect to the variable t is $i\vartheta$, ϑ is identified with $\|\vec{v}\|$.

5.3 Complex form of $\mathcal{H}_{\|\vec{v}\|}$

We define the new complex coordinates on \mathbb{R}^4 by $z_1 = y_0 + iy_1$ and $z_2 = y_2 + iy_3$. We get,

$$\begin{aligned} \mathcal{H}_{\|\vec{v}\|} &= -4 \left\{ \frac{\partial}{\partial z_1} \frac{\partial}{\partial \bar{z}_1} + \frac{\partial}{\partial z_2} \frac{\partial}{\partial \bar{z}_2} + \frac{\|\vec{v}\|}{2} \left[\left(z_1 \frac{\partial}{\partial z_1} - \bar{z}_1 \frac{\partial}{\partial \bar{z}_1} \right) \right. \right. \\ &\quad \left. \left. + \left(z_2 \frac{\partial}{\partial z_2} - \bar{z}_2 \frac{\partial}{\partial \bar{z}_2} \right) \right] - \left(\frac{\|\vec{v}\|}{2} \right)^2 \left(|z_1|^2 + |z_2|^2 \right) \right\}. \end{aligned} \quad (42)$$

The precedent expression (42) is much simpler to study than the expression (34).

From the second quantization formalism [7], we define the annihilation operators in the following manner:

$$a_1 = \frac{\partial}{\partial \bar{z}_1} + \frac{\|\vec{v}\|}{2} z_1, \quad a_2 = \frac{\partial}{\partial \bar{z}_2} + \frac{\|\vec{v}\|}{2} z_2$$

and the corresponding creation operators are

$$a_1^\dagger = -\frac{\partial}{\partial z_1} + \frac{\|\vec{v}\|}{2} \bar{z}_1, \quad a_2^\dagger = -\frac{\partial}{\partial z_2} + \frac{\|\vec{v}\|}{2} \bar{z}_2.$$

They satisfy the following commutation relations:

$$\begin{aligned} [a_i, a_j] &= [a_i^\dagger, a_j^\dagger] = 0 \\ [a_i, a_j^\dagger] &= \|\vec{v}\| \delta_{ij} \end{aligned}$$

A straightforward calculation gives

$$a_1^\dagger a_1 = -\frac{\partial}{\partial \bar{z}_1} \frac{\partial}{\partial z_1} + \frac{\|\vec{\nu}\|}{2} \left(z_1 \frac{\partial}{\partial z_1} - \bar{z}_1 \frac{\partial}{\partial \bar{z}_1} \right) + \left(\frac{\|\vec{\nu}\|}{2} \right)^2 |z_1|^2 - \frac{\|\vec{\nu}\|}{2}$$

$$a_2^\dagger a_2 = -\frac{\partial}{\partial \bar{z}_2} \frac{\partial}{\partial z_2} + \frac{\|\vec{\nu}\|}{2} \left(z_2 \frac{\partial}{\partial z_2} - \bar{z}_2 \frac{\partial}{\partial \bar{z}_2} \right) + \left(\frac{\|\vec{\nu}\|}{2} \right)^2 |z_2|^2 - \frac{\|\vec{\nu}\|}{2}$$

We have,

$$\mathcal{H}_{\|\vec{\nu}\|} = 4 \left(a_1^\dagger a_1 + a_2^\dagger a_2 \right) + 4\|\vec{\nu}\|$$

which is the Hamiltonian of two superposed uncoupled harmonic oscillators. The spectral analysis of this operator is well known [2] and references there in.

6 Symmetry group of the Landau operator

Here we examine a group theoretical aspect of the quantum system in a magnetic field. It is known that the translation symmetry group becomes noncommutative when a uniform magnetic field is introduced into the Euclidean space.

Theorem 6.1. *Let \mathcal{T}_a be the operator defined by*

$$(\mathcal{T}_a f)(x) = e^{i\langle A(a), x \rangle} f(x + a)$$

where $\langle \cdot, \cdot \rangle$ denotes the scalar product in \mathbb{R}^4 . Then

$$\mathcal{T}_a \circ \mathcal{H}_{\vec{\nu}} = \mathcal{H}_{\vec{\nu}} \circ \mathcal{T}_a.$$

Proof. We set, for every $x \in \mathbb{R}^4$, $g(x) = e^{i\langle A(a), x \rangle} f(x + a)$. Thus for $\alpha = 0, \dots, 3$, we have the following expressions:

$$\begin{aligned} \frac{\partial}{\partial x_\alpha} g(x) &= \frac{\partial}{\partial x_\alpha} (e^{i\langle A(a), x \rangle} f(x + a)) \\ &= iA_\alpha(a) e^{i\langle A(a), x \rangle} f(x + a) + e^{i\langle A(a), x \rangle} \frac{\partial f}{\partial x_\alpha}(x + a) \end{aligned}$$

$$\begin{aligned} \frac{\partial^2}{\partial x_\alpha^2} g(x) &= e^{i\langle A(a), x \rangle} \left\{ -A_\alpha^2(a) f(x + a) \right. \\ &\quad \left. + 2iA_\alpha(a) \frac{\partial f}{\partial x_\alpha}(x + a) + \frac{\partial^2 f}{\partial x_\alpha^2}(x + a) \right\} \end{aligned}$$

$$\begin{aligned}
\Delta^{\mathbb{R}^4} g(x) &= e^{i\langle A(a), a \rangle} \left\{ - \sum_{\alpha=0}^3 A_\alpha^2(a) f(x+a) \right. \\
&\quad \left. + 2i \sum_{\alpha=0}^3 A_\alpha(a) \frac{\partial f}{\partial x_\alpha}(x+a) + \Delta^{\mathbb{R}^4} f(x+a) \right\} \\
2i \sum_{\alpha=0}^3 A_\alpha(x) \frac{\partial}{\partial x_\alpha} g(x) &= 2i e^{i\langle A(a), x \rangle} \sum_{\alpha=0}^3 A_\alpha(x) \left\{ i A_\alpha(a) f(x+a) \right. \\
&\quad \left. + \frac{\partial f}{\partial x_\alpha}(x+a) \right\} \\
&= e^{i\langle A(a), x \rangle} \left\{ - 2 \sum_{\alpha=0}^3 A_\alpha(x) A_\alpha(a) f(x+a) \right. \\
&\quad \left. + 2i \sum_{\alpha=0}^3 A_\alpha(x) \frac{\partial f}{\partial x_\alpha}(x+a) \right\} \\
-\|\vec{\nu}\|_{\mathbb{R}^3}^2 \|x\|_{\mathbb{R}^4}^2 g(x) &= -\|\vec{\nu}\|_{\mathbb{R}^3}^2 \|x\|_{\mathbb{R}^4}^2 e^{i\langle A(a), x \rangle} f(x+a) \\
&= - \left(\sum_{\alpha=0}^3 A_\alpha^2(x) \right) e^{i\langle A(a), x \rangle} f(x+a)
\end{aligned}$$

so that

$$\begin{aligned}
(\mathcal{H}_{\vec{\nu}} \circ \mathcal{T}_a)(f)(x) &= (\mathcal{H}_{\vec{\nu}} g)(x) \\
&= e^{i\langle A(a), x \rangle} \left\{ \Delta^{\mathbb{R}^4} + 2i \sum_{\alpha=0}^3 A_\alpha(x+a) \frac{\partial}{\partial x_\alpha} \right. \\
&\quad \left. - \sum_{\alpha=0}^3 (A_\alpha^2(a) + 2A_\alpha(x)A_\alpha(a) + A_\alpha^2(x)) \right\} f(x+a).
\end{aligned}$$

Since

$$\begin{aligned}
\sum_{\alpha=0}^3 (A_\alpha^2(a) + 2A_\alpha(x)A_\alpha(a) + A_\alpha^2(x)) &= \sum_{\alpha=0}^3 A_\alpha^2(x+a) \\
&= \|\vec{\nu}\|_{\mathbb{R}^3}^2 \|x+a\|_{\mathbb{R}^4}^2,
\end{aligned}$$

we get finally

$$\begin{aligned}
(\mathcal{H}_{\vec{\nu}} \circ \mathcal{T}_a)(f)(x) &= e^{i\langle A(a), x \rangle} \left\{ \Delta^{\mathbb{R}^4} + 2i \sum_{\alpha=0}^3 A_\alpha(x+a) \frac{\partial}{\partial x_\alpha} \right. \\
&\quad \left. - \|\vec{\nu}\|_{\mathbb{R}^3}^2 \|x+a\|_{\mathbb{R}^4}^2 \right\} f(x+a) \\
&= (\mathcal{T}_a \circ \mathcal{H}_{\vec{\nu}})(f)(x).
\end{aligned}$$

□

Remark. For $\alpha = 0, \dots, 3$, we have

$$\begin{aligned}
\nabla_\alpha(e^{i\langle A(a), x \rangle} f(x+a)) &= \left(\frac{\partial}{\partial x_\alpha} + iA_\alpha(x) \right) (e^{i\langle A(a), x \rangle} f(x+a)) \\
&= e^{i\langle A(a), x \rangle} \left\{ \frac{\partial f}{\partial x_\alpha}(x+a) + iA_\alpha(a)f(x+a) \right. \\
&\quad \left. + iA_\alpha(x)f(x+a) \right\} \\
&= e^{i\langle A(a), x \rangle} \left\{ \frac{\partial f}{\partial x_\alpha}(x+a) + iA_\alpha(x+a)f(x+a) \right\} \\
&= e^{i\langle A(a), x \rangle} \left\{ \frac{\partial}{\partial x_\alpha} + iA_\alpha(x+a) \right\} f(x+a)
\end{aligned}$$

that is

$$\nabla_\alpha \circ \mathcal{T}_a(f) = \mathcal{T}_a \circ \nabla_\alpha(f).$$

Since $\mathcal{H}_{\vec{v}} = \sum_{\alpha=0}^3 \nabla_\alpha^2$, we obtain immediately

$$\mathcal{H}_{\vec{v}} \circ \mathcal{T}_a = \mathcal{T}_a \circ \mathcal{H}_{\vec{v}}.$$

It is to be noted that \mathcal{T}_a is a combination of a translation in the a -direction and a gauge transformation. We have unitary transformations, the set of these transformations form a noncommutative group, which is the magnetic translation group; the law of the group is:

$$\mathcal{T}_b \mathcal{T}_a f(x) = e^{i\langle A(a), b \rangle} e^{i\langle A(b+a), x \rangle} f(x+b+a)$$

where $a \in \mathbb{R}^4$ and $b \in \mathbb{R}^4$ are fixed vectors. It is also to be noted that the translation in the a -direction and the one in the b direction do not commute but satisfy

$$\mathcal{T}_b^{-1} \mathcal{T}_a^{-1} \mathcal{T}_b \mathcal{T}_a = e^{2i\langle A(a), b \rangle}.$$

The precedent relation means that the translation symmetry group becomes noncommutative when a uniform magnetic field is introduced into the Euclidean space.

7 Concluding remarks and comments

In the preceding sections, we have derived the Landau operator from a Lie group theoretical approach. We have shown that there exists a canonical

form of this operator which is easier to handle. An interesting investigation involving theta functions, which describes a charged particle moving in a uniform magnetic field on a lattice of \mathbb{R}^4 , was done in [4]. Other connections with this work via the magnetic translation group can be found in [6] and [7] and references therein. Tanimura [6] studied the magnetic translation groups in a n -dimensional torus and their representations. Brown [7] found that the translation symmetry of an electron in a lattice in a uniform magnetic field is noncommutative and that the quantum system obeys a projective representation of the translation group.

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