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Jean-Pierre Kahane

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# Winding numbers and Fourier series

Jean–Pierre Kahane

This is an expository talk on a topic of classical analysis, arising from the *BMO*–theory of topological degree (Brézis–Nirenberg 1995) [6]. We sketch the history of the subject and some of its recent developments.

## 1 The starting point

In October 1995 Haim Brézis visited Rutgers University and gave a lecture at the seminar of Israel Gelfand on his recent work with Louis Nirenberg on the extension of the notion of topological degree to a class of functions larger than the class of continuous functions, namely *VMO*, the class of functions with vanishing mean oscillation. Vanishing mean oscillation is expressed by the formula

$$\lim_{|B| \rightarrow 0} \frac{1}{|B|^2} \iint_{B \times B} |f(x) - f(y)| \, dx \, dy = 0$$

when  $f$  maps an open set  $\Omega \subset \mathbb{R}^m$  into  $\mathbb{R}^n$ ,  $B$  denotes a ball contained in  $\Omega$  and  $|B|$  its volume,  $dx = dx_1 \cdots dx_m$  and  $|z| = |z_1^2 + \cdots + z_n^2|^{1/2}$ . When  $f$  maps a manifold into a manifold, this definition should be translated accordingly.

Gelfand was eager to have examples, and Brézis had plenty of them. First,

$$W^{s,p}(\Omega) \subset VMO(\Omega)$$

when  $0 < s < 1$ ,  $1 < p < \infty$ ,  $sp = n$ , and  $f \in W^{s,p}(\Omega)$  means

$$\iint_{\Omega \times \Omega} \frac{|f(x) - f(y)|^p}{|x - y|^{n+sp}} \, dx \, dy < \infty.$$

Then, the Sobolev classes

$$H^{n/2}(\Omega) \subset VMO(\Omega)$$

since  $H^s = W^{s,2}$ . In particular, going to mapping of the  $n$ -sphere into itself,

$$H^1(S^2, S^2) \subset VMO(S^2),$$

the original motivation of Brézis [5]. In the same way,

$$H^{1/2}(S^1, S^1) \subset VMO(S^1).$$

Gelfand felt satisfied with  $S^1$ , but not with the definition of  $H^{1/2}$  by means of a double integral. Since  $f$  can be expressed as  $f(e^{it})$ ,  $t \in \mathbb{R}$ , and  $|f(e^{it})| = 1$ , how to write the definition in terms of the Fourier coefficients of  $f$ ,

$$a_n = \int f(e^{it}) e^{-int} \frac{dt}{2\pi}$$

(here and in the sequel we write  $\int$  instead of  $\int_0^{2\pi}$ ). Brézis had the answer immediately : the definition can be expressed as

$$\sum_{-\infty}^{\infty} |n| |a_n|^2 < \infty.$$

Then Gelfand asked another question. Can you express the topological degree by means of the  $a_n$  ? Brézis could not answer on the spot. He went home, made a little computation, and got the simple and beautiful formula

$$\deg f = \sum_{-\infty}^{\infty} n |a_n|^2.$$

Here  $\deg f$  denotes the usual topological degree if  $f$  is continuous, and the  $VMO$ -degree in general, that is,  $f \in H^{1/2}$ .

This was integrated in the article of Brézis and Nirenberg [6] and initiated a series of other questions. The first is already in [6]. Most of them can be found in the “mise au point” by Brézis in 2006 [3]. We shall see some answers and new questions.

**Question 1** [6].

What happens when  $f$  is continuous (then  $\deg f$  exists) and does not belong to  $H^{1/2}(S^1, S^1)$ , that is

$$\sum_{-\infty}^{\infty} |n| |a_n|^2 = \infty ?$$

Is there any summation process for the series

$$\sum_{-\infty}^{\infty} n |a_n|^2$$

such that  $\deg f$  can be computed in that way ?

The first answers were given by Jacob Korevaar in 1999 [10]. Korevaar considers two summation processes, namely

$$\lim_{n \rightarrow \infty} \sum_{-n}^n m |a_m|^2,$$

using symmetrical partial sums, and

$$\lim_{r \nearrow 1} \sum_{-\infty}^{\infty} r^{|m|} |a_m|^2,$$

the process of Abel–Poisson. The second is stronger than the first. Korevaar shows that they work when  $f$  has bounded variation,  $f \in C \cap BV$ , but that none of them work under the mere assumption  $f \in C$ . Actually they diverge for some  $f$ , and they converge to a value different from  $\deg f$  for some other  $f$ .

That led to another question.

**Question 2** [3].

Does  $\deg f$  depend on the absolute value of the Fourier coefficients  $a_n$  only ? Since the energy of  $f$  is defined by the  $|a_n|^2$ , and the topological degree of a mapping from  $S^1$  to  $S^1$  is nothing but the winding number, the question can be asked in a pleasant form : can we hear the winding number ?

The answer is negative. Jean Bourgain and Gaby Kozma were able to construct two functions belonging to  $C(S^1, S^1)$  with the same  $|a_n|$  and different degrees [2]. It is a very difficult construction.

**Question 3** [8, 9].

Let us return to the summation processes. Let us begin by the Abel–Poisson process, since it is stronger than most usual processes. If we assume that  $f$  satisfies a Hölder condition of order  $\alpha > 0$ , that is, in Zygmund’s notation,  $f \in \Lambda_\alpha(S^1, S^1)$  (Zygmund, as many authors, says “Lipschitz condition of order  $\alpha$ ”) [12], is it true that

$$\deg f = \lim_{r \uparrow 1} \sum_{-\infty}^{\infty} n r^{|n|} |a_n|^2 ?$$

The answer is positive for  $\alpha > \frac{1}{3}$  and negative for  $\alpha \leq \frac{1}{3}$ . This comes from a more precise statement, which involves the classes  $\lambda_\alpha^p$  of Zygmund ([12], p. 45), defined as

$$\lambda_\alpha^p = \left\{ g : \int |g(t+s) - g(s)|^p ds = o(t^\alpha), t \downarrow 0 \right\},$$

and a non-classical summation process, namely

$$\lim_{t \rightarrow 0} \sum_{-\infty}^{\infty} |a_n|^2 \frac{\sin nt}{t}.$$

This limit exists and is equal to  $\deg f$  when  $f \in C \cap \lambda_{1/3}^3(S^1, S^1)$  but there exists  $f \in \Lambda_{1/3}(S_1, S_1)$  such that it doesn't exist, or it exists and is different from  $\deg f$  [8].

The positive part of this statement is valid when  $C \cap \lambda_{1/3}^3(S^1, S^1)$  is replaced by  $W_{1/3}^3(S^1, S^1)$  [3]. We don't know if it is still valid under the assumption  $f \in VMO \cap \lambda_{1/3}^3(S^1, S^1)$ ; that would provide a common generalization to both statements.

**Question 4** [9].

We introduced three summation processes, and there are many others. It was a popular subject in the 1920's, and the best reference is Hardy's book of 1949, *Divergent Series* [7]. I returned to this topic in [9].

Hardy considers series of terms indexed by positive integers, say

$$\sum_1^{\infty} u_m.$$

In our situation

$$u_m = m(|a_m|^2 - |a_{-m}|^2).$$

Ordinary convergence to  $s$  means

$$(C) \quad s = \lim_{n \rightarrow \infty} \sum_1^n u_m$$

Cesàro summability of order  $k$  ( $k > -1$ ) to  $s$  means

$$(C, k) \quad s = \lim_{n \rightarrow \infty} \binom{n+k}{k}^{-1} \sum_m \binom{n+k-m}{k} u_m;$$

$(C, O)$  is the same as  $C$ , and  $(C, 1)$  deals with the arithmetic means of partial sums ; it is the process used by Fejér in Fourier series. The processes

$$(R, k) \quad s = \lim_{r \downarrow 0} \sum_1^{\infty} u_m \left( \frac{\sin mt}{mt} \right)^k$$

( $k$  being a positive integer) are called Riemann summation processes of order  $k$ . The original Riemann process is  $(R, 2)$  and it was used in order to study everywhere convergent trigonometric series. The process we just used is  $(R, 1)$ . Whenever we consider  $(R, k)$  we assume  $\sum_1^{\infty} |u_m| m^{-k} < \infty$ . The process

$$(A) \quad s = \lim_{n \uparrow 1} \sum_1^{\infty} r^m u_m$$

is the Abel, or Abel–Poisson, summation process.

It is classical, and easy to see, that  $(C, k')$  is stronger than  $(C, k)$  and  $(R, k')$  is stronger than  $(R, k)$  if  $k' > k$ , and that  $(A)$  is stronger than all  $(C, k)$  and stronger than  $(R, 2)$ . We write

$$\begin{aligned} k' > k &\implies (C, k) \longrightarrow (C, k') \\ k' > k &\implies (R, k) \longrightarrow (R, k') \\ (C, k) &\longrightarrow (A) \\ (R, 2) &\longrightarrow (A) \end{aligned}$$

Hardy mentioned some more subtle results :

$$\begin{aligned} (R, 2) &\longrightarrow (C, 2 + \delta) \quad (\delta > 0) \\ (R, 1) &\longrightarrow (C, 1 + \delta) \quad (\delta > 0) \end{aligned}$$

The first is due to Kuttner (1935) [11] and the second to Zygmund (1928) [13]. Moreover, Kuttner gave an example showing that

$$(R, 3) \not\rightarrow (A).$$

It is easy to see that

$$(R, 1) \not\rightarrow (C).$$

More interesting is the fact that

$$(R, 1) \not\rightarrow (C, 1).$$

In the opposite direction,

$$(C) \not\Rightarrow (R, 1).$$

These last results can be found at the end of [9].

**Question 5** [4, 1].

In September 2008, Brézis made a report on topological degree and asked the following question. Let  $f \in C(S^1, S^1)$ , with Fourier coefficients  $a_n (n \in \mathbb{Z})$ . Is it true that

$$\sum_{-\infty}^{\infty} |n| |a_n|^2 \leq |\deg f| + 2 \sum_1^{\infty} n |a_n|^2?$$

Since it is true when the first member is finite, the question can be written as

$$\sum_1^{\infty} n |a_n|^2 < \infty \implies \sum_{-\infty}^{\infty} |n| |a_n|^2 < \infty?$$

I could answer in a particular case ( $f \in \Lambda_\alpha, \alpha > 0$ ) and Bourgain in the general case.

The answer is positive. Moreover

$$\sum_{-\infty}^{\infty} |n| |a_n|^2 \leq 32 \sum_1^{\infty} n |a_n|^2.$$

The implication is valid in a more general situation. Let  $s > 0$ . Then

$$\sum_1^{\infty} n^{2s} |a_n|^2 < \infty \implies \sum_{-\infty}^{\infty} |n|^{2s} |a_n|^2 < \infty.$$

But, except when  $s = \frac{1}{2}$ , there is no constant  $C = C(s)$  such that

$$\sum_{-\infty}^{\infty} |n|^{2s} |a_n|^2 < C \sum_1^{\infty} n^{2s} |a_n|^2 \quad [1]$$

These results are valid when  $f$  is supposed to be *VMO* instead of continuous, but not when  $f$  is supposed to be bounded (counterexample : a Blaschke product).

The study of Fourier series of continuous or *VMO* unimodular functions is an interesting byproduct of the study of winding numbers.

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